Generalised GibbsU states for expanding maps

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Abstract

We show that for expanding maps there is a one-to-one correspondence between equilibrium states and Gibbs' states. In particular we show that families of multipliers are determined by suitable potentials.

1 Introduction

In this paper we consider the equivalence of Gibbs' and equilibrium states for expanding maps on compact metric spaces. A Gibbs' state is determined by a family of locally defined multipliers that satisfy a cocycle relation (see definition 2 below). On the other hand an equilibrium state is determined by a potential and satisfies a variational principle. It is easy to see that a (sufficiently regular) potential determines a family of multipliers (see equation (1) below) which then implies that equilibirium states satisfy the Gibbs' property with respect to that family of multipliers.

Here we show that a family of (local) multipliers gives rise to a potential and consequently the Gibbs measure for these multipliers turns out to be the equlibrium state for that potential. If everything is Hölder continuous then modifying the potential by a coboundaries yields a class of equivalent measures exactly one of which is invariant. Hence a modification of the multipliers by 'multiplicative coboundary' like terms yields one which is invariant under the map. See the remark in section 3.

Assuming the relation (1) applies, the papers [2, 4] established the equivalence of Gibbs and equilibrium states in various settings. For expanding

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maps, Ruelle showed [4] that generally if the multipiers are given by equation (1) for some f then the associated Gibbs' state is the equilibrium state for the potential f (which then is known to be characterised by the eigenfunctional and eigenvector to the largest eigenvalue of a transfer type operator).

In the special case of a subshift of finite type, we showed in [2] that a given family of Hölder continuous multipliers (with sufficiently large Hölder exponents) gives rise to a representation by an expression similar to equation (1) with a Hölder continuous potential, provided the family of multipliers is shift invariant. In that case the potential is determined up to a constant and additive coboudaries.

2 Multipliers

A map T on a compact metric space Ω is *expanding* if there exists a (expanding) constant $\lambda > 1$ such that $d(Tx, Ty) \geq \lambda d(x, y)$ for $d(x, y) \leq \varepsilon_0$, for some positive ε_0. For every $x \in \Omega$, the set $T^{-1}x = \{y \in \Omega : Ty = x\}$ is then finite, and, moreover, there exists a positive ζ such that for every $x \in \Omega$, the set $T^{-1}(B_\zeta(x))$ is the finite union of disjoint open subsets $A_1, A_2, \ldots, A_k$ of Ω on which T is one-to-one ($B_\zeta(x)$ denotes the ball of radius ζ centered at x). The sets $A_1, A_2, \ldots$ shall be called the components of $T^{-1}(B_\zeta(x))$ on which the restriction of T is a homeomorphism. In general, since T^{-1} does not increase distances (if they are small to begin with), the set $T^{-n}(B_\zeta(x))$ is (for every positive integer n and $x \in \Omega$) the disjoint union of a finite number of subsets (components) $A_1, A_2, \ldots$ of Ω. If z lies in some A_j and satisfies $T^n z = x$, then for every $y \in B_\zeta(x)$ there exists a unique $z' \in A_j$ satisfying $T^n z' = y$.

We shall require that T satisfies a mixing condition: For every positive ε and $x \in \Omega$, the set $T^{-n}(x)$ is ε-dense in Ω for all large enough n.

Definition 1 *(i) Two points $x, y \in \Omega$ are called* conjugate *or n-*conjugate *if $T^n x = T^n y$ for some positive integer n (and consequently also $T^m x = T^m y$ for all $m \geq n$.)*
(ii) A (local) homeomorphism $\varphi : U_\varphi \to \Omega$, $U_\varphi \subset \Omega$ open, is called conjugating, *if every $x \in U_\varphi$ is conjugate to $\varphi(x)$. In fact $T^j x = T^j \varphi(x)$ for all $x \in U_\varphi$ and $j \geq n$ for some positive integer n.*

The composition of conjugating homeomophisms is again conjugating: $\varphi = \varphi' \circ \varphi'$ is conjugating on the open set $U_\varphi = \varphi^{-1}(U_{\varphi''} \cap \varphi'(U_{\varphi'}))$ (if U_φ is non-empty). In particular, if φ' and φ'' are both n-conjugating then also φ is n-conjugating.

Definition 2 *[1, 3] A family of positive and continuous functions* $\{r_\varphi : U_\varphi \to (0,\infty) : \varphi \text{ conjugating}\}$ *is said to be a* family of multipliers *if for any two conjugating homeomorphisms φ' and φ'' the following cocyle equation is satisfied:*

$$(r_{\varphi''} \circ \varphi')\, r_{\varphi'} = r_{\varphi'' \circ \varphi'},$$

on $U'' = \varphi^{-1}(U_{\varphi''} \cap \varphi'(U_{\varphi'}))$, provided U'' is non-empty.

We introduce the function space $V(\Omega)$ [5] and say a function $f : \Omega \to \mathbf{R}$, belongs to it if it satisfies the following two conditions:
(i) For every positive $\delta < \zeta$ the norm $\|f\|_\delta = \sup_{d(x,x')<\delta} C(x,x')$ is finite, where $C(x,x')$ is the smallest number for which

$$\sup_{n\geq 1} \sup_{(y,y')\in T^{-n}x \times T^{-n}x'} \left| f^{(n)}(y) - f^{(n)}(y') \right| \leq C(x,x'),$$

where the supremum is over all pairs $(y,y') \in \Omega \times \Omega$ for which both points lie in the same component A_j of $T^{-n}(B_\zeta(x))$, that is, $y, y' \in A_j$, $j = 1, 2, \ldots$.
(ii) The constant $C(x,x')$ goes to zero as $\delta \to 0$.

We wrote $f^{(n)} = f + fT + fT^2 + \cdots + fT^{n-1}$ for the nth ergodic sum of f.

The prime example of a family of multipliers given as follows. Let f be a function in $V(\Omega)$ and φ an n-conjugating homeomorphism. Then put

$$r_\varphi = \exp\left(f^{(n)} \circ \varphi - f^{(n)}\right) = \exp \sum_{k\geq 0} (f \circ T^k \circ \varphi - f \circ T^k). \qquad (1)$$

The function is r_φ is defined on U_φ and one easily realises that, as φ runs through the entire set of conjugating homeomorphism, one obtains a family of multipliers which satisfies the cocyle equations of definition 2. The aim of this paper is to show that in general all families of multipliers are indeed of this form for a suitably chosen potential f. In the following we shall declare a function a member in the space $V(\Omega)$ if it satisfies the conditions (i) and (ii) on a set of full measure.

Definition 3 *[1] Let* $\{r_\varphi \in V(\Omega) : \varphi \text{ conjugating}\}$ *be a family of multipliers. A probability measure* μ *on* Ω *is called* Gibbs' *for the family* $\{r_\varphi : \varphi\}$ *if for every conjugating homeomorphism* $\varphi : U_\varphi \in \Omega$ *the following holds true:*
(i) $\varphi^*\mu$ *restricted to* U_φ *is absolutely continuous with respect to* μ.
(ii) $\frac{d\varphi^*\mu}{d\mu} = r_\varphi$, *that is*

$$\int \chi \circ \varphi \, r_\varphi \, d\mu = \int \chi \, d\mu$$

for measureable (test) functions χ *which are supported in* $\varphi(U_\varphi)$.

For $x \in \Omega$ let us write T_x for the homeomorphism which is given by the restriction of T to the ball $B_\zeta(x)$. Its inverse T_x^{-1} is then a homeomorphism as well. If μ is a measure on Ω such that $T_x^*\mu$ is absolutely continuous with respect to μ we write h for the Radon-Nikodym derivative $\frac{dT_x^*\mu}{d\mu}$ (for which we shall also write $\frac{d\mu(x)}{d\mu(Tx)} = \frac{d\mu}{d\mu T}(x)$).

3 Results

Theorem 4 *Let* μ *be a Gibbs' measure on* Ω *for a family of multipliers* $\{r_\varphi \in V(\Omega) : \varphi \text{ conjugating}\}$ *such that* μT *and* μ *are equivalent measures and suppose that there exists an* $h \in V(\Omega)$ *such that* $h = \frac{d\mu}{d\mu T}$ *a.e.. Then the multipliers* r_φ *are of the form given above in (1) with the potential function* $f(x) = \log h(x)$.

Let us note that the potential f is not unique. Any change by a constant will in fact yield the same equilibirium state (and multipliers). However to f is added a coboundary (see Remark below), then the measure will change to an equivalent measure.

The function h is positive and by [5] lies in the space $V(\Omega)$ if the quantity $\frac{d\mu T^n}{d\mu}(y')/\frac{d\mu T^n}{d\mu}(y)$ is uniformly bounded for all integers n and $y, y' \in A_j$ for which $d(T^n y, T^n y') < \zeta$, and converges to 1 as ζ decreases to zero, where the $A_j \subset \Omega$ are the finitely many disjoint component of $T^{-n}(B_\zeta(T^n y))$ (and have diameter less than 2ζ). For a function $f \in V(\Omega)$ we define Ruelle's Perron-Frobenius operator $\mathcal{L}_f : V(\Omega) \to V(\Omega)$ by

$$\mathcal{L}_f \chi(x) = \sum_{y \in T^{-1}x} e^{f(y)} \chi(y).$$

This operator is well-known [3] to have a largest, simple and positive eigenvalue λ (whose logarithm is the pressure of f) and an associated strictly positive eigenfunction k. In fact, if the probability measure ρ spans the (one-dimensional) eigenspace of the adjoint $\mathcal{L}_f^*$ to its largest eigenvalue λ, then by [5] lemma 14 f equals $\log \frac{d\rho}{d\rho T}$ a.e.. Moreover, $\lambda^{-n}\mathcal{L}_f^{*n}\chi$ converges to $k\rho(\chi)$ as n goes to infinity, for all $\chi \in V(\Omega)$. In the light of [5] and [4] we obtain the following corollary.

Corollary 5 *Let Ω be a metric space and T and expanding map as above. The following (I) and (II) are equivalent*
(I) μ is a Gibbs' measure for the family of multipliers

$$\{r_\varphi \in V(\Omega) : \varphi \text{ conjugating}\},$$

where μT equivalent to μ and $h = \frac{d\mu}{d\mu T}$ a.e. for some $h \in V(\Omega)$. Put $f = \log h$.
(II) (i) μ is the unique probability measure satisfying $\mathcal{L}_f^*\mu = \lambda\mu$, $\lambda > 0$.

(ii) $\nu = k\mu$ is the unique equilibrium state for the function f, where $\mathcal{L}_f k = \lambda k$ and $k \in V(\Omega)$, positive, is normalised such that $\mu(k) = 1$. That is, ν satisfies the variational principle

$$h(\nu) + \int f\, d\nu = \sup_\rho (h(\rho) + \int f\, d\rho),$$

where $h(\cdot)$ is the metric entropy and ρ ranges over T-invariant probability measures on Ω.

(iii)

$$\left\{ r_\varphi = \exp \sum_{k\geq 0} (fT^k\varphi - fT^k) \in V(\Omega) : \varphi \text{ conjugating} \right\}$$

is a family of multipliers for the measure μ.

Remark: If the potential f is Hölder continuous, then by Sinai's theorem there is a Hölder continuous function u so that the equilibrium state of $\tilde{f} = f + u - u \circ T$ is T-invariant. The sum $u - u \circ T$ is a coboundary. If $\tilde{r}_\varphi$ are the multipliers given by equation 1 for the modified potential $\tilde{f}$, then $\tilde{r}_\varphi = e^{u - u\circ\varphi} r_\varphi$, that is the multipliers change by a 'multiplicative coboundary' (conjugating homeomorphisms don't form a group because compositions do not always exist). In particular, the multipliers $\tilde{r}_\varphi$ are T-invariant, which means

$$\tilde{r}_{\tilde{\varphi}} = \tilde{r}_\varphi \circ T,$$

where $\tilde{\varphi} = T^{-1}\varphi T$ and T^{-1} is a local inverse to T. This shows that the invariant measure $\tilde{\mu}$ has density $\frac{d\tilde{\mu}}{d\mu} = e^u$, i.e. $\tilde{\mu}(\chi) = \mu(e^u\chi)$ for integrable χ.

4 Proof of theorem 4

We shall construct a sequence of approximating potentials $g_n \in V(\Omega)$ which will allow us for n-conjugating homeomorphisms φ to express the multipliers r_φ through equation (1). Then we shall show that they can be chosen to be equal to $f = \log h$. Let $\Phi(n)$ be the collection of all n-conjugating homeomorphism in Ω, and put

$$\omega_n(x) = \sum_{\varphi\in\Phi(n)} r_\varphi(x)$$

($\omega_n > 0$), where the sum is over all n-conjugating φ so that $x \in U_\varphi$ and $r_\varphi(x) = 0$ if $x \neq U_\varphi$. By assumption the sum is finite for every positive n (since $T^{-n}(T^n x)$ is a finite set) and for any n-conjugating ψ we have

$$\omega_n(x) = \sum_{\varphi'\in\Phi(n)} r_\varphi(\psi x)\, r_\psi(x) = r_\psi(x)\, \omega_n \circ \psi(x)$$

($x \in U_\psi$), where the sum is over all $\varphi' = \varphi \circ \psi^{-1}$ (φ' is n-conjugating). Now let us define:

$$g_n(x) = \log \frac{\omega_{n-1}T(x)}{\omega_n(x)},$$

for $n = 2, 3, \ldots$. One has $g_n \in V(\Omega)$.

Lemma 6 *If φ is n-conjugating then $r_\varphi = \exp\left(g_n^{(n)} \circ \varphi - g_n^{(n)}\right)$.*

Proof. Using the definition of g_n we get on U_φ:

$$\exp\left(g_n^{(n)} \circ \varphi - g_n^{(n)}\right) \quad = \quad \exp \sum_{0\le k<n} \left(\log \frac{\omega_{n-1}T}{\omega_n} T^k\varphi - \log \frac{\omega_{n-1}T}{\omega_n} T^k\right)$$

$$
\begin{aligned}
&= \prod_{0\le k<n} \frac{\omega_{n-1}T^{k+1}\varphi}{\omega_n T^{k+1}\varphi}\,\frac{\omega_n T^k}{\omega_{n-1}T^{k+1}} \\
&= \prod_{0\le k<n} \frac{\omega_{n-1}T^{k+1}\varphi}{\omega_{n-1}T^{k+1}}\,\frac{\omega_n T^k}{\omega_n T^{k+1}\varphi}.
\end{aligned}
$$

For $x \in U_\varphi$, the two points $T^k x$ and $T^k\varphi(x)$ are $(n-k)$-conjugate (and thus also n-conjugate). Let us put $\varphi^{(k)}$ for the conjugating homeomorphism defined on $T^k(U_\varphi)$ by the requirement that it maps $T^k x$ to $T^k\varphi(x)$ for all $x \in U_\varphi$. We obtain $\varphi^{(k)}T^k = T^k\varphi$ and

$$\frac{\omega_n T^k}{\omega_n T^k\varphi} = \frac{\omega_n}{\omega_n\varphi^{(k)}} \circ T^k = r_{\varphi^{(k)}} \circ T^k,$$

and also

$$\frac{\omega_{n-1}T^{k+1}\varphi}{\omega_{n-1}T^{k+1}} = \frac{\omega_{n-1}\varphi^{(k+1)}}{\omega_{n-1}} \circ T^{k+1} = \frac{1}{r_{\varphi^{(k+1)}}} \circ T^{k+1}.$$

Since $\varphi^{(0)} = \varphi$ ($k=0$) and $\varphi^{(n)} = \mathrm{id}$ (as $T^n x = T^n\varphi(x)$), this yields

$$\exp(g_n^{(n)} \circ \varphi - g_n^{(n)}) = \prod_{0\le k<n} \frac{r_{\varphi^{(k)}}T^k\varphi}{r_{\varphi^{(k+1)}}T^{k+1}} = r_\varphi.$$

□

Pick x and let δ be some positive number. Let η be small enough so that $|\log \frac{r_\varphi(y)}{r_\varphi(y')}| \le \varepsilon$ for almost every $y' \in B_\eta(y)$ and all n-conjugating homeomorphisms φ. For small enough η, we have $B_\eta(y) \in U_\varphi$ for every n-conjugating homeomorphism φ for which $x \in U_\varphi$. Let $C = T^n(B_\eta(y))$. If η is small enough then $T^{-n}(C)$ is the disjoint union of finitely many components $C_1(x), C_2(x), \dots$ of Ω such that T^n is a homeomorphism on each one of them. Obviously $\eta \to 0$ if ε tends to zero. Let $A \subset B_\eta(x)$ be a measurable set, and put $B = T^n(A)$. Then:

$$
\begin{aligned}
T^{*n}\mu(B) &= \mu(T^{-n}(B)) \\
&= \sum_\varphi \int_A r_\varphi(y)\, d\mu(y) \\
&= \int_A \omega_n(y)\, d\mu(y) \\
&= e^{c_1}\omega_n(x)\, \mu(A),
\end{aligned}
$$

where the constant c_1 satisfies $|c_1| \leq \varepsilon$, and where the summation is over all n-conjugating homeomorphism defined on $B_\eta(x)$ (choose η small enouth so that $B_\eta(x) \subset U_\varphi$ for all n-conjugating φ and $x \in U_\varphi$). We used the fact that $A \subset U_\varphi$ for every n-conjugating φ in the sum. Thus

$$\omega_n(x) = \frac{\mu(T^{-n}(B))}{\mu(A)} e^{c_1},$$

and similarly $\omega_{n-1}(Tx) = \frac{\mu(T^{-(n-1)}(B))}{\mu(T(A))} e^{c_2}$, with $|c_2| \leq \varepsilon$. Hence

$$g_n(x) = \log \frac{\omega_{n-1}(Tx)}{\omega_n(x)} = \log \frac{\mu(T^{-(n-1)}(B))}{\mu(T^{-n}(B))} + \log \frac{\mu(A)}{\mu(T(A))} + c_3,$$

where $|c_3| \leq 2\varepsilon$, and therefore

$$\mu(A) = \mu(T_x^{-1}(T_x(A))) = T_x^* \mu(T_x(A)) = \int_{T_x(A)} \frac{dT_x^* \mu}{d\mu} \, d\mu = h(x) e^{c_4} \mu(T(A)),$$

where T_x is the local homeomorphism obtained by the restriction of T to the ball $B_\eta(x)$, and $h = \frac{dT_x^* \mu}{d\mu}$ almost everywhere. For small enough η we can achieve $|c_4| \leq \varepsilon$. On the other hand we also have:

$$\begin{aligned} T^{*n} \mu(B) &= T^{*n-1}(T\mu)(B) \\ &= T^{*n-1} \mu(H \chi_B) \\ &= \int_{T^{-(n-1)}(B)} H \circ T^{n-1} \, d\mu \\ &= H(T^n x) \mu(T^{-(n-1)}(B)) e^{c_5}, \end{aligned}$$

(χ_B is the characteristic function of the set B) where $|c_5| \leq \varepsilon$ for η small enough such that $|\log \frac{H(T^n x)}{H(y)}| \leq \varepsilon$ for almost all $y \in B$. Here H is a function which is a.e. equal to the Radon-Nikodym-derivative $\frac{dT^* \mu}{d\mu}$. To verify' that $H \in V(\Omega)$, let us observe that

$$\frac{dT^* \mu}{d\mu}(y) = \sum_{Ty' = Ty} \frac{d\mu(y')}{d\mu(Ty')},$$

and therefore $H(y) = \sum_{Ty' = Ty} h(y')$ has the same regularity as h. Hence:

$$\log \frac{T^{*n} \mu(B)}{T^{*n-1} \mu(B)} = \log H(T^n x) + c_5,$$

and therefore $g_n(x) = \log h(x) - \log H(T^n x) + c_4 - c_5$. Since the constants c_4 and c_5 go to zero as $\eta \to 0$ one obtains

$$g_n(x) = \log h(x) - \log H(T^n x).$$

Finally, to see that in fact we can dispense with the term $\log H(T^n x)$, let x, x' be n-conjugate, that is $T^n x = T^n x'$, which implies that $\log H(T^n x) = \log H(T^n x')$; Hence by Lemma 6 we have for n-conjugating φ that indeed

$$r_\varphi = \exp(f^{(n)} \circ \varphi - f^{(n)}),$$

since $g_n^{(n)} \circ \varphi - g_n^{(n)} = f^{(n)} \circ \varphi - f^{(n)}$, where $f(x) = \log h(x)$. This concludes the proof of theorem 4. □

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