A CONSTRAINT ON THE ANOMALOUS GREEN'S FUNCTION

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It is shown that the physical constraint of the Anomalous Green's function gives a natural pairing condition. The resulting self-consistency equation is directly related to the BCS gap equation. Both inhomogeneous and homogeneous systems are considered to illustrate the importance of the constraint. Especially we find weak localization correction to the phonon-mediated interaction.

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1. Introduction

Recently Green's function treatment of impurity effects on the superconductors was shown to differ from that of the BCS theory.[1,2,3] The electron density of states change caused by nonmagnetic impurities is overestimated in the Abrikosov and Gor'kov's (AG) theory,[4] which is inconsistent with Anderson's theorem.[5] (Details of the strong coupling theory of impure superconductors are discussed elsewhere.[6]) For magnetic impurity effects, Kim and Overhauser[3] proposed a BCS type theory with different predictions: (i) The initial slope of T_c decrease depends on the superconductor and is not the universal constant.[4] (ii) The T_c reduction by exchange scattering is partially suppressed by potential scattering when the overall mean free path is smaller than the coherence length. This compensation has been confirmed in several experiments.[7,8,9,10,11]

In this letter we show that the physical constraint of the Anomalous Green's function gives a natural pairing condition in Gor'kov's formalism.[12] The resulting self-consistency equation is nothing but another form of the BCS gap equation. Accordingly the above discrepancies are settled in favor of the BCS theory.

To illustrate, we first consider inhomogeneous systems with nonmagnetic and magnetic impurities. Weak localization correction to the phonon-mediated interaction is derived. We also discuss a homogeneous system.

2. Inhomogeneous System: Nonmagnetic Impurity Case

Let's consider an electron gas in the presence of the ordinary impurities. In the second-quantized representation, the Hamiltonian is

$$H = \int d\mathbf{r} \sum_{\alpha} \Psi^{\dagger}(\mathbf{r}\alpha)[\frac{\vec{p}^2}{2m} + U_o(\mathbf{r})]\Psi(\mathbf{r}\alpha) - \frac{1}{2}V \int d\mathbf{r} \sum_{\alpha\beta} \Psi^{\dagger}(\mathbf{r}\alpha)\Psi^{\dagger}(\mathbf{r}\beta)\Psi(\mathbf{r}\beta)\Psi(\mathbf{r}\alpha). \quad (1)$$

where $U_o(\mathbf{r}) = \sum_i U_o\delta(\mathbf{r} - \mathbf{R}_i)$ is the impurity potential. The field operator, $\Psi(\mathbf{r}\alpha)$, may be expanded by the exact scattered states basis set $\psi_n(\mathbf{r})$, that is,

$$\Psi(\mathbf{r}\alpha) = \sum_n \psi_n(\mathbf{r}) c_{n\alpha}, \tag{2}$$

where $c_{n\alpha}$ is a destruction operator of the electron. The Gor'kov's Anomalous Green's function $F(\mathbf{r},\mathbf{r}')$ is defined[13]

$$F(\mathbf{r},\mathbf{r}') = - < \Psi(\mathbf{r}\uparrow)\Psi(\mathbf{r}'\downarrow) > . \tag{3}$$

Accordingly, the pair potential is given

$$\Delta(\mathbf{r}) = VF(\mathbf{r},\mathbf{r}) = V\sum_\omega F(\mathbf{r},\mathbf{r},\omega). \tag{4}$$

The ω's are $\omega_n = (2n+1)\pi T$ for all integer n. Near the transition temperature, it has been believed that[14]

$$F(\mathbf{r},\mathbf{r}',\omega) = \int \Delta(\mathbf{l}) G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l}) G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l}) d\mathbf{l}, \tag{5}$$

where

$$G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l}) = \sum_n \frac{\psi_{n\uparrow}(\mathbf{r})\psi^*_{n\uparrow}(\mathbf{l})}{i\omega - \epsilon_n}, \tag{6}$$

and

$$G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l}) = \sum_{n'} \frac{\psi_{n'\downarrow}(\mathbf{r}')\psi^*_{n'\downarrow}(\mathbf{l})}{-i\omega - \epsilon_{n'}}. \tag{7}$$

Substituting Eq. (5) in Eq. (4), we find a well-known self-consistency equation

$$\Delta(\mathbf{r}) = VT\sum_\omega \int \Delta(\mathbf{l}) G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l}) G^{\downarrow}_{-\omega}(\mathbf{r},\mathbf{l}) d\mathbf{l}. \tag{8}$$

2.1 A Constraint on the Anomalous Green's function

Now we consider the physical constraint of the Anomalous Green's function. If we average over the impurity positions, the system becomes homogeneous. Consequently, the Anomalous Green's function should be a function of $\mathbf{r}-\mathbf{r}'$ after the impurity average, i.e.,

$$\overline{F(\mathbf{r},\mathbf{r}')}^{imp} = \overline{F(\mathbf{r}-\mathbf{r}')}^{imp}. \tag{9}$$

$\overline{}^{imp}$ means an average over impurity positions $\mathbf{R}_i$.

However, if we substitute Eqs. (6) and (7) into Eq. (5) we find extra pairings between $n\uparrow$ and $n'(\neq\bar{n})\downarrow$, which violate the physical constraint of the anomalous Green's function. $\bar{n}$ denotes the time reversed partner of the scattered state n. Using the scattered states $\psi_{n(\vec{k})}(\mathbf{r})$,

$$\psi_{n(\vec{k})}(\mathbf{r}) = e^{i\vec{k}\cdot\mathbf{r}} + \sum_{\vec{q}} \frac{U_o}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}}}[\sum_i e^{-i\vec{q}\cdot\mathbf{R}_i}]e^{i(\vec{k}+\vec{q})\cdot\mathbf{r}} + \cdots, \tag{10}$$

it is easy to show that

$$\begin{aligned}\overline{\psi_{n(\vec{k})\uparrow}(\mathbf{r})\psi_{n'(\vec{k}')\downarrow}(\mathbf{r}')}^{imp} &= e^{i(\vec{k}\cdot\mathbf{r}+\vec{k}'\cdot\mathbf{r}')}[1 + V_o^2\sum_{\vec{q},i}\frac{e^{i\vec{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})(\epsilon_{\vec{k}'} - \epsilon_{\vec{k}'-\vec{q}})} + \cdots] \\ &\neq f(\mathbf{r}-\mathbf{r}'),\end{aligned} \tag{11}$$

and

$$\begin{aligned}\overline{\psi_{n(\vec{k})\uparrow}(\mathbf{r})\psi_{\bar{n}(-\vec{k})\downarrow}(\mathbf{r}')}^{imp} &= e^{i\vec{k}\cdot(\mathbf{r}-\mathbf{r}')}[1 + V_o^2\sum_{\vec{q},i}\frac{e^{i\vec{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})^2} + \cdots] \\ &\equiv g(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{12}$$

Note that even the zeroth order term of extra pairings between $n\uparrow$ and $n'(\neq\bar{n})\downarrow$ is not a function of $\mathbf{r}-\mathbf{r}'$. ¿From Eqs. (5) and (11) one finds that

$$\begin{aligned}\overline{F(\mathbf{r},\mathbf{r}',\omega)}^{imp} &\sim \overline{\psi_{n(\vec{k})\uparrow}(\mathbf{r})\psi_{n'(\vec{k}')\downarrow}(\mathbf{r}')}^{imp} \\ &\neq \overline{F(\mathbf{r}-\mathbf{r}',\omega)}^{imp},\end{aligned} \tag{13}$$

in the presence of the extra pairing terms.

It is clear that only Anderson's pairing between $n\uparrow$ and $\bar{n}\downarrow$ is compatible with the homogeneity condition of the Anomalous Green's function after averaging out the impurity positions. Accordingly, the physical constraint of the Anomalous Green's function gives a natural pairing condition. By eliminating the extra pairing, the 'corrected' Anomalous Green's function is given

$$F(\mathbf{r},\mathbf{r}',\omega) = \int \Delta(\mathbf{l})\{G_\omega^\uparrow(\mathbf{r},\mathbf{l})G_{-\omega}^\downarrow(\mathbf{r}',\mathbf{l})\}_{p.p.}d\mathbf{l}, \tag{14}$$

where

$$\{G_\omega^\uparrow(\mathbf{r},\mathbf{l})G_{-\omega}^\downarrow(\mathbf{r}',\mathbf{l})\}_{p.p.} = \sum_n \frac{\psi_{n\uparrow}(\mathbf{r})\psi_{n\uparrow}^*(\mathbf{l})}{i\omega - \epsilon_n} \sum_{n'} \frac{\psi_{n'\downarrow}(\mathbf{r}')\psi_{n'\downarrow}^*(\mathbf{l})}{-i\omega - \epsilon_{n'}} \delta_{n'=\bar{n}}, \qquad (15)$$

and $p.p.$ means proper pairing constraint, which dictates pairing between $n \uparrow$ and $\bar{n} \downarrow$. Consequently, the self-consistency equation also needs a proper pairing constraint derived from the Anomalous Green's function. The resulting equation is

$$\Delta(\mathbf{r}) = VT\sum_\omega \int \Delta(\mathbf{l})\{G_\omega^\uparrow(\mathbf{r},\mathbf{l})G_{-\omega}^\downarrow(\mathbf{r},\mathbf{l})\}_{p.p}d\mathbf{l}. \qquad (16)$$

Let's compare Eq. (16) with the BCS gap equation,

$$\Delta_n = -\sum_{n'} V_{nn'} \frac{\Delta_{n'}}{2\epsilon_{n'}} tanh(\frac{\epsilon_{n'}}{2T}), \qquad (17)$$

where[3,15]

$$V_{nn'} = -V \int \psi_{n'}^*(\mathbf{r})\psi_{\bar{n}'}^*(\mathbf{r})\psi_{\bar{n}}(\mathbf{r})\psi_n(\mathbf{r})d\mathbf{r}. \qquad (18)$$

In fact, the relation between $\Delta(\mathbf{r})$ and Δ_n was obtained by Ma and Lee,[15] (i.e.),

$$\Delta_n = \int \psi_n^*(\mathbf{r})\psi_{\bar{\mathbf{n}}}^*(\mathbf{r})\boldsymbol{\Delta}(\mathbf{r})\mathbf{dr}. \qquad (19)$$

Substitution of Eq. (16) into Eq. (19) leads to the BCS gap equation Eq. (17), since[16]

$$T\sum_\omega \frac{1}{\omega^2 + \epsilon_n^2} = \frac{1}{2\epsilon_n} tanh(\frac{\epsilon_n}{2T}). \qquad (20)$$

Note that because Eqs. (16) and (17) are basically equivalent, they give the same result for the impurity effect on the superconductors. Up to the first order in the impurity concentration, one finds[1,3] $V_{nn'} \cong -V$ and the density of states change is $O(1/E_F\tau)$. E_F is the Fermi energy and τ is the scattering time. Therefore Anderson's theorem is obtained. In the AG theory[4] the density of states change is $O(1/\omega_D\tau)$,[1] which gives a large decrease of T_c. ω_D is the Debye energy. The Dyson expansion of Green's function in the presence of impurities can not distinguish two energy scales ω_D and E_F coming from the phonon-mediated interaction and the virtual impurity scattering.

2.2 Weak localization correction

We are now ready to consider weak localization[17,18,19] correction to the phonon-mediated interaction. Mott and Kaveh[18] showed that the wavefunctions for the weakly localized states may be written as a mixture of power-law and extended wavefunctions, of the form, for three dimensions,

$$\psi_n^{3d}(\mathbf{r}) = A_3\Psi_{ext}^{I} + B_3\frac{\Psi_{ext}^{II}}{r^2}, \tag{21}$$

and, for two dimensions,

$$\psi_n^{2d}(\mathbf{r}) = A_2\Psi_{ext}^{I} + B_2\frac{\Psi_{ext}^{II}}{r}, \tag{22}$$

where

$$A_3^2 = 1 - 4\pi B_3^2(\frac{1}{\ell} - \frac{1}{L}), \quad B_3 = \frac{3}{4\pi}\frac{1}{k_F^2\ell}, \tag{23}$$

and

$$A_2^2 = 1 - 2\pi B_2^2 ln(L/\ell), \quad B_2 = \frac{1}{\pi^2 k_F\ell}. \tag{24}$$

ℓ and L are the elastic and inelastic mean free paths.

We can find the effective interaction by substituting Eqs. (21) and (22) into Eq. (18). However, we must be careful in calculating contributions from the power-law wavefunction. Because we are concerned with the bound state of Cooper pairs in a BCS condensate, only the power-law wavefunctions within the the BCS coherence length ξ_o are relevant.[3] This is analogous to the insensitivity of the localized states with the change of the boundary conditions.[20] Accordingly, it is obtained

$$\begin{aligned} V_{nn'}^{3d} &\cong -V[A_3^4 + (\frac{\xi_o^3}{\Omega})8\pi A_3^2 B_3^2(\frac{1}{\ell} - \frac{1}{L}) + O(B_3^4\xi_o^6/\Omega^2)] \\ &\cong -V[1 - \frac{1}{(k_F\ell)^2}(1 - \frac{\ell}{L})], \end{aligned} \tag{25}$$

$$\begin{aligned} V_{nn'}^{2d} &\cong -V[A_2^4 + (\frac{\xi_o^2}{\Omega})4\pi A_2^2 B_2^2 ln(\ell/L) + O(B_2^4\xi_o^4/\Omega^2)] \\ &\cong -V[1 - \frac{2}{\pi k_F\ell} ln(L/\ell)]. \end{aligned} \tag{26}$$

Note that ξ_o^3/Ω and ξ_o^2/Ω in the second terms of the right hand side of Eqs. (25) and (26) denote the probabilities of finding the centers of the power-law wavefunctions within the Cooper pair radius ξ_o in 3 and 2 dimensions. Ω is the volume of the system. It is remarkable that the same correction occurs in the conductivity and the phonon-mediated interaction by weak localization. Consequently, one expects in 1 dimension

$$V_{nn'}^{1d} \cong -V[1 - \frac{1}{(\pi k_F a)^2}(L/\ell - 1)], \tag{27}$$

where a is the radius of the wire.

There are many experimental results which show the reduction of T_c caused by weak localization.[21,22,23] Previously, it was interpreted by the enhanced Coulomb repulsion.[24] However, Dynes et al.[22] found a decrease of the Coulomb pseudo-potential μ^* with decreasing T_c. We believe that this signals the importance of weak localization correction to the phonon-mediated interaction. Comparisons with the experiments will be published elsewhere.

3. Inhomogeneous System: Magnetic Impurity Case

We now consider the effect of magnetic impurities on the superconductivity. Magnetic interaction is given by

$$U_m(\mathbf{r}) = \sum_i J\vec{s}\cdot\vec{S}_i\delta(\mathbf{r}-\mathbf{R}_i), \tag{28}$$

where $\vec{s} = \frac{1}{2}\vec{\sigma}$. The three components of $\vec{\sigma}$ are the Pauli matrices. For convenience, we consider only the z-component of $U_m(\mathbf{r})$. The effect of each x and y-component is basically the same with that of z-component.[3] Notice that up and down spin electrons feel different potential because of the Pauli spin matrix, σ_z, i.e.,

$$[\frac{\vec{p}^2}{2m} + \sum_i \frac{J}{2}\overline{S}cos\theta\delta(\mathbf{r}-\mathbf{R}_i)]\psi_{n\uparrow}^z(\mathbf{r}) = E_{n\uparrow}\psi_{n\uparrow}^z(\mathbf{r}), \tag{29}$$

$$[\frac{\vec{p}^2}{2m} - \sum_i \frac{J}{2}\overline{S}cos\theta\delta(\mathbf{r}-\mathbf{R}_i)]\psi_{n\downarrow}^z(\mathbf{r}) = E_{n\downarrow}\psi_{n\downarrow}^z(\mathbf{r}), \tag{30}$$

where $\overline{S}=\sqrt{S(S+1)}$ and θ is the polar angle of the fixed local-spin $\vec{S}$. As a result, one finds,[25]

$$\int \psi^z_{n\uparrow}(\mathbf{r})^* \psi^z_{m\downarrow}(\mathbf{r}) d\mathbf{r} \neq \delta_{nm}. \tag{31}$$

The Anomalous Green's function, near the transition temperature, is given by[4]

$$F(\mathbf{r},\mathbf{r}',\omega) = \int \Delta(\mathbf{l}) G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l}) G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l}) d\mathbf{l}, \tag{32}$$

where

$$G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l}) = \sum_n \frac{\psi^z_{n\uparrow}(\mathbf{r}) {\psi^z}^*_{n\uparrow}(\mathbf{l})}{i\omega - \epsilon_n}, \tag{33}$$

and

$$G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l}) = \sum_{n'} \frac{\psi^z_{n'\downarrow}(\mathbf{r}') {\psi^z}^*_{n'\downarrow}(\mathbf{l})}{-i\omega - \epsilon_{n'}}. \tag{34}$$

As in Sec. 2.1, we check the physical constraint of the Anomalous Green's function

$$\overline{F(\mathbf{r},\mathbf{r}',\omega)}^{imp} = \overline{F(\mathbf{r}-\mathbf{r}',\omega)}^{imp}. \tag{35}$$

Using the scattered states

$$\psi^z_{n(\vec{k})\uparrow}(\mathbf{r}) = e^{i\vec{k}\cdot\mathbf{r}} + \sum_{\vec{q}} \frac{J\overline{S}cos\theta/2}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}}} [\sum_i e^{-i\vec{q}\cdot\mathbf{R}_i}] e^{i(\vec{k}+\vec{q})\cdot\mathbf{r}} + \cdots, \tag{36}$$

and

$$\psi^z_{n(\vec{k})\downarrow}(\mathbf{r}) = e^{i\vec{k}\cdot\mathbf{r}} - \sum_{\vec{q}} \frac{J\overline{S}cos\theta/2}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}}} [\sum_i e^{-i\vec{q}\cdot\mathbf{R}_i}] e^{i(\vec{k}+\vec{q})\cdot\mathbf{r}} + \cdots, \tag{37}$$

it is straight forward to show that

$$\begin{aligned} \overline{\psi^z_{n(\vec{k})\uparrow}(\mathbf{r}) \psi^z_{n'(\vec{k}')\downarrow}(\mathbf{r}')}^{imp} &= e^{i(\vec{k}\cdot\mathbf{r}+\vec{k}'\cdot\mathbf{r}')} [1 - (J\overline{S}cos\theta/2)^2 \sum_{\vec{q},i} \frac{e^{i\vec{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})(\epsilon_{\vec{k}'} - \epsilon_{\vec{k}'-\vec{q}})} + \cdots] \\ &\neq f(\mathbf{r}-\mathbf{r}'), \end{aligned} \tag{38}$$

and

$$\begin{aligned} \overline{\psi^z_{n(\vec{k})\uparrow}(\mathbf{r}) \psi^z_{\bar{n}(-\vec{k})\downarrow}(\mathbf{r}')}^{imp} &= e^{i\vec{k}\cdot(\mathbf{r}-\mathbf{r}')} [1 - (J\overline{S}cos\theta/2)^2 \sum_{\vec{q},i} \frac{e^{i\vec{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})^2} + \cdots] \\ &\equiv g(\mathbf{r}-\mathbf{r}'). \end{aligned} \tag{39}$$

The minus sign in the perturbation correction term of Eqs. (37) and (39) is the noteworthy feature of exchange scattering. ¿From Eqs. (35), (38) and (39), it is obvious that only the pairing between the scattered states $n(\vec{k})\uparrow$ and $\bar{n}(-\vec{k})\downarrow$ is allowed. Note that $n(\vec{k})\uparrow$ and $\bar{n}(-\vec{k})\downarrow$ are degenerate,

$$\overline{E_{n\uparrow}}^{imp} = \overline{E_{\bar{n}\downarrow}}^{imp}. \tag{40}$$

The resulting self-consistency equation is

$$\Delta(\mathbf{r}) = VT\sum_{\omega}\int \Delta(\mathbf{l})\{G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l})G^{\downarrow}_{-\omega}(\mathbf{r},\mathbf{l})\}_{p.p}d\mathbf{l}. \tag{41}$$

where

$$\{G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l})G^{\downarrow}_{-\omega}(\mathbf{r},\mathbf{l})\}_{p.p.} = \sum_{n}\frac{\psi^{z}_{n\uparrow}(\mathbf{r}){\psi^{z}}^{*}_{n\uparrow}(\mathbf{l})}{i\omega-\epsilon_n}\sum_{n'}\frac{\psi^{z}_{n'\downarrow}(\mathbf{r}){\psi^{z}}^{*}_{n'\downarrow}(\mathbf{l})}{-i\omega-\epsilon_{n'}}\delta_{n'=\bar{n}}. \tag{42}$$

Accordingly, we obtain the BCS gap equation

$$\Delta_n = -\sum_{n'}V_{nn'}\frac{\Delta_{n'}}{2\epsilon_{n'}}tanh(\frac{\epsilon_{n'}}{2T}), \tag{43}$$

where

$$V_{nn'} = -V\int {\psi^{z}}^{*}_{n'\uparrow}(\mathbf{r}){\psi^{z}}^{*}_{\bar{n}'\downarrow}(\mathbf{r})\psi^{z}_{\bar{n}\downarrow}(\mathbf{r})\psi^{z}_{n\uparrow}(\mathbf{r})d\mathbf{r}. \tag{44}$$

By taking an impurity average of Eq. (44), we find that

$$\overline{V_{nn'}}^{imp} = -V[1-2Z_i\sum_{\vec{q}}\frac{(J\overline{S}cos\theta/2)^2}{(\epsilon_{\vec{k}}-\epsilon_{\vec{k}+\vec{q}})^2}-2Z_i\sum_{\vec{q}}\frac{(J\overline{S}cos\theta/2)^2}{(\epsilon_{\vec{k}'}-\epsilon_{\vec{k}'+\vec{q}})^2}], \tag{45}$$

where Z_i is the number of the magnetic impurities. Note that this result is the same as the z-component in Eq. (55) of Kim and Overhauser.[3] In other words, eliminating the extra pairing terms from the AG theory[4] gives rise to Kim and Overhauser's result.[3]

4. Homogeneous System

Near the transition temperature, the Anomalous Green's function is given by

$$F(\mathbf{r},\mathbf{r}',\omega) = \int \Delta(\mathbf{l})G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l})G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l})d\mathbf{l}, \tag{46}$$

where

$$G_{\omega}^{\uparrow}(\mathbf{r},\mathbf{l}) = \sum_{\vec{k}} \frac{\phi_{\vec{k}\uparrow}(\mathbf{r})\phi_{\vec{k}\uparrow}^{*}(\mathbf{l})}{i\omega - \epsilon_{\vec{k}}}, \tag{47}$$

and

$$G_{-\omega}^{\downarrow}(\mathbf{r}',\mathbf{l}) = \sum_{\vec{k}'} \frac{\phi_{\vec{k}'\downarrow}(\mathbf{r}')\phi_{\vec{k}'\downarrow}^{*}(\mathbf{l})}{-i\omega - \epsilon_{\vec{k}'}}. \tag{48}$$

Note that $\phi_{\vec{k}}(\mathbf{r}) = e^{i\vec{k}\cdot\mathbf{r}}$. Accordingly, the self-consistency equation is

$$\begin{aligned}\Delta(\mathbf{r}) &= VT\sum_{\omega}\int \Delta(\mathbf{l})G_{\omega}^{\uparrow}(\mathbf{r},\mathbf{l})G_{-\omega}^{\downarrow}(\mathbf{r},\mathbf{l})d\mathbf{l} \\ &= \int K(\mathbf{r},\mathbf{l})\Delta(\mathbf{l})d\mathbf{l}. \end{aligned}\tag{49}$$

For a homogeneous system, Gor'kov[12] pointed out correctly that $F(\mathbf{r},\mathbf{r}')$ should depend only on $\mathbf{r}-\mathbf{r}'$, that is,

$$F(\mathbf{r},\mathbf{r}') = F(\mathbf{r}-\mathbf{r}'). \tag{50}$$

However, observe that

$$\phi_{\vec{k}\uparrow}(\mathbf{r})\phi_{\vec{k}'\downarrow}(\mathbf{r}') = e^{i(\vec{k}\cdot\mathbf{r}+\vec{k}'\cdot\mathbf{r}')}, \tag{51}$$

and

$$\phi_{\vec{k}\uparrow}(\mathbf{r})\phi_{-\vec{k}\downarrow}(\mathbf{r}') = e^{i\vec{k}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{52}$$

Eqs. (46) and (49) include the extra pairing terms between $\vec{k}\uparrow$ and $\vec{k}'\downarrow$ ($\neq -\vec{k}\downarrow$), which do not satisfy the homogeneity condition of Eq. (50). In this case, the self-consistency condition of the pair potential happens to eliminate the extra pairing in Eq. (49) because of the orthogonality of the wavefunctions.

However, it is important to eliminate the extra pairing in the Anomalous Green's function from the beginning. Note that the kernel $K(\mathbf{r},\mathbf{l})$ is not for the pairing between $\vec{k}\uparrow$ and $-\vec{k}\downarrow$, but for the pairing between the states which are the linear combination of the plane wave states $\phi_{\vec{k}}(\mathbf{r})$.[26] ¿From the Bogoliubov-de Gennes equations, it can be shown that the corresponding unitary transformation leads to the vacuum state where the pairing occurs between the states, which are the linear combination of the plane wave states.[26] The proper kernel for the BCS pairing between $\vec{k}\uparrow$ and $-\vec{k}\downarrow$ is

$$K^c(\mathbf{r},\mathbf{l}) = VT\sum_{\omega}\{G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l})G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l})\}_{p.p.}, \tag{53}$$

where

$$\{G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l})G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l})\}_{p.p.} = \sum_{\vec{k}}\frac{\phi_{\vec{k}\uparrow}(\mathbf{r})\phi^*_{\vec{k}\uparrow}(\mathbf{l})}{i\omega-\epsilon_{\vec{k}}}\sum_{\vec{k}'}\frac{\phi_{\vec{k}'\downarrow}(\mathbf{r}')\phi^*_{\vec{k}'\downarrow}(\mathbf{l})}{-i\omega-\epsilon_{\vec{k}'}}\delta_{\vec{k}'=-\vec{k}}, \tag{54}$$

Consequently, the 'corrected' Anomalous Green's function is

$$F(\mathbf{r},\mathbf{r}',\omega) = \int \Delta(\mathbf{l})\{G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l})G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l})\}_{p.p.}d\mathbf{l}. \tag{55}$$

In fact, Eq. (55) is not new. Another form of this equation was already obtained as shown[27,28]

$$F(\mathbf{r},\mathbf{r}') = \sum_{\vec{\mathbf{k}}}\frac{\mathbf{\Delta}_{\vec{\mathbf{k}}}}{\mathbf{2}\epsilon_{\vec{\mathbf{k}}}}\mathbf{tanh}(\frac{\epsilon_{\vec{\mathbf{k}}}}{\mathbf{2T}})\mathbf{exp}[\mathbf{i}\vec{\mathbf{k}}\cdot(\mathbf{r}-\mathbf{r}')]. \tag{56}$$

By using

$$\Delta_{\vec{k}} = \int \phi^*_{\vec{k}}(\mathbf{r})\phi^*_{-\vec{\mathbf{k}}}(\mathbf{r})\mathbf{\Delta}(\mathbf{r})\mathbf{dr}, \tag{57}$$

and

$$T\sum_{\omega}\frac{1}{\omega^2+\epsilon^2_{\vec{k}}} = \frac{1}{2\epsilon_{\vec{k}}}tanh(\frac{\epsilon_{\vec{k}}}{2T}), \tag{58}$$

Eq. (56) is transformed to

$$\begin{aligned} F(\mathbf{r},\mathbf{r}') &= \sum_{\omega}F(\mathbf{r},\mathbf{r}',\omega), \\ &= \sum_{\omega}\int \Delta(\mathbf{l})\{G^{\uparrow}_{\omega}(\mathbf{r},\mathbf{l})G^{\downarrow}_{-\omega}(\mathbf{r}',\mathbf{l})\}_{p.p.}d\mathbf{l}. \end{aligned} \tag{59}$$

Therefore, Eq. (55) is confirmed.

5. Conclusion

It is shown that the physical constraint of the Anomalous Green's function gives a natural pairing constraint. The resulting self-consistency equation is nothing but another form of the BCS gap equation. Anderson's pairing between the time reversed scattered state partners are obtained in the presence of the ordinary impurities. Weak localization correction to the phonon-mediated interaction is calculated. In the presence of the magnetic impurities, the degenerate partners are paired, which give rise to the result of Kim and Overhauser. In the case of a homogeneous system, the BCS pairing is required.

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