DAMTP-98-36
DSF/16-98
OUTP-98-42P
hep-th/9706099

June 1998

String Geometry and the Noncommutative Torus

Giovanni Landi[1]

Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K.

Fedele Lizzi[2]

Dipartimento di Scienze Fisiche, Università di Napoli Federico II
and INFN, Sezione di Napoli, Mostra d'Oltremare Pad. 20, 80125 Napoli, Italy

Richard J. Szabo[3]

Department of Physics – Theoretical Physics, University of Oxford
1 Keble Road, Oxford OX1 3NP, U.K.

Abstract

We describe an intimate relationship between the noncommutative geometry associated with a lattice vertex operator algebra $\mathcal{A}$ and the noncommutative torus. We show that the tachyon subalgebra of $\mathcal{A}$ is naturally isomorphic to a class of twisted modules representing quantum deformations of the algebra of functions on the torus. We construct the corresponding even real spectral triples and determine their Morita equivalence classes using string duality arguments. These constructions yield simple proofs of the $O(d,d;\mathbb{Z})$ Morita equivalences between d-dimensional noncommutative tori and give a natural physical interpretation of them in terms of the target space duality group of toroidally compactified string theory. We classify the automorphisms of the twisted modules and construct the most general gauge theory which is invariant under the automorphism group. We compute bosonic and fermionic actions associated with these gauge theories and show that they are explicitly duality-symmetric. The duality-invariant gauge theory is manifestly covariant but contains highly non-local interactions. We show that it also admits a new sort of particle-antiparticle duality which enables the construction of instanton field configurations in any dimension. The duality non-symmetric on-shell projection of the field theory is shown to coincide with the standard non-abelian Yang-Mills gauge theory minimally coupled to massive Dirac fermion fields.

[1]On leave from Dipartimento di Scienze Matematiche, Università di Trieste, P.le Europa 1, I-34127, Trieste, Italy. Also INFN, Sezione di Napoli, Napoli, Italy.
E-mail: `landi@univ.trieste.it`

[2]E-mail: `lizzi@na.infn.it`

[3]Address after September 1, 1998: The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.
E-mail: `r.szabo1@physics.oxford.ac.uk`

1. Introduction

The noncommutative torus [1]–[3] is one of the basic examples of a noncommutative geometry [4, 5] which captures features of the difference between a manifold and a 'noncommutative space'. Recent interest in such geometries has occurred in the physics literature in the context of their relation to M-Theory [6]–[9]. As shown in the seminal paper [6], the most general solutions to the quotient conditions for toroidal compactifications of Matrix Theory satisfy the algebraic relations of gauge connections on noncommutative tori. This has led to, among other things, new physical insights into the structure of the supergravity sector of M-Theory by relating geometrical parameters of the noncommutative torus to physical parameters of the Matrix Theory gauge theories. In this paper we shall discuss the role played by the noncommutative torus in the short-distance structure of spacetime.

One way to describe the noncommutative torus is to promote the ordinary coordinates x^i, $i = 1, \ldots, d$, of a d-torus to non-commuting operators $\hat{x}^i$ acting on an infinite-dimensional Hilbert space which obey the commutation relations

$$\left[\hat{x}^i, \hat{x}^j\right] = 2\pi i\, \ell^{ij} \tag{1.1}$$

where ℓ^{ij} are real numbers. Defining the basic plane waves

$$U_i = \exp i\hat{x}^i \tag{1.2}$$

it follows from the Baker-Campbell-Hausdorff formula that these operators generate the algebra

$$U_i U_j = \mathrm{e}^{2\pi i \ell^{ij}}\, U_j U_i \tag{1.3}$$

which are the basic defining relations of the noncommutative torus, when viewed as an algebra of functions with generalized Fourier series expansions in terms of the plane waves (1.2). In the limit $\ell^{ij} \to 0$, one recovers the ordinary coordinates and the plane waves (1.2) become the usual ones of the torus. The antisymmetric matrix ℓ^{ij} can therefore be thought of as defining the Planck scale of a compactified spacetime. The noncommutative torus then realizes old ideas of quantum gravity that at distance scales below the Planck length the nature of spacetime geometry is modified. At large distances the usual (commutative) spacetime is recovered.

On the other hand, a viable model for Planck scale physics is string theory and its various extensions to D-brane field theory [10]–[12] and Matrix Theory [13]. In these latter extensions, the short-distance noncommutativity of spacetime coordinates is represented by viewing them as $N \times N$ matrices. The noncommutativity of spacetime in this picture is the result of massless quantum excitations yielding bound states of D-branes with broken supersymmetry [10], or alternatively of the quantum fluctuations in topology of the worldsheets of the D-branes [12]. However, it is also possible to view the spacetime described by ordinary string theory more directly as a noncommutative geometry [14]–[17]. The main idea in this context is to substitute the (commutative) algebra of continuous complex-valued functions on spacetime with the (noncommutative) vertex operator algebra of the underlying conformal field theory. Ordinary spacetime can be recovered by noticing that the algebra of continuous functions is a subalgebra of the vertex operator algebra. The fruitfulness of this approach is that the full vertex operator algebra is naturally invariant under target space duality transformations of the string theory. The simplest such mapping relates large and small radius circles to one another and leads directly to a fundamental length scale, usually the finite size of the string. Although the duality is a symmetry of the full noncommutative string spacetime, different classical spacetimes are identified under the transformation yielding a natural geometrical origin for these quantum symmetries of compactified string theory. This point of view therefore also describes physics at the Planck scale.

In this paper we will merge these two descriptions of the short-distance structure of spacetime as described by noncommutative geometry. We will show that a particular subalgebra, the tachyon algebra, which is obtained by projecting out the string oscillatory modes, of a vertex operator algebra associated with a compactification lattice naturally defines a twisted projective module over the noncommutative torus. The tachyon algebra is in this sense the smallest quantum deformation of ordinary, classical spacetime, and it represents the structure of spacetime below the Planck length. Thus strings compactified on a torus have a geometry which is already noncommutative at short distances. The noncommutative structure derives from the possibility to twist the vertex operator algebra by the canonical two-cocycle of its underlying lattice algebra. The remainder of the full vertex operator algebra acts to yield the non-trivial gauge transformations (including duality) of spacetime. This fact yields yet another interpretation for the noncommutativity of Planck scale spacetime. In string theory spacetime is a set of fields defined on a surface, and at short distances the interactions of the strings (described by the vertex operator algebra) causes the spacetime to become noncommutative. In the case of toroidally compactified string theory, spacetime below the Planck scale is a noncommutative torus. The results of this paper in this way merge the distinct noncommutative geometry formalisms for strings by connecting them all to the geometry of the noncommutative torus. At hand is therefore a unified setting for string theory in terms of (target space) D-brane field theory, Matrix Theory compactifications, and (worldsheet) vertex operator algebras.

Aside from these physically interesting consequences, we will show that the modules we obtain from the vertex operator algebra also bear a number of interesting mathematical characteristics. Most notably, the duality symmetries of the vertex operator algebra lead to a simple proof of the Morita equivalences of noncommutative tori with deformation parameters ℓ^{ij} which are related by the natural action of the discrete group $O(d,d;\mathbb{Z})$ on the space of real-valued antisymmetric $d \times d$ matrices. These Morita equivalences have been established recently using more direct mathematical constructions in [18]. In [19] it was shown that Matrix Theories compactified on Morita equivalent tori are physically equivalent to one another, in that the BPS spectra of states are the same and the associated field theories can be considered to be duals of each other. Here we shall find a direct manifestation of this duality equivalence in terms of the basic worldsheet theory itself. The relationship between vertex operator algebras, duality and the noncommutative torus was originally pointed out in [20] and discussed further in [21]. In a sense, this relationship shows that the duality properties of Morita equivalences in M-Theory are controlled by the stringy sector of the dynamics.

We will also show that the fairly complete classification of the automorphism group of the vertex operator algebra, given in [21], can be used to characterize the symmetries of the twisted modules that we find. This immediately leads us to the construction of an action functional for this particular noncommutative geometry which is naturally invariant under the automorphism group. Generally, the action functional in noncommutative geometry can be used to construct invariants of modules of the given algebra and it presents a natural geometrical origin for many physical theories, such as the standard model [22, 23] and superstring theory [15]. The spectral action principle of noncommutative geometry naturally couples gravitational and particle interactions from a very simple geometric perspective [24, 25]. In the following we shall construct both fermionic and bosonic actions for the twisted modules which possess the same properties as dictated by the spectral action principle. However, since we shall be neither concerned with coupling to gravity nor in renormalization effects, we shall use a somewhat simpler definition than that proposed in [25]. A consequence of the invariance of the action under automorphisms of the algebra is that the action is explicitly duality-symmetric. The construction of explicitly electric-magnetic duality symmetric action functionals has been of particular interest over the years [26, 27] and have had applications to the physics of black holes [28] and of D-branes [29, 30]. They are of special interest now because of the deep relevance of duality symmetries to the spacetime structure of superstring theory within the unified framework of M-Theory.

Because the derivation of the duality-symmetric action involves quite a bit of mathematics, it is worthwhile to summarize briefly the final result here. We shall show that a general gauge theory on the twisted module leads naturally to a target space Lagrangian of the form

$$\mathcal{L} = (F + {}^\star F)_{ij}\,(F + {}^\star F)^{ij} - i\,\overline{\psi_*}\,\gamma^i\left(\partial_i + i\overleftrightarrow{A_i}\right)\psi - i\,\overline{\psi}\,\gamma_i^*\left(\partial_*^i + i\overleftrightarrow{A_*^i}\right)\psi_* \tag{1.4}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i + i\left[A_i, A_j\right] - g_{ik}\,g_{jl}\left(\partial_*^k A_*^l - \partial_*^l A_*^k + i\left[A_*^k, A_*^l\right]\right) \tag{1.5}$$

and $i, j = 1, \ldots, d$. Here the field theory is defined on a Lorentzian spacetime $(x^i, x_j^*) \in \mathbb{R}^d \times (\mathbb{R}^d)^*$, where g_{ij} is the (flat) metric of $\mathbb{R}^d$ and the metric on the total spacetime is $(g_{ij}, -g_{ij})$. We have defined $\partial_i = \partial/\partial x^i$ and $\partial_*^j = \partial/\partial x_j^*$. The fields $A_i(x, x^*)$ and $A_*^j(x, x^*)$ are a dual pair of gauge fields, $\psi(x, x^*)$ and $\psi_*(x, x^*)$ are dual spinor fields, and γ^i and γ_i^* are Dirac matrices on $\mathbb{R}^d$ and $(\mathbb{R}^d)^*$, respectively. The field strength ${}^\star F_{ij}$ is a certain "dual" to the field strength F_{ij} with respect to the Lorentzian metric of the spacetime.

The bars on the fermion fields denote their "adjoints" and the double arrow on the gauge potentials in the fermionic part of the action denotes their left-right symmetric action on the fermion and anti-fermion fields (in a sense which we shall define more precisely in what follows). The commutators and actions of gauge potentials on fermion fields are defined using the noncommutative tachyon algebra structure, so that the action associated with (1.4) describes a certain nonabelian gauge theory coupled to fermions in a nontrivial representation of a gauge group (again this gauge group will be described more precisely in the following).

The action corresponding to (1.4) is explicitly invariant under the interchange of starred and un-starred quantities. This symmetry incorporates the T-duality transformation which inverts the metric g_{ij}, and it moreover contains a particle-antiparticle duality transformation $F_{ij} \leftrightarrow {}^\star F_{ij}$ that represents a certain topological instanton symmetry of the field theory in any dimension d. However, by its very construction, it is manifestly invariant under a much larger symmetry group, including the gauge group, which we shall describe in this paper. In this sense, we will see that the action functional corresponding to (1.4) measures the amount of duality symmetry as well as the strength of the string interactions present in the given spacetime theory. Like the usual formulations of electric-magnetic symmetric actions [26] (see also [20]), (1.4) involves an $O(2,\mathbb{R})$ doublet of vector potentials (A, A_*). The crucial difference between (1.4) and the usual actions is that it is also *manifestly covariant*, without the need of introducing auxiliary fields [27]. This general covariance follows from the fact that the diffeomorphism symmetries of the spacetime are encoded in the tachyon algebra as internal gauge symmetries, so that the gauge invariance of the action automatically makes it covariant. As a consequence of this feature, the on-shell condition for the field strengths is different than those in the usual formulations. Here it corresponds to a dimensional reduction $\mathbb{R}^d \times (\mathbb{R}^d)^* \to \mathbb{R}^d$ in which the field theory becomes ordinary Yang-Mills theory minimally coupled to massive Dirac fermions. This reduction thus yields a geometrical origin for colour degrees of freedom and fermion mass generation. The field theory (1.4) can thereby be thought of as a first stringy extension of many physical models, such as the standard model and Matrix Theory. As a conventional field theory, however, the Lagrangian (1.4) is highly non-local because it contains infinitely many orders of derivative interactions through the definition of the commutators (derived from the algebra (1.3)). This non-locality, and its origin as an associative noncommutative product, reflects the nature of the string interactions.

Thus, the natural action functional associated with the noncommutative geometry of string theory not only yields an explicitly duality-symmetric (non-local) field theory, but it also suggests a sort of noncommutative Kaluza-Klein mechanism for the origin of nonabelian gauge degrees of freedom and particle masses. The explicit invariance of (1.4) under the duality group of the spacetime thereby yields a physical interpretation of the mathematical notion of Morita equivalence. This gauge theory on the noncommutative torus is different from those discussed in the context of Matrix Theory [6]–[9] but it shares many of their duality properties. The formalism of the present paper may thus be considered as a step towards the formulation of Matrix Theory in terms of the framework of spectral triples in noncommutative geometry [11]. It is a remarkable feature that such target space dynamics can be induced so naturally at the level of a worldsheet formalism.

The structure of the remainder of this paper is as follows. All ideas and results of noncommutative geometry which we use are briefly explained throughout the paper. In section 2 we will briefly define the vertex operator algebra associated with toroidally compactified string theory. In section 3 we study the tachyon subalgebra of the lattice vertex operator algebra and show that it defines a particular twisted module over the noncommutative torus. In section 4 we introduce a set of spectral data appropriate to the noncommutative geometry of string theory. In section 5 we exploit the duality symmetries of this noncommutative geometry to study some basic properties of the twisted module, including its Morita equivalence classes and its group of automorphisms. In section 6 the most general gauge theory on the module is constructed, and in section 7 that gauge theory is used to derive the duality-symmetric action functional. In section 8 we describe some heuristic, physical aspects of these twisted modules along the lines described in this section. For completeness, an appendix at the end of the paper gives a brief overview of the definition and relevant mathematical significance of Morita equivalence in noncommutative geometry.

2. Lattice Vertex Operator Algebras

In this section we will briefly review, mainly to introduce notation, the definition of a lattice vertex operator algebra (see [17, 31] and references therein for more details). Let L be a free infinite discrete abelian group of rank d with $\mathbb{Z}$-bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle_L : L \times L \to \mathbb{R}^+$ which is symmetric and nondegenerate. Given a basis $\{e_i\}_{i=1}^d$ of L, the symmetric nondegenerate tensor

$$g_{ij} = \langle e_i, e_j\rangle_L \tag{2.1}$$

defines a Euclidean metric on the flat d-dimensional torus $T_d \equiv \mathbb{R}^d/2\pi L$. We can extend the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_L$ to the complexification $L^c = L \otimes_{\mathbb{Z}} \mathbb{C}$ by $\mathbb{C}$-linearity. The dual lattice to L is then

$$L^* = \{p \in L^c \mid \langle p, w\rangle_L \in \mathbb{Z} \;\; \forall w \in L\} \tag{2.2}$$

which is also a Euclidean lattice of rank d with bilinear form g^{ij} inverse to (2.1) that defines a metric on the dual torus $T_d^* \equiv \mathbb{R}^d/2\pi L^*$.

Given the lattice L and its dual (2.2), we can form the free abelian group

$$\Lambda = L^* \oplus L \tag{2.3}$$

If $\{e^i\}_{i=1}^d$ is a basis of L^* dual to a basis $\{e_i\}_{i=1}^d$ of L, then the chiral basis of Λ is $\{e^i_\pm\}_{i=1}^d$, where

$$e^i_\pm = \tfrac{1}{\sqrt{2}}\left(e^i \pm g^{ij} e_j\right) \tag{2.4}$$

with an implicit sum over repeated indices always understood. Given $p \in L^*$ and $w \in L$, we write the corresponding elements of Λ with respect to the basis (2.4) as $p^\pm$ with components

$$p_i^\pm = \tfrac{1}{\sqrt{2}}\Big(p_i \pm \langle e_i, w\rangle_L\Big) \tag{2.5}$$

Then with $q_i^\pm = \frac{1}{\sqrt{2}}(q_i \pm \langle e_i, v\rangle_L)$, we can define a $\mathbb{Z}$-bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle_\Lambda : \Lambda\times\Lambda\to\mathbb{Z}$ by

$$\langle p,q\rangle_\Lambda \equiv p_i^+ g^{ij} q_j^+ - p_i^- g^{ij} q_j^- = \langle p,v\rangle_L + \langle q,w\rangle_L \tag{2.6}$$

which makes Λ an integral even self-dual Lorentzian lattice of rank $2d$ and signature (d,d) which is called the Narain lattice. Note that, in the chiral basis (2.4), the corresponding metric tensor is

$$\eta_{\alpha\beta}^{ij} \equiv \Big\langle e_\alpha^i, e_\beta^j\Big\rangle_\Lambda = \begin{cases} \pm\, g^{ij} & , \quad \alpha=\beta=\pm \\ 0 & , \quad \alpha\neq\beta \end{cases} \tag{2.7}$$

The commutator map

$$c_\Lambda(p^+,p^-;q^+,q^-) = \mathrm{e}^{i\pi\langle q,w\rangle_L} \tag{2.8}$$

on $\Lambda^c\times\Lambda^c\to\mathbb{Z}_2$ is a two-cocycle of the group algebra $\mathbb{C}[\Lambda]$ generated by the vector space Λ^c, the complexification of Λ. This two-cocycle corresponds to a central extension $\widetilde{\Lambda^c}$ of Λ^c,

$$1 \;\to\; \mathbb{Z}_2 \;\to\; \widetilde{\Lambda^c} \;\to\; \Lambda^c \;\to\; 1 \tag{2.9}$$

where $\widetilde{\Lambda^c} = \mathbb{Z}_2\times\Lambda^c$ as a set and the multiplication is given by

$$\Big(\rho\,;\,q^+,q^-\Big)\cdot\Big(\sigma\,;\,r^+,r^-\Big) = \Big(c_\Lambda(q^+,q^-;r^+,r^-)\rho\sigma\,;\,q^++r^+,q^-+r^-\Big) \tag{2.10}$$

for $\rho,\sigma\in\mathbb{Z}_2$ and $(q^+,q^-),(r^+,r^-)\in\Lambda^c$. This can be used to define the twisted group algebra $\mathbb{C}\{\Lambda\}$ associated with the double cover $\widetilde{\Lambda^c}$ of Λ^c. A realization of $\mathbb{C}\{\Lambda\}$ in terms of closed string modes is given as follows. Viewing L^c and $(L^*)^c$ as abelian Lie algebras of dimension d, we can consider the corresponding affine Lie algebras $\widehat{L^c}$ and $\widehat{(L^*)^c}$, and also the affinization $\widehat{\Lambda^c} = \widehat{L^c}\oplus\widehat{(L^*)^c}$. In the basis (2.4), we then have

$$\widehat{\Lambda^c} \cong \widehat{u(1)}{}_+^d \oplus \widehat{u(1)}{}_-^d \tag{2.11}$$

where the generators $\alpha_n^{(\pm)i}$, $n\in\mathbb{Z}$, of $\widehat{u(1)}{}_\pm^d$ satisfy the Heisenberg algebra

$$\Big[\alpha_n^{(\pm)i},\alpha_m^{(\pm)j}\Big] = n\,g^{ij}\,\delta_{n+m,0} \tag{2.12}$$

The basic operators of interest to us are the chiral Fubini-Veneziano fields

$$X_\pm^i(z_\pm) = x_\pm^i + ig^{ij}p_j^\pm\log z_\pm + \sum_{n\neq 0}\frac{1}{in}\,\alpha_n^{(\pm)i}z_\pm^{-n} \tag{2.13}$$

and the chiral Heisenberg fields

$$\alpha_\pm^i(z_\pm) = -i\partial_{z_\pm}X_\pm^i(z_\pm) = \sum_{n=-\infty}^{\infty}\alpha_n^{(\pm)i}z_\pm^{-n-1} \qquad ; \qquad \alpha_0^{(\pm)i} \equiv g^{ij}p_j^\pm \tag{2.14}$$

where $z_\pm \in \mathbb{C} \cup \{\infty\}$. Classically, the fields (2.13) are maps from the Riemann sphere into the torus T_d, with $x_\pm \in T_d$. When they are interpreted as classical string embedding functions from a spherical worldsheet into a toroidal target space, the first two terms in (2.13) represent the center of mass (zero mode) motion of a closed string while the Laurent series represents its oscillatory (vibrational) modes. The w^i in (2.5) represent the winding modes of the string about each of the cycles of T_d, while the fields (2.14) are the conserved currents which generate infinitesimal reparametrizations of the torus.

Upon canonical quantization, the oscillatory modes satisfy (2.12) and the zero modes form a canonically conjugate pair,

$$\left[x_\alpha^i, p_j^\beta\right] = i\,\delta_\alpha^\beta\,\delta_i^j \tag{2.15}$$

where $\alpha, \beta = \pm$. Then the multiplication operators

$$\varepsilon_{q^+q^-}(p^+,p^-) \equiv \mathrm{e}^{-iq_i^+x_+^i - iq_i^-x_-^i}\; c_\Lambda(p^+,p^-;q^+,q^-) \tag{2.16}$$

generate the twisted group algebra $\mathbb{C}\{\Lambda\}$. The fields $X_\pm$ now become quantum operators which act formally on the infinite-dimensional Hilbert space

$$\mathbf{h} = L^2\left(T_d \times T_d^*\,,\,\prod_{i=1}^d \tfrac{dx^i\,dx_i^*}{(2\pi)^2}\right) \otimes \mathcal{F}^+ \otimes \mathcal{F}^- \tag{2.17}$$

where $x^i \equiv \frac{1}{\sqrt{2}}(x_+^i + x_-^i)$ and $x_i^* \equiv \frac{1}{\sqrt{2}}\,g_{ij}(x_+^j - x_-^j)$ define, respectively, local coordinates on the torus T_d and on the dual torus T_d^*. The L^2 space in (2.17) is generated by the canonical pairs (2.15) of zero modes and is subject to the various natural isomorphisms

$$\begin{aligned} L^2\left(T_d \times T_d^*\,,\,\prod_{i=1}^d \tfrac{dx^i\,dx_i^*}{(2\pi)^2}\right) &\cong L^2\left(T_d\,,\,\prod_{i=1}^d \tfrac{dx^i}{2\pi}\right) \otimes_{\mathbb{C}} L^2\left(T_d^*\,,\,\prod_{i=1}^d \tfrac{dx_i^*}{2\pi}\right) \\ &\cong \bigoplus_{w\in L} L^2\left(T_d\,,\,\prod_{i=1}^d \tfrac{dx^i}{2\pi}\right) \\ &\cong \bigoplus_{p\in L^*} L^2\left(T_d^*\,,\,\prod_{i=1}^d \tfrac{dx_i^*}{2\pi}\right) \end{aligned} \tag{2.18}$$

The Hilbert space (2.17) is thus a module for the group algebras $\mathbb{C}[L]$ and $\mathbb{C}[L^*]$, and also for $\mathbb{C}\{\Lambda\}$. The dense subspace $C^\infty(T_d \times T_d^*) \subset L^2(T_d \times T_d^*\,,\,\prod_{i=1}^d \frac{dx^i\,dx_i^*}{(2\pi)^2})$ of smooth complex-valued functions on $T_d \times T_d^*$ is a unital $*$-algebra which is spanned by the eigenstates $|q,v\rangle = |q^+,q^-\rangle = \mathrm{e}^{-iq_ix^i - iv^ix_i^*}$ of the operators $-i\frac{\partial}{\partial x^i}$ and $-i\frac{\partial}{\partial x_i^*}$ on $T_d \times T_d^*$, where $q \in L^*$ and $v \in L$. The spaces $\mathcal{F}^\pm$ are bosonic Fock spaces generated by the oscillatory modes $\alpha_n^{(\pm)i}$ which act as annihilation operators for $n > 0$ and as creation operators for $n < 0$ on some vacuum states $|0\rangle_\pm$.

The basic single-valued quantum fields which act on the Hilbert space (2.17) are the chiral tachyon operators

$$V_{q^\pm}(z_\pm) = {}^\circ_\circ\, \mathrm{e}^{-iq_i^\pm X_\pm^i(z_\pm)}\, {}^\circ_\circ \tag{2.19}$$

where $(q^+,q^-) \in \Lambda$ and ${}^\circ_\circ \cdot {}^\circ_\circ$ denotes the Wick normal ordering defined by reordering the operators (if necessary) so that all $\alpha_n^{(\pm)i}$, $n < 0$, and $x_\pm^i$ occur to the left of all $\alpha_n^{(\pm)i}$,

$n>0$, and $p_i^\pm$. The twisted dual $\widetilde{\mathbf{h}^*}$ of the Hilbert space (2.17) is spanned by operators of the form

$$\Psi=\varepsilon_{q^+q^-}(p^+,p^-)\otimes\prod_k\alpha_{-n_k}^{(+)i_k}\otimes\prod_l\alpha_{-m_l}^{(-)j_l} \tag{2.20}$$

To (2.20) we associate the vertex operator

$$V(\Psi;z_+,z_-)={}^\circ_\circ\, V_{q^+q^-}(z_+,z_-)\prod_k\frac{1}{(n_k-1)!}\,\partial_{z_+}^{n_k-1}\alpha_+^{i_k}(z_+)\prod_l\frac{1}{(m_l-1)!}\,\partial_{z_-}^{m_l-1}\alpha_-^{j_l}(z_-)\,{}^\circ_\circ \tag{2.21}$$

where

$$\begin{aligned}V_{q^+q^-}(z_+,z_-)&\equiv V\left(\varepsilon_{q^+q^-}(p^+,p^-)\otimes\mathbb{I};z_+,z_-\right)\\&=c_\Lambda(p^+,p^-;q^+,q^-)\;{}^\circ_\circ\, V_{q^+}(z_+)\,V_{q^-}(z_-)\,{}^\circ_\circ\\&=\mathrm{e}^{i\pi\langle q,w\rangle_L}\;{}^\circ_\circ\,\mathrm{e}^{-iq_i^+X_+^i(z_+)-iq_i^-X_-^i(z_-)}\,{}^\circ_\circ\end{aligned} \tag{2.22}$$

are the basic tachyon vertex operators. This gives a well-defined linear map $\Psi\mapsto V(\Psi;z_+,z_-)$ on $\widetilde{\mathbf{h}^*}\to(\mathrm{End}\,\mathbf{h})[z_+^{\pm1},z_-^{\pm1}]$, the latter being the space of endomorphism-valued Laurent polynomials in the variables $z_\pm$. This mapping is known as the operator-state correspondence.

With an appropriate smearing (see the next section), the vertex operators (2.21) yield well-defined and densely-defined operators acting on $\mathbf{h}$. They generate a noncommutative unital $*$-algebra $\mathcal{A}$ with the usual Hermitian conjugation and operator norm (defined on an appropriate dense domain of bounded operators in $\mathcal{A}$). The various algebraic properties of $\mathcal{A}$ can be found in [17, 31], for example. It has the formal mathematical structure of a vertex operator algebra. In this paper we shall not discuss these generic properties of $\mathcal{A}$, but will instead analyse in detail a particular subalgebra of it which possesses some remarkable properties.

It is important to note that $\mathcal{A}\cong\widetilde{\mathcal{E}_\Lambda}$ is the $\mathbb{Z}_2$-twist of the algebra $\mathcal{E}_\Lambda=\mathcal{E}_+\otimes_{\mathbb{C}}\mathcal{E}_-$, where $\mathcal{E}_\pm$ are the chiral algebras generated by the operators (2.19). The lattice L^* itself yields the structure of a vertex operator algebra $\mathcal{E}_{L^*}$ without any reference to its dual lattice or the Narain lattice with its chiral sectors. Indeed, given a basis $\{e^i\}_{i=1}^d$ of L^* and two-cocycles $c_L(p,q)=\mathrm{e}^{i\pi\langle q,p\rangle_{L^*}}$ corresponding to an S^1 covering group of the dual lattice, the operators

$$\widetilde{V}_q(z)=\mathrm{e}^{i\pi\langle q,p\rangle_{L^*}}\;{}^\circ_\circ\,\mathrm{e}^{-iq_iX^i(z)}\,{}^\circ_\circ \tag{2.23}$$

generate a vertex operator algebra, acting on the Hilbert space $L^2(T_d^*,\prod_{i=1}^d\frac{dx_i^*}{2\pi})\otimes\mathcal{F}$, in an analogous way that the operators (2.22) do [31]. Similarly one defines a vertex operator algebra $\mathcal{E}_L$ associated with the lattice L. Modulo the twisting factors, the algebras $\mathcal{A}$ and $\mathcal{E}_{L^*}\otimes_{\mathbb{C}}\mathcal{E}_L$ are isomorphic since the latter representation comes from changing basis on Λ from the chiral one (2.4) to the canonical one $\{e^i\}_{i=1}^d\oplus\{e_j\}_{j=1}^d$. Physically, the difference between working with the full chirally symmetric algebra $\mathcal{A}$ and only the chiral ones or $\mathcal{E}_{L^*}$, $\mathcal{E}_L$ is that the former represents the algebra of observables of closed strings while the latter ones are each associated with open strings. Furthermore, the algebra $\mathcal{A}$ is that which encodes the duality symmetries of spacetime and maps the various open string algebras among each other via duality transformations. Much of what is said in the following will therefore not only apply as symmetries of a closed string theory, but also as mappings among various open string theories. It is in this way that these results are applicable to D-brane physics and M-Theory.

3. Tachyon Algebras and Twisted Modules over the Noncommutative Torus

The operator product algebra of the chiral tachyon operators (2.19) can be evaluated using standard normal ordering properties to give [17]

$$V_{q^\pm}(z_\pm)\,V_{r^\pm}(z'_\pm) = \left(z_\pm - z'_\pm\right)^{q_i^\pm g^{ij} r_j^\pm} \;\,^\circ_\circ\, V_{q^\pm}(z_\pm)\,V_{r^\pm}(z'_\pm)\,^\circ_\circ \tag{3.1}$$

which leads to the operator product

$$\begin{aligned} V_{q^+q^-}(z_+,z_-)\,V_{r^+r^-}(z'_+,z'_-) \;&=\; \left(z_+ - z'_+\right)^{q_i^+ g^{ij} r_j^+}\left(z_- - z'_-\right)^{q_i^- g^{ij} r_j^-} \\ &\qquad \times \,^\circ_\circ\, V_{q^+q^-}(z_+,z_-)\,V_{r^+r^-}(z'_+,z'_-)\,^\circ_\circ \end{aligned} \tag{3.2}$$

of the full vertex operators (2.22). These expressions are singular at $z_\pm = z'_\pm$ and are therefore to be understood only as formal relationships valid away from coinciding positions of the operators on $\mathbb{C}\cup\{\infty\}$. They are the defining characteristics of the vertex operator algebra which in the mathematics literature are collectively referred to as the Jacobi identity of $\mathcal{A}$ [17, 31].

The tachyon vertex operators thus form a $*$-subalgebra of $\mathcal{A}$ which, according to the operator-state correspondence of the previous section, is associated with the subspace $\mathbf{h}_0 \subset \mathbf{h}$ defined by

$$\mathbf{h}_0 \;\equiv\; \bigotimes_{n>0}\bigotimes_{i=1}^{d}\left(\ker\alpha_n^{(+)i}\otimes\ker\alpha_n^{(-)i}\right) \;\cong\; L^2\left(T_d\times T_d^*\,,\,\textstyle\prod_{i=1}^d \frac{dx^i\,dx_i^*}{(2\pi)^2}\right) \tag{3.3}$$

Algebraically, $\mathbf{h}_0$ is the subspace of highest weight vectors for the representation of $\widehat{u(1)}{}_+^d \oplus \widehat{u(1)}{}_-^d$ on the L^2 space, and it is spanned by the vectors $|q^+;q^-\rangle$. The irreducible (highest weight) representations of this current algebra are labelled by the $u(1)_\pm^d$ charges $q_i^\pm$, and $\mathbf{h}_0$ carries a representation of $u(1)_+^d \oplus u(1)_-^d$ given by the actions of $\alpha_0^{(\pm)i}$. Then,

$$\mathbf{h} = \left(\bigoplus_{(q^+,q^-)\in\Lambda}\mathcal{V}_{q^+q^-}\right)\otimes\mathcal{F}^+\otimes\mathcal{F}^- \tag{3.4}$$

where $q_i^\pm = \frac{1}{\sqrt{2}}(q_i \pm \langle e_i, v\rangle_L)$ and $\mathcal{V}_{q^+q^-} \cong \mathcal{V}_q\otimes\mathcal{V}_v$ are the irreducible $u(1)_+^d \oplus u(1)_-^d$ modules for the representation of the Kac-Moody algebra on $L^2(T_d\times T_d^*\,,\,\prod_{i=1}^d\frac{dx^i\,dx_i^*}{(2\pi)^2})$. In this sense, the tachyon operators generate the full vertex operator algebra $\mathcal{A}$, and the algebraic relations between any set of vertex operators (2.21) can be deduced from the operator product formula (3.2) [17]. In the following we will therefore focus on the tachyon sector of the vertex operator algebra.

If $\mathcal{P}_0 : \mathbf{h} \to \mathbf{h}_0$ is the orthogonal projection onto the subspace (3.3), then the tachyon subalgebra is defined to be

$$\mathcal{A}_0 = \mathcal{P}_0\,\mathcal{A}\,\mathcal{P}_0 \tag{3.5}$$

To describe this algebra, we need to define an appropriate smearing which makes the vertex operators well-defined. For this, we parametrize the coordinates on the Riemann sphere as

$$z_\pm = \mathrm{e}^{\tau \pm i\sigma} \qquad , \qquad (\tau,\sigma) \in \overline{\mathbb{R}} \times S^1 \tag{3.6}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. The smeared tachyon vertex operators are then defined as

$$V_{q^+q^-}(\sigma; f) = \int_{-\infty}^{\infty} d\tau \;V_{q^+q^-}(z_+, z_-)\,f(\tau) \tag{3.7}$$

where $f \in \mathcal{S}(\mathbb{R})$ is a suitably normalized Schwartz space test function. The algebra $\mathcal{A}_0$, with this smearing, is thus a quantized algebra of functions on the loop space $C^\infty(S^1, T_d \times T_d^*)$. Given a chiral basis (2.4) of Λ, we define

$$V_i^\pm(\sigma; f) = \int_{-\infty}^{\infty} d\tau \;V_{e^i_\pm}(z_\pm)\,f(\tau) \tag{3.8}$$

The operator product formula (3.1) implies a cocycle relation among the operators $V_i^\pm$. Interchanging the order of the two operators in (3.1), and choosing the branch on $\mathbb{C} \cup \{\infty\}$ of the logarithm function for which the imaginary part of $\log(-1)$ lies in the interval $[-i\pi, +i\pi]$, it follows that $\mathcal{A}_0$ is generated by the elements $V_i^\pm(\sigma_i^\pm; f)$ subject to the relations

$$\begin{aligned} V_i^\pm(\sigma_i^\pm; f)\,V_j^\mp(\sigma_j^\mp; f) &= V_j^\mp(\sigma_j^\mp; f)\,V_i^\pm(\sigma_i^\pm; f) && (3.9)\\ V_i^\pm(\sigma_i^\pm; f)\,V_i^\pm(\sigma_i^\pm; f)^\dagger &= V_i^\pm(\sigma_i^\pm; f)^\dagger\,V_i^\pm(\sigma_i^\pm; f) \;=\; \mathbb{I} && (3.10)\\ V_i^\pm(\sigma_i^\pm; f)\,V_j^\pm(\sigma_j^\pm; f) &= \mathrm{e}^{2\pi i \omega_\pm^{ij}}\;V_j^\pm(\sigma_j^\pm; f)\,V_i^\pm(\sigma_i^\pm; f) \quad , \quad i \neq j && (3.11) \end{aligned}$$

where $\dagger$ is the $*$-involution on $\mathcal{A}$, $\sigma_1^\pm, \dots, \sigma_d^\pm$ are distinct points on the circle S^1, and

$$\omega_\pm^{ij} = \pm\, g^{ij}\,\mathrm{sgn}(\sigma_i^\pm - \sigma_j^\pm) \qquad , \qquad i \neq j \tag{3.12}$$

with $\omega_\pm^{ii} \equiv 0$ are two-cocyles of the vector space $(L^*)^c$. The cocycles $\omega_\pm$ depend only on the relative orientations of the phases of the given tachyon operators. If we order these phases so that $\pm\sigma_1^\pm > \pm\sigma_2^\pm > \ldots > \pm\sigma_d^\pm$, then $\omega_\pm^{ij} = \omega^{ij} \equiv \mathrm{sgn}(j-i)\,g^{ij}$ for $i \neq j$. This choice is unique up to a permutation in S_{2d} of the coordinate directions of $T_d \times T_d^*$.

The mutually commuting pair of (identical) algebras (3.10),(3.11) consists of two copies $\mathcal{A}_+^{(\omega)} \cong \mathcal{A}_-^{(\omega)} \cong \mathcal{A}^{(\omega)}$ of the algebra of the noncommutative d-torus T_ω^d. Choosing an appropriate closure, the algebra $\mathcal{A}^{(\omega)}$ can be identified with the abstract unital $*$-algebra generated by elements U_i , $i = 1, \dots, d$, subject to the cocycle relations (3.10),(3.11),

$$\begin{aligned} U_i\,U_i^\dagger &= U_i^\dagger\,U_i \;=\; \mathbb{I} && (3.13)\\ U_i\,U_j &= \mathrm{e}^{2\pi i \omega^{ij}}\;U_j\,U_i \quad , \quad i \neq j && (3.14) \end{aligned}$$

The presentation of a generic 'smooth' element of $\mathcal{A}^{(\omega)}$ in terms of the U_i is

$$f=\sum_{p\in L^*} f_p\, U_1^{p_1}U_2^{p_2}\cdots U_d^{p_d} \qquad , \qquad f_p\in\mathcal{S}(L^*) \tag{3.15}$$

where $\mathcal{S}(L^*)$ is a Schwartz space of rapidly decreasing sequences. When $\omega=0$, one can identify the operator U_i with the multiplication by $U_i=\mathrm{e}^{-ix^i}$ on the ordinary torus T_d, so that $\mathcal{A}^{(0)}\cong C^\infty(T_d)$, the algebra of smooth complex-valued functions on T_d. In this case the expansion (3.15) reduces to the usual Fourier series expansion $f(x)=\sum_{p\in L^*} f_p\,\mathrm{e}^{-ip_ix^i}$.

For a general non-vanishing anti-symmetric bilinear form ω, we shall think of the algebra $\mathcal{A}^{(\omega)}$ as a quantum deformation of the algebra $C^\infty(T_d)$. While the $*$-involution is the usual complex-conjugation $f^\dagger(x)=\overline{f(x)}$, given $f,g\in C^\infty(T_d)$ their deformed product is defined to be

$$(f\star_\omega g)(x)=\exp\left(i\pi\omega^{ij}\,\tfrac{\partial}{\partial x^i}\tfrac{\partial}{\partial x'^j}\right) f(x)g(x')\,\Big|_{x'=x} \tag{3.16}$$

Furthermore, the unique normalized trace $\tau:\mathcal{A}^{(\omega)}\to\mathbb{C}$ is represented by the classical average [32]

$$\tau(f)=\int_{T_d}\prod_{i=1}^d\frac{dx^i}{2\pi}\,f(x) \tag{3.17}$$

which is equivalent to projecting onto the zero mode f_0 in the Fourier series expansion (3.15), $\tau(f)=f_0$. This trace will be used later on to construct a duality-invariant gauge theory.

The $*$-algebra $\mathcal{A}_0$ therefore defines a module for the noncommutative torus which possesses some remarkable properties that distinguish it from the usual modules for T^d_ω. Although it is related to the projective regular representation of the twisted group algebra $\mathbb{C}\{\Lambda\}$ of the Narain lattice, this is not the algebraic feature which determines the cocycle relation (3.11). If we localize the loop functionals (3.7) on constant loops $\sigma\mapsto(x_0,x_0^*)\in T_d\times T_d^*\ \ \forall\sigma\in S^1$, then the corresponding algebra generators are given by

$$\mathcal{P}_0\,V_{q^+q^-}(0;f)\,\mathcal{P}_0=\mathrm{e}^{i\pi\langle q,w\rangle_L}\ \mathrm{e}^{-iq_ix^i-iv^ix_i^*}=\varepsilon_{q^+q^-}(p^+,p^-) \tag{3.18}$$

and coincide with the generators (2.16) of $\mathbb{C}\{\Lambda\}$. From the multiplication property

$$\varepsilon_{q^+q^-}(p^+,p^-)\,\varepsilon_{r^+r^-}(p^+,p^-)=c_\Lambda(q^+,q^-;r^+,r^-)\ \varepsilon_{(q^++r^+)(q^-+r^-)}(p^+,p^-) \tag{3.19}$$

we find the clock algebra

$$\varepsilon_{q^+q^-}(p^+,p^-)\,\varepsilon_{r^+r^-}(p^+,p^-)=\mathrm{e}^{i\pi\gamma_\Lambda(q^+,q^-;r^+,r^-)}\ \varepsilon_{r^+r^-}(p^+,p^-)\,\varepsilon_{q^+q^-}(p^+,p^-) \tag{3.20}$$

where

$$\gamma_\Lambda(q^+,q^-;r^+,r^-)=\langle r,v\rangle_L-\langle q,w\rangle_L \tag{3.21}$$

is a two-cocycle of Λ^c. The clock algebra (3.20) is very different from the algebra defined by (3.11) which arises from the operator product (3.1). The latter algebra is actually associated with the twisted chiral operators $\mathrm{e}^{\pm i\pi q_i^\pm g^{ij}p_j^\pm}\mathcal{P}_0V_{q^\pm}(0;f)\mathcal{P}_0$, although their chiral-antichiral products do not yield the $\mathbb{Z}_2$-twist of the tachyon vertex operators (3.18). Thus the product of the two algebras in (3.9)–(3.11) have deformation matrix associated with the bilinear form (2.7). However, the minus sign which appears in the antichiral sector is irrelevant since $T^d_{-\omega} \cong T^d_\omega$ by a relabeling of the generators U_i.

The tachyon algebra thus defines a $2d$-dimensional $\mathbb{Z}_2$-twisted module $\widetilde{\mathcal{T}}^d_\omega$ over the noncommutative torus, where $\mathcal{T}^d_\omega$ carries a double representation $T^{(+)d}_\omega \times T^{(-)d}_\omega$ of T^d_ω. The non-trivial $\mathbb{Z}_2$-twist of this module is important from the point of view of the noncommutative geometry of the string spacetime. Note that its algebra is defined by computing the operator product relations in (3.9)–(3.11) in the full vertex operator algebra $\mathcal{A}$, and then *afterwards* projecting onto the tachyon sector $\mathcal{A}_0$. Otherwise we arrive at the (trivial) clock algebra, so that within the definition of $\widetilde{\mathcal{T}}^d_\omega$ there is a particular ordering that must be carefully taken into account. This module is therefore quite different from the usual projective modules over the noncommutative torus [2, 9], and we shall exploit this fact dramatically in what follows. Note that if we described $\mathcal{A}$ using instead the canonical basis $\{e^i\}_{i=1}^d \oplus \{e_j\}_{j=1}^d$ of the Narain lattice Λ, i.e. taking as generators for $\mathcal{A}$ twisted products of operators of the form (2.23) and their duals, then we would have arrived at a twisted representation of $T^d_\omega \times T^d_{\omega^{-1}}$ associated with quantum deformations of the d-torus and its dual. That $T^d_{\omega^{-1}} \cong T^d_\omega$ will be a consequence of a set of Morita equivalences that we shall prove in section 5. In particular, this observation shows that for *any* vertex operator algebra associated with a positive-definite lattice L of rank d, there corresponds a module over the d-dimensional noncommutative torus T^d_ω, with deformation matrix ω given as above in terms of the bilinear form of L, which is determined by an S^1-twisted projective regular representation of the group algebra $\mathbb{C}[L]$.

As mentioned in section 1, there is a nice heuristic interpretation of the noncommutativity of T^d_ω in the present case. The ordinary torus T_d is formally obtained from T^d_ω by eliminating the deformation parameter matrix $\omega \to 0$. In the case at hand, $\omega \sim g^{-1}$, so that formally we let the metric $g \to \infty$ become very large. As all distances in this formalism are evaluated in terms of a fundamental length which for simplicity we have set equal to 1, this simply means that at distance scales larger than this unit of length (which is usually identified with the Planck length), we recover ordinary, classical spacetime T_d. Thus the classical limit of this quantum deformation of the Lie algebra $u(1)^d_+ \oplus u(1)^d_-$ coincides with the decompactification limit in which the dual coordinates x^*, representing the windings of the string around the spacetime, delocalize and become unobservable. At very short distances ($g \to 0$), spacetime becomes a noncommutative manifold with all of the exotic duality symmetries implied by string theory. This representation of the noncommutative torus thus realizes old ideas in string theory about the nature of spacetime below the fundamental length scale l_s determined by the finite size of the string. In this case l_s is determined by the lattice spacing of L. Therefore, within the framework of toroidally compactified string theory, spacetime at very short length scales is a noncommutative torus.

4. Spectral Geometry of Toroidal Compactifications

From the point of view of constructing a 'space', the pair $(\mathcal{A},\mathcal{H})$, i.e. a $*$-algebra $\mathcal{A}$ of operators acting on a Hilbert space $\mathcal{H}$, determines only the topology and differentiable structure of the 'manifold'. To put more structure on the space, such as an orientation and a metric, we need to construct a larger set of data. This is achieved using, for even-dimensional spaces, an even real spectral triple $(\mathcal{A},\mathcal{H},D,J,\Gamma)$ [24], where D is a generalized Dirac operator acting on $\mathcal{H}$ which determines a Riemannian structure, Γ defines a Hochschild cycle for the geometry which essentially determines an orientation or Hodge duality operator, and J determines a real structure for the geometry which is used to define notions such as Poincaré duality. In this section we will construct a set of spectral data to describe a particular noncommutative geometry appropriate to the spacetime implied by string theory.

To define $(\mathcal{A},\mathcal{H},D,J,\Gamma)$, we need to introduce spinors. For this, we fix a spin structure on $T_d\times T_d^*$ and augment the L^2 space in (2.17) to the space $L^2(T_d\times T_d^*,S)$ of square integrable spinors, where $S\to T_d\times T_d^*$ is the spin bundle which carries an irreducible left action of the Clifford bundle $Cl(T_d\times T_d^*)$. We shall take as $\mathcal{H}$ the corresponding augmented Hilbert space of (2.17), so that, in the notation of the previous section (see (3.3)), $\mathcal{H}_0=L^2(T_d\times T_d^*,S)$. A dense subspace of $L^2(T_d\times T_d^*,S)$ is provided by the smooth spinor module $\mathcal{S}=C^\infty(T_d\times T_d^*,S)$ which is an irreducible Clifford module of rank 2^d over $C^\infty(T_d\times T_d^*)$. There is a $\mathbb{Z}_2$-grading $\mathcal{S}=\mathcal{S}^+\oplus\mathcal{S}^-$ arising from the chirality grading of $Cl(T_d\times T_d^*)$ by the action of a grading operator Γ with $\Gamma^2=\mathbb{I}$. As a consequence, the augmented Hilbert space splits as

$$\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^- \tag{4.1}$$

into the ±1 (chiral-antichiral) eigenspaces of Γ.

The spinor module $\mathcal{S}$ carries a representation of the double toroidal Clifford algebra

$$\left\{\gamma_i^\pm,\gamma_j^\pm\right\}=\pm\,2g_{ij}\quad,\quad\left\{\gamma_i^\pm,\gamma_j^\mp\right\}=0\quad,\quad\left(\gamma_i^\pm\right)^\dagger=\gamma_i^\pm \tag{4.2}$$

The $\mathbb{Z}_2$-grading is determined by the chirality matrices

$$\Gamma_c^\pm=\frac{1}{d!\sqrt{\det g}}\,\epsilon^{i_1i_2\cdots i_d}\,\gamma_{i_1}^\pm\gamma_{i_2}^\pm\cdots\gamma_{i_d}^\pm \tag{4.3}$$

where $\epsilon^{i_1\cdots i_d}$ is the antisymmetric tensor with the convention $\epsilon^{12\cdots d}=+1$. The matrices $\Gamma_c^\pm$ have the properties

$$\Gamma_c^\pm\gamma_i^\pm=-(-1)^d\,\gamma_i^\pm\Gamma_c^\pm\quad,\quad\Gamma_c^\pm\gamma_i^\mp=(-1)^d\,\gamma_i^\mp\Gamma_c^\pm\quad,\quad\left(\Gamma_c^\pm\right)^2=(-1)^{d(d-1)/2}\,\mathbb{I} \tag{4.4}$$

so that the Klein operator

$$\Gamma\equiv(-1)^d\,\Gamma_c^+\Gamma_c^- \tag{4.5}$$

then satisfies the desired properties for a grading operator. The representation of the vertex operator algebra $\mathcal{A}$ on $\mathcal{H}$ acts diagonally with respect to the chiral decomposition (4.1), i.e.

$$[\Gamma, V] = 0 \qquad \forall V \in \mathcal{A} \tag{4.6}$$

It was shown in [17] that the two natural Dirac operators D associated with a lattice vertex operator algebra are the fields

$$\not{D}^{\pm} = \gamma_i^{\pm} \otimes z_{\pm}\alpha_{\pm}^i(z_{\pm}) \tag{4.7}$$

They are self-adjoint operators on $\mathcal{H}$ with compact resolvent on an appropriate dense domain (after smearing). They each act off-diagonally on (4.1) taking $\mathcal{H}^{\pm} \to \mathcal{H}^{\mp}$, i.e.

$$\left\{\Gamma, \not{D}^{\pm}\right\} = 0 \tag{4.8}$$

We shall use the chirally symmetric and antisymmetric self-adjoint combinations

$$\not{D} = \not{D}^{+} + \not{D}^{-} \qquad , \qquad \overline{\not{D}} = \not{D}^{+} - \not{D}^{-} \tag{4.9}$$

The final quantity we need is the operator J, which is needed to define a real structure. Since the representation of $\mathcal{A}$ on $\mathbf{h}$ is faithful (after smearing), we naturally have an injective map $\mathcal{A} \to \mathbf{h}$ defined by

$$V \mapsto |\psi_V\rangle \equiv \lim_{z_{\pm}\to 0} V(z_+, z_-)|0;0\rangle \otimes |0\rangle_+ \otimes |0\rangle_- \tag{4.10}$$

so that $V(\psi_V; z_+, z_-) \equiv V(z_+, z_-)$. This means that the (unique) vacuum state $|0;0\rangle \otimes |0\rangle_+ \otimes |0\rangle_-$ is a cyclic separating vector for $\mathbf{h}$ (by the operator-state correspondence), and we can therefore define an antilinear self-adjoint unitary isometry $J_c : \mathcal{H} \to \mathcal{H}$ by

$$J_c|\psi_V\rangle = |\psi_{V^\dagger}\rangle \qquad , \qquad J_c\gamma_i^{\pm} = \pm\gamma_i^{\pm} J_c \tag{4.11}$$

Then J is defined with respect to the decomposition (4.1) by acting off-diagonally as J_c on $\mathcal{H}^+ \to \mathcal{H}^-$ and as $(-1)^d J_c$ on $\mathcal{H}^- \to \mathcal{H}^+$. Note that on spinor fields $\psi \in L^2(T_d \times T_d^*, S) \subset \mathcal{H}$ the action of the operator J is given by

$$J\psi = \mathcal{C}\,\overline{\psi} \tag{4.12}$$

where $\mathcal{C}$ is the charge conjugation matrix acting on the spinor indices.

The antilinear unitary involution J satisfies the commutation relations

$$J\not{D} = \not{D}J \quad , \quad J\overline{\not{D}} = \overline{\not{D}}J \quad , \quad J\Gamma = (-1)^d\,\Gamma J \quad , \quad J^2 = \epsilon(d)\,\mathbb{I} \tag{4.13}$$

and

$$\left[V, JW^\dagger J^{-1}\right] = 0 \qquad , \qquad \left[[\not{D}, V], JW^\dagger J^{-1}\right] = \left[[\overline{\not{D}}, V], JW^\dagger J^{-1}\right] = 0 \tag{4.14}$$

for all $V,W\in\mathcal{A}$. The mod 4 periodic function $\epsilon(d)$ is given by [23]

$$\epsilon(d)=(1,-1,-1,1) \tag{4.15}$$

The dimension-dependent $\pm$ signs in the definition of J arise from the structure of real Clifford algebra representations. The first condition in (4.14) implies that for all $V\in\mathcal{A}$, $JV^\dagger J^{-1}$ lies in the commutant $\mathcal{A}'$ of $\mathcal{A}$ on $\mathcal{H}$, while the second condition is a generalization of the statement that the Dirac operators are first-order differential operators. The algebra $\mathcal{A}'$ defines an anti-representation of $\mathcal{A}$, and J can be thought of as a charge conjugation operator.

The spectral data $(\mathcal{A},\mathcal{H},D,J,\Gamma)$, with the relations above and D taken to be either of the two Dirac operators (4.9), determines an even real spectral triple for the geometry of the space we shall work with. As shown in [17], the existence of two "natural" Dirac operators (metrics) for the noncommutative geometry is not an ambiguous property, because the two corresponding spectral triples are in fact unitarily equivalent, i.e. there exist many unitary isomorphisms $U:\mathcal{H}\to\mathcal{H}$ which define automorphisms of the vertex operator algebra ($U\mathcal{A}U^{-1}=\mathcal{A}$) and for which

$$U\not{D}=\overline{\not{D}}U \tag{4.16}$$

This means that the two spaces are naturally isomorphic at the level of their spectral triples,

$$\Bigl(\mathcal{A}\,,\,\mathcal{H}\,,\,\not{D}\,,\,J\,,\,\Gamma\Bigr)~\cong~\Bigl(\mathcal{A}\,,\,\mathcal{H}\,,\,\overline{\not{D}}\,,\,UJU^{-1}\,,\,U\Gamma U^{-1}\Bigr) \tag{4.17}$$

and so the two Dirac operators determine the same geometry. This feature, along with the even-dimensionality of the spectral triple, will have important consequences in what follows. An isomorphism of the form (4.17) is called a duality symmetry of the noncommutative string spacetime [17].

We now consider the tachyon subalgebra $\mathcal{A}_0$ defined in (3.5) in terms of the orthogonal projection $\mathcal{P}_0:\mathbf{h}\to\mathbf{h}_0$ onto the subspace (3.3). The corresponding restrictions of the Dirac operators (4.7),(4.9) are

$$\begin{aligned}\not{\partial}~&\equiv~\mathcal{P}_0\,\not{D}\,\mathcal{P}_0~=~\tfrac{1}{2}\Bigl[g^{ij}\Bigl(\gamma_i^+ +\gamma_i^-\Bigr)\otimes p_j+\Bigl(\gamma_i^+ -\gamma_i^-\Bigr)\otimes w^i\Bigr]\\ \overline{\not{\partial}}~&\equiv~\mathcal{P}_0\,\overline{\not{D}}\,\mathcal{P}_0~=~\tfrac{1}{2}\Bigl[g^{ij}\Bigl(\gamma_i^+ -\gamma_i^-\Bigr)\otimes p_j+\Bigl(\gamma_i^+ +\gamma_i^-\Bigr)\otimes w^i\Bigr]\end{aligned} \tag{4.18}$$

Note that the operators J and Γ preserve both $\mathcal{A}_0$ and $\mathcal{H}_0$. We can therefore define subspaces $(\mathcal{A}_0,\mathcal{H}_0,\not{\partial},J,\Gamma)$ and $(\mathcal{A}_0,\mathcal{H}_0,\overline{\not{\partial}},J,\Gamma)$ of the geometries represented by the (isomorphic) spectral triples in (4.17).

5. Duality Transformations and Morita Equivalence

We shall now describe some basic features of the twisted modules constructed in section 3. An important property of $\widetilde{\mathcal{T}}^d_\omega$ is that there is no distinction between the torus T_d and its dual T_d^* within the tachyon algebra. These two commutative subspaces of the tachyon spacetime are associated with the subspaces

$$
\begin{aligned}
\mathbf{h}_0^{(-)} &\equiv \bigotimes_{i=1}^{d} \ker\left(\alpha_+^i \otimes \mathbb{I} + \mathbb{I} \otimes \alpha_-^i\right) \cong L^2\left(T_d\,,\,\prod_{i=1}^d \frac{dx^i}{2\pi}\right) \\
\mathbf{h}_0^{(+)} &\equiv \bigotimes_{i=1}^{d} \ker\left(\alpha_+^i \otimes \mathbb{I} - \mathbb{I} \otimes \alpha_-^i\right) \cong L^2\left(T_d^*\,,\,\prod_{i=1}^d \frac{dx_i^*}{2\pi}\right)
\end{aligned}
\tag{5.1}
$$

of (3.3). The subspace $\mathbf{h}_0^{(\pm)}$ is the projection of $\mathbf{h}_0$ onto those states $|q^+;q^-\rangle$ with $q^+ = \pm q^-$ (equivalently $v = 0$ and $q = 0$, respectively), and it contains only those highest-weight modules which occur in complex conjugate pairs of left-right representations of the current algebra $\widehat{u(1)}_+^d \oplus \widehat{u(1)}_-^d$. From (3.18) we see that if $\mathcal{P}_0^{(\pm)} : \mathbf{h}_0 \to \mathbf{h}_0^{(\pm)}$ are the respective orthogonal projections, then the corresponding vertex operator subalgebras are

$$
\begin{aligned}
\mathcal{A}_0^{(-)} &\equiv \mathcal{P}_0^{(-)}\,\mathcal{A}_0\,\mathcal{P}_0^{(-)} \cong C^\infty(T_d) \\
\mathcal{A}_0^{(+)} &\equiv \mathcal{P}_0^{(+)}\,\mathcal{A}_0\,\mathcal{P}_0^{(+)} \cong C^\infty(T_d^*)
\end{aligned}
\tag{5.2}
$$

and represent ordinary (commutative) spacetimes. If we choose a spin structure on $T_d \times T_d^*$ such that the spinors are periodic along the elements of a homology basis, then the associated spin bundle is trivial ($\gamma_i^+ = \gamma_i^- \equiv \gamma_i \;\; \forall i$) and the corresponding Dirac operators (4.18) are

$$
\not{\partial} = -ig^{ij}\gamma_i \otimes \frac{\partial}{\partial x^j} \qquad , \qquad \overline{\not{\partial}} = -i\gamma_i \otimes \frac{\partial}{\partial x_i^*} \tag{5.3}
$$

where we have used the coordinate space representations $p_i = -i\frac{\partial}{\partial x^i}$ on $L^2(T_d, \prod_{i=1}^d \frac{dx^i}{2\pi})$ and $w^i = -i\frac{\partial}{\partial x_i^*}$ on $L^2(T_d^*, \prod_{i=1}^d \frac{dx_i^*}{2\pi})$ given by the adjoint actions (2.15). The Dirac operators (5.3) represent the canonical geometries of the (noncommutative) torus and its dual.

Thus, as subspaces of the tachyon algebra, tori are identified with their duals, since, as mentioned in the previous section, there exists a unitary isomorphism, that interchanges $p \leftrightarrow w$ and $g \leftrightarrow g^{-1}$, of the Hilbert space $\mathcal{H}$ which exchanges the two subspaces in (5.1) and (5.2) and also the two Dirac operators (5.3). Since this isomorphism does not commute with the projection operators $\mathcal{P}_0^{(\pm)}$, distinct classical spacetimes are identified. The equivalence of them from the point of view of noncommutative geometry is the celebrated T-duality symmetry of quantum string theory. Different choices of spin structure on T_d induce twistings of the spin bundle, and, along with some modifications of the definitions of the subspaces (5.1) and (5.2), they induce projections onto different dual tori with appropriate modifications of (5.3). But, according to (4.17), these spectral geometries are all isomorphic [17]. We can summarize these results in the following

Theorem 1. *Let L be a positive-definite lattice of rank d with bilinear form g_{ij}, and let $\mathcal{A}$ be the vertex operator algebra associated with L. Then the tachyon subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ defines a unitary equivalence class of finitely-generated self-dual $\mathbb{Z}_2$-twisted projective modules $\widetilde{\mathcal{T}}_\omega^d$ of the double noncommutative torus $T_\omega^{(+)d} \times T_\omega^{(-)d}$ with generators*

$U_i = \mathcal{P}_0 V_{e^i_\pm}(0;f)\mathcal{P}_0$ and antisymmetric deformation matrix

$$\omega^{ij} = \begin{cases} \mathrm{sgn}(j-i)\, g^{ij} & , \quad i \neq j \\ 0 & , \quad i = j \end{cases}$$

We now turn to a more precise description of the isomorphism classes in $\widetilde{\mathcal{T}}^d_\omega$. We note first of all that the string geometry is *not* contained entirely within the tachyon sector of the vertex operator algebra $\mathcal{A}$. This is because the explicit inner automorphisms of $\mathcal{A}$ which implement the duality symmetries are constructed from higher-order perturbations of the tachyon sector (by, for example, graviton operators). The basic tachyon vertex operators V_i together with the Heisenberg fields $\alpha^i_\pm(z_\pm) = -iV(\mathbb{I} \otimes \alpha^{(\pm)i}_{-1}; 1, z_\pm)$ generate an affine Lie group $\widehat{\mathrm{Inn}}^{(0)}(\mathcal{A})$ of inner automorphisms of the vertex operator algebra which is in general an enhancement of the generic affine $U(1)^d_+ \times U(1)^d_-$ gauge symmetry [21]. This property is crucial for the occurrence of string duality as a gauge symmetry of the noncommutative geometry, and the isomorphisms described above only occur when the relevant structures are embedded into the full spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$.

It is well known that various noncommutative tori with different deformation parameters are equivalent to each other (For completeness, a brief summary of the general definition and relevance of Morita equivalence for C^*-algebras is presented in appendix A). For instance, when $d = 2$ it can be shown that the abelian group $\mathbb{Z} \oplus \omega\mathbb{Z}$ is an isomorphism invariant of $\mathcal{A}^{(\omega)}$. This means, for example, that the tori $T^2_\omega \cong T^2_{-\omega} \cong T^2_{\omega+1}$ are unitarily equivalent. Moreover, it can be shown in this case that the algebras $\mathcal{A}^{(\omega)}$ and $\mathcal{A}^{(\omega')}$ are Morita equivalent if and only if they are related by a discrete Möbius transformation [1]

$$\omega' = \frac{a\omega + b}{c\omega + d} \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}) \tag{5.4}$$

where $GL(2, \mathbb{Z})$ is the group of 2×2 integer-valued matrices with determinant ± 1. This then also identifies the tori $T^2_\omega \cong T^2_{-1/\omega}$. Another copy of the discrete group $SL(2, \mathbb{Z})$ appears naturally by requiring that the transformation

$$U_1 \mapsto U_1^a U_2^b \quad , \quad U_2 \mapsto U_1^c U_2^d \tag{5.5}$$

be an automorphism of $\mathcal{A}^{(\omega)}$. Indeed, the transformation (5.5) preserves the product (3.14), and so extends to an (outer) automorphism of $\mathcal{A}^{(\omega)}$, if and only if $ad - bc = \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$.

The duality symmetries of the string spacetime described above can be used to identify certain isomorphisms among the modules $\widetilde{\mathcal{T}}^d_\omega$. These incorporate the usual isomorphisms between noncommutative tori described above as gauge symmetries, and they also naturally contain, for any d, the discrete geometrical automorphism group $SL(d, \mathbb{Z})$ of T_d. When the algebra $\mathcal{A}^{(\omega)}$ of the noncommutative torus T^d_ω is embedded as described in section 3 into the full vertex operator algebra $\mathcal{A}$, these isomorphisms are induced by the discrete inner automorphisms of $\mathcal{A}$ which generate the isometry group of the Narain lattice,

$$\mathrm{Aut}(\Lambda) = O(d,d;\mathbb{Z}) \supset SL(d,\mathbb{Z}) \tag{5.6}$$

They act on the metric tensor g_{ij} by matrix-valued Möbius transformations and induce an $O(d,d;\mathbb{Z})$ symmetry on the Hilbert space acting unitarily on the Dirac operators in the sense of (4.16). Therefore, with the embedding $\mathcal{A}^{(\omega)} \hookrightarrow \mathcal{A}$, a much larger class of tori are identified by the action of the full duality group[1],

$$T^d_\omega \cong T^d_{\omega^*} \qquad \text{with } \omega^* = (A\omega + B)(C\omega + D)^{-1} \quad , \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(d,d;\mathbb{Z}) \tag{5.7}$$

where A, B, C, D are $d \times d$ integer-valued matrices satisfying the relations

$$A^\top C + C^\top A = 0 = B^\top D + D^\top B \qquad , \qquad A^\top D + C^\top B = \mathbb{I} \tag{5.8}$$

To see this, we embed the two algebras $\mathcal{A}^{(\omega)}$ and $\mathcal{A}^{(\omega^*)}$ into $\mathcal{A}$. Then the unitary equivalence (4.17) implies that, in $\mathcal{A}$, there is a finitely-generated projective right $\mathcal{A}^{(\omega)}$-module $\mathcal{E}^{(\omega)}$ with $\mathcal{A}^{(\omega^*)} \cong \mathrm{End}_{\mathcal{A}^{(\omega)}}\mathcal{E}^{(\omega)}$ where the $*$-isomorphism is implemented by the unitary operator U in (4.16). Thus the projections back onto the tachyon subalgebras establishes the Morita equivalence (5.7) between the twisted modules. In other words, the tori in (5.7), being Morita equivalent, are indistinguishable as subalgebras of $\mathcal{A}$.

To summarize, string duality implies the

Theorem 2. *There is a natural Morita equivalence*

$$T^d_\omega \cong T^d_{\omega^*} \qquad ; \qquad \omega^* \equiv (A\omega + B)(C\omega + D)^{-1} \quad , \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(d,d;\mathbb{Z})$$

of spectral geometries.

A similar result for multi-dimensional noncommutative tori has been established recently in [18] using more explicit formal constructions of projective modules. Here we see the power of duality in establishing this strong result. It is intriguing that the symmetry group (5.6) contains the $SL(2,\mathbb{Z})$ S-duality symmetry of type-IIB superstring theory. In [19] it is shown that Matrix Theory compactifications on Morita equivalent noncommutative tori are physically equivalent, in that the associated quantum theories are dual. Here we find a similar manifestation of this property, directly in the language of string geometry. The construction in the present case exposes the natural relationship between target space duality and Morita equivalence of noncommutative geometries. It also demonstrates explicitly in what sense compactifications on Morita equivalent tori are physically equivalent,

[1] As emphasized in [18] this $O(d,d;\mathbb{Z})$ transformation is only defined on the dense subspace of the space of antisymmetric real-valued tensors ω where $C\omega + D$ is invertible. We shall implicitly assume this here.

as conjectured in [6]. For instance, the equivalence $\omega \leftrightarrow \omega^{-1}$ represents the unobservability of small distances in the physical spacetime, while the equivalence $\omega \leftrightarrow \omega+1$ (for $d=2$) represents the invariance of the spacetime under a change of complex structure.

As we discussed above, string duality can be represented as a gauge symmetry on the full vertex operator algebra $\mathcal{A}$ and, when projected onto the tachyon sector, one obtains intriguing equivalences between the twisted realizations of the noncommutative torus. In this representation of T^d_ω, the automorphisms of the algebra $\mathcal{A}^{(\omega)}$ are determined in large part by gauge transformations, i.e. the elements of $\widehat{\rm Inn}^{(0)}(\mathcal{A})$,[2] consistent with the corresponding results for the ordinary noncommutative torus. In particular, the usual diffeomorphism symmetries are given by inner automorphisms, i.e. gravity becomes a gauge theory on this space. The outer automorphisms of $\mathcal{A}$ are given by the full duality group of toroidally compactified string theory which is the semi-direct product $O(d,d;\mathbb{Z}) \otimes O(2,\mathbb{R})$, where $O(2,\mathbb{R})$ is a worldsheet symmetry group that acts on the algebra $\mathcal{A}$ by rotating the two chiral sectors on $\mathbb{C}\cup\{\infty\}$ among each other (this part of the duality group does not act on the metric tensor of T_d). After applying the appropriate projections, we obtain canonical actions of these automorphisms on the tachyon sector. Thus, again using string duality, we arrive at the

Proposition 1. *There is a natural subgroup of automorphisms of the twisted module $\widetilde{\mathcal{T}}^d_\omega$ given by the duality group*

$$\mathrm{Aut}^{(0)}(\mathcal{A}_0) = \mathcal{P}_0\left(\widehat{\rm Inn}^{(0)}(\mathcal{A}) \otimes \mathrm{Out}(\mathcal{A})\right)\mathcal{P}_0$$

where $\widehat{\rm Inn}^{(0)}(\mathcal{A})$ is the affine Lie group generated by the tachyon vertex operators V_i and the Heisenberg fields $\alpha^i_\pm$, and $\mathrm{Out}(\mathcal{A}) \cong O(d,d;\mathbb{Z}) \otimes O(2,\mathbb{R})$.

6. Gauge Theory on $\widetilde{\mathcal{T}}^d_\omega$

In noncommutative geometry a finitely-generated projective module of an algebra replaces the classical notion of a vector bundle over a manifold [4, 5]. The geometry of gauge theories can therefore also be cast into the natural algebraic framework of noncommutative geometry. In this section we will classify the gauge theories on $\widetilde{\mathcal{T}}^d_\omega$ which in the next section will enable the construction of an associated action functional that is naturally invariant under the automorphism group given by Proposition 1, and in particular under duality and diffeomorphism symmetries. The possibility of such a characterization comes from the fairly complete mathematical theory of projective modules and of connections on these modules for the noncommutative torus T^d_ω [2].

On $\mathcal{A}^{(\omega)}$ there is a natural set of linear derivations Δ_i, $i=1,\ldots,d$, defined by

$$\Delta_i(U_j) = \delta_{ij}\,U_j \tag{6.1}$$

[2] This Lie group is described in detail in [21].

These derivations come from the derivative operators $p_i = -i\frac{\partial}{\partial x^i}$ acting on the algebra $C^\infty(T_d)$ of Fourier series (see section 3). They generate a $u(1)^d$ Lie algebra. There is a a natural representation of the operators Δ_i in the twisted module. The tachyon vertex operators $V_{q^+q^-}(z_+,z_-)$ have charge $g^{ij}q_j^\pm$ under the action of the $u(1)_\pm^d$ current algebra generated by the Heisenberg fields (2.14). As such $\alpha_\pm^i$ generate an affine Lie algebra $\widehat{\Lambda^c}$ of automorphisms of $\mathcal{A}_0$ (Proposition 1). We can define a set of linear derivations $\nabla_\pm^{(0)i}$, $i=1,\dots,d$, acting on $\mathcal{A}_0$ by the infinitesimal adjoint action of the Heisenberg fields [17, 21] which is given by

$$\nabla_\pm^{(0)i}\,V_j(0;f) \equiv \Bigl[\alpha_\pm^i(0;f),V_j(0;f)\Bigr] = \delta_j^i\,V_j(0;f) \tag{6.2}$$

This leads us to the notion of a connection on the tachyon algebra $\mathcal{A}_0$. We shall start by considering this algebra as a finitely-generated projective left module over itself. A (chiral) left connection on $\mathcal{A}_0$ is then defined as a set of $\mathbb{C}$-linear operators $\nabla_\pm^i$, $i=1,\dots,d$, acting on $\mathcal{A}_0$ and satisfying the left Leibniz rule

$$\nabla_\pm^i(VW) = V\,\nabla_\pm^i W + \Bigl[\alpha_\pm^i,V\Bigr]\,W \qquad , \qquad \forall V,W\in\mathcal{A}_0 \tag{6.3}$$

If $\nabla_\pm^i$ and $\nabla_\pm'^i$ are two connections on $\mathcal{A}_0$, then their difference $\nabla_\pm^i-\nabla_\pm'^i$ commutes with the left action of $\mathcal{A}_0$ on itself, i.e. $(\nabla_\pm^i-\nabla_\pm'^i)(VW)=V(\nabla_\pm^i-\nabla_\pm'^i)W$. Thus $\nabla_\pm^i-\nabla_\pm'^i$ is an element of the endomorphism algebra $\mathrm{End}_{\mathcal{A}_0}\mathcal{A}_0$ which we identify with the commutant $\mathrm{End}_{\mathcal{A}_0}\mathcal{A}_0\cong\mathcal{A}_0'=J\mathcal{A}_0J^{-1}$ of the algebra $\mathcal{A}_0$, where J is the real structure introduced in section 4. The derivations defined in (6.2) also satisfy the Leibniz rule (6.3) and thus define fixed fiducial elements in the space of connections on $\mathcal{A}_0$. It follows that $\Omega_\pm^i=\nabla_\pm^i-\nabla_\pm^{(0)i}$ obeys

$$\Omega_\pm^i VW = V\,\Omega_\pm^i W \qquad , \qquad \forall V,W\in\mathcal{A}_0 \tag{6.4}$$

so that we can write an arbitrary connection in the form

$$\nabla_\pm^i = \nabla_\pm^{(0)i} + \Omega_\pm^i \qquad \text{with} \;\; \Omega_\pm^i\in\mathrm{End}_{\mathcal{A}_0}\mathcal{A}_0 \tag{6.5}$$

We can introduce a Hermitian structure on $\mathcal{A}_0$ via the $\mathcal{A}_0$-valued positive-definite inner product $\langle\cdot\;,\;\cdot\rangle_{\mathcal{A}_0}:\mathcal{A}_0\times\mathcal{A}_0\to\mathcal{A}_0$ defined by

$$\langle V,W\rangle_{\mathcal{A}_0} = V^\dagger W \qquad , \qquad V,W\in\mathcal{A}_0 \tag{6.6}$$

The compatibility condition with respect to this Hermitian structure for left connections,

$$-\Bigl\langle\nabla_\pm^i V,W\Bigr\rangle_{\mathcal{A}_0} + \Bigl\langle V,\nabla_\pm^i W\Bigr\rangle_{\mathcal{A}_0} = \nabla_\pm^{(0)i}\Bigl(\langle V,W\rangle_{\mathcal{A}_0}\Bigr) \tag{6.7}$$

implies that $\Omega_\pm^i=(\Omega_\pm^i)^\dagger$ is self-adjoint. The minus sign in (6.7) arises from the fact that the fiducial connection defined in (6.2) anticommutes with the $*$-involution, $(\nabla_\pm^{(0)i}V)^\dagger = -\nabla_\pm^{(0)i}V^\dagger$.

Since the connection coefficients $\Omega_\pm^i$ in (6.5) are elements of the commutant $\mathcal{A}_0'\cong\mathrm{End}_{\mathcal{A}_0}\mathcal{A}_0$, it is natural to introduce an $\mathcal{A}_0$-bimodule structure. We first define a right

$\mathcal{A}_0$-module which we denote by $\overline{\mathcal{A}_0}$. Elements of $\overline{\mathcal{A}_0}$ are in bijective correspondence with those of $\mathcal{A}_0$, $\overline{\mathcal{A}_0} \equiv \{\overline{V} \mid V \in \mathcal{A}_0\}$, and the right action of $V \in \mathcal{A}_0$ on $\overline{W} \in \overline{\mathcal{A}_0}$ is given by $\overline{W} \cdot V = \overline{V^\dagger W}$. Associated with the connection $\nabla^i_\pm$ on $\mathcal{A}_0$ there is then a right connection $\overline{\nabla}^i_\pm$ on $\overline{\mathcal{A}_0}$ defined by

$$\overline{\nabla}^i_\pm \overline{V} = -\overline{\nabla^i_\pm V} \qquad , \qquad \forall \overline{V} \in \overline{\mathcal{A}_0} \tag{6.8}$$

The operator (6.8) is $\mathbb{C}$-linear and obeys a right Leibniz rule. Indeed, with $\overline{V} \in \overline{\mathcal{A}_0}$ and $W \in \mathcal{A}_0$, we have

$$\begin{aligned} \overline{\nabla}^i_\pm(\overline{V} \cdot W) &= \overline{\nabla}^i_\pm(\overline{W^\dagger V}) \\ &= -\overline{\nabla^i_\pm(W^\dagger V)} \\ &= -\overline{W^\dagger \nabla^i_\pm V} - \overline{\left(\nabla^{(0)i}_\pm W^\dagger\right) V} \\ &= -\overline{\nabla^i_\pm V} \cdot W + \overline{\left(\nabla^{(0)i}_\pm W\right)^\dagger V} \\ &= \left(\overline{\nabla}^i_\pm \overline{V}\right) \cdot W + \overline{V} \cdot \left[\alpha^i_\pm, W\right] \end{aligned} \tag{6.9}$$

which expresses the right Leibniz rule for $\overline{\nabla}^i_\pm$.

We can now combine the two connections $\nabla^i_\pm$ and $\overline{\nabla}^i_\pm$ to get a symmetric connection $\widetilde{\nabla}^i_\pm$ on $\overline{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \mathcal{A}_0$ defined by

$$\widetilde{\nabla}^i_\pm(\overline{V} \otimes W) = \overline{\nabla}^i_\pm \overline{V} \otimes W + \overline{V} \otimes \nabla^i_\pm W \qquad , \qquad \forall \overline{V} \otimes W \in \overline{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \mathcal{A}_0 \tag{6.10}$$

The $\mathcal{A}_0$-bimodule we are describing is just the Hilbert space $\mathcal{H}_0$ which carries both a left representation of $\mathcal{A}_0$ and a (right) anti-representation of $\mathcal{A}_0$ given by $V \mapsto JV^\dagger J^{-1} \in \mathcal{L}(\mathcal{H}_0)$, where $\mathcal{L}(\mathcal{H}_0)$ is the algebra of bounded linear operators on $\mathcal{H}_0$. In the present interpretation, the right module structure of $\mathcal{H}_0$ comes from the right $\mathcal{A}_0$-module $\overline{\mathcal{A}_0}$ when mapping $\overline{\mathcal{A}_0}$ into $\mathcal{L}(\mathcal{H}_0)$ by $\overline{V} \mapsto JV^\dagger J^{-1}$. Thus, when representing $\overline{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \mathcal{A}_0$ on $\mathcal{H}_0$, using (6.5) and (6.8) we find that the action of the connection (6.10) restricted to $\mathcal{A}_0$ can be expressed in terms of the fiducial connection (6.2) and the connection coefficients as

$$\widetilde{\nabla}^i_\pm(\mathbb{I} \otimes V) = \mathbb{I} \otimes \left(\nabla^{(0)i}_\pm + \Omega^i_\pm + J(\Omega^i_\pm)^\dagger J^{-1}\right) V \tag{6.11}$$

The extra connection term in (6.11) achieves the desired left-right symmetric representation and can be thought of as enforcing CPT-invariance.

The operator

$$\widetilde{\nabla\!\!\!/}^{\pm} \equiv \gamma^\pm_i \otimes \widetilde{\nabla}^i_\pm \tag{6.12}$$

is then a map on $\overline{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \mathcal{A}_0 \to \overline{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \Omega^1_{D\!\!\!/^\pm}(\mathcal{A}_0) \otimes_{\mathcal{A}_0} \mathcal{A}_0$, where

$$\Omega^1_{D\!\!\!/^\pm}(\mathcal{A}_0) = \mathrm{span}_{\mathbb{C}} \left\{ V \left[D\!\!\!/^\pm, W\right] \;\middle|\; V, W \in \mathcal{A}_0 \right\} \tag{6.13}$$

are linear spaces of one-forms which carry a natural $\mathcal{A}_0$-bimodule structure. The action of the operator $\widetilde{\nabla\!\!\!/}^{\pm}$ on $\mathcal{A}_0$ as defined in (6.11) can be expressed in terms of the adjoint actions of the Dirac operators (4.7) as $\widetilde{\nabla}^{\pm}|_{\mathcal{A}_0}=\mathrm{Ad}_{\not{D}^{\pm}}+A^{\pm}+J(A^{\pm})^{\dagger}J^{-1}$, where $A^{\pm}\in\Omega^1_{\not{D}^{\pm}}(\mathcal{A}_0)$ are self-adjoint operators (by the compatibility condition (6.7)) which are called gauge potentials. In fact, all of this structure is induced by a covariant Dirac operator which is associated with the full data $(\mathcal{A},\mathcal{H},J,\Gamma)$ and is defined as

$$\not{D}^{\pm}_{\nabla}=\not{D}^{\pm}+A^{\pm}+J(A^{\pm})^{\dagger}J^{-1}\qquad,\qquad A^{\pm}\in\Omega^1_{\not{D}^{\pm}}(\mathcal{A})\tag{6.14}$$

where $\Omega^1_{\not{D}^{\pm}}(\mathcal{A})$ are the $\mathcal{A}$-bimodules of one-forms defined as in (6.13) but with the tachyon algebra $\mathcal{A}_0$ replaced by the full vertex operator algebra $\mathcal{A}$. The Dirac operator $\not{D}^{\pm}_{\nabla}$ is regarded as an internal perturbation of $\not{D}^{\pm}$ and it yields a geometry that is unitarily equivalent to that determined by $\not{D}^{\pm}$ [24], i.e. the geometries with fixed data $(\mathcal{A},\mathcal{H},J,\Gamma)$ form an affine space modelled on $\Omega^1_{\not{D}^{\pm}}(\mathcal{A})$.

The spaces $\Omega^1_{\not{D}^{\pm}}(\mathcal{A})$ are free $\mathcal{A}$-bimodules with bases $\{\gamma_i^{\pm}\}_{i=1}^d$ [17]. The gauge potentials in (6.14) can therefore be decomposed as

$$A^{\pm}=g^{ij}\gamma_i^{\pm}\otimes A_j^{\pm}\qquad\text{with}\quad A_i^{\pm}\in\mathcal{A}\tag{6.15}$$

Defining the self-adjoint elements $A_i\equiv\mathcal{P}_0(A_i^++A_i^-)\mathcal{P}_0$ and $A_*^i=\mathcal{P}_0\,g^{ij}(A_j^+-A_j^-)\mathcal{P}_0$ of $\mathcal{A}_0$, the covariant versions of the Dirac operators (4.18), obtained from the restrictions of (6.14) to $\mathcal{A}_0$, are then

$$\begin{aligned}\partial\!\!\!/_{\nabla}\equiv\mathcal{P}_0\left(\not{D}^+_{\nabla}+\not{D}^-_{\nabla}\right)\mathcal{P}_0&=\tfrac12\Big[g^{ij}\left(\gamma_i^++\gamma_i^-\right)\otimes\left(p_j+A_j+g_{jk}\,JA_*^kJ^{-1}\right)\\&\qquad+\left(\gamma_i^+-\gamma_i^-\right)\otimes\left(w^i+A_*^i+g^{ij}JA_jJ^{-1}\right)\Big]\\\overline{\partial\!\!\!/}_{\nabla}\equiv\mathcal{P}_0\left(\not{D}^+_{\nabla}-\not{D}^-_{\nabla}\right)\mathcal{P}_0&=\tfrac12\Big[g^{ij}\left(\gamma_i^+-\gamma_i^-\right)\otimes\left(p_j+A_j+g_{jk}\,JA_*^kJ^{-1}\right)\\&\qquad+\left(\gamma_i^++\gamma_i^-\right)\otimes\left(w^i+A_*^i+g^{ij}JA_jJ^{-1}\right)\Big]\end{aligned}\tag{6.16}$$

where we have used (4.11).

The final ingredient we need for a gauge theory on $\widetilde{\mathcal{T}}^d_\omega$ is some definition of an invariant integration in order to define an action functional. The trace (3.17) yields a natural normalized trace $\mathrm{Tr}:\mathcal{A}_0\to\mathbb{C}$ on the twisted module defined by

$$\mathrm{Tr}\,V=\int_{T_d}\int_{T_d^*}\prod_{i=1}^d\frac{dx^i\;dx_i^*}{(2\pi)^2}\;V(x,x^*)\tag{6.17}$$

This trace is $\widehat{\Lambda^c}$-invariant, $\mathrm{Tr}(\nabla_{\pm}^{(0)i}V)=0\quad\forall V\in\mathcal{A}_0$, and the corresponding Gelfand-Naimark-Segal representation space $L^2(\mathcal{A}_0,\mathrm{Tr})$ is, by the operator-state correspondence, canonically isomorphic to the Hilbert space $\mathbf{h}_0$. Using the trace (6.17) and the $\mathcal{A}_0$-valued inner product (6.6) we obtain a usual complex inner product $(\cdot\;,\;\cdot)_{\mathcal{A}_0}:\mathcal{A}_0\times\mathcal{A}_0\to\mathbb{C}$ defined by

$$(V,W)_{\mathcal{A}_0}=\mathrm{Tr}\,\langle V,W\rangle_{\mathcal{A}_0}=\langle\psi_V|\psi_W\rangle_{\mathbf{h}_0}\tag{6.18}$$

which coincides with the inner product on the Hilbert space $\mathbf{h}_0$. Being functions which are constructed from invariant traces, the quantities (6.17) and (6.18) are naturally invariant under unitary transformations of the tachyon algebra $\mathcal{A}_0$, and, in particular, under the action of the inner automorphism group given in Proposition 1. This property immediately implies the manifest duality-invariance of any action functional constructed from them.

7. Duality-symmetric Action Functional

The constructions of the preceeding sections yield a completely duality-symmetric formalism, and we are now ready to define a manifestly duality-invariant action functional

$$I[\psi,\widehat{\psi};\nabla,\nabla_*]=I_B[\nabla,\nabla_*]+I_F[\psi,\widehat{\psi};\nabla,\nabla_*] \tag{7.1}$$

associated with a generic gauge theory on the twisted module. The bosonic part of the action is defined as

$$I_B[\nabla,\nabla_*]\equiv\tfrac{1}{2^d}\,\mathrm{Tr}_S\left[\Pi\left(\partial\!\!\!/_\nabla^2-\overline{\partial\!\!\!/}_\nabla^2\right)\right]^2 \tag{7.2}$$

where Tr_S is the trace (6.17) including a trace over the Clifford module, and Π is the projection operator onto the space of antisymmetric tensors (two-forms). This projection is equivalent to the quotienting by junk forms of the representation of the universal forms on the Hilbert space [4, 5]. The operator $\partial\!\!\!/_\nabla^2-\overline{\partial\!\!\!/}_\nabla^2$ is the lowest order polynomial combination of the two Dirac operators (6.16) which lies in the endomorphism algebra $\mathrm{End}_{\mathcal{A}_0}\mathcal{A}_0$. Thus the action (7.2) comes from the lowest order polynomial multiplication operator two-form which is invariant under the duality symmetries represented by an interchange of the Dirac operators. The fermionic action is of the form of a Dirac action. Let $\psi=(\psi^a)_{a=1}^{2^d}$ be a square-integrable section of the spin bundle over $T_d\times T_d^*$. Then using a flat metric δ_{ab} for the spinor indices, we define

$$I_F[\psi,\widehat{\psi};\nabla,\nabla_*]\equiv\mathrm{Tr}\sum_{a=1}^{2^d}\left\langle J^{-1}V(\psi^a)J\,,\,[\partial\!\!\!/_\nabla,V(\psi^a)]\right\rangle_{\mathcal{A}_0}=\left\langle\widehat{\psi}\right|\partial\!\!\!/_\nabla\left|\psi\right\rangle_{\mathcal{H}_0} \tag{7.3}$$

where

$$\widehat{\psi}=J^{-1}\,\psi=\epsilon(d)\,\mathcal{C}\,\overline{\psi} \tag{7.4}$$

is the corresponding anti-fermion field.

The actions (7.2) and (7.3) are both gauge-invariant and depend only on the spectral properties of the Dirac K-cycles $(\mathcal{H}_0,\partial\!\!\!/)$ and $(\mathcal{H}_0,\overline{\partial\!\!\!/})$. A gauge transformation is an inner automorphism $\sigma_u:\mathcal{A}\to\mathcal{A}$, parametrized by a unitary element u of $\mathcal{A}$, i.e. an element of the unitary group $\mathcal{U}(\mathcal{A})=\{u\in\mathcal{A}\mid u^\dagger u=uu^\dagger=\mathbb{I}\}$ of $\mathcal{A}$, and defined by

$$\sigma_u(V)=uVu^\dagger\qquad,\qquad V\in\mathcal{A} \tag{7.5}$$

It acts on gauge potentials as

$$A^\pm\;\mapsto\;(A^\pm)^u\equiv uA^\pm u^\dagger+u\left[D\!\!\!\!/^\pm,u^\dagger\right] \tag{7.6}$$

and, using the $\mathcal{A}$-bimodule structure on $\mathcal{H}$, on spinor fields by the adjoint representation

$$\psi \mapsto \psi^u \equiv u\,\psi\,u^\dagger = U\psi \tag{7.7}$$

where

$$U = uJuJ^{-1} \tag{7.8}$$

The transformation (7.7) preserves the inner product on the Hilbert space $\mathcal{H}$, $\langle\psi_1^u|\psi_2^u\rangle = \langle\psi_1|\psi_2\rangle$, since both u and J are isometries of $\mathcal{H}$. Moreover, one easily finds that, under a gauge transformation (7.6),

$$\nabla\!\!\!\!/^{\pm} \mapsto (\nabla\!\!\!\!/^{\pm})^u \equiv u\,\nabla\!\!\!\!/^{\pm}\,u^\dagger \qquad , \qquad D\!\!\!\!/^{\pm}_{\nabla} \mapsto D\!\!\!\!/^{\pm}_{\nabla^u} = U\,D\!\!\!\!/^{\pm}_{\nabla}\,U^\dagger \tag{7.9}$$

which immediately shows that

$$I[\psi^u,\widehat{\psi}^u;\nabla^u,\nabla_*^u] = I[\psi,\widehat{\psi};\nabla,\nabla_*] \tag{7.10}$$

Let us write the action (7.1) in a form which makes its duality symmetries explicit. Using the double Clifford algebra (4.2) and the coordinate space representations of the momentum and winding operators in (6.16), after some algebra we find

$$\begin{aligned}\partial\!\!\!/_{\nabla}^2 - \overline{\partial\!\!\!/}_{\nabla}^2 = -\frac{i}{2}\,\gamma_+^i\gamma_-^j\otimes\Biggl(&\frac{\partial a_j}{\partial x^i}-\frac{\partial a_i}{\partial x^j}-g_{ik}g_{jl}\frac{\partial a_*^l}{\partial x_k^*}+g_{ik}g_{jl}\frac{\partial a_*^k}{\partial x_l^*}+i\bigl[a_i,a_j\bigr]-ig_{ik}g_{jl}\left[a_*^k,a_*^l\right]\\&+g_{jk}\frac{\partial a_*^k}{\partial x^i}+g_{ik}\frac{\partial a_*^k}{\partial x^j}+ig_{jk}\left[a_i,a_*^k\right]-g_{jk}\frac{\partial a_i}{\partial x_k^*}-g_{ik}\frac{\partial a_j}{\partial x_k^*}+ig_{ik}\left[a_j,a_*^k\right]\Biggr)\end{aligned} \tag{7.11}$$

where we have defined $a_i = A_i + g_{ij}JA_*^jJ^{-1}$ and $a_*^i = A_*^i + g^{ij}JA_jJ^{-1}$. The projection operator Π acting on (7.11) sends the gamma-matrix product $\gamma_+^i\gamma_-^j$ into its antisymmetric component $\frac{1}{2}[\gamma_+^i,\gamma_-^j]$, thus eliminating from (7.11) the symmetric part. Since conjugation of (the components of) a gauge potential by the real structure J produces elements of the commutant $\mathcal{A}_0'$ (see (4.14)), we can write the bosonic action (7.2) in the form of a symmetrized Yang-Mills type functional

$$\begin{aligned}I_B[\nabla,\nabla_*] &= \int_{T_d}\int_{T_d^*}\prod_{i=1}^d\frac{dx^i\,dx_i^*}{(2\pi)^2}\;g^{ik}g^{jl}\left(F_{ij}^{\nabla,\nabla_*}F_{kl}^{\nabla,\nabla_*}+J\,\widetilde{F}_{ij}^{\nabla,\nabla_*}\widetilde{F}_{kl}^{\nabla,\nabla_*}\,J^{-1}\right.\\&\qquad\left.+\;2\,F_{ij}^{\nabla,\nabla_*}\,J\,\widetilde{F}_{kl}^{\nabla,\nabla_*}\,J^{-1}\right)\end{aligned} \tag{7.12}$$

where

$$F_{ij}^{\nabla,\nabla_*} = \frac{\partial A_j}{\partial x^i}-\frac{\partial A_i}{\partial x^j}+i\bigl[A_i,A_j\bigr]-g_{ik}g_{jl}\left(\frac{\partial A_*^l}{\partial x_k^*}-\frac{\partial A_*^k}{\partial x_l^*}+i\left[A_*^k,A_*^l\right]\right) \tag{7.13}$$

$$\begin{aligned}\widetilde{F}_{ij}^{\nabla,\nabla_*} &= g_{jk}\frac{\partial A_*^k}{\partial x^i}-g_{ik}\frac{\partial A_*^k}{\partial x^j}+ig_{ik}g_{jl}\left[A_*^k,A_*^l\right]\\&\qquad-\left(g_{ik}\frac{\partial A_j}{\partial x_k^*}-g_{jk}\frac{\partial A_i}{\partial x_k^*}+i\bigl[A_i,A_j\bigr]\right)\end{aligned} \tag{7.14}$$

Note that the field strength (7.14) is obtained from (7.13) by interchanging the gauge potentials $A_i \leftrightarrow g_{ij}A_*^j$, but *not* the local coordinates (x, x^*). According to the description of section 3 (see Theorem 1), the commutators in (7.13) and (7.14) can be defined using the Moyal bracket

$$[A, B] \equiv \{A, B\}_\omega = A \star_\omega B - B \star_\omega A \tag{7.15}$$

where

$$(A \star_\omega B)(x, x^*) = \exp\left[i\pi\omega^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x'^j} - g_{ik}g_{jl}\frac{\partial}{\partial x_k^*}\frac{\partial}{\partial x_l'^*}\right)\right] A(x, x^*)B(x', x'^*)\Big|_{(x',x'^*)=(x,x^*)} \tag{7.16}$$

is the deformed product on $C^\infty(T_d \times T_d^*)$.

The fermionic action can be written in a more transparent form as follows. We fix a spin structure such that any spinor field on $T_d \times T_d^*$ can be decomposed into a periodic spinor χ and an antiperiodic spinor χ_*,

$$\psi = \chi \oplus \chi_* \tag{7.17}$$

with respect to a homology basis. They are defined by the conditions

$$\left(\gamma_i^+ - \gamma_i^-\right)\chi = 0 \qquad , \qquad \left(\gamma_i^+ + \gamma_i^-\right)\chi_* = 0 \tag{7.18}$$

for all $i = 1, \ldots, d$. It is important to note that this periodic-antiperiodic decomposition is not the same as the chiral-antichiral one in (4.1), although its behaviour under the action of the charge conjugation operator J is very similar. From (4.11) it follows that the corresponding anti-spinors $\widehat{\chi} = J^{-1}\chi$ and $\widehat{\chi}_* = J^{-1}\chi_*$ obey, respectively, antiperiodic and periodic conditions. Furthermore, as there are 2^d possible choices of spin structure on the d-torus T_d, there are many other analogous decompositions that one can make. However, these choices are all related by "partial" T-duality transformations of the noncommutative geometry [17] and hence the fermionic action is independent of the choice of particular spin structure. Here we choose the one which makes its duality symmetries most explicit. Defining, with the usual conventions, the gamma-matrices $\gamma_i = \frac{1}{2}(\gamma_i^+ + \gamma_i^-)$ and $\gamma_*^i = \frac{1}{2}g^{ij}(\gamma_j^+ - \gamma_j^-)$, after some algebra we find that the fermionic action (7.3) can be written in terms of the decomposition (7.17) as

$$\begin{aligned} &I_F[\chi, \chi_*, \widehat{\chi}, \widehat{\chi}_*; \nabla, \nabla_*] \\ &\quad = \int_{T_d}\int_{T_d^*}\prod_{i=1}^d \frac{dx^i\ dx_i^*}{(2\pi)^2}\ \left[-i\widehat{\chi}_*^\dagger\ g^{ij}\gamma_i\left(\frac{\partial}{\partial x^j} + iA_j\right)\chi + \chi_*^\dagger\ g_{ij}\gamma_*^i A_*^j\ \widehat{\chi}\right. \\ &\qquad\qquad \left. - i\widehat{\chi}^\dagger\ g_{ij}\gamma_*^i\left(\frac{\partial}{\partial x_j^*} + iA_*^j\right)\chi_* + \chi^\dagger\ g^{ij}\gamma_i A_j\ \widehat{\chi}_*\right] \end{aligned} \tag{7.19}$$

The (left) action of gauge potentials on fermion fields in (7.19) is given by the action of the tachyon generators

$$\left(V_{q^\pm}f\right)_{r^\pm} = \mathrm{e}^{2\pi i q_i^\pm g^{ij} r_j^\pm}\ f_{r^\pm + q^\pm} \tag{7.20}$$

on functions $f \in \mathcal{S}(\Lambda)$. Note that (7.19) naturally includes the (left) action of gauge potentials on anti-fermion fields.

The duality transformation is defined by interchanging starred quantities with unstarred ones. In terms of the fields of the gauge theory this is the mapping

$$A_i \leftrightarrow g_{ij} A_*^j \quad , \quad \chi \leftrightarrow \chi_* \quad , \quad \widehat{\chi} \leftrightarrow \widehat{\chi}_* \tag{7.21}$$

while in terms of the geometry of the space $T_d \times T_d^*$ we have

$$x^i \leftrightarrow g^{ij} x_j^* \quad , \quad \gamma_i \leftrightarrow g_{ij} \gamma_*^j \tag{7.22}$$

for all $i = 1, \ldots, d$. It is easily seen that both the bosonic and fermionic actions above are invariant under this transformation, so that

$$I[\chi, \chi_*, \widehat{\chi}, \widehat{\chi}_*; \nabla, \nabla_*] = I[\chi_*, \chi, \widehat{\chi}_*, \widehat{\chi}; \nabla_*, \nabla] \tag{7.23}$$

More general duality transformations can also be defined by (7.21) and (7.22) (as well as the spinor conditions (7.18)) taken over only a subset of all of the coordinate directions $i = 1, \ldots, d$. In all cases we find a manifestly duality invariant gauge theory. In a four-dimensional spacetime, the left-right symmetric combination $F + JFJ^{-1}$ of a field strength is relevant to the proper addition of a topological term for the gauge field to the Yang-Mills action yielding a gauge theory that has an explicit (anti-)self-dual form [33]. As we will describe in the next section, the extra terms in (7.12) incorporate, in a certain sense to be explained, the "dual" $J\widetilde{F}_{ij}^{\nabla,\nabla_*}J^{-1}$ to the field strength F_{ij}^{∇,∇_*}. The action (7.12) is therefore also naturally invariant under the symmetry $F_{ij}^{\nabla,\nabla_*} \leftrightarrow J\widetilde{F}_{ij}^{\nabla,\nabla_*}J^{-1}$ which can be thought of as a particle-antiparticle duality on $T_d \times T_d^*$ with respect to the chiral Lorentzian metric (2.7).

More generally, the action (7.1) is invariant under the automorphism group given in Proposition 1. The gauge group is the affine Lie group $\mathcal{P}_0 \, \widehat{\mathrm{Inn}}^{(0)}(\mathcal{A})\mathcal{P}_0$ which contains the duality symmetries and also the diffeomorphisms of $T_d \times T_d^*$ generated by the Heisenberg fields. Thus the gauge invariance of (7.1) also naturally incorporates the gravitational interactions in the target space. The "diffeomorphism" invariance of the action under the group $\mathcal{P}_0 \, \mathrm{Out}(\mathcal{A})\mathcal{P}_0$ naturally incorporates the $O(d,d;\mathbb{Z})$ Morita equivalences between classically distinct theories. The discrete group $O(d,d;\mathbb{Z})$ acts on the gauge potentials as

$$\begin{pmatrix} A_i \\ A_*^i \end{pmatrix} \mapsto \begin{pmatrix} (A^\top)_i{}^j & (C^\top)_{ij} \\ (B^\top)^i{}_j & (D^\top)_{ij} \end{pmatrix} \begin{pmatrix} A_j \\ A_*^j \end{pmatrix} \tag{7.24}$$

where the $d \times d$ matrices A, B, C, D are defined as in Theorem 2. Again, this symmetry is the statement that compactifications on Morita equivalent tori are physically equivalent. The $O(2,\mathbb{R})$ part of this group yields the discrete duality symmetries described above and in general it rotates the two gauge potentials among each other as

$$\begin{pmatrix} A_i + g_{ij}A_*^j \\ A_i - g_{ij}A_*^j \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_i + g_{ij}A_*^j \\ A_i - g_{ij}A_*^j \end{pmatrix} \quad , \quad \theta \in [0, 2\pi) \tag{7.25}$$

for each $i = 1, \ldots, d$. The action is in this sense a complete isomorphism invariant of the twisted module $\widetilde{\mathcal{T}}^d_\omega$. The key feature leading to this property is that (7.1) is a spectral invariant of the Dirac operators (4.18).

8. Physical Characteristics of $\widetilde{\mathcal{T}}^d_\omega$

Having obtained a precise duality-symmetric characterization of the twisted module $\widetilde{\mathcal{T}}^d_\omega$ over the noncommutative torus, we now discuss some heuristic aspects of it using the gauge theory developed in the previous section. We remark first of all that the operators (7.13) and (7.14), which can be interpreted as Yang-Mills curvatures, change sign under the above duality transformation. This signals a change of orientation of the vector bundle (represented by the finitely-generated projective module) over the tachyon algebra. Such changes of orientation of vector bundles under duality are a common feature of explicitly duality-symmetric quantum field theories [30].

It is interesting to note that in this duality symmetric framework there are analogs of the usual Yang-Mills instantons in *any* dimension. They are defined by the curvature condition

$$F_{ij}^{\nabla,\nabla_*} = -J\,\widetilde{F}_{ij}^{\nabla,\nabla_*}\,J^{-1} \tag{8.1}$$

Since the bosonic action functional (7.12) is the "square" of the operator $F^{\nabla,\nabla_*}+J\widetilde{F}^{\nabla,\nabla_*}J^{-1}$, the equations (8.1) determine those gauge field configurations at which the bosonic action functional attains its global minimum of 0. They therefore define instanton-like solutions of the duality-symmetric gauge theory. It is in this sense that the operator J acts to map the field strength $\widetilde{F}^{\nabla,\nabla_*}$ into the dual of F^{∇,∇_*}. That the instanton charge (or Chern number) here is 0 follows from the fact that the gauge theory we constructed in section 6 was built on a trivial vector bundle where the module of sections is the algebra itself. To obtain instanton field configurations with non-trivial topological charges one needs to use non-trivial bundles which are constructed using non-trivial projectors. It would be very interesting to generalize the gauge theory of this paper to twisted and also non-abelian modules.

Although the general solutions of the equations (8.1) appear difficult to deduce, there is one simple class that can be immediately identified. For this, we consider the diagonal subgroup $\Lambda_{\rm diag}$ of the Narain lattice (2.3), which we can decompose into two subgroups $\Lambda^\pm_{\rm diag}$ that are generated by the bases $\{e^i\oplus(\pm g^{ij}e_j)\}_{i=1}^d$, respectively. These rank d lattices define, respectively, self-dual and anti-self-dual d-dimensional tori $T_d^\pm \equiv \mathbb{R}^d/2\pi\Lambda^\pm_{\rm diag} \subset T_d \times T_d^*$. Then, on these tori, the gauge potentials obey the self-duality and anti-self-duality conditions

$$x^i = \pm\,g^{ij}x_j^* \qquad , \qquad A_i = \pm\,g_{ij}A_*^j \tag{8.2}$$

On these gauge field configurations the curvatures (7.13) and (7.14) vanish identically,

$$F_{ij}^{\nabla,\pm\nabla} = \widetilde{F}_{ij}^{\nabla,\pm\nabla} = 0 \tag{8.3}$$

Thus the self-dual and anti-self-dual gauge field configurations provide the analog of the instanton solutions which minimize the usual Euclidean Yang-Mills action functional. The conditions (8.2) can be thought of as projections onto the classical sector of the theory in which there is only a single, physical gauge potential in a d-dimensional spacetime. In the classical theory duality symmetries are absent and so the gauge field action vanishes identically. The Yang-Mills type action functional (7.12) can therefore be thought of as measuring the amount of asymmetry between a given connection and its dual on $\widetilde{\mathcal{T}}^d_\omega$. As such, it measures how much duality symmetry is present in the target space and hence how far away the stringy perturbation is from ordinary classical spacetime. The action (7.12) can thus be regarded as an effective measure of distance scales in spacetime.

There are other interesting physical projections of the theory that one can make which, unlike the relations (8.2), break the duality symmetries explicitly. For instance, consider the projection $T_d \times T_d^* \to T_d^+$ along with the freezing out of the dual gauge degrees of freedom. This means that the fields now depend only on the local coordinates $x^i = g^{ij} x_j^*$, and the dual gauge potential A_*^i is frozen at some constant value,

$$(x, x^*) \;\to\; (x, x) \qquad , \qquad A_*^i \;\to\; \text{const.} \tag{8.4}$$

Since each A_*^i is constant, it follows that $[A_*^i, A_*^j] = 0$. The field strength (7.14) is then identical to (7.13) which becomes the usual Yang-Mills curvature of the d-dimensional gauge field A_i over T_d^+,

$$F_{ij}[A] \;\equiv\; F_{ij}^{\nabla,\nabla_*}\Big|_{T_d^+} \;=\; -\,\widetilde{F}_{ij}^{\nabla,\nabla_*}\Big|_{T_d^+} \;=\; \partial_i A_j - \partial_j A_i + i[A_i, A_j] \tag{8.5}$$

The bosonic action functional can be easily read off from (7.11),

$$\Pi\left(\not{\partial}_\nabla^2 - \overline{\not{\partial}}_\nabla^2\right)\Big|_{T_d^+} = -\tfrac{i}{4}\left[\gamma_+^i, \gamma_-^j\right] \otimes \left(F_{ij}[A] - J F_{ij}[A] J^{-1}\right) \tag{8.6}$$

When the operator (8.6) is squared, the resulting bosonic action has the form of a symmetrized Yang-Mills functional for the gauge field A_i on T_d^+. It is remarkable that the projection (8.4) reduces the bosonic action functional (7.2) to the standard Yang-Mills action used in noncommutative geometry [33].

In the infrared limit $g \to \infty$ ($\omega \to 0$), the Moyal bracket (7.15) vanishes and the gauge theory generated by (8.6) becomes the usual electrodynamics on the commutative manifold T_d^+. Thus at large-distance scales, we recover the usual commutative classical limit with the canonical abelian gauge theory defined on it [4, 5]. The continuous "internal" space T_d^* of the string spacetime acts to produce a sort of Kaluza-Klein mechanism by inducing nonabelian degrees of freedom when the radii of compactification are made very small. This nonabelian generating mechanism is rather different in spirit than the usual ones of noncommutative geometry which extend classical spacetime, represented by the commutative algebra $C^\infty(T_d)$, by a discrete internal space, represented typically by a noncommutative finite-dimensional matrix algebra. In the present case the "internal" space comes from the natural embedding of the classical spacetime into the noncommutative string spacetime represented by the tachyon sector of the vertex operator algebra. In this context we find that the role of the noncommutativity of spacetime coordinates at very short distance scales is to induce internal (nonabelian) degrees of freedom.

As for the fermionic sector of the field theory, the projection above applied to the spinor fields is defined as

$$\chi \Big|_{T_d^+} = \chi_* \Big|_{T_d^+} \qquad , \qquad \widehat{\chi} \Big|_{T_d^+} = \widehat{\chi}_* \Big|_{T_d^+} \tag{8.7}$$

with $\gamma_i = g_{ij}\,\gamma_*^j$. Then, denoting the constant value of A_*^i by M^i, it follows that the fermionic action (7.19) becomes the usual gauged Dirac action for the fermion fields $(\widehat{\chi},\chi)$ minimally coupled to the nonabelian Yang-Mills gauge field A_i and with mass parameters M^i,

$$I_F[\chi,\widehat{\chi};\nabla,\nabla_*] \Big|_{T_d^+} = I_{\rm Dirac}[\chi,\widehat{\chi},M;A] \tag{8.8}$$

Thus the internal symmetries of the string geometry also induce fermion masses, and so the explicit breaking of the duality symmetries, required to project onto physical spacetime, of the twisted module acts as a sort of geometrical mass generating mechanism. Again in the classical limit $g\to\infty$ the left action (7.20) becomes ordinary multiplication and (8.8) becomes the Dirac action for $U(1)$ fermions coupled to electrodynamics. The fact that the Kaluza-Klein modes coming from T_d^- induce nonabelian degrees of freedom and fermion masses could have important ramifications for string phenomenology. In particular, when $d=4$, the above projections suggest a stringy origin for the canonical action of the standard model. We remark again that the nonabelian gauge group thus induced is the natural enhancement of the generic abelian $U(1)^d$ gauge symmetry within the vertex operator algebra [21]. It is intriguing that both nonabelian gauge degrees of freedom and masses are induced so naturally by string geometry, and it would be interesting to study the physical consequences of this feature in more depth.

It would also be very interesting to give a physical origin for the duality-symmetric noncommutative gauge theory developed here using either standard model or M-Theory physics. For instance, in [7] it is argued that the characteristic interaction term for gauge theory on the noncommutative torus, within the framework of compactified Matrix Theory [6], appears naturally as the worldline field theories of N D-particles. This observation follows from careful consideration of the action of T-duality on superstring data in the presence of background fields. What we have shown here is that the natural algebraic framework for the noncommutative geometry of string theory (i.e. vertex operator algebras) has embedded within it a very special representation of the noncommutative torus, and that the particular module $\widetilde{\mathcal{T}}_\omega^d$ determines a gauge theory which is manifestly duality-invariant. Thus we can derive an explicitly duality-symmetric field theory based on very basic principles of string geometry. This field theory is manifestly covariant, at the price of involving highly non-local interactions. The non-locality arises from the algebraic relations of the vertex operator algebra and as such it reflects the nature of the string interactions. The fact that string dynamics control the very structure of the field theory should mean that it has a more direct relationship to M-Theory dynamics. One possible scenario would be to interpret the dimension N of the matrices which form the dynamical variables of Matrix Theory as the winding numbers of strings wrapping around the d-torus T_d. In the 11-dimensional light-cone frame, N is related to the longitudinal momentum as $p_+ \propto N$, so that the vertex operator algebra is a dual model for the M-Theory dynamics in which the light-cone momenta are represented by winding numbers. The limit $N \to \infty$ of infinite winding number corresponds to the usual Matrix Theory description of M-Theory dynamics in the infinite momentum frame. The origin of the noncommutative torus as the infinite winding modes of strings wrapping around T_d is naturally contained within the tachyon sector of the vertex operator algebra and yields a dual picture of infinite momentum frame dynamics.

We have also seen that the relationship between lattice vertex operator algebras and the noncommutative torus implies a new physical interpretation of Morita equivalence in terms of target space duality transformations. It would be interesting to investigate the relationship between this duality and the non-classical Nahm duality which maps instantons on one noncommutative torus to instantons on another (dual) one [6, 19]. It appears that the duality interpretations in the case of the twisted module $\widetilde{\mathcal{T}}^d_\omega$ are somewhat simpler because of the special relationship between the deformation parameters ω^{ij} and the metric of the compactification lattice (see Theorem 1). This relation in essence breaks some of the symmetries of the space. In any case, it remains to study more the gauge group of the present module given the results of [21] and hence probe more in depth the structure of the gauge theory developed here.

Acknowledgements: We thank A. Connes, S. Majid and L. Pilo for helpful discussions. G.L. thanks all members of DAMTP and in particular Prof. M. Green for the kind hospitality in Cambridge. G.L. is a fellow of the Italian National Council of Research (CNR) under grant CNR-NATO 215.29/01. The work of R.J.S. was supported in part by the Particle Physics and Astronomy Research Council (U.K.).

Appendix A. Morita Equivalence of C^*-algebras

In this appendix we shall briefly describe the notion of (strong) Morita equivalence for C^*-algebras [34]. Additional details can be found in [5], for example. Throughout this appendix $\mathcal{A}$ is an arbitrary unital C^*-algebra whose norm we denote by $\|\cdot\|$.

A right Hilbert module over $\mathcal{A}$ is a right $\mathcal{A}$-module $\mathcal{E}$ [3] endowed with an $\mathcal{A}$-valued Hermitian structure, i.e. a sesquilinear form $\langle\cdot\,,\,\cdot\rangle_{\mathcal{A}} : \mathcal{E}\times\mathcal{E}\to\mathcal{A}$ which is conjugate linear in its first argument and satisfies

$$\langle\eta_1,\eta_2 a\rangle_{\mathcal{A}} \;=\; \langle\eta_1,\eta_2\rangle_{\mathcal{A}}\, a \tag{A.1}$$

[3]In more simplistic terms this means that $\mathcal{E}$ carries a right action of $\mathcal{A}$.

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{A}}^* = \langle \eta_2, \eta_1 \rangle_{\mathcal{A}} \tag{A.2}$$

$$\langle \eta, \eta \rangle_{\mathcal{A}} \geq 0 \quad , \quad \langle \eta, \eta \rangle_{\mathcal{A}} = 0 \Leftrightarrow \eta = 0 \tag{A.3}$$

for all $\eta_1, \eta_2, \eta \in \mathcal{E}$, $a \in \mathcal{A}$. We define a norm on $\mathcal{E}$ by $\|\eta\|_{\mathcal{A}} = \sqrt{\| \langle \eta, \eta \rangle_{\mathcal{A}} \|}$ for any $\eta \in \mathcal{E}$ and require that $\mathcal{E}$ be complete with respect to this norm. We also demand that the module be full, i.e. that the ideal $\mathrm{span}_{\mathbb{C}}\{\langle \eta_1, \eta_2 \rangle_{\mathcal{A}} \mid \eta_1, \eta_2 \in \mathcal{E}\}$ is dense in $\mathcal{A}$ with respect to the norm closure. A left Hilbert module structure on a left $\mathcal{A}$-module $\mathcal{E}$ is provided by an $\mathcal{A}$-valued Hermitian structure $\langle \cdot \, , \, \cdot \rangle_{\mathcal{A}}$ on $\mathcal{E}$ which is conjugate linear in its second argument and with the condition (A.1) replaced by

$$\langle a\eta_1, \eta_2 \rangle_{\mathcal{A}} = a \, \langle \eta_1, \eta_2 \rangle_{\mathcal{A}} \qquad \forall \eta_1, \eta_2 \in \mathcal{E}, \ a \in \mathcal{A} \tag{A.4}$$

Given a Hilbert module $\mathcal{E}$, its compact endomorphisms are obtained as usual from the 'endomorphisms of finite rank'. For any $\eta_1, \eta_2 \in \mathcal{E}$ an endomorphism $|\eta_1\rangle \langle \eta_2|$ of $\mathcal{E}$ is defined by

$$|\eta_1\rangle \langle \eta_2| \, (\xi) = \eta_1 \, \langle \eta_2, \xi \rangle_{\mathcal{A}} \quad , \quad \forall \xi \in \mathcal{E} \tag{A.5}$$

which is right $\mathcal{A}$-linear,

$$|\eta_1\rangle \langle \eta_2| \, (\xi a) = \Big(|\eta_1\rangle \langle \eta_2| \, (\xi) \Big) a \quad , \quad \forall \xi \in \mathcal{E}, \ a \in \mathcal{A} \tag{A.6}$$

Its adjoint endomorphism is given by

$$\Big(|\eta_1\rangle \langle \eta_2| \Big)^* = |\eta_2\rangle \langle \eta_1| \quad , \quad \forall \eta_1, \eta_2 \in \mathcal{E} \tag{A.7}$$

It can be shown that for $\eta_1, \eta_2, \xi_1, \xi_2 \in \mathcal{E}$ one has the expected composition rule

$$|\eta_1\rangle \langle \eta_2| \circ |\xi_1\rangle \langle \xi_2| = |\eta_1 \, \langle \eta_2, \xi_1 \rangle_{\mathcal{A}} \rangle \langle \xi_2| = |\eta_1\rangle \langle \langle \eta_2, \xi_1 \rangle_{\mathcal{A}} \, \xi_2| \tag{A.8}$$

It turns out that if $\mathcal{E}$ is a finitely-generated projective module then the norm closure $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ (with respect to the natural operator norm which yields a C^*-algebra) of the $\mathbb{C}$-linear span of the endomorphisms of the form (A.5) coincides with the endomorphism algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ of $\mathcal{E}$. In fact, this property completely characterizes finitely-generated projective modules. For then, there are two finite sequences $\{\xi_k\}$ and $\{\zeta_k\}$ of elements of $\mathcal{E}$ such that the identity endomorphism $\mathbb{I}_{\mathcal{E}}$ can be written as $\mathbb{I}_{\mathcal{E}} = \sum_k |\xi_k\rangle \langle \zeta_k|$. For any $\eta \in \mathcal{E}$, we then have

$$\eta = \mathbb{I}_{\mathcal{E}} \, \eta = \sum_k |\xi_k\rangle \langle \zeta_k| \, \eta = \sum_k \xi_k \, \langle \zeta_k, \eta \rangle_{\mathcal{A}} \tag{A.9}$$

and hence $\mathcal{E}$ is finitely-generated by the sequence $\{\xi_k\}$. If N is the length of the sequences $\{\xi_k\}$ and $\{\zeta_k\}$, we can embed $\mathcal{E}$ as a direct summand of $\mathcal{A}^N \equiv \bigoplus_{n=1}^N \mathcal{A}$, proving that it is projective. The embedding and surjection maps are defined, respectively, by

$$\begin{aligned} \lambda : \mathcal{E} \to \mathcal{A}^N \quad &, \quad \lambda(\eta) = \Big(\langle \zeta_1, \eta \rangle_{\mathcal{A}}, \dots, \langle \zeta_N, \eta \rangle_{\mathcal{A}} \Big) \\ \rho : \mathcal{A}^N \to \mathcal{E} \quad &, \quad \rho\Big((a_1, \dots, a_N) \Big) = \sum_k \xi_k a_k \end{aligned} \tag{A.10}$$

Then, given any $\eta \in \mathcal{E}$, we have

$$\rho \circ \lambda(\eta) = \rho\Big(\big(\langle \zeta_1, \eta \rangle_{\mathcal{A}}, \ldots, \langle \zeta_N, \eta \rangle_{\mathcal{A}}\big)\Big) = \sum_k \xi_k \langle \zeta_k, \eta \rangle_{\mathcal{A}} = \sum_k |\xi_k\rangle \langle \zeta_k| (\eta) = \mathbb{I}_{\mathcal{E}}(\eta) \quad \text{(A.11)}$$

so that $\rho \circ \lambda = \mathbb{I}_{\mathcal{E}}$, as required. The projector $p = \lambda \circ \rho$ identifies $\mathcal{E}$ as $p\mathcal{A}^N$.

For any full Hilbert module $\mathcal{E}$ over a C^*-algebra $\mathcal{A}$, the latter is *(strongly) Morita equivalent* to the C^*-algebra $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ of compact endomorphisms of $\mathcal{E}$. If $\mathcal{E}$ is finitely-generated and projective, so that $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E}) = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$, then the algebra $\mathcal{A}$ is strongly Morita equivalent to the whole of $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$. The equivalence is expressed as follows. The idea is to construct an $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$-valued Hermitian structure on $\mathcal{E}$ which is compatible with the Hermitian structure $\langle \cdot\ ,\ \cdot \rangle_{\mathcal{A}}$. Consider then a full *right* Hilbert module $\mathcal{E}$ over the algebra $\mathcal{A}$ with $\mathcal{A}$-valued Hermitian structure $\langle \cdot\ ,\ \cdot \rangle_{\mathcal{A}}$. It follows that $\mathcal{E}$ is a *left* module over the C^*-algebra $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$. A left Hilbert module structure is constructed by inverting the definition (A.5) so as to produce an $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$-valued Hermitian structure on $\mathcal{E}$,

$$\langle \eta_1, \eta_2 \rangle_{\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})} = |\eta_1\rangle \langle \eta_2| \quad , \quad \forall \eta_1, \eta_2 \in \mathcal{E} \qquad \text{(A.12)}$$

It is straightforward to check that (A.12) satisfies all the properties of a left Hermitian structure including conjugate linearity in its second argument. It follows from the definition of a compact endomorphism that the module $\mathcal{E}$ is also full as a module over $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$. Furthermore, from the definition (A.5) we have a compatibility condition between the two Hermitian structures on $\mathcal{E}$,

$$\langle \eta_1, \eta_2 \rangle_{\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})}\, \xi = |\eta_1\rangle \langle \eta_2| (\xi) = \eta_1 \langle \eta_2, \xi \rangle_{\mathcal{A}} \quad , \quad \forall \eta_1, \eta_2, \xi \in \mathcal{E} \qquad \text{(A.13)}$$

The Morita equivalence is also expressed by saying that the module $\mathcal{E}$ is an $\mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$-$\mathcal{A}$ equivalence Hilbert bimodule.

A C^*-algebra $\mathcal{B}$ is said to be Morita equivalent to the C^*-algebra $\mathcal{A}$ if $\mathcal{B} \cong \mathrm{End}^0_{\mathcal{A}}(\mathcal{E})$ for some $\mathcal{A}$-module $\mathcal{E}$. Morita equivalent C^*-algebras have equivalent representation theories. If the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent with $\mathcal{B}$-$\mathcal{A}$ equivalence bimodule $\mathcal{E}$, then given a representation of $\mathcal{A}$, using $\mathcal{E}$ we can construct a unitarily equivalent representation of $\mathcal{B}$. For this, let $(\mathcal{H}, \pi_{\mathcal{A}})$ be a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. The algebra $\mathcal{A}$ acts by bounded operators on the left on $\mathcal{H}$ via $\pi_{\mathcal{A}}$. This action can be used to construct another Hilbert space

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \quad , \quad \eta a \otimes_{\mathcal{A}} \psi - \eta \otimes_{\mathcal{A}} \pi_{\mathcal{A}}(a)\psi = 0 \ , \ \forall a \in \mathcal{A},\ \eta \in \mathcal{E},\ \psi \in \mathcal{H} \qquad \text{(A.14)}$$

with scalar product

$$(\eta_1 \otimes_{\mathcal{A}} \psi_1, \eta_2 \otimes_{\mathcal{A}} \psi_2)_{\mathcal{H}'} = (\psi_1, \langle \eta_1, \eta_2 \rangle_{\mathcal{A}}\, \psi_2)_{\mathcal{H}} \quad , \quad \forall \eta_1, \eta_2 \in \mathcal{E},\ \psi_1, \psi_2 \in \mathcal{H} \qquad \text{(A.15)}$$

A representation $(\mathcal{H}', \pi_{\mathcal{B}})$ of the algebra $\mathcal{B}$ is then defined by

$$\pi_{\mathcal{B}}(b)(\eta \otimes_{\mathcal{A}} \psi) = (b\eta) \otimes_{\mathcal{A}} \psi \quad , \quad \forall b \in \mathcal{B},\ \eta \otimes_{\mathcal{A}} \psi \in \mathcal{H}' \qquad \text{(A.16)}$$

This representation is unitarily equivalent to the representation $(\mathcal{H}, \pi_{\mathcal{A}})$. Conversely, starting with a representation of $\mathcal{B}$, we can use a conjugate $\mathcal{A}$-$\mathcal{B}$ equivalence bimodule $\overline{\mathcal{E}}$ to construct an equivalent representation of $\mathcal{A}$. Morita equivalence also yields isomorphic K-groups and cyclic homology, so that Morita equivalent algebras determine the same noncommutative geometry. However, the physical characteristics can be drastically different. For example, the algebras can have different (unitary) gauge groups and hence determine physically inequivalent gauge theories.

References

[1] M.A. Rieffel, *C^*-algebras associated with Irrational Rotations*, Pac. J. Math. **93** (1981) 415.

[2] A. Connes, *C^*-algèbres et Géométries Différentielle*, C. R. Acad. Sci. Paris **A290** (1980) 599;
A. Connes and M.A. Rieffel, *Yang-Mills for Noncommutative Two-tori*, Contemp. Math. **62** (1987) 237;
M.A. Rieffel, *Projective Modules over Higher-dimensional Noncommutative Tori*, Can. J. Math. **40** (1988) 257.

[3] M.A. Rieffel, *The Cancellation Theorem for Projective Modules over Irrational Rotation C^*-algebras*, Proc. London Math. Soc. **47** (1983) 285.

[4] A. Connes, *Noncommutative Geometry* (Academic Press, 1994).

[5] G. Landi, *An Introduction to Noncommutative Spaces and their Geometries* (Springer, 1997).

[6] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative Geometry and Matrix Theory: Compactification on Tori*, JHEP **02** (1998) 003.

[7] M.R. Douglas and C.M. Hull, *D-branes and the Noncommutative Torus*, JHEP **02** (1998) 008;
M. Li, *Comments on Supersymmetric Yang-Mills Theory on a Noncommutative Torus*, hep-th/9802052;
M. Berchooz, *Nonlocal Field Field Theories and the Noncommutative Torus*, hep-th/9802073;
Y.-K.E. Cheung and M. Krogh, *Noncommutative Geometry from 0-branes in a Background B-field*, hep-th/9803031;
D. Bigatti, *Noncommutative Geometry and Super Yang-Mills Theory*, hep-th/9804120.

[8] P.-M. Ho, Y.-Y. Wu and Y.-S. Wu, *Towards a Noncommutative Geometric Approach to Matrix Compactification*, hep-th/9712201;
T. Kawano and K. Okuyama, *Matrix Theory on Noncommutative Torus*, hep-th/9803044;
F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Mixed Branes and Matrix Theory on Noncommutative Torus*, hep-th/9803067;
P.-M. Ho, *Twisted Bundle on Quantum Torus and BPS States in Matrix Theory*, hep-th/9803166.

[9] P.-M. Ho and Y.-S. Wu, *Noncommutative Gauge Theories in Matrix Theory*, hep-th/9801147;
R. Casalbuoni, *Algebraic Treatment of Compactification on Noncommutative Tori*, hep-th/9801170, to appear in Phys. Lett. **B**

[10] E. Witten, *Bound States of Strings and p-branes*, Nucl. Phys. **B460** (1996) 335;
M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, *D-branes and Short Distances in String Theory*, Nucl. Phys. **B485** (1997) 85–127.

[11] P.-M. Ho and Y.-S. Wu, *Noncommutative Geometry and D-branes*, Phys. Lett. **B398** (1997) 52–60.

[12] F. Lizzi, N.E. Mavromatos and R.J. Szabo, *Matrix σ-models for Multi D-brane Dynamics*, Mod. Phys. Lett. **A13** (1998) 829–842.

[13] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M Theory as a Matrix Model: A Conjecture*, Phys. Rev. **D55** (1997) 5112.

[14] J. Fröhlich and K. Gawędzki, *Conformal Field Theory and Geometry of Strings*, CRM Proc. Lecture Notes **7** (1994) 57.

[15] A.H. Chamseddine, *The Spectral Action Principle in Noncommutative Geometry and the Superstring*, Phys. Lett. **B400** (1997) 87; *An Effective Superstring Spectral Action*, Phys. Rev. **D56** (1997) 3555.

[16] F. Lizzi and R.J. Szabo, *Target Space Duality in Noncommutative Geometry*, Phys. Rev. Lett. **79** (1997) 3581.

[17] F. Lizzi and R.J. Szabo, *Duality Symmetries and Noncommutative Geometry of String Spacetimes*, hep-th/9707202, to appear in Commun. Math. Phys.

[18] M.A. Rieffel and A. Schwarz, *Morita Equivalence of Multidimensional Noncommutative Tori*, math.QA/9803057.

[19] A. Schwarz, *Morita Equivalence and Duality*, hep-th/9805034.

[20] F. Lizzi and R.J. Szabo, *Electric-magnetic Duality in Noncommutative Geometry*, Phys. Lett. **B417** (1998) 303.

[21] F. Lizzi and R.J. Szabo, *Noncommutative Geometry and Spacetime Gauge Symmetries of String Theory*, hep-th/9712206, to appear in Chaos, Solitons and Fractals.

[22] A. Connes and J. Lott, *Particle Models and Noncommutative Geometry*, Nucl. Phys. B (Proc. Suppl.) **18B** (1990) 29;
C.P. Martín, J.M. Gracia-Bondía and J.C. Várilly, *The Standard Model as a Noncommutative Geometry: The Low Energy Regime*, Phys. Rep. **294** (1998) 363.

[23] A. Connes, *Noncommutative Geometry and Reality*, J. Math. Phys. **36** (1995) 619.

[24] A. Connes, *Gravity Coupled with Matter and the Foundation of Noncommutative Geometry*, Commun. Math. Phys. **182** (1996) 155.

[25] A.H. Chamseddine and A. Connes, *Universal Formula for Noncommutative Geometry Actions: Unification of Gravity and the Standard Model*, Phys. Rev. Lett. **77** (1996) 4868; *The Spectral Action Principle*, Commun. Math. Phys. **186** (1997) 731.

[26] D. Zwanziger, Phys. Rev. **D3** (1971) 880;
S. Deser and C. Teitelboim, *Duality Transformations of Abelian and Nonabelian Gauge Fields*, Phys. Rev. **D13** (1976) 1592–1597;
J.H. Schwarz and A. Sen, *Duality Symmetric Actions*, Nucl. Phys. **B411** (1994) 35;
S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, *Duality, Selfduality, Sources and Charge Quantization in Abelian N-form Theories*, Phys. Lett. **B400** (1997) 80.

[27] P. Pasti, D. Sorokin and M. Tonin, *Duality Symmetric Actions with Manifest Space-time Symmetries*, Phys. Rev. **D52** (1995) 4277;
N. Berkovits, *Manifest Electromagnetic Duality in Closed Superstring Field Theory*, Phys. Lett. **B388** (1996) 743; *Local Actions with Electric and Magnetic Sources*, **B395** (1997) 28; *Super-Maxwell Actions with Manifest Duality*, **B398** (1997) 79;
R. Medina and N. Berkovits, *Pasti-Sorokin-Tonin Actions in the Presence of Sources*, Phys. Rev. **D56** (1997) 6388–6390.

[28] S. Deser, M. Henneaux and C. Teitelboim, *Electric-magnetic Black Hole Duality*, Phys. Rev. **D55** (1997) 826;
D.D. Song and R.J. Szabo, *Black String Entropy from Anomalous D-brane Couplings*, hep-th/9805027.

[29] M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *Dual D-brane Actions*, Nucl. Phys. **B496** (1997) 215–230;
A. Nurmagambetov, *Duality-symmetric Three-brane and its Coupling to Type IIB Supergravity*, hep-th/9804157.

[30] Y.-K.E. Cheung and Z. Yin, *Anomalies, Branes and Currents*, Nucl. Phys. **B517** (1998) 69.

[31] I.B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure Appl. Math. **134** (Academic Press, New York, 1988);
R.W. Gebert, *Introduction to Vertex Algebras, Borcherds Algebras and the Monster Lie Algebra*, Intern. J. Mod. Phys. **A8** (1993) 5441.

[32] S. De Bièvre, *Chaos, Quantization and the Classical Limit on the Torus*, Proc. XIV Workshop on Geometrical Methods in Physics, July 1995, Bialowieza, Poland, mp-arc 96–191;
J.C. Várilly, *An Introduction to Noncommutative Geometry*, Lectures at the EMS Summer School on Noncommutative Geometry and Applications, September 1997, Portugal, physics/9709045.

[33] J.M. Gracia-Bondía, B. Iochum and T. Schücker, *The Standard Model in Noncommutative Geometry and Fermion Doubling*, Phys. Lett. **B416** (1998) 123.

[34] M.A. Rieffel, *Induced Representations of C^*-algebras*, Bull. Amer. Math. Soc. **78** (1972) 606–609; Adv. Math. **13** (1974) 176–257; *Morita Equivalence for Operator Algebras*, in: Operator Algebras and Applications, Proc. Symp. Pure Math. **38**, R.V. Kadison ed. (American Mathematical Society, 1982) 285–298.