Loss network representation of Peierls contours

Roberto Fernández
Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina and Universidade de São Paulo

Pablo A. Ferrari
Universidade de São Paulo

Nancy L. Garcia
Universidade Estadual de Campinas

Abstract

We present a probabilistic approach for the study of systems with exclusions, in the regime traditionally studied via cluster-expansion methods. In this paper we focus on its application for the gases of Peierls contours found in the study of the Ising model at low temperatures, but most of the results are general. We realize the equilibrium measure as the invariant measure of a loss-network process whose existence is ensured by a subcriticality condition of a dominant branching process. In this regime, the approach yields, besides existence and uniqueness of the measure, properties such as exponential space convergence and mixing, and a central limit theorem. The loss network converges exponentially fast to the equilibrium measure, without metastable traps. This convergence is faster at low temperatures, where it leads to the proof of an asymptotic Poisson distribution of contours. Our results on the mixing properties of the measure are comparable to those obtained with "duplicated-variables expansion", used to treat systems with disorder and coupled map lattices. It works in a larger region of validity than usual cluster-expansion formalisms, and it is not tied to the analyticity of the pressure. In fact, it does not lead to any kind of expansion for the latter, and the properties of the equilibrium measure are obtained without resorting to combinatorial or complex analysis techniques.

Key words: Peierls contours. Animal models. Loss networks. Ising model. Oriented percolation. Central limit theorem. Poisson approximation.
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1 Introduction

In this paper we develop a probabilistic approach to the study of the equilibrium measure of systems with exclusions —like hard-core gases, contours, polymers or animals— in the low-density or extreme-temperature regime. This regime has traditionally been studied via cluster-expansion methods, which relied either on sophisticated combinatorial estimations [Malyshev (1980), Seiler (1982), Brydges (1984)] or on astute inductive hypotheses plus complex analysis [Kotecký and Preiss (1986), Dobrushin (1996, 1996a)].

In contrast, we realize the equilibrium measure as the invariant measure of a loss network process that can be studied using standards tools and notions from probabilistic models and processes. Loss networks were introduced by Erlang in 1917, see Kelly (1991) and references therein. In particular, we show (Section 3.4) that the existence of the loss network, and hence of the equilibrium measure, can be related to the absence of percolation in an oriented percolation process. This condition also yields other properties, like uniqueness, convergence through sequences of finite volumes and mixing properties (Theorem 4.7). More precise results can be obtained by resorting to a dominant multitype branching process (Section 5). Roughly speaking, the mean number of branches of this process becomes the driving parameter: Subcriticality is a sufficient condition for the construction to work. Time and space rates of convergence and mixing rates are explicitly obtained in terms of this parameter (Theorem 2.11).

For concreteness, we analyze in this paper the gases of Peierls contours used, for instance, for low-temperature studies of the Ising model [Peierls (1936), Dobrushin (1965), Griffiths (1964)]. The subcriticality condition of the corresponding branching process is

$$\sup_{\gamma} \frac{1}{|\gamma|} \sum_{\theta:\theta\not\sim\gamma} |\theta| w(\theta) \;<\; 1 \;, \tag{1.1}$$

where $|\gamma|$ indicates the length (perimeter, surface area) of a contour γ, $w(\gamma)$ is its weight and "$\sim$" stands for the volume-exclusion (=non-intersection) condition.

Condition (1.1) is considerably weaker than those obtained by the most developed cluster-expansion approaches [Kotecký and Preiss (1986), Dobrushin (1996, 1996a)] in which the factor $|\theta|$ in the right-hand side is replaced by a function that grows roughly exponentially with $|\theta|$.

The weakening of the condition has a price: unlike previous approaches, ours does not yield analyticity properties of the expectations. Condition (1.1) is similar to conditions obtained in the study of systems for which nonanalyticity is known [von Dreifus, Klein and Perez (1995)] or suspected [Bricmont and Kupiainen (1996, 1997)]. Our results on space convergence and mixing rates can basically be obtained with the "duplicate-system" expansions of Bricmont and Kupiainen (1996, 1997).

The novelty of our approach lies in the following features: First, it offers a completely different framework for the study of hard core measures, based on well known stochastic processes. This can conceivably lead to new insights and stronger results. In particular the condition (1.1) can probably be weakened by studing conditions for the sub-criticality of the oriented percolation process instead of the dominating multitype branching process. Improvements like that have been done for the contact process and oriented percolation, see Liggett (1995), for instance. Second, our construction involves a stochastic process (the loss network) that converges exponentially fast to the sought-after measure. This process is, in principle, easy to simulate and its potential as a computational tool deserves to be explored. Its rate of convergence to the equilibrium measure increases as the temperature decreases and, unlike spin-flip dynamics, it does not present meta-stable traps (any contour lives an exponential time of mean one). Finally, the construction permits a rather straightforward proof of the asymptotic Poisson distribution of contours at low temperature (Theorem 2.18). It is not obvious to us how such a result can be obtained through the standard statistical mechanical expansions.

The present approach does not lead to a series expansion for the pressure (or free-energy density). In particular, it does not yield the "surface-tension bounds" that play such a crucial role in some applications of cluster expansions [see eg. Zahradníck (1984), Borgs and Imbrie (1989)]. In fact, the approach is designed so to *bypass* expansions of this type. It is a probabilistic approach designed to answer probabilistic questions —existence of expectations, properties of correlation functions— in a direct way, without combinatorial or complex-analysis techniques that are quite unnecessary for this purpose. It is not an alternative to cluster expansions: it has a different regime of validity and it has different aims.

Our work was motivated in part by the posthumous review of Roland Dobrushin (1996), where he complained that while "perturbation methods are intensively used by mathematical physicists, they are not so popular as correlation inequalities among the probabilists". He called for a "systematical exposition oriented to the mathematicians". In this paper we follow his call: we concentrate on probabilistic issues and exploit probabilistic arguments.

In this paper we refine and complete the theory presented in Fernández, Ferrari and Garcia (1998). In particular we present (a) a proof of exponential mixing, (b) an enlargement of the region of validity of the properties and (c) the proof of convergence to a Poisson process.

2 Definitions and results

2.1 The contour model

We consider the d-dimensional lattice $\mathbb{Z}^d$ and call *plaquettes* the $(d-1)$-dimensional unit cubes centered at points of $\mathbb{Z}^d$. We identify each plaquette with its center. A set of plaquettes is called *surface*. Two plaquettes are *adjacent* if they share a $(d-2)$-dimensional face. This defines a notion of connection: a set γ of plaquettes forms a *connected* (hyper) surface if for every two plaquettes x,y there is a sequence of pairwise adjacent plaquettes starting at x and ending at y. A surface is *closed* if every $(d-2)$-dimensional face is shared by an even number of plaquettes in the surface. A *contour* γ is a connected and closed family of plaquettes. Two contours γ and θ are incompatible if they share some $(d-2)$-dimensional face. In this case we denote $\gamma \not\sim \theta$.

Denote by $\mathbf{G}^\Lambda$ the set of all contours in the volume Λ. The subset $\mathcal{X}^\Lambda \subset \mathbb{N}^{\mathbf{G}^\Lambda}$ of *compatible* configurations is defined as

$$\mathcal{X}^\Lambda = \{\eta \in \{0,1\}^{\mathbf{G}^\Lambda}\, ;\, \eta(\gamma)\,\eta(\theta) = 0 \text{ if } \gamma \not\sim \theta\} \tag{2.1}$$

that is, a configuration of contours is compatible if it does not contain two incompatible contours. We denote $\mathcal{X} = \mathcal{X}^{\mathbb{Z}^d}$ and $\mathbf{G} = \mathbf{G}^{\mathbb{Z}^d}$. As usual, we endowed $\mathcal{X}$ with the product topology.

For each fixed $\beta \in \mathbb{R}^+$, a parameter usually called the inverse temperature, and for each finite Λ define the measure μ^Λ on $\mathcal{X}^\Lambda$ by

$$\mu^\Lambda(\eta) = \frac{\exp\Big(-\beta \sum_{\gamma:\eta(\gamma)=1} |\gamma|\Big)}{Z^\Lambda} \tag{2.2}$$

where $|\gamma|$ is the number of plaquettes in γ and Z^Λ is a renormalization constant making μ^Λ a probability.

2.2 The exclusion dynamics — loss network

We introduce a Markov process called *loss network* in the set of compatible contours. This process was introduced by Erlang in 1917 (see Brockmeyer, Halstrom and Jensen (1948), p. 139). An account of its properties can be found in Kelly (1991). For a finite or infinite set $\Lambda \subset \mathbb{Z}^d$ and f real continuous function on $\mathcal{X}^\Lambda$, the generator of the process is defined by:

$$A^\Lambda f(\eta) = \sum_{\gamma \in \mathbf{G}^\Lambda} e^{-\beta|\gamma|}\, \mathbf{1}\{\eta + \delta_\gamma \in \mathcal{X}^\Lambda\}\, [f(\eta+\delta_\gamma) - f(\eta)] + \sum_{\gamma \in \mathbf{G}^\Lambda} \eta(\gamma)\, [f(\eta - \delta_\gamma) - f(\eta)] \tag{2.3}$$

where the sum of configurations is given by $(\eta+\xi)(\gamma) = \eta(\gamma) + \xi(\gamma)$ and $\delta_\gamma(\theta) = \mathbf{1}\{\theta = \gamma\}$. In words, each contour γ attempts to appear at rate $e^{-\beta|\gamma|}$ but it does so only if no incompatible contour is present. Present contours disappear at rate 1. Our first result is the following sufficient condition for the existence of a process with generator A^Λ for any (infinite) Λ.

Theorem 2.4 *If*

$$\alpha(\beta) = \sup_\gamma \frac{1}{|\gamma|} \sum_{\theta:\theta \not\sim \gamma} |\theta|\, e^{-\beta|\theta|} < \infty \tag{2.5}$$

then for any (infinite) Λ the Markov process η_t^Λ with generator A^Λ exists and admits at least one invariant measure.

This theorem is proven in Section 6. In what follows we use the notation

$$\eta_t = \eta_t^{\mathbb{Z}^d} \qquad A = A^{\mathbb{Z}^d} \tag{2.6}$$

and when the initial configuration is fixed to be η, we denote

$$\eta_t^{\Lambda,\eta}. \tag{2.7}$$

2.3 Results on the invariant measure

We say that f has *support* in $\Upsilon \subset \mathbb{Z}^d$ if f depends only on contours intersecting Υ (not necessarily contained in Υ). Let $|\mathrm{Supp}\,(f)| = \min\{|\Upsilon| : f \text{ has support in } \Upsilon\}$. When we write $\mathrm{Supp}\,(f)$ we mean any Υ such that $|\Upsilon| = |\mathrm{Supp}\,(f)|$ and f has support in Υ. For instance, if $f(\eta) = \eta(\gamma)$, $\mathrm{Supp}\,(f)$ may be set as $\{x\}$ for any $x \in \gamma$. Let

$$\beta^* \text{ solution of } \alpha(\beta) = 1, \tag{2.8}$$

$$\alpha_0(\beta) = \sum_{\gamma \ni 0} |\gamma|\, e^{-\beta|\gamma|} \tag{2.9}$$

and the distance between two subsets of $\mathbb{Z}^d$ be

$$d(\Lambda, \Upsilon) = \min\{|x-y| : x \in \Lambda, y \in \Upsilon\}. \tag{2.10}$$

Theorem 2.11 *If $\beta > \beta^*$ (that is, $\alpha(\beta) < 1$) then the following statements hold:*

1. *Uniqueness. There is a unique invariant measure for the process η_t. We denote it $\mu = \mu^{\mathbb{Z}^d}$.*

2. *Exponential time convergence. For any $\Lambda \subset \mathbb{Z}^d$ and for measurable f on $\mathcal{X}_\Lambda$,*

$$\sup_{\eta \in \mathcal{X}_\Lambda} \left| \mu^\Lambda f - \mathbb{E} f(\eta_t^{\Lambda,\eta}) \right| \;\le\; 2\,\|f\|_\infty\, |\mathrm{Supp}\,(f)|\, \frac{\alpha_0}{\rho}\, e^{-\rho t} \tag{2.12}$$

   where $\rho = \frac{1-\alpha}{2-\alpha}$.

3. *Exponential space convergence. Let Λ be a (finite or infinite) subset of $\mathbb{Z}^d$ and f a measurable function depending on contours contained in Λ. Then*

$$|\mu f - \mu^\Lambda f| \;\le\; 2\,\|f\|_\infty\, \alpha_0\, M_2 \sum_{x \in \mathrm{Supp}\,(f)} e^{-M_3\, d(\{x\}, \Lambda^c)} \tag{2.13}$$

   where $M_2 = (1 - \alpha(\widetilde{\beta}))^{-1}$ and $M_3 = (\beta - \widetilde{\beta})$, for any $\widetilde{\beta} \in (\beta^*, \beta)$.

4. *Exponential mixing. For measurable functions f and g depending on contours contained in an arbitrary set $\Lambda \subset \mathbb{Z}^d$:*

$$|\mu^\Lambda(fg) - \mu^\Lambda f\, \mu^\Lambda g| \le 2\,\|f\|_\infty\, \|g\|_\infty\, (M_2)^2 \sum_{\substack{x \in \mathrm{Supp}\,(f),\\ y \in \mathrm{Supp}\,(g)}} |x-y| e^{-M_3|x-y|} \tag{2.14}$$

   where M_2 ad M_3 are the same of (2.13).

5. *Central limit theorem. Let f be a measurable function on $\mathcal{X}$ with finite support such that $\mu f = 0$ and $\mu(|f|^{2+\delta}) < \infty$ for some $\delta > 0$. Assume $D = \sum_x \mu(f\tau_x f) > 0$. Then $D < \infty$ and*

$$\frac{1}{\sqrt{|\Lambda|}} \sum_{x\in\Lambda} \tau_x f \underset{\Lambda \to \mathbb{Z}^d}{\Longrightarrow} \text{Normal}(0, D) \tag{2.15}$$

where the double arrow means convergence in distribution.

This theorem strengthens the results **R1**–**R5** of Fernández, Ferrari and Garcia (1998). In that paper β^* was replaced by a value β_M strictly bigger than β^*. Item 5 generalizes (the central limit) Theorem 7.4 of Dobrushin (1996a), where only functions depending on a finite number of contours are considered. The above theorem will be proven in Section 7.

Finally, we prove a Poisson approximation. Consider the equivalence relation induced by the translation of contours. Let $\widetilde{\mathbf{G}}_j$ be a the set formed by one representative containing the origin from each class of equivalence of contours with length j. For each Borel set $V \subset \mathbb{R}^d$ and $a \in \mathbb{R}$ let

$$V \cdot a = \Big\{x \in \mathbb{Z}^d : \Big[\frac{x-1/2}{a}, \frac{x+1/2}{a}\Big]^d \subset V\Big\}. \tag{2.16}$$

Fix a contour length $j > 0$. For each $\gamma \in \widetilde{\mathbf{G}}_j$, let $M_{\gamma,\beta}$ be defined by

$$M_{\gamma,\beta}(V) = \sum_{x\in V\cdot e^{\beta|\gamma|/d}} \eta^\beta(\tau_x\gamma) \tag{2.17}$$

where η^β is distributed according to the invariant measure for the given β. Let $(M_{\gamma,\infty},\ \gamma \in \widetilde{\mathbf{G}}_j)$ be a family of independent unit Poisson processes in $\mathbb{R}^d$.

Theorem 2.18 *For each contour length j, it is possible to jointly construct the d-dimensional processes $\{M_{\gamma,\beta} : \beta^* < \beta \le \infty,\ \gamma \in \widetilde{\mathbf{G}}_j\}$ in such a way that for all V = product of intervals:*

$$\mathbb{P}(M_{\gamma,\beta}(V) \neq M_{\gamma,\infty}(V)) \;\le\; c(|\gamma|, V)\Big(\alpha(\beta) + e^{-\beta|\gamma|/d}\Big) \;\sim\; \exp\Big(-\beta\,\min\{2d, |\gamma|/d\}\Big). \tag{2.19}$$

where $c(|\gamma|, V)$ is a computable constant. As a consequence $\{M_{\gamma,\beta} : \gamma \in \widetilde{\mathbf{G}}_j\}$ converges in distribution to a family consisting of $|\widetilde{\mathbf{G}}_j|$ independent Poisson processes with mean $e^{-\beta j}$.

Theorem 2.18 is proven in Section 8.

3 Graphical representation of loss networks

3.1 Marked Poisson processes

To each contour $\gamma \in \mathbf{G}$ we associate an independent (of everything) marked Poisson process N_γ with rate $e^{-\beta|\gamma|}$. We call $T_k(\gamma)$, $\gamma \in \mathbf{G}$, the ordered time-events of N_γ with the convention that $T_0(\gamma) < 0 < T_1(\gamma)$. For each occurrence time $T_i(\gamma)$ of the process N_γ we choose an independent mark $S_i(\gamma)$ exponentially distributed with mean 1. At the Poisson time-event $T_i(\gamma)$ a contour γ appears and it lasts $S_i(\gamma)$ time units.

The random family $\mathbf{C} = \{\{(\gamma, T_i(\gamma), S_i(\gamma)) : i \in \mathbb{Z}\} : \gamma \in \mathbf{G}\}$ consists of independent marked Poisson processes. A marked point $(\gamma, T_k(\gamma), S_k(\gamma)) \in \mathbf{C}$ is identified with $\gamma \times [T_k(\gamma), T_k(\gamma) + S_k(\gamma)]$, the *cylinder* with *basis* γ, *birth time* $T_k(\gamma)$ and *lifetime* $S_k(\gamma)$. The *life* of the cylinder is the time interval $[T_k(\gamma), T_k(\gamma) + S_k(\gamma)]$. Calling $C = (\gamma, t, s)$, we use the notation

$$\text{Basis}\,(C) = \gamma, \quad \text{Birth}\,(C) = t, \quad \text{Life}\,(C) = [t, t+s]. \tag{3.1}$$

Define incompatibility between cylinders C and C' by

$$C' \not\sim C \quad \text{if and only if} \quad \text{Basis}\,(C) \not\sim \text{Basis}\,(C') \ \text{ and } \ \text{Life}\,(C) \cap \text{Life}\,(C') \neq \emptyset, \tag{3.2}$$

otherwise $C' \sim C$ (compatible).

3.2 The free network

Let

$$\xi_t(\gamma) = \sum_{C \in \mathbf{C}} \mathbf{1}\{\text{Basis}\,(C) = \gamma, \text{Life}\,(C) \ni t\}. \tag{3.3}$$

The above process, called the *free network*, is a product of independent birth-death processes on $\mathbb{N}^{\mathbf{G}^\Lambda}$ whose generator is given by

$$A^0 f(\xi) = \sum_{\gamma \in \mathbf{G}} e^{-\beta|\gamma|}[f(\xi + \delta_\gamma) - f(\xi)] + \sum_{\gamma \in \mathbf{G}} \xi(\gamma)[f(\xi - \delta_\gamma) - f(\xi)]. \tag{3.4}$$

The invariant (and reversible) measure for this process is the product measure μ^0 on $\mathcal{X}$ with Poisson marginals

$$\mu^0(\xi(\gamma) = k) = e^{e^{-\beta|\gamma|}} \frac{(e^{-\beta|\gamma|})^k}{k!}. \tag{3.5}$$

In terms of loss networks, ξ_t is the process for which all the calls are accepted, that is $\xi_t(\gamma)$ is the number of calls on route γ at time t when there is no restriction on the number of calls a circuit can accept.

3.3 Finite-volume construction of a loss network

The construction of a loss network in a finite volume Λ with an initial condition η_0 using the Poisson processes is straightforward. We use only the finite set of processes $\{N_\gamma : \gamma \subset \Lambda\}$. Let $\mathbf{C}^\Lambda = \{C \in \mathbf{C} : \text{Basis}\,(C) \subset \Lambda\}$. To each contour θ present in the initial configuration η_0 we independently associate an exponential time $S(\theta)$ and a cylinder $(\theta, 0, S(\theta))$. The collection of initial cylinders is called $\mathbf{C}_0^\Lambda$. We realize the dynamics η_t as a (deterministic) function of $\mathbf{C}^\Lambda$ and $\mathbf{C}_0^\Lambda$.

We construct inductively $\mathbf{K}_{[0,t]}^\Lambda$, the set of *kept* cylinders at time t. The complementary set corresponds to *erased* cylinders. First include all cylinders of $\mathbf{C}_0^\Lambda$ in $\mathbf{K}_{[0,t]}^\Lambda$. Then, move forward in time and consider the first Poisson mark: The corresponding cylinder is erased if it is incompatible with any of the cylinders already in $\mathbf{K}_{[0,t]}^\Lambda$, otherwise it is kept. This procedure is successively performed mark by mark until all cylinders born before t are considered. Define $\eta_t^\Lambda \in \mathcal{X}^\Lambda$ as

$$\eta_t^\Lambda(\gamma) \quad = \quad \mathbf{1}\Big\{\gamma \in \{\text{Basis}\,(C) : C \in \mathbf{K}_{[0,t]}^\Lambda,\, \text{Life}\,(C) \ni t\}\Big\}, \tag{3.6}$$

that is, η_t^Λ signals all contours which are basis of a kept cylinder that is alive at time t. We leave to the reader to show that η_t^Λ has generator A^Λ defined as in (2.3) restricting the sums to the set of contours contained in Λ. It is immediate that μ^Λ defined in (2.2) is reversible for this process. Since we are dealing with an irreducible Markov process in a finite state space, η_t^Λ converges in distribution to μ^Λ for any initial configuration η. This in particular implies that μ^Λ is the unique invariant measure for η_t^Λ. Later in the paper we determine the speed of convergence.

Using the same Poisson marks for η_t^Λ and ξ_t, we have

$$\eta_t^\Lambda(\gamma) \;\le\; \xi_t(\gamma), \tag{3.7}$$

because in the free network ξ_t all cylinders are kept. This implies

$$\mu^\Lambda\{\eta : \eta(\gamma) = 1\} = \mathbb{E}\eta_t^\Lambda(\gamma) \;\le\; \mathbb{E}\xi_t(\gamma) = e^{-\beta|\gamma|}. \tag{3.8}$$

3.4 Backwards oriented percolation

If we try to perform an analogous construction in infinite volume we are confronted with the problem that there is not a first mark. To overcome this we follow the original approach of Harris (1972) (see also Durrett (1997)) and introduce the notion of percolation. The goal is to partition the set of cylinders in finite subsets to which the previous mark-by-mark construction can be applied.

We come back to the infinite volume construction of Section 3.1. For an arbitrary space-time point (x, t) define the set

$$\mathbf{A}_1^{x,t} \;=\; \{C \in \mathbf{C}\,;\, \mathrm{Basis}\,(C) \ni x,\ \mathsf{Life}\,(C) \ni t\} \tag{3.9}$$

the set of cylinders containing the point (x, t).

For any cylinder C define the set of *ancestors* of C as the set

$$\mathbf{A}_1^C = \{C' \in \mathbf{C}\,;\, C' \not\sim C\} \tag{3.10}$$

Notice that the definition of ancestor does not depend on the lifetime of C. Recursively for $n \ge 1$, the nth generation of ancestors are defined as

$$\mathbf{A}_n^{x,t} = \{C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^{x,t}\}. \tag{3.11}$$

and for a given cylinder C,

$$\mathbf{A}_n^C = \{C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^C\}. \tag{3.12}$$

We say that there is *backward oriented percolation* in $\mathbf{C}$ if there exists a space-time point (x, t) such that $\mathbf{A}_n^{x,t} \ne \emptyset$ for all n, that is, there exists a point with infinitely many generations of ancestors. Let the *clan* of the space-time point (x, t) be the union of its ancestors:

$$\mathbf{A}^{x,t} = \bigcup_{n\ge1} \mathbf{A}_n^{x,t} \tag{3.13}$$

and $\mathbf{C}[0,t] = \{C \in \mathbf{C} : \mathrm{Birth}\,(C) \in [0,t]\}$.

In the next theorem we give a sufficient condition for the existence of the infinite-volume process in any finite time interval in terms of backwards percolation.

Theorem 3.14 *If with probability one $\mathbf{A}^{x,t} \cap \mathbf{C}[0,t]$ is finite for any $x, \in \mathbb{Z}^d$ and $t \geq 0$, then for any $\Lambda \subset \mathbb{Z}^d$, the process with generator A^Λ is well defined and has at least one invariant measure μ^Λ.*

Proof. We construct the process for $\Lambda = \mathbb{Z}^d$. The construction for other Λ is analogous. The initial distribution is denoted $\eta_0 \in \mathcal{X}$. For each γ such that $\eta_0(\gamma) = 1$ let $S(\gamma)$ be an independent exponentially distributed random time of mean 1. The time $S(\gamma)$ represents the lifetime of the cylinder with basis γ and birth time 0. We call $\mathbf{C}(0)$ the set of cylinders $(\gamma, 0, S(\gamma))$. Since the cylinders in $\mathbf{C}(0)$ have no ancestors in $\mathbf{C}[0,t]$, under the hypothesis of the theorem, every cylinder in $\mathbf{C}(0) \cup \mathbf{C}[0,t]$ has a finite number of ancestors in $\mathbf{C}[0,t]$. We partition this set as follows. As in the finite-volume case, we construct a set of *kept* cylinders, denoted by $\mathbf{K}$, and a set of *deleted* cylinders denoted by $\mathbf{D}$. Let

$$\mathbf{K}_0[0,t] = \mathbf{C}(0) \cup \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C = \emptyset\} \tag{3.15}$$

be the set of cylinders with no ancestors and $\mathbf{U}_0[0,t] = \mathbf{K}_0[0,t]$.

$$\mathbf{D}_0[0,t] = \{C \in \mathbf{C}[0,t] : \mathbf{A}_1^C \in \mathbf{K}_0[0,t]\} \tag{3.16}$$

be the set of cylinders with ancestors in $\mathbf{K}_0[0,t]$. Inductively, for $k \geq 1$ let

$$\begin{aligned} \mathbf{K}_k[0,t] &= \mathbf{K}_{k-1}[0,t] \cup \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] = \emptyset\Big\} && (3.17)\\ \mathbf{D}_k[0,t] &= \mathbf{D}_{k-1}[0,t] \cup \Big\{C \in \mathbf{U}_k[0,t] : \mathbf{A}_1^C \cap \mathbf{K}_{k-1}[0,t] \neq \emptyset\Big\} && (3.18) \end{aligned}$$

where

$$\mathbf{U}_k[0,t] = \Big\{C \in \mathbf{C}[0,t] \setminus \Big[\cup_{i=0}^{k-1} \mathbf{U}_i[0,t]\Big] : \mathbf{A}_1^C \cap \mathbf{U}_{k-1}[0,t] \neq \emptyset\Big\} \tag{3.19}$$

is the set of cylinders being classified at step k. Defining

$$\mathbf{K}[0,t] = \cup_{k\geq 0} \mathbf{K}_k[0,t], \qquad \mathbf{D}[0,t] = \cup_{k\geq 0} \mathbf{D}_k[0,t], \tag{3.20}$$

we have

$$\mathbf{C}[0,t] = \mathbf{K}[0,t] \,\dot{\cup}\, \mathbf{D}[0,t]. \tag{3.21}$$

The process is now defined as in (3.6) by

$$\eta_t(\gamma) = \mathbf{1}\{\gamma \in \{\mathrm{Basis}\,(C) : C \in \mathbf{K}[0,t],\ \mathrm{Life}\,(C) \ni t\}\}. \tag{3.22}$$

The reader can check that if we apply the above construction to the set of cylinders in $C^{\Lambda}[0,t]$ we obtain $K^{\Lambda}[0,t]$ as defined in Section 3.3.

Display (3.22) says that the presence in η_t of a contour $\gamma \ni x$ can be established by classifying the ancestors of (x,t) in kept and erased cylinders. Since there is no backwards oriented percolation, this classification can be accomplished in a finite number of steps. This idea is used in Ferrari and Garcia (1998) to prove ergodicity of a one-dimensional loss network.

To show that η_t has generator A, write

$$\begin{aligned}
&[f(\eta_{t+h}) - f(\eta_t)] \\
&= \sum_{C \in \mathbf{K}[0,t+h]} \mathbf{1}\{\mathrm{Birth}\,(C) \in [t,t+h]\}[f(\eta_t + \delta_{\mathrm{Basis}\,(C)}) - f(\eta_t)] \\
&\quad + \sum_{C \in \mathbf{K}[0,t]} \mathbf{1}\{\mathrm{Life}\,(C) \ni t,\ \mathrm{Life}\,(C) \not\ni t+h\}[f(\eta_t - \delta_{\mathrm{Basis}\,(C)}) - f(\eta_t)] \\
&\quad + \{\text{other things}\},
\end{aligned} \tag{3.23}$$

where {other things} refer to events with more than one Poisson mark in the time interval $[t,t+h]$ for the contours in the (finite) support of f. Since the total rate of the Poisson marks in this set is finite, the event {other things} has a probability proportional to h^2. Now, denoting

$$N^+_\gamma(t,s) = \#\{k : T_k(\gamma) \in (t,s)\} \tag{3.24}$$

$$N^-_\gamma(t,s) = \#\{k : T_k(\gamma) + S_k(\gamma) \in (t,s)\} \tag{3.25}$$

we have

$$\begin{aligned}
&\sum_{C \in \mathbf{C}} \mathbf{1}\{\mathrm{Birth}\,(C) \in [t,t+h]\}\mathbf{1}\{C \in \mathbf{K}[0,t+h]\}[f(\eta_t + \delta_{\mathrm{Basis}\,(C)}) - f(\eta_t)] \\
&= \sum_{C \in \mathbf{C}} \mathbf{1}\{N^+_{\mathrm{Basis}\,(C)}[t,t+h] = 1\}\mathbf{1}\{\mathrm{Basis}\,(C) \sim \mathrm{Basis}\,(C'), \forall C' \in \mathbf{K}[0,t] : \mathrm{Life}\,(C') \ni t\}
\end{aligned}$$

$$\times [f(\eta_t + \delta_{\mathrm{Basis}\,(C)}) - f(\eta_t)]$$

$$= \sum_\gamma \mathbf{1}\{N^+_\gamma[t, t+h] = 1\}\mathbf{1}\{\eta_t + \delta_\gamma \in \mathcal{X}\}[f(\eta_t + \delta_\gamma) - f(\eta_t)] \tag{3.26}$$

and

$$\begin{aligned}
&\sum_C \mathbf{1}\{C \in \mathbf{K}[0,t]\}\mathbf{1}\{\mathrm{Life}\,(C) \ni t,\ \mathrm{Life}\,(C) \not\ni t+h\}[f(\eta_t - \delta_{\mathrm{Basis}\,(C)}) - f(\eta_t)] \\
&\quad = \sum_C \mathbf{1}\{C \in \mathbf{K}[0,t]\}\mathbf{1}\{N^-_{\mathrm{Basis}\,(C)}[t,t+h] = 1\}[f(\eta_t - \delta_{\mathrm{Basis}\,(C)}) - f(\eta_t)] \\
&\quad = \sum_\gamma \eta_t(\gamma)\mathbf{1}\{N^-_\gamma[t,t+h] = 1\}[f(\eta_t - \delta_\gamma) - f(\eta_t)].
\end{aligned} \tag{3.27}$$

Substituting (3.26) and (3.27) and taking expectations we get

$$\begin{aligned}
&\mathbb{E}[f(\eta_{t+h}) - f(\eta_t)|] \\
&\quad = \sum_\gamma h e^{-\beta|\gamma|}\mathbb{E}(\mathbf{1}\{\eta_t + \delta_\gamma \in \mathcal{X}\}[f(\eta_t + \delta_\gamma) - f(\eta_t)]) + o(h) \\
&\quad\quad + \sum_\gamma h\,\mathbb{E}\Big(\eta_t(\gamma)\,[f(\eta_t - \delta_\gamma) - f(\eta_t)]\Big) + o(h)
\end{aligned} \tag{3.28}$$

which dividing by h and taking limit gives the desired result.

The existence of an invariant measure follows as in Chapter 1 of Liggett (1985), as our process is defined in the compact space $\mathcal{X}$. □

With a little more work under stronger hypothesis, we show in the next theorem that the process can be constructed for times in the whole real line. Since the construction is time-translation invariant, the distribution of η_t will be invariant.

Theorem 3.29 *If with probability one there is no backwards oriented percolation in* $\mathbf{C}$*, then the process with generator* A *can be constructed in* $(-\infty, \infty)$ *in such a way that the marginal distribution of* η_t *is invariant.*

Proof. The lack of percolation allows us to construct a set $\mathbf{K} \subset \mathbf{C}$ as $\mathbf{K}[0,t]$ was constructed from $\mathbf{C}(0) \cup \mathbf{C}[0,t]$ in the proof of the previous theorem. Note that $\mathbf{K}$ is both space and time-translation invariant by construction. Analogously to the previous theorem we define η_t as the section of $\mathbf{K}$ at time t:

$$\eta_t(\gamma) = \mathbf{1}\{\gamma \in \{\mathrm{Basis}\,(C) : C \in \mathbf{K},\ \mathrm{Life}\,(C) \ni t\}\}. \tag{3.30}$$

By construction, the distribution of η_t does not depend on t, hence its distribution is an invariant measure for the process. □

Definition 3.31 *The distribution of η_t is called μ.*

As in the finite case, for all $\gamma \in \mathbf{G}$,

$$\eta_t(\gamma) \;\leq\; \xi_t(\gamma). \tag{3.32}$$

This implies that the distribution μ inherits property (3.8):

$$\mu(\eta_t^\Lambda(\gamma) = 1) = \mathbb{E}\eta_t(\gamma) \;\leq\; \mathbb{E}\xi_t(\gamma) = e^{-\beta|\gamma|}. \tag{3.33}$$

Let the *clan* of $\Upsilon \subset \mathbb{Z}^d$ (at time 0), be

$$\mathbf{A}(\Upsilon) = \cup_{x\in\Upsilon}\mathbf{A}^{x,0}. \tag{3.34}$$

Remarks. 1. It follows from (3.30) that for any $t \in \mathbb{R}$ and continuous f,

$$\mu f = \mathbb{E}f(\eta_t). \tag{3.35}$$

2. The presence/absence of a contour γ at time t depends only on the set of ancestors of (x,t) for any $x \in \gamma$ through a certain function. More generally, for each f there exists a function Φ such that

$$f(\eta_t) = \Phi(\cup_{x\in\text{Supp}\,(f))}\mathbf{A}^{x,t}). \tag{3.36}$$

For instance, $\eta_t(\gamma) = 1$ if and only if $\mathbf{A}_1^{x,t}$ contains a cylinder in $\mathbf{K}$ with basis γ whose life contains t. This depends only on the set $\mathbf{A}^{x,t}$. In particular, with the notation (3.34),

$$f(\eta_0) = \Phi(\mathbf{A}(\text{Supp}\,(f))). \tag{3.37}$$

3. Mimicking the proof of Theorem 3.29 one concludes that the loss network in a finite volume Λ can be constructed for all $t \in \mathbb{R}$. In this case one constructs the set $\mathbf{K}^\Lambda$ using the same specification used to construct $\mathbf{K}$ but using only cylinders in $\mathbf{C}^\Lambda$. In this case we can write

$$\mu^\Lambda f = \mathbb{E}f(\eta_t^\Lambda). \tag{3.38}$$

An argument using the domination of the finite volume loss network by the (ergodic) finite volume free network shows that this construction in the finite case can be performed without imposing conditions on the rates. But we are not going to use this.

4 Percolation, space-time convergence and mixing

4.1 The key theorem

The following theorem gives a more precise version of Theorem 2.11. It shows that all that is needed is the absence of backwards and non-oriented percolation. We need a continuous-time construction of the backwards percolation clan. To do this in the infinite-volume case, we need to introduce a notion of non-oriented percolation in a time interval. Fix a time interval (s, t) and for any space-time point (x, t') define

$$\mathbf{G}_0^{x,t'}[s,t] = \{C' \in \mathbf{C}[s,t] : \text{Basis}\,(C) \ni x, \text{Life}\,(C') \ni t'\} \tag{4.1}$$

and

$$\mathbf{G}_n^{x,t'}[s,t] = \{C \in \mathbf{C}[s,t] : \text{Basis}\,(C) \not\sim \text{Basis}\,(C'),\ \text{ for some } C' \in \mathbf{G}_{n-1}^{x,t'}\}. \tag{4.2}$$

Notice that in the definition of $\mathbf{G}_n$ there is no exigency that the birth time of C' be previous to the birth time of C or that the lifetimes intersect. Let

$$\mathbf{G}^{x,t'}[s,t] = \cup_{k\geq 0}\mathbf{G}_k^{x,t'}. \tag{4.3}$$

We say that there is no (non-oriented) percolation in $[s, t]$ if for any space-time point (x, t'), $\mathbf{G}^{x,t'}[s,t]$ contains a finite number of cylinders.

We will show later that the condition $\alpha < \infty$ is sufficient for the existence of an h such that the probability that there is no non-oriented percolation in $[0, h]$ is one.

Let the time-length and the space-width of the family of cylinders $\mathbf{A}^{x,t}$ be respectively

$$\begin{aligned} \text{TL}\,(\mathbf{A}^{x,t}) &= t - \sup\{s : \text{Life}\,(C) \ni s,\ \text{for some } C \in \mathbf{A}^{x,t}\}, & (4.4)\\ \text{SW}\,(\mathbf{A}^{x,t}) &= |\cup_{C\in\mathbf{A}^{x,t}} \text{Basis}\,(C)|. & (4.5) \end{aligned}$$

Let $\mathbf{A}^{\Lambda}(\text{Supp}\,(f))$ be the set of ancestors of $(\text{Supp}\,(f))$ constructed from $\mathbf{C}^{\Lambda}$ as $\mathbf{A}(\text{Supp}\,(f))$ was constructed from $\mathbf{C}$. (Notice that this is *not* the same as $\mathbf{A}(\text{Supp}\,(f)) \cap \mathbf{C}^{\Lambda}$).

We say that two sets of cylinders $\mathbf{A}$ and $\mathbf{A}'$ are incompatible if there is a cylinder in $\mathbf{A}$ incompatible with a cylinder in $\mathbf{A}'$:

$$\mathbf{A} \not\sim \mathbf{A}' \ \text{ if and only if }\ C \not\sim C' \text{ for some } C \in \mathbf{A} \text{ and } C' \in \mathbf{A}'. \tag{4.6}$$

Theorem 4.7 *Assume that there is no backwards oriented percolation with probability one. Then,*

1. *Uniqueness. The measure μ is the unique invariant measure for the process η_t.*

2. *Time convergence. For any function f with finite support,*

$$\lim_{t\to\infty}\sup_{\eta\in\mathcal{X}}|\mathbb{E}f(\eta_t^\eta)-\mu f|=0. \tag{4.8}$$

Furthermore,

$$\begin{aligned}&\sup_{\eta\in\mathcal{X}}|\mu f-\mathbb{E}f(\eta_t^\eta)|\\ &\quad\le\; 2\,\|f\|_\infty\,\mathbb{P}(\cup_{x\in\mathrm{Supp}\,(f)}\{\mathbf{A}^{x,t}\not\sim\mathbf{C}(0)\ \text{or}\ \mathrm{TL}\,(\mathbf{A}^{x,t})>t\}) &(4.9)\\ &\quad\le\; 2\,\|f\|_\infty\sum_{x\in\mathrm{Supp}\,(f)}\Big(\mathbb{P}(\mathrm{TL}\,(\mathbf{A}^{x,0})>bt)+e^{-(1-b)t}\,\mathbb{E}(\mathrm{SW}\,(\mathbf{A}^{x,0}))\Big) &(4.10)\end{aligned}$$

for any $b\in(0,1)$.

3. *Space convergence. As $\Lambda\to\mathbb{Z}^d$, μ^Λ converges weakly to μ. More precisely, if f is a function depending on contours contained in a finite set Λ, then*

$$|\mu f-\mu^\Lambda f|\;\le\;2\,\|f\|_\infty\,\mathbb{P}\Big(\mathbf{A}(\mathrm{Supp}\,(f))\neq\mathbf{A}^\Lambda(\mathrm{Supp}\,(f))\Big)\,. \tag{4.11}$$

4. *Mixing. If in adition there exists a value h such that there is no (non-oriented) percolation in $(0,h)$ with probability one, then for f and g with finite support,*

$$\lim_{|x|\to\infty}|\mu(f\tau_x g)-\mu f\,\mu g|\;=\;0 \tag{4.12}$$

where τ_x is translation by x and $|x|$ is (for instance) the Manhattan distance. More precisely,

$$|\mu(fg)-\mu f\,\mu g|\;\le\;2\,\|f\|_\infty\|g\|_\infty\,\mathbb{P}\Big(\mathbf{A}(\mathrm{Supp}\,(f))\;\not\sim\;\widehat{\mathbf{A}}(\mathrm{Supp}\,(g))\Big) \tag{4.13}$$

where $\widehat{\mathbf{A}}(\mathrm{Supp}\,(g))$ has the same distribution as $\mathbf{A}(\mathrm{Supp}\,(g))$ but is independent of $\mathbf{A}(\mathrm{Supp}\,(f))$.

Existence of μ has been proven in Theorem 3.29. In the rest of the section we prove the other properties.

4.2 Time convergence and uniqueness

We use the same Poisson marks to construct simultaneously the stationary process η_t and a process starting at time zero with an arbitrary initial configuration η. The second process is called η_t^η, where $\eta_0^\eta = \eta$. The process η_t^η ignores the cylinders in $\mathbf{C}$ with birth times less than 0 and considers $\mathbf{C}(0) = \{(\theta, 0, S_\theta) : \eta(\theta) = 1\}$, the set of cylinders with basis given by the contours of the initial configuration η and birth time zero —the times S_θ are exponentially distributed with mean 1 and independent of everything.

By (3.35) and (3.38),

$$\sup_{\eta\in\mathcal{X}} |\mathbb{E}f(\eta_t^\eta) - \mu f| \;\le\; \sup_{\eta\in\mathcal{X}} \Big|\mathbb{E}\Big(f(\eta_t^\eta) - f(\eta_t)\Big)\Big| . \tag{4.14}$$

Since we are using $\mathbf{C}$ to construct η_t and $\mathbf{C}[0,t] \cup \mathbf{C}(0)$ to construct η_t^η, it follows

$$|f(\eta_t^\eta) - f(\eta_t)| \;\le\; 2\,\|f\|_\infty\, \mathbf{1}\Big\{\cup_{x\in \mathrm{Supp}\,(f)}\Big(\mathbf{A}^{x,t} \not\sim \mathbf{C}(0) \text{ or } \mathrm{TL}\,(\mathbf{A}^{x,t}) > t\Big)\Big\} \tag{4.15}$$

This shows (4.9).

To prove the weak convergence (4.8) we fix $b \in [0,1]$ and bound the right hand side of (4.15) by

$$\mathbf{1}\{\mathrm{TL}\,(\mathbf{A}^{x,t}) > bt\} + \mathbf{1}\{\mathrm{TL}\,(\mathbf{A}^{x,t}) < bt, \mathbf{C}^{x,t} \not\sim \mathbf{C}(0)\}. \tag{4.16}$$

The expected value of the first term in (4.16) goes to zero because $\mathbf{A}^{x,t}$ has a finite number of cylinders with probability one. Since at most $\mathrm{SW}\,(\mathbf{A}^{x,t})$ cylinders of $C(0)$ can be incompatible with cylinders in $\mathbf{A}^{x,t}$, the second term in (4.16) is bounded by

$$\mathbf{1}\Big\{\max\{S_z : z = 1,\dots,\mathrm{SW}\,(\mathbf{A}^{x,t})\} \;>\; (1-b)\,t\Big\} \tag{4.17}$$

where S_z are independent exponential random variables of mean one also independent of $\mathrm{SW}\,(\mathbf{A}^{x,t})$. The expectation of the indicator function in (4.17) is bounded by

$$\begin{aligned} &\sum_{n\ge 1} \mathbb{P}(\mathrm{SW}\,(\mathbf{A}^{x,t}) = n)\,\mathbb{P}(\max\{S_z : z = 1,\dots,n\} > (1-b)t) \\ &\qquad = \sum_{n\ge 1} \mathbb{P}(\mathrm{SW}\,(\mathbf{A}^{x,0}) = n)\,(1 - (1 - e^{-(1-b)t})^n) \end{aligned} \tag{4.18}$$

because the distribution of $\mathbf{A}^{x,t}$ does not depend on t. Since $\mathbf{A}^{x,0}$ contains a finite number of (finite) contours, $\sum_{n\geq 1}\mathbb{P}(\mathrm{SW}\,(\mathbf{A}^{x,0}) = n) = 1$ and by dominated convergence (4.18) goes to zero as $t\to\infty$. This proves (4.8).

To prove (4.10) start from the expectation of (4.16) and use (4.18) to bound the expected value of the second term by

$$\begin{aligned} & e^{-(1-b)t}\sum_{n\geq 1}\mathbb{P}(\mathrm{SW}\,(\mathbf{A}^{x,0}) = n)\sum_{k=0}^{n-1}(1-e^{-(1-b)t})^k \\ \leq\; & e^{-(1-b)t}\sum_{n\geq 1} n\,\mathbb{P}(\mathrm{SW}\,(\mathbf{A}^{x,0}) = n) \\ \leq\; & e^{-(1-b)t}\,\mathbb{E}(\mathrm{SW}\,(\mathbf{A}^{x,0})). \end{aligned} \tag{4.19}$$

The above arguments prove that the process converges, uniformly in the initial configuration, to the invariant measure μ. An immediate consequence is that μ is the unique invariant measure. This concludes the proof of (1) and (2) of Theorem 4.7. □

4.3 Finite-volume effects

To prove inequality (4.11) we use (3.35), (3.38) and (3.36) to get

$$|\mu f - \mu^\Lambda f| \;=\; \mathbb{E}f(\eta_0) - \mathbb{E}f(\eta_0^\Lambda) \;=\; \mathbb{E}\Big[\Phi(\mathbf{A}(\mathrm{Supp}\,(f))) - \Phi(\mathbf{A}^\Lambda(\mathrm{Supp}\,(f)))\Big], \tag{4.20}$$

where Φ is the function referred to in (3.37). By definition, $\Phi(\mathbf{A}(\mathrm{Supp}\,(f))) \leq \|f\|_\infty$. Hence, inequality (4.11) follows from (4.20).

Since the spatial projections of the set of ancestors of Supp (f) are finite, the right hand side of (4.11) goes to zero, proving, in particular, the weak convergence of μ^Λ to μ. □

4.4 Mixing

The proof of (4) of Theorem 4.7 is very similar in spirit to the above proof but it requires a somewhat more delicate argument based on the coupling of two continuous-time versions of the backwards percolation process.

4.4.1 A continuous time construction of the backwards percolation clan

We construct the backwards percolation clan of a given set $\Upsilon \subset \mathbb{Z}^d$, by combing the Poisson marks backwards in time in a continuous manner. Define

$$\mathbf{A}_t(\Upsilon) = \{C' \in \mathbf{A}(\Upsilon) : 0 > \text{Birth}\,(C') > -t\}, \tag{4.21}$$

that is, the set of cylinders in $A(\Upsilon)$ with birth time posterior to $-t$. This process is Markovian. In fact the inclusion of a new cylinder in the time interval $[t, t+h]$ depends on the existence of a Poisson mark in $[-t-h, -t]$ whose corresponding cylinder is incompatible with some $C' \in \mathbf{A}_t(\Upsilon)$.

Finite volume case. We start with contours contained in a finite set $\Lambda \subset \mathbb{Z}^d$. In this case there is only a finite set of possible bases for the cylinders and the Poisson marks are well ordered with probability one. The construction proceeds mark by mark backwards in time. Set $A_0(\Upsilon) = \Upsilon$. If there is a Poisson mark at time $-t$ whose corresponding cylinder is called C'', then

- if $C'' \not\sim C'$ for some $C' \in A_{t-}(\Upsilon)$, then set $A_t(\Upsilon) = A_{t-}(\Upsilon) \cup \{C''\}$.
- if $C'' \sim C'$ for all $C' \in A_{t-}(\Upsilon)$, then set $A_t(\Upsilon) = A_{t-}(\Upsilon)$.

where the incompatibility between cylinders is defined in (3.2).

By definition $\mathbf{A}_t(\Upsilon)$ is a function of the marked Poisson processes between $-t$ and 0:

$$\mathbf{A}_t(\Upsilon) = \text{Function of } \{(\gamma, T_k(\gamma), S_k(\gamma)) : \gamma \subset \Lambda, T_k(\gamma) \in [-t, 0]\} \text{ and } \Upsilon. \tag{4.22}$$

Infinite volume case. In infinite volume the construction can be performed using a percolation argument as in Section 3.4. By hypothesis, there exists an h such that each cylinder born in the interval $[-h, 0]$ belongs to a finite non-oriented clan. Hence the set of cylinders born in the interval $[-h, 0]$ can be partitioned as follows:

$$\mathbf{C}[-h, 0] = \cup_{k \geq 0} \mathbf{H}_k[-h, 0] \tag{4.23}$$

where $\mathbf{H}_k[s,t]$ are finite sets of cylinders satisfying: for $k \neq j$,

$$C \in \mathbf{H}_k[-t,0], C' \in \mathbf{H}_j[-t,0] \quad \text{implies} \quad C \sim C'. \tag{4.24}$$

Therefore we can well-order the birth time of the cylinders inside each one of the $\mathbf{H}_k$ and proceed to the construction as we did in the finite case.

4.4.2 A coupling between two interacting and two independent clans

We take two independent marked Poisson process. For simplicity we call blue the marks and corresponding cylinders of the first process and red those of the second. Now using these marks we construct simultaneously $(\mathbf{A}_t(\Lambda_1), \mathbf{A}_t(\Lambda_2), \widehat{\mathbf{A}}_t(\Lambda_1), \widehat{\mathbf{A}}_t(\Lambda_2))$, where Λ_1 Λ_2 are subsets of $\mathbb{Z}^d$. $\mathbf{A}_t(\Lambda_1)$ and $\mathbf{A}_t(\Lambda_2)$ are constructed using the blue marks as described in Subsection 4.4.1 (ignoring the red marks). Hence, they are the clans of Λ_1 and Λ_2 respectively. On the other hand, $\widehat{\mathbf{A}}_t(\Lambda_1)$ is constructed with the blue marks, hence it coincides with $\mathbf{A}_t(\Lambda_1)$. Finally $\widehat{\mathbf{A}}_t(\Lambda_2)$ is constructed with a precise combination of blue and red marks in such a way that (a) it coincides with $\mathbf{A}_t(\Lambda_2)$ for an (as long as possible) time interval; (b) it is independent of $\widehat{\mathbf{A}}_t(\Lambda_1)$, and (c) it has the same marginal distribution as $\mathbf{A}_t(\Lambda_2)$.

Finite-volume case. We order marks by appearance and introduce the flag variable: $\text{Flag}(t) \in \{0,1\}$ which indicates if some cylinder of $\widehat{\mathbf{A}}_t(\Lambda_1)$ is incompatible with some cylinder of $\widehat{\mathbf{A}}_t(\Lambda_2)$:

$$\text{Flag}(t) = \mathbf{1}\{C' \cap C'' \neq \emptyset \text{ for some } C' \in \widehat{\mathbf{A}}_t(\Lambda_1), C'' \in \widehat{\mathbf{A}}_t(\Lambda_2)\}. \tag{4.25}$$

To accomplish the coupling we proceed as follows, mark by mark backwards in time. Set the Flag at time 0 equal to 0.

- If at time $-t$ a (blue or red) mark is present and $\text{Flag}(t-) = 0$, then
  - If the mark is blue and the corresponding cylinder can be included in $\mathbf{A}_{t-}(\Lambda_1)$ but not in $\mathbf{A}_t(\Lambda_2)$, then include it in $\mathbf{A}_t(\Lambda_1)$ and $\widehat{\mathbf{A}}_t(\Lambda_1)$. Analogously, if it can be included in $\mathbf{A}_{t-}(\Lambda_2)$ but not in $\mathbf{A}_t(\Lambda_1)$, then include it in $\mathbf{A}_t(\Lambda_2)$ and $\widehat{\mathbf{A}}_t(\Lambda_2)$. Keep the Flag $= 0$.

- If the mark is blue and the corresponding cylinder can be included in both $\mathbf{A}_{t-}(\Lambda_1)$ and $\mathbf{A}_{t-}(\Lambda_2)$, then include it in both $\mathbf{A}_t(\Lambda_1)$ and $\mathbf{A}_t(\Lambda_2)$ but include it only in $\widehat{\mathbf{A}}_t(\Lambda_1)$. Set the Flag $= 1$.
- If the mark is red and the corresponding cylinder can be included in both $\mathbf{A}_{t-}(\Lambda_1)$ and $\mathbf{A}_{t-}(\Lambda_2)$, then include it only in $\widehat{\mathbf{A}}_t(\Lambda_2)$. Set the Flag $= 1$.
- If the mark is red and the corresponding cylinder can be included either in $\mathbf{A}_{t-}(\Lambda_1)$ or in $\mathbf{A}_{t-}(\Lambda_2)$ but not in both of them, then ignore the mark. Keep the Flag $= 0$.
- If the mark is red of blue but the corresponding cylinder can not be included in neither $\mathbf{A}_{t-}(\Lambda_1)$ nor in $\mathbf{A}_{t-}(\Lambda_2)$ then ignore the mark. Keep the Flag $= 0$.

• If at time $-t$ a (blue or red) mark appears and Flag$(t-) = 1$, then use blue marks for $\mathbf{A}_t(\Lambda_1)$, $\mathbf{A}_t(\Lambda_2)$, and $\widehat{\mathbf{A}}_t(\Lambda_1)$ and red marks for $\widehat{\mathbf{A}}_t(\Lambda_2)$. Keep the Flag $= 1$.

To verify that the above coupling give the right marginals, it suffices to notice that the rate of inclusion of cylinders in each one of the marginals is the correct one. This defines the coupling when the Poisson processes are considered only in a finite box of $\mathbb{Z}^d$.

Infinite volume case. Define $\overline{\mathbf{H}}_k$ analogously to $\mathbf{H}_k$ given in (4.23) but now use the time interval $[-h/2, 0]$ and both red and blue marks. Therefore in the combined set of cylinders there is no non-oriented percolation and we can construct the coupling working in a finite set of $\overline{\mathbf{H}}_k$'s at a time.

We enlarge our probability space and continue using $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation with respect to the probability space generated by the product of the blue and red Poisson processes.

4.4.3 Proof of (4) of Theorem 4.7

Using (3.35) and (3.36) we get

$$|\mu(fg) - \mu f \, \mu g|$$

$$
\begin{aligned}
&= \mathbb{E}(f(\eta_0)g(\eta_0)) - \mathbb{E}f(\eta_0)\,\mathbb{E}g(\eta_0) \\
&= \mathbb{E}\Big[\Phi(\mathbf{A}(\mathrm{Supp}\,(f)))\Phi(\mathbf{A}(\mathrm{Supp}\,(g))) - \Phi(\widehat{\mathbf{A}}(\mathrm{Supp}\,(f)))\Phi(\widehat{\mathbf{A}}(\mathrm{Supp}\,(g)))\Big], \qquad (4.26)
\end{aligned}
$$

where Φ is the function referred to in (3.36). It is clear that

$$
\mathbf{A}(\mathrm{Supp}\,(f)) = \lim_{t\to\infty} \mathbf{A}_t(\mathrm{Supp}\,(f)) \qquad (4.27)
$$

and similarly for $\mathrm{Supp}\,(g)$ and $\widehat{\mathbf{A}}_t$. Now we use the coupling of Section 4.4.2 with $\Lambda_1 = \mathrm{Supp}\,(f)$ and $\Lambda_2 = \mathrm{Supp}\,(g)$. Since $\mathbf{A}_t(\mathrm{Supp}\,(f)) = \widehat{\mathbf{A}}_t(\mathrm{Supp}\,(f))$ for all $t \geq 0$ and $\mathrm{Flag}(t) = 0$ implies $\mathbf{A}_t(\mathrm{Supp}\,(g)) = \widehat{\mathbf{A}}_t(\mathrm{Supp}\,(g))$, then

$$
\begin{aligned}
&\mathrm{Flag}(\infty) = 0 \text{ implies} \\
&\qquad \Phi(\mathbf{A}(\mathrm{Supp}\,(f)))\,\Phi(\mathbf{A}(\mathrm{Supp}\,(g))) - \Phi(\widehat{\mathbf{A}}(\mathrm{Supp}\,(f)))\,\Phi(\widehat{\mathbf{A}}(\mathrm{Supp}\,(g))) = 0. \qquad (4.28)
\end{aligned}
$$

By definition,

$$
\Phi(\mathbf{A}(\mathrm{Supp}\,(f))) \leq \|f\|_\infty; \quad \Phi(\mathbf{A}(\mathrm{Supp}\,(g))) \leq \|g\|_\infty. \qquad (4.29)
$$

This and (4.28) imply that the absolute value of the right hand side of (4.26) is dominated by

$$
2\|f\|_\infty\,\|g\|_\infty \mathbb{P}(\mathrm{Flag}(\infty) = 1). \qquad (4.30)
$$

To conclude, observe that the flag changes from 0 to 1 (and remains 1 forever) when a cylinder of $\mathbf{A}_t(\mathrm{Supp}\,(g))$ is incompatible with a cylinder of $\widehat{\mathbf{A}}_t(\mathrm{Supp}\,(f))$. This implies

$$
\mathrm{Flag}(\infty) = \mathbf{1}\{\mathbf{A}(\mathrm{Supp}\,(f)) \not\sim \widehat{\mathbf{A}}(\mathrm{Supp}\,(g))\}. \qquad (4.31)
$$

Substituting in (4.30) we get (4.13).

To prove (4.12) notice that under the absence of backwards percolation, the spatial projections of the set of ancestors of Supp (f) and Supp (g) are finite. This implies that the right hand side of (4.13) goes to zero. $\square$

5 Branching processes. Time length and space width

In this section we introduce the tools to show that the conditions of Theorems 2.4 and 2.11 imply the hypothesis of Theorems 3.14 and 4.7 respectively. We also show that under those conditions there is an exponential upperbound for the time length and space width of $\mathbf{A}^{x,t}$. The tool is a domination of the backwards percolation process with a multitype branching process.

5.1 Multitype branching processes

We introduce a multitype branching process $\mathbf{B}_n$ in the set of cylinders which dominates $\mathbf{A}_n$. Consider the total order $\prec$ in the set of cylinders induced by the birth times. That is $C \prec C'$ if and only if $\text{Birth}\,(C) \leq \text{Birth}\,(C')$. For any finite set of cylinders $\{C_1, \dots, C_k\}$ such that $C_i \prec C_{i+1}$, $i = 1, \dots, k-1$, define

$$\widetilde{\mathbf{A}}_1^{C_j} = \mathbf{A}_1^{C_j} \setminus \left(\bigcup_{\ell=1}^{j-1} \mathbf{A}_1^{C_\ell} \right). \tag{5.1}$$

This implies that $\widetilde{\mathbf{A}}_1^{C_j}$ are independent sets and

$$\bigcup_{i=1}^{k} \mathbf{A}_1^{C_i} = \bigcup_{i=1}^{k} \widetilde{\mathbf{A}}_1^{C_i}. \tag{5.2}$$

On the other hand, for any C, $\widetilde{\mathbf{A}}_1^{C}$ is stochastically dominated by $\mathbf{A}_1^{C}$ (that is, there exists a joint realization $(\widetilde{\mathbf{A}}_1^{C}, \mathbf{A}_1^{C})$ such that $\mathbb{P}(\widetilde{\mathbf{A}}_1^{C} \subset \mathbf{A}_1^{C}) = 1$). Hence, we can enlarge our probability space in such a way that for any set $\{C_1, \dots, C_k\}$,

$$\widetilde{\mathbf{A}}_1^{C_j} \subset \mathbf{B}_1^{C_j} \tag{5.3}$$

where $\mathbf{B}_1^{C_i}$ are independent sets with the same marginal distribution as $\mathbf{A}_1^{C_i}$. We then have

$$\bigcup_{i=1}^{k} \mathbf{A}_1^{C_i} \subset \bigcup_{i=1}^{k} \mathbf{B}_1^{C_i}. \tag{5.4}$$

The procedure defined by $\mathbf{B}_1$ naturally induces a multitype branching process in the space of cylinders. We define the n-th generation of the branching process by

$$\mathbf{B}_n^{C} = \{\mathbf{B}_1^{C'} : C' \in \mathbf{B}_{n-1}^{C}\} \tag{5.5}$$

where for all C', $\mathbf{B}_1^{C'}$ has the same distribution as $\mathbf{A}_1^{C'}$ and are independent random sets depending only on C'. Inductively,

$$\mathbf{A}_n^{C} \subset \mathbf{B}_n^{C}. \tag{5.6}$$

Indeed, assume

$$\mathbf{A}_i^C \subset \mathbf{B}_i^C, \text{ for } i = 1, \dots, n-1. \tag{5.7}$$

Now

$$\mathbf{A}_n^C = \bigcup_{C' \in \mathbf{A}_{n-1}^C} \mathbf{A}_1^{C'} = \bigcup_{C' \in \mathbf{A}_{n-1}^C} \widetilde{\mathbf{A}}_1^{C'} \tag{5.8}$$

where in the definition of $\widetilde{\mathbf{A}}_1^{C'}$ we use $\{C_1, \dots, C_k\} = \cup_{i=0}^{n-1} \mathbf{A}_i^C$. Using (5.3) we get (5.6). As before we denote

$$\mathbf{B}^C = \cup_{n \geq 0} \mathbf{B}_n^C \quad ; \quad \mathbf{B}^{x,t} = \cup_{n \geq 0} \mathbf{B}_n^{x,t} \quad ; \quad \mathbf{B}^\Upsilon = \cup_{x \in \Upsilon} \mathbf{B}^{x,0}, \tag{5.9}$$

the branching clans of C, (x,t) and Υ (at time 0) respectively. By (5.6),

$$\mathbf{A}^C \subset \mathbf{B}^C; \quad \mathbf{A}^{x,t} \subset \mathbf{B}^{x,t}; \quad \mathbf{A}^\Upsilon \subset \mathbf{B}^\Upsilon. \tag{5.10}$$

Defining the time length and space width of this clan as in (4.4) and (4.5), we get

$$\mathrm{TL}\,(\mathbf{A}^C) \leq \mathrm{TL}\,(\mathbf{B}^C); \quad \mathrm{TL}\,(\mathbf{A}^{x,t}) \leq \mathrm{TL}\,(\mathbf{B}^{x,t}); \quad \mathrm{TL}\,(\mathbf{A}^\Upsilon) \leq \mathrm{TL}\,(\mathbf{B}^\Upsilon), \tag{5.11}$$

and similarly for the respective space widths.

The (multitype) branching process $\mathbf{B}_n$ induces naturally a multitype branching process in the set of contours. For a cylinder C with basis γ and birth time 0, define $\mathbf{b}_n^\gamma \in \mathbb{N}^{\mathbf{G}}$ as

$$\mathbf{b}_n^\gamma(\theta) = |\{C' \in \mathbf{B}_n^C : \mathrm{Basis}\,(C') = \theta\}| \tag{5.12}$$

as the number of cylinders in the nth generation of ancestors of C with basis θ. We have the following relationship:

$$\sum_\theta \mathbf{b}_n^\gamma(\theta) = |\mathbf{B}_n^C|. \tag{5.13}$$

The process $\mathbf{b}_n$ is a multitype branching process whose offspring distributions are Poisson with means

$$\begin{aligned} m(\gamma, \theta) &= \mathbf{1}\{\gamma \not\sim \theta\}\, e^{-\beta|\theta|} \int_0^\infty e^{-t} dt \\ &= \mathbf{1}\{\gamma \not\sim \theta\} e^{-\beta|\theta|}. \end{aligned} \tag{5.14}$$

To see this, notice that the cylinders C' with basis θ that are potential ancestors of C (with basis γ) form a Poisson process of rate $\mathbf{1}\{\gamma \not\sim \theta\}e^{-\beta|\theta|}$. Each of those cylinders will be an ancestor of C if its lifetime is bigger than the difference between the birth time of C and C'. The lifetimes of different cylinders are independent exponentially distributed random variables of rate 1. The probability that the lifetime of any given cylinder is bigger than t is given by e^{-t}. Hence, the birth times of the ancestors of C with basis θ form a (non homogeneous) Poisson process of rate depending on t given by $\mathbf{1}\{\gamma \not\sim \theta\}e^{-\beta|\theta|}e^{-t}$. The mean number of births is given by (5.14).

Lemma 5.15 *The means (5.14) satisfy*

$$\sum_{\theta} m^n(\gamma,\theta) \;\leq\; |\gamma|\,\alpha^n, \tag{5.16}$$

where α is defined in (2.5).

Proof.

$$\begin{aligned}
\sum_{\theta} m^n(\gamma,\theta) &= \sum_{\gamma_1:\gamma_1\not\sim\gamma} e^{-\beta|\gamma_1|} \sum_{\gamma_2:\gamma_2\not\sim\gamma_1} e^{-\beta|\gamma_2|} \cdots \sum_{\theta:\theta\not\sim\gamma_{n-1}} e^{-\beta|\theta|} \\
&\leq |\gamma| \sum_{\gamma_1:\gamma_1\not\sim\gamma} \frac{|\gamma_1|}{|\gamma|} e^{-\beta|\gamma_1|} \sum_{\gamma_2:\gamma_2\not\sim\gamma_1} \frac{|\gamma_2|}{|\gamma_1|} e^{-\beta|\gamma_2|} \cdots \sum_{\theta:\theta\not\sim\gamma_{n-1}} \frac{|\theta|}{|\gamma_{n-1}|} e^{-\beta|\theta|} \\
&\leq |\gamma| \Big(\sup_{\gamma} \sum_{\theta:\theta\not\sim\gamma} \frac{|\theta|}{|\gamma|} e^{-\beta|\theta|}\Big)^n . \;\square
\end{aligned} \tag{5.17}$$

5.2 Continuous-time branching process

Let C be a cylinder with basis γ and birth time 0. Combing backwards continuously in time the branching clan $\mathbf{B}^C$ we define a continuous time multitype branching process $\zeta_t^\gamma(\theta)$ = number of contours of type θ present at time t (of this process) whose initial configuration is δ_γ. Each $C' \in \mathbf{B}^C$ belongs to the first generation of ancestors of a unique cylinder $U(C')$ in $\mathbf{B}^C$; notice that Life$(C') \ni$ Birth$(U(C'))$. In the branching process ζ_t the basis of $C' \in \mathbf{B}^C$ is counted at time t if $-t$ is in the time interval [Birth(C'), Birth$(U(C'))$] and ignored if $-t$ is in the interval [Birth$(U(C'))$, Death(C')] :

$$\zeta_t^\gamma(\theta) = |\{C' \in \bar{\mathbf{B}}^C : \text{Basis}\,(C') = \theta, \text{Birth}\,(C') < -t < \text{Birth}\,(U(C')), \text{Life}\,(C') \ni t\}|. \tag{5.18}$$

It is clear that

$$\sum_\theta \zeta_t^\gamma(\theta) = 0 \text{ implies } \mathrm{TL}\,(\mathbf{B}^C) < t. \tag{5.19}$$

In the process ζ_t, each contour γ lives a mean-one exponential time after which it dies and gives birth to k_θ contours θ, $\theta \in \mathbf{G}$, with probability

$$\prod_\theta \frac{e^{m(\gamma,\theta)}\,(m(\gamma,\theta))^{k_\theta}}{k_\theta!} \tag{5.20}$$

for $k_\theta \geq 0$. These are independent Poisson distributions of mean $m(\gamma,\theta)$. The infinitesimal generator of the process is given by

$$Lf(\zeta) = \sum_{\gamma\in\mathbf{G}} \zeta(\gamma) \sum_{\eta\in\mathcal{Y}_0(\gamma)} \prod_{\theta:\eta(\theta)\geq 1} \frac{e^{m(\gamma,\theta)}\,(m(\gamma,\theta))^{\eta(\theta)}}{\eta(\theta)!}\,[f(\zeta+\eta-\delta_\gamma)-f(\zeta)] \tag{5.21}$$

where $\zeta,\eta \in \mathcal{Y}_0 = \{\zeta \in \mathbb{N}^{\mathbf{G}}; \sum_\theta \zeta(\theta) < \infty\}$ and $\mathcal{Y}_0(\gamma) = \{\zeta \in \mathcal{Y}_0; \zeta(\theta) \geq 1 \text{ implies } \theta \not\sim \gamma\}$ and $f : \mathcal{Y}_0 \to \mathbb{N}$.

Let $M_t(\gamma,\theta)$ be the mean number of contours of type θ in ζ_t and $R_t(\gamma)$ its sum over θ:

$$M_t(\gamma,\theta) = \mathbb{E}\zeta_t^\gamma(\theta); \qquad R_t(\gamma) = \sum_\theta M_t(\gamma,\theta). \tag{5.22}$$

This can be also written as $R_t(\gamma) = \mathbb{E}(\sum_\theta \zeta_t^\gamma(\theta))$. Using the generator given by (5.21), we get the Kolmogorov backwards equations for $R_t(\gamma)$ which read

$$\frac{d}{dt}R_t(\gamma) = \sum_{\gamma'} m(\gamma,\gamma')R_t(\gamma') - R_t(\gamma). \tag{5.23}$$

Since $R_0(\gamma') \equiv 1$, the solution is

$$R_t(\gamma) = \sum_{\gamma'} \Big[\exp[t(m-I)]\Big](\gamma,\gamma') \tag{5.24}$$

where m is the matrix with entries $m(\gamma,\gamma')$ and I is the identity matrix.

Lemma 5.25 *The mean number of branches, $R_t(\gamma)$ satisfies*

$$\mathbb{P}(\sum_\theta \zeta_t^\gamma(\theta) > 0) \;\leq\; R_t(\gamma) \;\leq\; |\gamma|\, e^{(\alpha-1)t}. \tag{5.26}$$

Proof. The first inequality is immediate because $\sum_\theta \zeta_t^\gamma(\theta)$ assumes non-negative integer values and $R_t(\gamma)$ is its mean value. To show the second rewrite (5.24) as

$$R_t(\gamma) = e^{-t}\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\gamma'} m^n(\gamma,\gamma') \tag{5.27}$$

$$\leq e^{-t}\sum_{n\geq 0}\frac{t^n}{n!}\,|\gamma|\,\alpha^n. \tag{5.28}$$

by (5.16). □

5.3 Time length and space width

In this section we provide bounds for the time length and space width of the percolation clan.

Theorem 5.29 *If $\beta > \beta^*$ (i.e. $\alpha(\beta) < 1$), then*

(i) The probability of backward oriented percolation is zero.

(ii) For any positive b,

$$\mathbb{P}(\mathrm{TL}\,(\mathbf{A}^{x,t}) > bt) \;\leq\; \alpha_0\, e^{-(1-\alpha)bt} \tag{5.30}$$

(iii)

$$\mathbb{E}(\mathrm{SW}\,(\mathbf{A}^{x,t})) \;\leq\; \frac{\alpha_0(\beta)}{1-\alpha(\beta)} \tag{5.31}$$

(iv)

$$\mathbb{E}(\exp(a\,\mathrm{SW}\,(\mathbf{A}^{x,t}))) \;\leq\; \frac{\alpha_0(\beta-a)}{1-\alpha(\beta-a)} \tag{5.32}$$

(v)

$$\mathbb{P}(\mathrm{SW}\,(\mathbf{A}^{x,t}) \geq \ell) \;\leq\; \frac{\alpha_0(\widetilde{\beta})}{1-\alpha(\widetilde{\beta})}\, e^{-(\beta-\widetilde{\beta})\,\ell} \tag{5.33}$$

for any $\widetilde{\beta} \in (\beta^*, \beta)$.

Proof (i) We follow an idea of Hall (1985). For each $C \in \mathbf{C}$ we use the domination (5.6) and the identity (5.13). Therefore, to prove that there is no backward oriented percolation it is enough to prove that, for fixed γ

$$\mathbb{P}\Big(\sum_\theta \mathbf{b}_n^\gamma(\theta) \neq 0 \text{ i.o.}\Big) = 0. \tag{5.34}$$

Since $\mathbf{b}_n^\gamma(\theta)$ assumes non negative integer values, by Borel-Cantelli lemma, a sufficient condition for (5.34) is

$$\sum_n \sum_\theta m^n(\gamma, \theta) < \infty. \tag{5.35}$$

But this follows from Lemma 5.15.

(ii) By (5.19) and (5.11) for each $x \in \gamma$ and $s \leq t$

$$\sum_\theta \zeta_t^\gamma(\theta) = 0 \ \text{ implies } \ \mathrm{TL}\,(\mathbf{A}^{x,0}) \leq t. \tag{5.36}$$

Hence,

$$\mathbb{P}(\mathrm{TL}\,(\mathbf{A}^{x,0}) > t) \ \leq \ \sum_{\gamma \ni x} \mathbb{P}(\eta_0(\gamma) = 1)\, R_t(\gamma) \leq \sum_{\gamma \ni x} e^{-\beta|\gamma|}\, R_t(\gamma) \ \leq \ \alpha_0\, e^{-(1-\alpha)t} \tag{5.37}$$

by the rightmost inequality in (5.26).

(iii) We find upperbounds for the space diameter of the backwards percolation clan through upperbounds for the total number of occupied points by the multitype branching process $\mathbf{b}_n$ defined by (5.12). In fact,

$$\mathrm{SW}\,(\mathbf{A}^{x,0}) \leq \sum_{\gamma \ni x} \eta_0(\gamma) \sum_n \sum_\theta |\theta|\, \mathbf{b}_n^\gamma(\theta). \tag{5.38}$$

By (3.33), $\mathbb{E}\eta_0(\gamma) \leq e^{-\beta|\gamma|}$, hence by (5.16)

$$\begin{aligned} \mathbb{E}\Big(\sum_{\gamma \ni x} \eta_0(\gamma) \sum_n \sum_\theta |\theta|\, \mathbf{b}_n^\gamma(\theta)\Big) \ &\leq \ \sum_{\gamma \ni x} e^{-\beta|\gamma|} \sum_n \sum_\theta |\theta|\, m^n(\gamma, \theta) \\ &\leq \ \sum_{\gamma \ni x} e^{-\beta|\gamma|}\, |\gamma| \sum_n \Big(\sup_\gamma \sum_{\theta:\theta \not\sim \gamma} \frac{|\theta|}{|\gamma|}\, e^{-\beta|\theta|}\Big)^n \\ &= \ \alpha_0 \sum_n \alpha^n. \end{aligned} \tag{5.39}$$

(iv) Write

$$\begin{aligned} \mathbb{E}(e^{a\mathrm{SW}}) \ &= \ \sum_\ell e^{a\ell}\, \mathbb{P}(\mathrm{SW} = \ell) \\ &\leq \ \sum_\ell e^{a\ell} \sum_k \sum_{\gamma_1, \ldots, \gamma_k} \mathbf{1}\{|\gamma_1 \cup \ldots \cup \gamma_k| = \ell\}\, \mathbb{P}(\gamma_1 \ni 0, \mathbf{b}_1^{\gamma_1}(\gamma_2) \geq 1, \ldots, \mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \geq 1) \end{aligned} \tag{5.40}$$

By the Markovian property of $\mathbf{b}_n$ we get

$$\begin{aligned}
&\mathbb{P}(\gamma_1 \ni 0, \mathbf{b}_1^{\gamma_1}(\gamma_2) \geq 1, \ldots, \mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \geq 1) \\
&= \mathbb{P}(\gamma_1 \ni 0)\, \mathbb{P}(\mathbf{b}_1^{\gamma_1}(\gamma_2) \geq 1) \ldots \mathbb{P}(\mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \geq 1) \\
&\leq \mathbf{1}\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \ldots, \gamma_{k-1} \not\sim \gamma_k\} \prod_{i=1}^{k} e^{-\beta|\gamma_i|} \qquad (5.41)
\end{aligned}$$

Substituting this in (5.40) and using that

$$e^{a\ell}\, \mathbf{1}\{|\gamma_1 \cup \ldots \cup \gamma_k| = \ell\} \;\leq\; \mathbf{1}\{|\gamma_1 \cup \ldots \cup \gamma_k| = \ell\} \exp\Big(a \sum_{i=1}^{k} |\gamma_i|\Big) \qquad (5.42)$$

we get

$$\begin{aligned}
\mathbb{E}(e^{a\mathrm{SW}}) &\leq \sum_k \sum_{\gamma_1,\ldots,\gamma_k} \mathbf{1}\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \ldots, \gamma_{k-1} \not\sim \gamma_k\} \exp\Big(-(\beta - a) \sum_{i=1}^{k} |\gamma_i|\Big) \\
&= \sum_k \sum_{\gamma_1 \ni 0} |\gamma_1|\, e^{-(\beta-a)|\gamma_1|} \frac{1}{|\gamma_1|} \sum_{\gamma_2 \not\sim \gamma_1} |\gamma_2| e^{-(\beta-a)|\gamma_2|} \ldots \frac{1}{|\gamma_{k-1}|} \sum_{\gamma_k \not\sim \gamma_{k-1}} e^{-(\beta-a)|\gamma_k|} \\
&\leq \alpha_0(\beta - a) \sum_{k \geq 0} (\alpha(\beta - a))^k. \qquad (5.43)
\end{aligned}$$

(v) It suffices to use (iv) and the exponential Chevichev inequality and to notice that a must be less than $\beta - \beta^*$ to avoid a zero in the denominator of (5.32). □

Remarks. **1.** Part (ii) can in fact be proven by a more elementary argument similar to the one used to prove (v). The argument gives the same rate of decay as in (5.30) but a worse leading constant. Let us sketck it.

$$\mathbb{P}(\mathrm{TL}\,(\mathbf{A}^{x,0} > t) \leq \sum_k \sum_{\gamma_1,\ldots,\gamma_k} \mathbf{1}\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \ldots, \gamma_{k-1} \not\sim \gamma_k\}\, \mathbb{P}(S_1 + \cdots + S_k > t) \qquad (5.44)$$

where S_i are independent mean one exponentially distributed random variables and independent of γ_i. The time S_i represents the period between the birth of γ_i and γ_{i+1}. As the sum of independent exponentials is a gamma distribution with parameters k and 1,

$$\mathbb{P}(S_1 + \cdots + S_k > t) = e^{-t} \sum_{i=0}^{k} \frac{t^i}{i!} \qquad (5.45)$$

Therefore (5.44) is bounded by

$$e^{-t} \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{k \geq i} \sum_{\gamma_1 \ni 0} |\gamma_1|\, e^{-\beta|\gamma_1|} \frac{1}{|\gamma_1|} \sum_{\gamma_2 \not\sim \gamma_1} |\gamma_2| e^{-\beta|\gamma_2|} \ldots \frac{1}{|\gamma_{k-1}|} \sum_{\gamma_k \not\sim \gamma_{k-1}} e^{-\beta|\gamma_k|}$$

$$\begin{aligned} &\le e^{-t}\sum_{i=0}^{\infty}\frac{t^i}{i!}\sum_{k\ge i}\alpha_0\,\alpha^{k-1} \\ &= \frac{\alpha_0}{\alpha(1-\alpha)}\,e^{-(1-\alpha)t}. \end{aligned} \tag{5.46}$$

2. In Fernández, Ferrari and Garcia (1998) we offered an alternative proof of part (iii), based on a the computation of the exponential moment of the total population of a subcritical single-type branching process, which dominates the space width. However this proof works in a smaller range of β.

6 Proof of Theorem 2.4

In Theorem 6.1 below we show that the condition $\alpha < \infty$ implies the hypothesis of Theorem 3.14. This proves Theorem 2.4.

Theorem 6.1 *If $\alpha < \infty$, then for all x and positive t the set $A^{x,0}[-t,0]$ has a finite number of cylinders with probability one.*

Proof. Let C be a cylinder with basis γ and birth time 0. Recall the definition of $U(C')$ just before display (5.18) and define

$$\widetilde{\zeta}_t^{\gamma}(\theta) = |\{C' \in \bar{\mathbf{B}}^C : \text{Basis}\,(C') = \theta, -t < \text{Birth}\,(U(C'))\}|. \tag{6.2}$$

The process $\widetilde{\zeta}_t^{\gamma}$ signals all contours born in $[0,t]$ in the process ζ_t^{γ}. Notice that for $x \in \gamma$,

$$|\mathbf{A}^{x,0}[-t,0]| \le |\mathbf{B}^{x,0}[-t,0]| = |\{C \in \mathbf{B}^{x,0} : \text{Birth}\,(C) \in [-t,0]\}| \le \sum_{\theta}\widetilde{\zeta}_t^{\gamma}(\theta) \tag{6.3}$$

We prove that this is finite with probability one by showing it has a finite mean. Defining

$$\widetilde{R}_t(\gamma) = \mathbb{E}\left(\sum_{\theta}\widetilde{\zeta}_t^{\gamma}(\theta)\right), \tag{6.4}$$

a reasoning as in the previous section gives us

$$\widetilde{R}_t(\gamma) = \sum_{\theta}(e^{mt})(\gamma,\theta) \le |\gamma|\,e^{t\alpha} < \infty \tag{6.5}$$

if $\alpha < \infty$. This and (6.3) prove the theorem. □

7 Proof of Theorem 2.11

We prove that the hypothesis of Theorem 2.11 imply those of Theorem 4.7. The different parts of Theorem 2.11 follow by combining Theorem 4.7 and the space width and time length estimations of Theorem 5.29.

Existence and uniqueness The first hypothesis of Theorem 4.7 only requires $\alpha < \infty$ and is proven in the next theorem. The second hypothesis, which requires $\alpha < 1$, was proven in Theorem 5.29. This proves (1) of Theorem 2.11.

Theorem 7.1 *For all $h > 0$ such that*

$$\alpha\, h < 1, \tag{7.2}$$

the probability that there is no (non-oriented) percolation in $(0, h)$ is one.

Proof. Analogously to Section 5.1, we dominate the construction of the set ancestors of the non oriented percolation process by a multitype branching process. In this branching process, the mean number of ancestors θ of a contour γ is

$$\overline{m}(\gamma, \theta) = h\, \mathbf{1}\{\theta \not\sim \gamma\}\, e^{-\beta|\theta|} \tag{7.3}$$

As in Lemma 5.15, this branching process is subcritical if $\overline{\alpha} = \alpha\, h < 1$. □

Exponential time convergence Inequality (2.12) follows from (5.30), (5.31) and (4.10) by choosing $b = (2 - \alpha)^{-1}$. □

Exponential space convergence In view of (4.20) it suffices to bound

$$\mathbb{P}\Big(\mathrm{SW}\,(\mathbf{A}(\mathrm{Supp}\,(f)))\, \geq\, d(\mathrm{Supp}\,(f), \Lambda^c)\Big) \leq \sum_{x \in \mathrm{Supp}\,(f)} \mathbb{P}\Big(\mathrm{SW}\,(\mathbf{A}^{x,0})\, \geq\, d(\{x\}, \Lambda^c)\Big). \tag{7.4}$$

By (5.33), this is bounded by

$$\sum_{x \in \mathrm{Supp}\,(f)} \frac{\alpha_0(\widetilde{\beta})}{1 - \alpha(\widetilde{\beta})}\, e^{-(\beta - \widetilde{\beta})\, d(\{x\}, \Lambda^c)}. \;\square \tag{7.5}$$

Exponential mixing We prove now (2.14). From (3.35) and (3.38):

$$\mu_\Lambda(fg) - \mu_\Lambda f\, \mu_\Lambda g = \mathbb{E}(f(\eta_0)g(\eta_0)) - \mathbb{E}f(\eta_0)\,\mathbb{E}g(\eta_0). \tag{7.6}$$

By (4.13) it is enough to bound

$$\mathbb{P}\Big(\mathbf{A}(\mathrm{Supp}\,(f)) \not\sim \widehat{\mathbf{A}}(\mathrm{Supp}\,(g))\Big) \tag{7.7}$$

where $\widehat{\mathbf{A}}(\mathrm{Supp}\,(g))$ has the same distribution as $\mathbf{A}(\mathrm{Supp}\,(g))$ but is independent of $\mathbf{A}(\mathrm{Supp}\,(f))$. This is bounded by

$$\sum_{\substack{x\in \mathrm{Supp}\,(f),\\ y\in\mathrm{Supp}\,(g)}} \mathbb{P}\Big(\mathbf{A}^{x,0} \not\sim \widehat{\mathbf{A}}^{y,0}\Big) \le \sum_{\substack{x\in \mathrm{Supp}\,(f),\\ y\in\mathrm{Supp}\,(g)}} \mathbb{P}\Big(\mathrm{SW}\,(\mathbf{A}^{x,0}) + \mathrm{SW}\,(\widehat{\mathbf{A}}^{y,0}) \ge |x-y|\Big) \tag{7.8}$$

Using the following inequality which is valid for independent random variables S_1 and S_2,

$$\mathbb{P}(S_1 + S_2 \ge \ell) \le \sum_{j=1}^{\ell} \mathbb{P}(S_1 \ge j)\mathbb{P}(S_2 \ge \ell - j) \tag{7.9}$$

and the exponential decay of (5.33) we get the decay stated in (2.14). □

Central Limit Theorem We apply the central limit theorem for stationary mixing random fields proven by Bolthausen (1982). Let $X_x = \tau_x f$. Let $\mathcal{A}_\Lambda$ be the sigma algebra generated by $\{X_x : x \in \Lambda\}$. Define

$$\begin{aligned}\alpha_{k,\ell}(n) \;=\; \sup\Big\{ & |\mathbb{P}(A_1\cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| : A_1 \in \mathcal{A}_{\Lambda_1}, A_2 \in \mathcal{A}_{\Lambda_2}, \\ & |\Lambda_1| \le k, |\Lambda_2| \le \ell, \mathrm{distance}(\Lambda_1,\Lambda_2) \ge n\Big\}.\end{aligned} \tag{7.10}$$

The simplified version of Bolthausen theorem stated in Remark 1 pag 1049 of his paper says that if there exists a $\delta > 0$ such that $||X_x||_{2+\delta} < \infty$ and

$$\sum_{n=1}^{\infty} n^{d-1} (\alpha_{2,\infty}(n))^{\delta/(2+\delta)} < \infty \tag{7.11}$$

then $D < \infty$ and (2.15) holds. Hence, it suffices to show that $\alpha_{2,\infty}(n)$ decays exponentially fast with n. We can write

$$\alpha_{2,\infty}(n) = \sup_{a, g_1, g_2} |\mu(g_1 g_2) - \mu g_1\, \mu g_2| \tag{7.12}$$

where the supremum is taken over the set of $a \in \mathbb{Z}^d$, g_1 in the set of indicator functions with support on $\mathrm{Supp}\,(f) \cup \tau_a \mathrm{Supp}\,(f)$ and g_2 in the set of indicator functions with support in

$$\bigcup \Big\{ \tau_y \mathrm{Supp}\,(f) : y \in \mathbb{Z}^d \text{ and } |y - x| \geq n \, \forall x \in \mathrm{Supp}\,(f) \Big\}. \tag{7.13}$$

By (2.14)

$$\begin{aligned} \alpha_{2,\infty}(n) &\leq 2\,(M_2)^2 \sum_{\substack{x \in \mathrm{Supp}\,(g_1),\\ y \in \mathrm{Supp}\,(g_2)}} |x-y| e^{-M_3|x-y|} \\ &\leq 4\,(M_2)^2\,|\mathrm{Supp}\,(f)| \sum_{|y| \geq n - 2|\mathrm{Supp}\,(f)|} e^{-M_3|y|} \end{aligned} \tag{7.14}$$

because $|\mathrm{Supp}\,(g_1)| \leq 2\,|\mathrm{Supp}\,(f)|$ and $\|g_1\|_\infty = \|g_2\|_\infty = 1$. Hence, $\alpha_{2,\infty}(n)$ decreases exponentially fast with n. $\square$

8 Proof of Theorem 2.18. Poisson approximation

We construct first the probability space where the $\{\eta^\beta : \beta* < \beta \leq \infty\}$ are constructed. For each $\gamma \in \widetilde{\mathbf{G}} = \cup_j \widetilde{\mathbf{G}}_j$ let $\mathbf{N}_\gamma$ be a marked Poisson process on $\mathbb{R}^{d+2}$ of rate 1. The event points of this process are denoted (u, t, r, s), where $r \in \mathbb{R}^d$, $t, r \in \mathbb{R}$ and $s \in \mathbb{R}^+$. The coordinate t is interpreted as time while the coordinate r is later used to tune the rate of the projected process (u, t). The coordinate s —the *mark*— is an exponential random variable with mean one independent of everything (used later to determine the lifetime of the corresponding point/cylinder). Denote by $\mathbb{P}$ the product measure generated by $(\mathbf{N}_\gamma : \gamma \in \widetilde{\mathbf{G}})$, and by $\mathbb{E}$ the corresponding expectation. We identify the random counting measure $\mathbf{N}_\gamma$ with the corresponding discrete random subset of $R^{d+1} \times \mathbb{R}^+$.

Fix a contour length j and an inverse temperature β. By counting only those points in $\mathbf{N}_\theta$ whose r coordinate is in $[0, e^{-\beta(|\theta|-j)}]$ we generate the $(d+1)$-dimensional marked process

$$\mathbf{N}_{\theta,\beta} \;=\; \text{marked Poisson process of rate } e^{-\beta(|\theta|-j)}. \tag{8.1}$$

The life of each point $(u, t, s) \in \mathbf{N}_{\theta,\beta} \times \mathbb{R}^+$ is the interval $[t, t+s]$.

Define a family of marked point processes indexed by β and $\tau_x\theta$, $x \in \mathbb{Z}^d$, $\theta \in \widetilde{G}$ for Borel sets $I \subset \mathbb{R}$ by

$$N_{\tau_x\theta,\beta}(I) \;=\; \mathbf{N}_{\theta,\beta}(O(xe^{-\beta j/d}, e^{-\beta j/d}/2) \times I) \tag{8.2}$$

where for $y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$, O is the d-dimensional "rectangle"

$$O(y,\rho) = [y_1 - \rho, y_1 + \rho] \times \ldots \times [y_d - \rho, y_d + \rho] \tag{8.3}$$

Since the volume of $O(xe^{-\beta j/d}, e^{-\beta j/d}/2)$ is $e^{-\beta j}$, and $\mathbf{N}_{\theta,\beta}$ has rate $e^{-\beta(|\theta|-j)}$, the resulting process $N_{\tau_x\theta,\beta}(I)$ is a one dimensional marked Poisson process of rate $e^{-\beta|\theta|}$. The marks are the independent exponentially distributed random variables of mean one, inherited from $\mathbf{N}_{\theta,\beta}$. The point of this construction is that all these Poisson processes are constructed simultaneously as a function of the original $d+2$-dimensional Poisson processes.

Now we use the processes $N_{\tau_x\theta,\beta}$ to perform the graphical construction of Section 3.1. We call $\mathbf{C}^\beta$ be the family of cylinders so obtained. Let ξ_t^β be the free network of Section 3.2. and η_t^β the loss networks of (3.3). As in (3.5) and (3.35) these processes have invariant distributions μ_β^0 and μ_β respectively.

Let V be a d-dimensional rectangle as in the statement of the theorem. Let

$$M^0_{\gamma,\beta}(V) = \sum_{x \in V\cdot e^{\beta|\gamma|/d}} \xi_0^\beta(\tau_x\gamma) = \sum_{x \in V\cdot e^{\beta|\gamma|/d}} \sum_{C \in \mathbf{C}^\beta} \mathbf{1}\{\mathrm{Basis}\,(C) = \tau_x\gamma, \mathrm{Life}\,(C) \ni 0\} \tag{8.4}$$

as in (3.3). The super-label zero in the left hand side indicates we are dealing with the free process ξ_t^β, while the sublabel zero in the right hand side indicates time zero. The family $(M^0_{\gamma,\beta}(V) : \gamma \in \widetilde{G}_j)$ consists of $|\widetilde{G}_j|$ independent Poisson random variables with mean:

$$\mathbb{E}M^0_{\gamma,\beta}(V) = |V \cdot e^{\beta|\gamma|/d}|\, e^{-\beta|\gamma|}. \tag{8.5}$$

By (3.30) $\eta_0^\beta(\gamma)$ constructed with the cylinders in $\mathbf{C}^\beta$ is μ_β distributed. Thus we can use $\eta_0^\beta(\gamma)$ in the definition (2.17) of $M_{\gamma,\beta}$. By (3.32)

$$\eta_0^\beta(\gamma) \le \xi_0^\beta(\gamma). \tag{8.6}$$

Hence, $M_{\gamma,\beta}(V) \le M^0_{\gamma,\beta}(V)$.

The joint construction also implies

$$\begin{aligned}\mathbb{P}(M^0_{\gamma,\beta}(V) - M_{\gamma,\beta}(V) \geq 1) &\leq \sum_{x\in V\cdot e^{\beta|\gamma|/d}} \mathbb{P}\Big(\xi_0^\beta(\tau_x\gamma) - \eta_0^\beta(\tau_x\gamma) \geq 1\Big) \\ &\leq \sum_{x\in V\cdot e^{\beta|\gamma|/d}} \Big[\mathbb{P}\Big(\xi_0^\beta(\tau_x\gamma) \geq 1, \eta_0^\beta(\tau_x\gamma) = 0\Big) \\ &\qquad + \mathbb{P}\Big(\xi_0^\beta(\tau_x\gamma) \geq 2, \eta_0^\beta(\tau_x\gamma) = 1\Big)\Big]. \end{aligned} \tag{8.7}$$

¿From the construction, for any $\theta \in \mathbf{G}$,

$$\Big\{\xi_0^\beta(\theta) \geq 1, \eta_0^\beta(\theta) = 0\Big\} \subset \Big\{\mathbf{C}^\beta : \mathbf{C}^\beta \ni C \text{ with Basis}\,(C) = \theta\,\text{Life}\,(C) \ni 0, \mathbf{A}_1^C \neq \emptyset\Big\}. \tag{8.8}$$

The probability of this last event is bounded by

$$\mathbb{P}(\xi_0^\beta(\theta) \geq 1)\ \mathbb{P}\Big(\sum_{\theta'} \mathbf{b}_1^\theta(\theta') \geq 1\Big). \tag{8.9}$$

Since $\mathbb{P}(\xi_0^\beta(\theta) \geq 1) = 1 - \exp(e^{-\beta|\theta|}) \leq e^{-\beta|\theta|}$, the rhs of (8.9) is bounded above by

$$e^{-\beta|\theta|} \sum_{\theta':\theta'\not\sim\theta} e^{-\beta|\theta'|} \ \leq\ e^{-\beta|\theta|}\,|\theta|\,\alpha(\beta). \tag{8.10}$$

On the other hand,

$$\mathbb{P}\Big(\xi_0^\beta(\theta) \geq 2, \eta_0^\beta(\theta) = 1\Big) \ \leq\ \mathbb{P}\Big(\xi_0^\beta(\theta) \geq 2\Big) \ \leq\ \frac{1}{2}\,e^{-2\beta|\theta|} \tag{8.11}$$

¿From (8.7)–(8.11) we get

$$\mathbb{P}(M^0_{\gamma,\beta}(V) - M_{\gamma,\beta}(V) \geq 1) \ \leq\ \Big|V\cdot e^{\beta|\gamma|/d}\Big|\,e^{-\beta|\gamma|}\,\Big(|\gamma|\,\alpha(\beta)\ +\ \frac{1}{2}\,e^{-\beta|\gamma|}\Big)\ \sim\ e^{-2d\beta}. \tag{8.12}$$

To finish the proof of (2.19) we must show that $M^0_{\gamma,\beta}$ is close to a Poisson process. For $|\gamma| = j$, let $M_{\gamma,\infty}$ count those points of the Poisson process $\mathbf{N}_{\gamma,\infty}$ whose life contains the origin. The process $M_{\gamma,\infty}$ is a Poisson process in $\mathbb{R}^d$ of rate one. This is because the lifetimes are independent exponentials of mean one and for $|\gamma| = j$, $\mathbf{N}_{\gamma,\infty}$ is a Poisson process of rate one. The family $\{M_{\gamma,\infty} : \gamma \in \widetilde{G}_j\}$ inherits independence from $\{\mathbf{N}_\gamma : \gamma \in \widetilde{\mathbf{G}}\}$. For $J \subset \mathbb{Z}^d$ let

$$J : a \ =\ \{r \in \mathbb{R}^d : ra \in J + [1/2\,, 1/2]^d\} \ \subset\ \mathbb{R}^d. \tag{8.13}$$

By definition,

$$M^0_{\gamma,\beta}(V) \;=\; M_{\gamma,\infty}\Big((V\cdot e^{\beta j/d}):e^{\beta j/d}\Big). \tag{8.14}$$

Then, as $(V\cdot e^{\beta j/d}):e^{\beta j/d}\subset V$,

$$\begin{aligned}\mathbb{P}\Big(M^0_{\gamma,\beta}(V)-M_{\gamma,\infty}(V)\neq 0\Big) &\le \mathbb{P}\Big(M_{\gamma,\infty}\Big(V\setminus[(V\cdot e^{\beta j/d}):e^{\beta j/d}]\Big)>0\Big)\\ &\le \Big|V\setminus[(V\cdot e^{\beta j/d}):e^{\beta j/d}]\Big|\\ &\le 2d\,|V|^{(d-1)/d}\,e^{-\beta j/d}. \end{aligned}\tag{8.15}$$

Inequality (2.19) follows from (8.12) and (8.15).

Proposition I.2 of Neveu (1977) and the comments below the statement of the Proposition say that the distribution on finite unions of d- dimensional finite volume rectangles is enough to characterize a point process. Since the estimates (2.19) can be easily extended to finite unions of rectangles, the weak convergence follows. □

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Roberto Fernández,
IEA USP,
Av. Prof. Luciano Gualberto,
Travessa J, 374 Térreo
05508-900 - São Paulo,
BRAZIL
email: `rf@ime.usp.br`

Pablo A. Ferrari
IME USP,
Caixa Postal 66281,
05389-970 - São Paulo,
BRAZIL
email: `pablo@ime.usp.br`
http://www.ime.usp.br/~pablo

Nancy L. Garcia
IMECC, UNICAMP, Caixa Postal 6065,
13081-970 - Campinas SP
BRAZIL
email: `nancy@ime.unicamp.br`
http://www.ime.unicamp.br/~nancy