Asymptotic Dynamical Difference between the Nonlocal and Local Swift-Hohenberg Models *

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Abstract

In this paper the difference in the asymptotic dynamics between the nonlocal and local two-dimensional Swift-Hohenberg models is investigated. It is shown that the bounds for the dimensions of the global attractors for the nonlocal and local Swift-Hohenberg models differ by an absolute constant, which depends only on the Rayleigh number, and upper and lower bounds of the kernel of the nonlocal nonlinearity. Even when this kernel of the nonlocal operator is a constant function, the dimension bounds of the global attractors still differ by an absolute constant depending on the Rayleigh number.

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1 Introduction

Fluid convection due to density gradients arises in geophysical fluid flows in the atmosphere, oceans and the earth's mantle. The Rayleigh-Benard convection is a prototypical model for fluid convection, aiming at predicting spatio-temporal convection patterns. The mathematical model for the Rayleigh-Benard convection involves nonlinear Navier-Stokes partial differential equations coupled with the temperature equation. When the Rayleigh number is near the onset of the convection, the Rayleigh-Benard convection model may be approximately reduced to an amplitude or order parameter equation, as derived by Swift and Hohenberg ([15]).

In the current literature, most work on the Swift-Hohenberg model deals with the following one-dimensional equation for $w(x,t)$, which is a localized, one-dimensionalized version of the model originally derived by Swift and Hohenberg ([15]),

$$w_t \;=\; \mu w - (1+\partial_{xx})^2 w - w^3. \tag{1}$$

The cubic term w^3 is used as an approximation of a nonlocal integral term. For the (local) one-dimensional Swift-Hohenberg equation (1), there has been some recent research on propagating or steady patterns (e.g., [1], [6], [9]). Mielke and Schneider([10]) proved the existence of the global attractor in a weighted Sobolev space on the whole real line. Hsieh et al. ([7], [8]) remarked that the elemental instability mechanism is the negative diffusion term $-w_{xx}$.

Roberts ([12], [13]) recently re-examined the rationale for using the Swift-Hohenberg model as a reliable model of the spatial pattern evolution in specific physical systems. He argued that, although the localization approximation used in (1) makes some sense in the one-dimensional case, this approximation is deficient in the two-dimensional convection problem and one should use the nonlocal Swift-Hohenberg model ([15], [12], [13]):

$$u_t = \mu u - (1+\Delta)^2 u - u \int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})\, u^2(\xi,\eta,t)\, d\xi d\eta, \tag{2}$$

where $u = u(x,y,t)$ is the unknown amplitude function, μ measures the difference of the Rayleigh number from its critical onset value, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator, and $G(r)$ is a given radially symmetric function ($r = \sqrt{x^2+y^2}$). The equation is defined for $t > 0$ and $(x,y) \in D$, where D is a bounded planar domain with smooth boundary ∂D.

The two-dimensional version of the local Swift-Hohenberg equation for $u(x,y,t)$ is

$$u_t \;=\; \mu u - (1+\Delta)^2 u - u^3. \tag{3}$$

Here u^3 is used to approximate the nonlocal term in (2).

Roberts ([12], [13]) noted that the range of Fourier harmonics generated by the nonlinearities is fundamentally different in two-dimensions than in one-dimension. This difference requires a more sophisticated treatment of two-dimensional convection problem, which leads to nonlocal nonlinearity in the Swift-Hohenberg model. He also argued that nonlocal operators naturally appear in systematic derivation of simplified models for pattern evolution, and nonlocal operators also permit symmetries which are consisitent with physical considerations.

In this paper, we discuss the difference between nonlocal and local two-dimensional Swift-Hohenberg models (2), (3), from a viewpoint of asymptotic dynamics. We show that the bounds for the dimensions of the global attractors for the nonlocal and local Swift-Hohenberg models differ by an absolute constant, which depends only on the the Rayleigh number, and upper and lower bounds of the kernel of the nonlocal nonlinearity. Even when this kernel is a constant function, the dimension bounds of the global attractors still differ by a constant depending on the Rayleigh number. In §2 and §3, we will consider the nonlocal and local Swift-Hohenberg models, respectively. Finally in §4, we summarize the results.

2 Nonlocal Swift-Hohenberg Model

In this section, we discuss the global attractor and its dimension estimate for the nonlocal Swift-Hohenberg model (2). In the following we use the abbreviations $L^2 = L^2(D)$, $L^\infty = L^\infty(D)$, $H^k = H^k(D)$ and $H_0^k = H_0^k(D)$ (k is a non-negative integer) for the standard Sobolev spaces. Let $(\cdot,\cdot)$, $\|\cdot\| \equiv \|\cdot\|_2$ denote the standard inner product and norm in L^2, respectively. The norm for H_0^k is $\|\cdot\|_{H_0^k}$. Due to the Poincaré inequality, $\|D^k u\|$ is an equivalent norm in H_0^k.

We rewrite the two-dimensional nonlocal Swift-Hohenberg equation (2) as

$$u_t + \alpha u + 2\Delta u + \Delta^2 u + u\int_D G(\sqrt{(x-\xi)^2 + (y-\eta)^2})u^2(\xi,\eta,t)d\xi d\eta = 0, \tag{4}$$

where $\alpha = 1 - \mu$. This equation is supplemented with the initial condition

$$u(x,y,0) = u_0(x,y), \tag{5}$$

and the boundary conditions

$$u|_{\partial D} = 0, \ \ \frac{\partial u}{\partial n}|_{\partial D} = 0, \tag{6}$$

where n denotes the unit outward normal vector of the boundary ∂D.

In this paper, we assume the following conditions for every $t \geq 0$ and $(x,y) \in D$,

$$0 < b \leq G(\sqrt{x^2+y^2}) \leq a, \text{ and } G,\ \nabla G,\ \Delta G \in L^\infty(D), \tag{7}$$

where $a, b > 0$ are some positive constants and $\nabla = (\partial_x, \partial_y)$ is the gradient operator. Denote $K_1 = \|\nabla G\|_\infty$ and $K_2 = \|\Delta G\|_\infty$.

To study the global attractor, we need to derive some a priori estimates about solutions.

Lemma 1 *Suppose u is a solution of (4)-(6). Then u is uniformly (in time) bounded, and the following estimates hold for $t > 0$*

$$\|u(x,y,t)\|^2 \leq \|u_0(x,y)\|^2 exp(-2\mu t) + \frac{\mu}{b}, \tag{8}$$

and thus

$$\limsup_{t\to+\infty} \|u(x,y,t)\| \leq \sqrt{\frac{\mu}{b}} \equiv R, \tag{9}$$

where $R = \sqrt{\frac{\mu}{b}}$.

Proof. Taking the inner product of (4) with u, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + 2(\Delta u, u) + \alpha\|u\|^2 \\ &+(u^2, \int_D G(\sqrt{(x-\xi)^2 + (y-\eta)^2})u^2(\xi,\eta)d\xi d\eta) = 0. \end{aligned} \tag{10}$$

Note that

$$2|(\Delta u, u)| \leq 2\|\Delta u\|\|u\| \leq \|\Delta u\|^2 + \|u\|^2,$$

$$\begin{aligned} &(u^2, \int_D G(\sqrt{(x-\xi)^2 + (y-\eta)^2})u^2(\xi,\eta)d\xi d\eta) \\ &= \int_D u^2 (\int_D G(\sqrt{(x-\xi)^2 + (y-\eta)^2})u^2(\xi,\eta)d\xi d\eta) dxdy \\ &\geq b\int_D u^2(x,y)dxdy \int_D u^2(\xi,\eta)d\xi d\eta = b\|u\|^4. \end{aligned}$$

Then from (10) we get

$$\frac{d}{dt}\|u\|^2 + 2(\alpha-1)\|u\|^2 + 2b\|u\|^4 \leq 0. \tag{11}$$

It is easy to see that if $\alpha \geq 1$, i.e., $\mu \leq 0$, then all solutions approach zero in L^2. We will not consider this simple dynamical case. In the rest of this paper we assume that $\mu > 0$, i.e., $\alpha < 1$.

Thus we have, for any constant $\epsilon > 0$,

$$\frac{d}{dt}\|u\|^2 + 2\epsilon\|u\|^2 + 2(\alpha - 1 - \epsilon)\|u\|^2 + 2b\|u\|^4 \leq 0, \tag{12}$$

or

$$\frac{d}{dt}\|u\|^2 + 2\epsilon\|u\|^2 + [\frac{(\alpha-1-\epsilon)}{\sqrt{2b}} + \sqrt{2b}\|u\|^2]^2 \leq \frac{(\alpha-1-\epsilon)^2}{2b}. \tag{13}$$

So

$$\frac{d}{dt}\|u\|^2 + 2\epsilon\|u\|^2 \leq \frac{(\alpha - 1 - \epsilon)^2}{2b}. \tag{14}$$

By the usual Gronwall inequality ([17]) we obtain

$$\|u\|^2 \leq \|u_0\|^2 exp(-2\epsilon t) + \frac{(\alpha - 1 - \epsilon)^2}{4b\epsilon}. \tag{15}$$

When $\epsilon = 1 - \alpha = \mu$, we get the optimal or tight estimate

$$\|u\|^2 \leq \|u_0\|^2 exp(-2\mu t) + \frac{\mu}{b}. \tag{16}$$

This completes the proof of Lemma 1. ■

Moreover, higher order derivatives of u are also uniformly bounded.

Lemma 2 *Suppose u is a solution of (4)-(6). Then $\|\nabla u\|$ and $\|\Delta u\|$ are uniformly (in time) bounded.*

In order to prove this lemma, we recall a few useful inequalities.

Uniform Gronwall inequality ([17]). Let g, h, y be three positive locally integrable functions on $[t_0, +\infty)$ satisfying the inequalities

$$\frac{dy}{dt} \leq gy + h,$$

with $\int_t^{t+1} gds \leq a_1$, $\int_t^{t+1} hds \leq a_2$ and $\int_t^{t+1} yds \leq a_3$ for $t \geq t_0$, where the a_i(i=1,2,3) are positive constants. Then

$$y(t+1) \leq (a_2 + a_3)exp(a_1), \text{for } t \geq t_0.$$

Gagliardo-Nirenberg inequality ([11]). Let $w \in L^q \cap W^{m,r}(D)$, where $1 \leq q, r \leq \infty$. For any integer j, $0 \leq j \leq m$, $\frac{j}{m} \leq \lambda \leq 1$.

$$\|D^j w\|_p \leq C_0 \|w\|_q^{1-\lambda} \|D^m w\|_r^{\lambda}$$

provided

$$\frac{1}{p} = \frac{j}{n} + \lambda(\frac{1}{r} - \frac{m}{n}) + \frac{1-\lambda}{q},$$

and $m - j - \frac{n}{r}$ is not a nonnegative integer If $m - j - \frac{n}{r}$ is a nonnegative integer, then the inequality (2) holds for $\lambda = \frac{j}{m}$.

Poincaré inequality ([2]). For $w \in H_0^1(D)$,

$$\lambda_1 \|w\|^2 \leq \|\nabla w\|^2,$$

where λ_1 is the first eigenvalue of $-\Delta$ on the domain D, with zero Dirichlet boundary condition on ∂D.

Proof of Lemma 2. Due to the boundary condition (6) on ∇u and the Poincaré inequality, we get $\|\nabla u\|^2 \leq \lambda_1^{-1}\|\Delta u\|^2$. Hence it is sufficient to prove that $\|\Delta u\|$ is bounded. We first show that $\int_t^{t+1}\|\Delta u\|^2 ds$ is bounded. In fact, using

$$2|(\Delta u, u)| \leq 2\|\Delta u\|\|u\| \leq \frac{1}{2}\|\Delta u\|^2 + 2\|u\|^2,$$

in (10), we get

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + 2(\alpha-2)\|u\|^2 + 2b\|u\|^4 \leq 0. \tag{17}$$

Since

$$\begin{aligned} 2b\|u\|^4 + 2(\alpha-2)\|u\|^2 &= b\|u\|^2 + 2\beta(\|u\|^4 + \frac{2\alpha-4-\beta}{2\beta}\|u\|^2) \\ &= b\|u\|^2 + 2\beta(\|u\|^2 + \frac{2\alpha-4-\beta}{4\beta})^2 - \frac{(2\alpha-4-\beta)^2}{8\beta} \\ &\geq b\|u\|^2 - \frac{(2\alpha-4-\beta)^2}{8\beta}, \end{aligned}$$

we conclude

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + b\|u\|^2 \leq \frac{(2\alpha-4-\beta)^2}{8\beta} = \frac{(2+2\mu+\beta)^2}{8\beta}. \tag{18}$$

Integrating (18) with respect to t from t to $t+1$ and noting Lemma 1, we see that $\int_t^{t+1}\|\Delta u\|^2 ds$ is bounded.

Now, multiplying (4) by $\Delta^2 u$ and integrating over D, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 + 2\int_D \Delta u \Delta^2 u dxdy + \alpha\|\Delta u\|^2 \\ &+ \int_D (u\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u^2(\xi,\eta)d\xi d\eta)\Delta^2 u dxdy = 0. \end{aligned} \tag{19}$$

Note that

$$2|\int_D \Delta u \Delta^2 u dxdy| \leq \frac{1}{2}\|\Delta^2 u\|^2 + 2\|\Delta u\|^2, \tag{20}$$

and

$$\begin{aligned} &|\int_D (u\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u^2(\xi,\eta)d\xi d\eta)\Delta^2 u dxdy| \\ &= |\int_D (\Delta u)^2(\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u^2(\xi,\eta)d\xi d\eta)dxdy \end{aligned}$$

$$+\int_D u\Delta u(\int_D (\Delta G(\sqrt{(x-\xi)^2+(y-\eta)^2})))u^2(\xi,\eta)d\xi d\eta)dxdy$$

$$+2\int_D \nabla u\Delta u(\int_D (\nabla G(\sqrt{(x-\xi)^2+(y-\eta)^2})))u^2(\xi,\eta)d\xi d\eta)dxdy|$$

$$\leq (a\|u\|^2+2\lambda_1^{-\frac{1}{2}}\|\nabla G\|_\infty\|u\|^2+\frac{1}{2}\|\Delta G\|_\infty\|u\|^2)\|\Delta u\|^2+\frac{1}{2}\|\Delta G\|_\infty\|u\|^4$$

$$\leq (a+2\lambda_1^{-\frac{1}{2}}K_1+\frac{1}{2}K_2)\|u\|^2\|\Delta u\|^2+\frac{1}{2}K_2\|u\|^4,$$

where a, K_1, K_2 are various upper bounds of G defined in (7), and R is the L^2 bound of the solution u as in Lemma 1. Hence by (19) we get

$$\frac{d}{dt}\|\Delta u\|^2 \leq 2[(a+2\lambda_1^{-\frac{1}{2}}K_1+\frac{1}{2}K_2)\|u\|^2-\alpha+2]\|\Delta u\|^2+K_2\|u\|^4. \tag{21}$$

Finally, applying the uniform Gronwall inequality (21) and noting Lemma 1, we conclude that $\|\Delta u\|^2$ is uniformly bounded for all $t \geq 0$. This proves Lemma 2. ■

We now have the following global existence and uniqueness result.

Theorem 1 *Let $u_0(x,y) \in L^2(D)$ and G satisfies (7), then the initial-boundary value problem (2), (5), (6) has a unique global solution $u \in L^\infty(0,\infty;H_0^2(D))$. Moreover, the corresponding solution semigroup $S(t)$, defined by*

$$u = S(t)u_0,$$

has a bounded absorbing set

$$B_0 = \{u \in H_0^2(D) : (\|u\|^2+\|\nabla u\|^2+\|\Delta u\|^2)^{\frac{1}{2}} \leq \tilde{R}\},$$

where $\tilde{R}$ is a postive constant which depending on the uniform bound of $\|u\|, \|\nabla u\|, \|\Delta u\|$. Finally, the solution semigroup $S(t)$, when restricted on $H_0^2(D)$, is continuous from $H_0^2(D)$ into $H_0^2(D)$ for $t > 0$.

Proof. The global existence, uniqueness and absorbing property follow from standard arguments (e.g., [17]) together with Lemmas 1, 2 above. The absorbing property also follows from these two lemmas.

We now prove that $S(t)$ is continuous in $H^2(D)\cap H_0^1(D)$. Suppose that $u_0, v_0 \in H_0^2(D)$ with $\|\Delta u_0\|, \|\Delta v_0\| \leq 2R_1$, we denote by $u(t), v(t)$ the corresponding solutions, i.e., $u(t) = S(t)u_0, v(t) = S(t)v_0$. Let $w(t) = u(t) - v(t)$. Then $w(t)$ satisfies

$$w_t+\Delta^2 w+2\Delta w+\alpha w+w\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u^2(\xi,\eta)d\xi d\eta+$$

$$v\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})(u(\xi,\eta)+v(\xi,\eta))w(\xi,\eta)d\xi d\eta = 0. \tag{22}$$

Applying the Gagliardo-Nirenberg inequality

$$\|u\|_\infty \leq C_0\|\Delta u\|,$$

and the Poincaré inequality

$$\|w\| \leq \frac{1}{\lambda_1}\|\Delta w\|,$$

we obtain (similar to the proof of Lemma 2),

$$\frac{d}{dt}\|\Delta w\|^2 \leq C_1\|\Delta w\|^2,$$

which implies that $\|\Delta w(t)\|^2 \leq \|\Delta w_0\|^2 exp(C_1 t)$ for some positive constant C_1. This shows that $S(t)$ is continuous. ∎

This theorem implies that (4)-(6) defines an infinite dimensional nonlocal dynamical system.

In the rest of this section, we consider the global attractor for the nonlocal dynamical system (4)-(6). We will establish the following result about the global attractor.

Theorem 2 *There exists a global attractor $\mathcal{A}$ for the nonlocal dynamical system* (2), (5), (6). *The global attractor is the $\omega-$limit set of the absorbing set B_0 (as in Theorem 1), and it has the following properties:*

(i) A is compact and $S(t)\mathcal{A} = \mathcal{A}$, for $t > 0$;

(ii) for every bounded set $B \subset H_0^2(D)$, $\lim_{t\to\infty} d(S(t)B, \mathcal{A}) = 0$;

(iii)$\mathcal{A}$ is connected in $H_0^2(D)$, where $d(X,Y) = \sup_{x\in X}\inf_{y\in Y}\|x-y\|_{H_0^2(D)}$ is the Hausdorff distance.

Moreover, the global attractor $\mathcal{A}$ has finite Hausdorff dimension $d_H(\mathcal{A}) \leq m$, where

$$m \sim C(1 + \sqrt{\mu + (2a-b)\frac{\mu}{b}}),$$

where $C > 0$ is a constant depending only on the domain D, and $a > 0, b > 0$ are the upper, lower bounds of the kernel G, respectively.

Proof. The existence and properties of $\mathcal{A}$ are quite standard now (see [17] and references therein). We omit this part, and only estimate the dimensions below.

As in [17], we may use the so-called Constantin-Foias-Temam trace formula (which works for the semiflow $S(t)$ here) to estimate the sum of the global Lyapunov exponents of $\mathcal{A}$. The sum of these Lyapunov exponents can then be used to estimate the upper bounds of $\mathcal{A}$'s Hausdorff dimension, $d_H(\mathcal{A})$. To this end, we linearize equation (4) about a solution $u(t)$ in the global attractor to obtain an equation for $v(t)$ and then use the trace formula to estimate the sum of the global Lyapunov exponents. Doing so, we obtain

$$v_t + L(u(t))v = 0, \tag{23}$$

where

$$L(u(t))v = \Delta^2 v + 2\Delta v + \alpha v + v\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u^2(\xi,\eta)d\xi d\eta$$

$$+2u\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u(\xi,\eta)v(\xi,\eta)d\xi d\eta.$$

This equation is supplemented with $v(x,y,0)=\xi(x,y)\in H_0^2(D)$. Denote by $\xi_1(x,y),\dots,\xi_m(x,y)$, m linearly independent functions in $H_0^2(D)$, and $v_i(x,y,t)$ the solution of (23) satisfying $v_i(x,y,0)=\xi_i(x,y)$, $i=1,\dots,m$. Let $Q_m(t)$ represent the orthogonal projection of $H_0^2(D)$ onto the subspace spanned by $\{v_1(x,y,t),\dots,v_m(x,y,t)\}$.

We need to estimate the lower bound of $Tr(L(u(t)Q_m(t)))$, which gives bounds on the sum of global Lyapunov exponents. Note that in [17], the linearized equation like (23) is written as $v_t = L(u(t))v$ and in that case one needs to estimate the upper bound of $Tr(L(u(t)Q_m(t)))$. Suppose that $\phi_1(t),...,\phi_m(t)$ is an orthonormal basis ($\|\phi_j\|=1$) of the subspace $Q_m(t)H_0^2(D)$ for any $t>0$.

Now we estimate the lower bound of $Tr(L(u(t)Q_m(t)))$. It is easy to see that

$$Tr(L(u(t)Q_m))$$

$$=\sum_{j=1}^m(\Delta^2\phi_j + 2\Delta\phi_j + \alpha\phi_j + \phi_j\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u^2(\xi,\eta)d\xi d\eta, \phi_j)$$

$$+\sum_{j=1}^m(2u\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u(\xi,\eta)\phi_j d\xi d\eta, \phi_j).$$

Since $(2\Delta\phi_j,\phi_j)\geq -(\frac{1}{\epsilon}\|\Delta\phi_j\|^2+\epsilon\|\phi_j\|^2)$ for any constant $\epsilon>1$, we get

$$Tr(L(u(t)Q_m)) \geq \sum_{j=1}^m[(1-\frac{1}{\epsilon})\|\Delta\phi_j\|^2 + b\|\phi_j\|^2\|u\|^2 + (\alpha-\epsilon)\|\phi_j\|^2]$$

$$+\sum_{j=1}^m 2\int_D u\phi_j dxdy(\int_D G(\sqrt{(x-\xi)^2+(y-\eta)^2})u(\xi,\eta)\phi_j d\xi d\eta)$$

$$\geq \sum_{j=1}^m(1-\frac{1}{\epsilon})\|\Delta\phi_j\|^2 + \sum_{j=1}^m(b\|u\|^2+\alpha-\epsilon-2a\|u\|^2)$$

$$=\sum_{j=1}^m(1-\frac{1}{\epsilon})\|\Delta\phi_j\|^2 + [1-\mu-\epsilon+(b-2a)\|u\|^2]m. \tag{24}$$

We introduce notation $f(x,y) = \sum_{j=1}^{m} |\phi_j|^2$. Note that $m = \int_D f(x,y)dxdy$. By the generalized Sobolev-Lieb-Thirring inequality ([17], page 462),

$$\int_D f^3(x,y)dxdy \le K_0 \sum_{j=1}^{m} \|\Delta\phi_j\|^2,$$

where $K_0 > 0$ depending only on the domain D. Moreover, due to the fact that $L^3(D) \hookrightarrow L^1(D)$,

$$\begin{aligned} m^3 = (\int_D f(x,y)dxdy)^3 &\le C_2 \int_D f^3(x,y)dxdy \\ &\le K_0 C_2 \sum_{j=1}^{m} \|\Delta\phi_j\|^2 \\ &= C \sum_{j=1}^{m} \|\Delta\phi_j\|^2 \end{aligned}$$

for some constants $C_2 > 0, C > 0$ depending only on the domain D.

Thus

$$(1 - \frac{1}{\epsilon}) \sum_{j=1}^{m} \|\Delta\phi_j\|^2 \ge (1 - \frac{1}{\epsilon}) \frac{1}{C} m^3. \tag{25}$$

Therefore, by (24)-(25) we have

$$\begin{aligned} Tr(L(u(t)Q_m)) &\ge \frac{1-\frac{1}{\epsilon}}{C} m^3 - (\mu - 1 + \epsilon + (2a-b)\|u\|^2)m \\ &\ge \frac{1-\frac{1}{\epsilon}}{C} m^3 - (\mu - 1 + \epsilon + (2a-b)\frac{\mu}{b})m \\ &> 0 \end{aligned} \tag{26}$$

whenever

$$m > \sqrt{[\mu - 1 + \epsilon + (2a-b)\frac{\mu}{b}]\frac{C}{1-\frac{1}{\epsilon}}}. \tag{27}$$

The right hand side of (27) has the minimal value of

$$m \sim C(1 + \sqrt{\mu + (2a-b)\frac{\mu}{b}}) \tag{28}$$

when $\epsilon = 1 + \sqrt{\mu + (2a-b)\frac{\mu}{b}}$.

As in [17], we conclude that the Hausdorff dimension of $\mathcal{A}$ is estimated as in (28). This proves Theorem 2. ■

3 Local Swift-Hohenberg Model

Similarly, for the two-dimensional local Swift-Hohenberg equation (3), we can obtain the existence of the global attractor $\tilde{\mathcal{A}}$. We omit this part and will only estimate the dimension of $\tilde{\mathcal{A}}$.

Theorem 3 *There exists the global attractor $\tilde{\mathcal{A}}$ for the local dynamical system (3), (5), (6). The Hausdorff dimension of $\tilde{\mathcal{A}}$ is finite, and $d_H(\tilde{\mathcal{A}}) \leq m_1 \sim C(1+\sqrt{\mu})$, where C is a constant depending only on the domain D.*

Proof. As in the proof of Ttheorem 2, we consider the linearized equation of (3), defined by

$$v_t + L_1(u(t))v = 0,$$

where

$$L_1(u(t))v = \Delta^2 v + 2\Delta v + \alpha v + 3u^2 v.$$

Then we estimate

$$Tr(L_1(u(t)Q_m))$$

$$= \sum_{j=1}^{m} (\Delta^2\phi_j + 2\Delta\phi_j + \alpha\phi_j + 3u^2\phi_j, \phi_j)$$

$$= \sum_{j=1}^{m} [\|\Delta\phi_j\|^2 + 2(\Delta\phi_j, \phi_j) + \alpha\|\phi_j\|^2 + 3(u^2\phi_j, \phi_j)]$$

$$\geq \sum_{j=1}^{m} (1 - \frac{1}{\epsilon})\|\Delta\phi_j\|^2 + \sum_{j=1}^{m} (\alpha - \epsilon),$$

where we have used the fact that $3(u^2\phi_j, \phi_j) \geq 0$. Noting again that $m^3 \leq C \sum_{j=1}^{m} \|\Delta\phi_j\|^2$ and $\alpha = 1 - \mu$, we have

$$Tr(L_1(u(t)Q_m)) \geq \frac{1-\frac{1}{\epsilon}}{C} m^3 - (\mu - 1 + \epsilon)m > 0 \tag{29}$$

whenever

$$m > \sqrt{(\mu - 1 + \epsilon)\frac{C}{1-\frac{1}{\epsilon}}}. \tag{30}$$

The right hand side of (30) has the minimal value of

$$m \sim C(1+\sqrt{\mu}) \tag{31}$$

when $\epsilon = 1 + \sqrt{\mu}$. This completes the proof. ∎

4 Discussions

In this paper, we have discussed the Hausdorff dimension estimates for the global attractors of the two-dimensional nonlocal and local Swift-Hohenberg model for Rayleigh-Benard convection.

The Hausdorff dimension for the global attractor of the nonlocal model is estimated as

$$m \sim C(1 + \sqrt{\mu + (2a - b)\frac{\mu}{b}}),$$

while for the local model this estimate is

$$m \sim C(1 + \sqrt{\mu}),$$

where $C > 0$ is an absolute constant depending only on the fluid convection domain, and $\mu > 0$ measures the difference of the Rayleigh number from its critical convection onset value. Note that $a, b > 0$ are the upper and lower bounds, respectively, of the kernel G of the nonlocal nonlinearity in (2).

The two dimension estimates above differ by an absolute constant $(2a - b)\frac{\mu}{b}$, which depends only on the the Rayleigh number through μ, and upper and lower bounds of the kernel G of the nonlocal nonlinearity. Moreover, if the kernel G is a constant function (thus, $a = b = G$), then the dimension estimate for the nonlocal model becomes

$$m \sim C(1 + \sqrt{2\mu}),$$

which still differs from the dimension estimate for the local model by a constant depending on the Rayleigh number through μ.

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