Integral estimates for the spectral shift function*

A. B. Pushnitski

Abstract

The spectral shift function $\xi(\lambda)$ is considered for the pair of operators H_0, H_0+V, where H_0 is the Schrödinger operator with a variable Riemannian metric and an electro-magnetic field, and V is the operator of multiplication by the potential $V(x)$. For integrals of the type $\int \xi(\lambda)f(\lambda)d\lambda$, where $f(\lambda)$ is a weight, the estimates in terms of the integral characteristics of the potential V are obtained. These estimates are of an asymptotically "correct" order in λ and V; they will be used in a subsequent paper in the problem of asymptotics of $\xi(\lambda)$ in the large coupling constant limit.

0 Introduction

0.1 Let H_0 and H be selfadjoint operators in a Hilbert space $\mathcal{H}$ and let their difference, V, be a trace-class operator:

$$V := H - H_0 \in \mathbf{S}_1(\mathcal{H}). \tag{0.1}$$

Then the *Lifshits—Krein trace formula* holds [21, 19]:

$$\operatorname{Tr}(\phi(H) - \phi(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda; H, H_0)\phi'(\lambda)d\lambda. \tag{0.2}$$

Here ϕ is any function of some class and $\xi(\lambda; H, H_0)$ is the *spectral shift function* (SSF) for the pair H_0, H, which is given by the *Krein* formula

$$\xi(\lambda; H, H_0) = \frac{1}{\pi} \lim_{\varepsilon\to+0} \arg\det(I + V(H_0 - (\lambda + i\varepsilon)I)^{-1}), \text{ a. e. } \lambda \in \mathbf{R}. \tag{0.3}$$

The branch of the argument in (0.3) is fixed by the condition $\arg\det(I+V(H_0-zI)^{-1}) \to 0$, $\operatorname{Im} z \to +\infty$. A historical review and description of the modern state of the SSF theory can be found in [8, 36]. Among the most important facts of the SSF theory are (see [19]) the *Krein inequality*

$$\int_{-\infty}^{\infty} |\xi(\lambda; H, H_0)|d\lambda \le \|V\|_{\mathbf{S}_1} \tag{0.4}$$

*submitted to Algebra i Analiz (St.Petersburg Math. J.)

and the *monotonicity* of the SSF:

$$\pm V \geq 0 \quad \Rightarrow \quad \pm\xi(\lambda; H, H_0) \geq 0. \tag{0.5}$$

It follows from (0.5) that

$$\xi(\lambda; H_0 - V_-, H_0) \leq \xi(\lambda; H_0 + V, H_0) \leq \xi(\lambda; H_0 + V_+, H_0), \tag{0.6}$$

where $2V_\pm = |V| \pm V$.

0.2 In applications, instead of (0.1), it is usually possible to check the inclusion

$$h(H) - h(H_0) \in \mathbf{S}_1(\mathcal{H}), \tag{0.7}$$

where $h : \mathbf{R} \to \mathbf{R}$ is a smooth enough monotonic function. In this case, the SSF for the pair H_0, H is *defined* by the natural formula

$$\xi(\lambda; H, H_0) := \operatorname{sign} h' \cdot \xi(h(\lambda); h(H), h(H_0)). \tag{0.8}$$

The trace formula (0.2) is, of course, still valid; only the class of admissible functions ϕ has to be changed. The relation (0.5) sometimes can be also justified. But this problem is far from being trivial if the SSF is defined via (0.8). In [18] (see also [36, §8.10]), (0.5) has been proved for

$$h(\lambda) = (\lambda - \lambda_0)^{-k}, \quad k > 0, \quad \lambda_0 < \inf(\sigma(H_0) \cup \sigma(H)) \tag{0.9}$$

for all k, λ_0 specified. This is sufficient for most applications.

0.3 Let

$$H_0 = -\triangle \text{ in } L_2(\mathbf{R}^d), \quad d \geq 1, \quad V = V(x), \quad H = H_0 + V. \tag{0.10}$$

If the potential V decays at infinity rapidly enough, then the inclusion (0.7) can be verified for the function h of the type (0.9). This enables one to define the SSF via (0.8); the relations (0.5), (0.6) are valid in this case due to the results of [18].

In [35], estimates for the SSF for the pair H_0, H of the form (0.10) were studied. The potential V was subjected to the condition

$$|V(x)| \leq (1 + |x|)^{-\rho}, \quad \rho > d.$$

The following estimate (see [35, theorem 4.2]) was obtained:

$$\xi(\lambda; H_0 + \alpha V, H_0) \leq C(\alpha^{(d/2)} + \alpha\lambda^{(d/2)-1}(|\log\lambda| + 1)), \quad \forall\alpha > 0, \lambda \geq c; \tag{0.11}$$

for $V \geq 0$ it can be improved up to

$$\xi(\lambda; H_0 + \alpha V, H_0) \leq C\alpha\lambda^{(d/2)-1}(|\log\lambda| + 1), \quad \forall\alpha > 0, \lambda \geq c. \tag{0.12}$$

Here C, c are some positive numbers which do not depend on the spectral parameter λ and the *coupling constant* α, but may depend on V (this dependence has not been studied in [35]).

Besides the estimates (0.11), (0.12), there are many results concerning the asymptotics of $\xi(\lambda; H_0 + \alpha V, H_0)$ for $\lambda \to \infty$ and fixed α (high energy asymptotics — here the initial results were obtained in [9]) and for $\alpha \to \infty$ and $\lambda \to \infty$ and fixed λ/α (semiclassical asymptotics) — see, e.g., [30] and references therein.

0.4 The above mentioned results on the *pointwise* estimates and asymptotics for the SSF of the pair (0.10) are based on the fact that the spectrum of H_0 is absolutely continuous and V is *smooth* with respect to H_0 in some sense. In this paper we follow a different approach and consider the SSF entirely in the the framework of the *trace class* perturbation theory. Starting from the Krein inequality (0.4), we obtain *integral* estimates for the SSF. As an example, we present the weighted estimates for the SSF of the pair (0.10) for $d \geq 3$ (see Theorem 6.4(ii), Corollary 6.8(ii) below and (0.6)):

$$\int_0^\infty |\xi(\lambda; H_0 + V, H_0)| f(\lambda) d\lambda \leq C_1 \int_0^\infty f(\lambda) d\lambda \int V_-^{d/2}(x) dx +$$

$$+ C_2 \int_0^\infty \lambda^{(d/2)-1} f(\lambda) d\lambda \int |V(x)| dx; \tag{0.13}$$

if $V \geq 0$, then, obviously, the first term in the right hand side vanishes and we obtain

$$\int_0^\infty \xi(\lambda; H_0 + V, H_0) f(\lambda) d\lambda \leq C_2 \int_0^\infty \lambda^{(d/2)-1} f(\lambda) d\lambda \int V(x) dx. \tag{0.14}$$

Here $f = f(\lambda)$ is any nonnegative monotone decreasing function and C_1, C_2 are the constants which do not depend on λ, V, f. The precise statements are given in §6.2–6.3.

Obviously, the pointwise estimates (0.11), (0.12) are stronger than the integral ones (0.13), (0.14) (up to constants and a logarithmic factor). But our method of deriving the estimates is not "sensible" to the quality of the spectrum of H_0. This allows us to consider a wide class of H_0's (Schrödinger operators with a variable metric and an electromagnetic field — see §5.2 below) and to avoid imposing too restrictive conditions on V. Besides, the estimates (0.13), (0.14), in contrast to (0.11), (0.12), explicitly indicate the dependence on V. We also obtain some analogues of (0.13), (0.14) for $H_0 = (-\triangle)^l$ in $L_2(\mathbf{R}^d)$, $d \geq 1$; here l is not necessarily an integer.

Note that in [12], for the quantities $\int_0^\lambda \xi(t; H_0 + \alpha V, H_0) dt$ and $\int_0^\infty e^{-t\lambda} \xi(\lambda; H_0 + \alpha V, H_0) d\lambda$, related to the pair (0.10) for $d = 3$, the questions of concavity and subadditivity with respect to V were studied.

0.5 The present paper is mostly aimed at the applications to the problem of asymptotics of the SSF in the large coupling constant limit (the asymptotics itself will be considered in the subsequent paper). To a large extent, this has determined the style of the paper. Let $V \geq 0$; then for $\lambda < \inf \sigma(H_0)$ the SSF $\xi(\lambda; H_0 - \alpha V, H_0)$ equals minus the quantity $N_+(\lambda; H_0, \sqrt{V}, \alpha)$, which is the number of eigenvalues of $H_0 - \alpha V$, located on the left from the point λ on the real axis (see, e. g., [8]). The question of asymptotics of $N_+(\lambda; H_0, \sqrt{V}, \alpha)$ as $\alpha \to \infty$ is well studied in applications — see, e. g., [7] and references therein. In particular, a description of a class of perturbations V (*regular* perturbations), such that the leading term of the asymptotics does not depend on λ, is known. For this reason, in order to study the SSF in the large coupling constant limit for regular perturbations $V \geq 0$, it is natural to consider the quantity

$$\xi(\lambda; H_0 - \alpha V, H_0) + N_+(\Lambda_-; H_0, \sqrt{V}, \alpha), \tag{0.15}$$

for some fixed $\Lambda_- < \inf \sigma(H_0)$. The initial estimates for $\xi(\lambda; H_0 - \alpha V, H_0)$ are formulated in terms of this quantity (see Theorems 6.6, 6.9 below). From this follow inequalities of the form (0.13) (see Corollary 6.8).

0.6 In order to obtain estimates for the SSF, one can, due to (0.6), consider the case of perturbations of a definite sign. Here we use the following approach. Let a nonnegative operator V be factored as $V = G^*G$. In [25], the following new representation for the SSF was obtained (see Propositions 2.6, 2.7 below):

$$\xi(\lambda; H_0 \pm V, H_0) = \pm \mathcal{N}_\mp(\lambda; H_0, G). \tag{0.16}$$

Here $\mathcal{N}_\pm(\lambda; H_0, G)$ is an integral (see (2.4)) of a counting function of eigenvalues of a family of compact operators, related to G and to the resolvent of H_0. The relation (0.16) is of an abstract character. In applications, it appears that the existence of $\mathcal{N}_\pm$ can be established under somewhat more general and natural conditions than the existence of the SSF (see §6.1). For this reason, we consider the quantities $\mathcal{N}_\pm$ as a basic object and define them independently of SSF. To analize $\mathcal{N}_\pm$, we use a mixed approach: both straightforward analysis of the integral representation for $\mathcal{N}_\pm$ and the equality (0.16) together with the estimate (0.4) for "intermediate" objects are employed. This yields integral estimates for $\mathcal{N}_\pm(\lambda; H_0, G)$. The corresponding inequalities for the SSF are obtained by using Propositions 2.6 and 2.7, which give sufficient conditions for the validity of (0.16).

0.7 The paper is organized as follows. In §1 the necessary notation and definitions are introduced. In §2, the quantities $\mathcal{N}_\pm(\lambda; H_0, G)$ are defined and their basic properties and connections with the SSF and the "counting function" $N_\pm(\lambda; H_0, G, \alpha)$ are discussed. The estimates for $\mathcal{N}_\pm$ are obtained in §3 and for the quantity (0.15) — in §4. The results of §§2–4 are of an abstract character. In §5, the Schrödinger and polyharmonic operators (which, in applications, play the part of H_0) are introduced and some relevant statements on this operators are presented. Finally, in §6, on the basis of the results of §3–4 the integral estimates for $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V})$ are derived, where H_0 is the Schrödinger or polyharmonic operator and V is the operator of multiplication by the potential $V(x) \geq 0$.

0.8 Acknowledgments The author is deeply grateful to his scientific advisor M. Sh. Birman, who introduced him into the SSF theory, for many valuable discussions and remarks. The author is grateful to V. S. Buslaev, who attracted his attention to the question of integral estimates for the SSF. The author is grateful to A. Laptev and G. Rozenblioum for consultations and to N. D. Filonov for useful remarks on the text of the paper. Finally, the author expresses his gratitude to the Department of Mathematics of the Royal Institute of Technology, Stockholm, for the hospitality and to the ISF foundation for the financial support.

1 Notation and preliminaries

1.1 Notation

1) Integral without the domain of integration explicitly indicated implies integration over $\mathbf{R}^d$. We denote $\omega_d := \mathrm{vol}\,\{x \in \mathbf{R}^d \mid |x| \le 1\}$. Statements with double indices ($\pm$ and $\mp$) should be considered as a pair of statements, in one of which *all* the indices take the upper values, and in another — the lower ones. In the statements which involve upper estimates we suppose all the quantities (norms and integrals) in the right hand side to be finite. A constant which appears for the first time in formula $(i.j)$ is denoted by $C_{i.j}$.

2) *Functions.* The spaces $L_p(\mathbf{R}^d)$, $L_{p,\mathrm{loc}}(\mathbf{R}^d)$ are defined in a usual way. The space $l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d))$ consists of functions $u \in L_2(\mathbf{R}^d)$ such that the functional

$$\|u\|_{l_1(L_2)} := \sum_{j\in\mathbf{Z}^d} \left(\int_{\mathbf{Q}^d+j} |u|^2 dx\right)^{1/2}, \quad \mathbf{Q}^d = (0,1)^d \subset \mathbf{R}^d,$$

is finite. For a real-valued function F we put $2F_\pm := |F| \pm F$. The characteristic function of a set M is denoted by χ_M.

3) *Operators.* Below $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. By $\mathrm{Dom}\,A$, $\mathrm{Ran}\,A$ and $\mathrm{Ker}\,A$ we denote the domain, range and kernel of a linear operator A; $|A| := \sqrt{A^*A}$; I is the identity operator. For a selfadjoint operator A the symbols $\sigma(A)$, $\rho(A) = \mathbf{C} \setminus \sigma(A)$ and $E_A(\delta)$ denote respectively spectrum, resolvent set and a spectral measure of a Borel set $\delta \subset \mathbf{R}$; $2A_\pm := |A| \pm A$. Resolvent of a selfadjoint operator H_0 is denoted by $R_0(z) = (H_0 - zI)^{-1}$.

By $\mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ we denote the space of compact operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$; $\mathbf{S}_\infty(\mathcal{H}) := \mathbf{S}_\infty(\mathcal{H}, \mathcal{H})$. For $T = T^* \in \mathbf{S}_\infty(\mathcal{H})$ and $s > 0$ we denote $n_\pm(s,T) := \dim \mathrm{Ran}\, E_{T_\pm}(s, +\infty)$, and for $T \in \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ we put $n(s,T) := n_+(s^2, T^*T)$. For compact selfadjoint operators T_1, T_2 the following estimates hold (see, e.g., [3]):

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2), \quad s_1, s_2 > 0, \tag{1.1}$$

which can be written as

$$n_\pm(s_1, T_1 + T_2) \ge n_\pm(s_1 + s_2, T_1) - n_\mp(s_2, T_2), \quad s_1, s_2 > 0. \tag{1.2}$$

For $1 \le p < \infty$ the Neumann-Schatten class $\mathbf{S}_p(\mathcal{H}_1, \mathcal{H}_2) \subset \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is defined as a set of operators V such that the norm

$$\|T\|_{\mathbf{S}_p} := \left(p \int_0^\infty s^{p-1} n(s,T)\, ds\right)^{1/p}$$

is finite. In particular, $\mathbf{S}_1$ is the trace class, $\mathbf{S}_2$ is the Hilbert–Schmidt class.

1.2 Auxiliary facts of the perturbation theory

Let $\mathcal{H}$ be a "basic" and $\mathcal{K}$ — an "auxiliary" Hibert space, H_0 be a selfadjoint operator in $\mathcal{H}$ and G be a linear operator acting from $\mathcal{H}$ into $\mathcal{K}$. Assume that

$$G \text{ is closed}, \quad \mathrm{Dom}\, G \supset \mathrm{Dom}\, |H_0|^{1/2}, \quad G(|H_0| + I)^{-1/2} \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}). \tag{1.3}$$

For $z \in \rho(H_0)$ define the operator

$$T(z; H_0, G) := (G(|H_0| + I)^{-1/2})(|H_0| + I)R_0(z)(G(|H_0| + I)^{-1/2})^*, \tag{1.4}$$

which is compact in $\mathcal{K}$. We shall write $T(z)$ instead of $T(z; H_0, G)$ if the choice of H_0, G is clear from the context. On the dense in $\mathcal{K}$ set $\operatorname{Dom} G^*$ the operator $T(z)$ can be defined by the expression

$$T(z)\psi = GR_0(z)G^*\psi, \quad \psi \in \operatorname{Dom} G^*. \tag{1.5}$$

It is easy to check that for all $z \in \mathbf{C} \setminus \mathbf{R}$ the operator $I \pm T(z; H_0, G)$ has a bounded inverse. Let $V := G^*G$. Below we introduce the operator which corresponds to the formal sum $H_0 \pm V$. For some $z \in \mathbf{C} \setminus \mathbf{R}$ define a bounded operator $R_\pm(z)$ in $\mathcal{H}$ by

$$R_\pm(z) = R_0(z) \mp (GR_0(\overline{z}))^*(I \pm T(z; H_0, G))^{-1}(GR_0(z)). \tag{1.6}$$

Proposition 1.1 (see [14] or [36]) *The operator $R_\pm(z)$ coincides with the resolvent of a selfadjoint in $\mathcal{H}$ operator $H_\pm = H_\pm(H_0, G)$ taken at the point z. This operator does not depend on z. For all $z \in \rho(H_0) \cap \rho(H_\pm)$ the operator $I \pm T(z; H_0, G)$ has a bounded inverse and the equality* (1.6) *holds true for $R_\pm(z) = (H_\pm - zI)^{-1}$. If H_0 is semibounded from below, then $H_\pm$ coincides with the sum $H_0 \pm V$ in the form sense. If V is H_0-bounded with the relative bound $\gamma < 1$, then $H_\pm = H_0 \pm V$ in the sense of the Kato-Rellich theorem.*

It follows from (1.6) and (1.3) that the difference $R_\pm(z) - R_0(z)$ is compact; thus, the essential spectra of $H_\pm$ and H_0 coincide. It is easy to check that $H_\pm(H_0, G) = H_\pm(H_0, |G|)$.

Proposition 1.2 [17, Lemma 1] *Let the conditions* (1.3) *hold true. Then, for every $\lambda \in \rho(H_0) \cap \mathbf{R}$,*

$$\dim \operatorname{Ker}(H_\pm(H_0, G) - \lambda I) = \dim \operatorname{Ker}(T(\lambda; H_0, G) \pm I). \tag{1.7}$$

Proposition 1.2, in particular, implies that the eigenvalues of $H_\pm(H_0, \sqrt{\alpha}G)$, $\alpha > 0$, are monotonic functions of α. For $\lambda \in \rho(H_0) \cap \mathbf{R}$ and $\alpha > 0$ the "counting function" $N_\pm(\lambda; H_0, G, \alpha)$ is defined as the number of eigenvalues (counting multiplicities) of $H_\mp(H_0, \sqrt{t}G)$, which cross the point λ as t grows monotonically in the interval $(0, \alpha)$. From Proposition 1.2 follows the equality which is known as the *Birman-Schwinger principle*:

$$N_\pm(\lambda; H_0, G, \alpha) = n_\pm(\alpha^{-1}; T(\lambda; H_0, G)), \quad \lambda \in \rho(H_0) \cap \mathbf{R}. \tag{1.8}$$

In this paper, the scope of applications is restricted to the semibounded from below operators H_0 (Schrödinger and polyharmonic operator). In this case, $H_\pm$ is defined as a form sum, which simplifies the arguments. However, in the abstract part of the paper we do not suppose H_0 to be semibounded from below (with the exception of §4, where it is dictated by the nature of the question).

Note that the condition of closedness of G is introduced only for the sake of simplicity. The contents of the present section, as well as that of §2–4, can be reformulated for the case of non-closable G's. It is only important that that the second and the third of the conditions (1.3) hold true.

2 The functions $\mathcal{N}_\pm$

2.1 Definition of the functions $\mathcal{N}_\pm$

In this and the next two sections $\mathcal{H}$ is a "basic" and $\mathcal{K}$ — an "auxiliary" Hilbert space, H_0 is a selfadjoint operator in $\mathcal{H}$ and G — the operator acting from $\mathcal{H}$ into $\mathcal{K}$ and satisfying conditions (1.3). For $z \in \rho(H_0)$ define compact in $\mathcal{K}$ operators $T(z; H_0, G)$ (see (1.4)) and put

$$A(z; H_0, G) := \operatorname{Re} T(z; H_0, G), \quad K(z; H_0, G) := \operatorname{Im} T(z; H_0, G). \tag{2.1}$$

We shall write $A(z)$, $K(z)$ instead of $A(z; H_0, G)$, $K(z; H_0, G)$ if the choice of H_0, G is clear from the context. Suppose that for some $\lambda \in \mathbf{R}$ for a pair of operators H_0, G the following condition holds.

Condition 2.1 *The limit*

$$\lim_{\varepsilon \to +0} T(\lambda + i\varepsilon; H_0, G) =: T(\lambda + i0; H_0, G), \tag{2.2}$$

exists in the operator norm and

$$K(\lambda + i0; H_0, G) \in \mathbf{S}_1(\mathcal{K}). \tag{2.3}$$

Then define

$$\mathcal{N}_\pm(\lambda; H_0, G) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\pm(1; A(\lambda + i0) + tK(\lambda + i0)). \tag{2.4}$$

It is easy to see that under the condition (2.3) the integral (2.4) converges.

Lemma 2.2 *Fix an open interval $\delta \subset \mathbf{R}$ and a number $\lambda \in \delta$ and put $G_\delta := GE_{H_0}(\delta)$. Then Condition 2.1 at the point λ is satisfied (or not satisfied) for the pairs H_0, G and H_0, G_δ simultaneously.*

Proof Denote $\tilde{G}_\delta := GE_{H_0}(\mathbf{R} \setminus \delta)$. Obviously, for every $z \in \rho(H_0)$:

$$T(z; H_0, G) = T(z; H_0, G_\delta) + T(z; H_0, \tilde{G}_\delta).$$

It is clear that the limit $T(\lambda + i0; H_0, \tilde{G}_\delta)$ exists in the operator norm and $K(\lambda + i0; H_0, \tilde{G}_\delta) = 0$. From here follows the assertion of the lemma. $\square$

Lemma 2.3 *For any $\lambda \in \mathbf{R}$ Condition 2.1 is satisfied (or not satisfied) for the pairs H_0, G and H_0, $|G|$ simultaneously and*

$$\mathcal{N}_\pm(\lambda; H_0, G) = \mathcal{N}_\pm(\lambda; H_0, |G|). \tag{2.5}$$

Proof Let $G = \Phi|G|$ be a polar decomposition of G; Φ acts unitarily from $\overline{\operatorname{Ran}|G|}$ into $\overline{\operatorname{Ran} G}$. It is easy to see that

$$\Phi T(z; H_0, |G|) = T(z; H_0, G)\Phi, \quad \forall z \in \rho(H_0).$$

From here follows the first assertion of the lemma and the equality

$$n_\pm(1; A(\lambda+i0; H_0, G)+tK(\lambda+i0; H_0, G)) = n_\pm(1; A(\lambda+i0; H_0, |G|)+tK(\lambda+i0; H_0, |G|)).$$

Integrating the last equality over t with the weight $\pi^{-1}(1+t^2)^{-1}$, we arrive at (2.5). □

Lemma 2.4 *Let V_1, V_2 be selfadjoint operators in $\mathcal{H}$ such that $0 \le V_1 \le V_2$ and suppose that for some $\lambda \in \mathbf{R}$ Condition 2.1 holds for the pair H_0, $V_2^{1/2}$. Then it also holds for the pair H_0, $V_1^{1/2}$ and*

$$\mathcal{N}_\pm(\lambda; H_0, V_1^{1/2}) \le \mathcal{N}_\pm(\lambda; H_0, V_2^{1/2}).$$

Proof Write $V_1 = (BV_2^{1/2})^* BV_2^{1/2}$, where $\|B\| \le 1$. Then

$$T(z; H_0, V_1^{1/2}) = BT(z; H_0, V_2^{1/2})B^*, \quad \forall z \in \rho(H_0).$$

From here it follows that Condition 2.1 is satisfied for the pair H_0, $V_1^{1/2}$ and

$$n_\pm(1; A(\lambda+i0; H_0, V_1^{1/2}) + tK(\lambda+i0; H_0, V_1^{1/2})) \le n_\pm(1; A(\lambda+i0; H_0, V_2^{1/2}) + \\ +tK(\lambda+i0; H_0, V_2^{1/2})).$$

Integrating the last inequality over t with the weight $\pi^{-1}(1+t^2)^{-1}$, we get the desired result. □

2.2 Connection between $\mathcal{N}_\pm$ and the "counting function" $N_\pm$

Let $\lambda \in \rho(H_0) \cap \mathbf{R}$. Comparing (1.8) and (2.4) and observing that $K(\lambda+i0; H_0, G) = 0$ and $A(\lambda+i0; H_0, G) = T(\lambda+i0; H_0, G)$, we arrive at the following

Proposition 2.5 *For $\lambda \in \rho(H_0) \cap \mathbf{R}$*

$$\mathcal{N}_\pm(\lambda; H_0, G) = N_\pm(\lambda; H_0, G, 1). \tag{2.6}$$

2.3 Connection between $\mathcal{N}_\pm$ and the SSF

Here we formulate some results of [25] which will be used below.

1) Trace class perturbations. Let H_0 be a selfadjoint operator in $\mathcal{H}$ and $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$. It is well-known that under these conditions for a.e. $\lambda \in \mathbf{R}$ the limits (2.2) exist in the Hilbert-Schmidt norm and the inclusion (2.3) holds (see [2] or [36]).

Proposition 2.6 [25] *Let H_0 be a selfadjoint operator in $\mathcal{H}$ and $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$; then*

$$\xi(\lambda; H_0 \pm G^*G, H_0) = \pm\mathcal{N}_\mp(\lambda; H_0, G), \quad \text{a. e. } \lambda \in \mathbf{R}. \tag{2.7}$$

2) Relatively trace class perturbations. Let H_0 be a selfadjoint in $\mathcal{H}$ semibounded from below operator:

$$H_0 = H_0^* \text{ in } \mathcal{H}, \quad -\infty < \inf\sigma(H_0). \tag{2.8}$$

Let the operator $G : \mathcal{H} \to \mathcal{K}$ satisfy the conditions (1.3) and for some $m > 0$

$$GR_0^m(\mu) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}), \quad \mu < \inf\sigma(H_0). \tag{2.9}$$

Then for a. e. $\lambda \in \mathbf{R}$ Condition 2.1 holds for the pair H_0, G (see, e. g., Corollary 3.8 below). Let $H_\pm = H_\pm(H_0, G)$ (see Proposition 1.1). In order to define the SSF for the pair H_0, $H_\pm$, assume that for some $k > 0$ and $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm))$ the inclusion

$$(H_\pm - \lambda_0 I)^{-k} - (H_0 - \lambda_0 I)^{-k} \in \mathbf{S}_1(\mathcal{H}) \tag{2.10$\pm$}$$

holds. The inclusion (2.10) enables one to define the SSF via (0.8) for $h(\lambda) = (\lambda - \lambda_0)^{-k}$.

Proposition 2.7 [25] *Let the conditions* (1.3), (2.8)–(2.10) *hold true. Then*

$$\xi(\lambda; H_\pm, H_0) = \pm\mathcal{N}_\mp(\lambda; H_0, G), \quad \text{a. e. } \lambda \in \mathbf{R}. \tag{2.11}$$

Finally, we need two sufficient conditions for the inclusion (2.10).

Proposition 2.8 (i) *Assume the conditions* (1.3) *and* (2.9) *for* $m = 1$. *Then, for all* $\lambda_0 \in \rho(H_0) \cap \rho(H_\pm)$ *and* $k = 1$ *the inclusion* (2.10) *holds.*

(ii) *Assume the conditions* (1.3), (2.8) *and for some* $m > 0$ *let*

$$(GR_0^{1/2}(\mu))^*(GR_0^m(\mu)) \in \mathbf{S}_1(\mathcal{H}), \quad \mu < \inf\sigma(H_0). \tag{2.12}$$

Then for all integers $k > m + \frac{1}{2}$ *and all large enough* $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm))$ *the inclusion* (2.10) *holds.*

(i) follows directly from the definition (1.6). (ii) has been proved in [28, Theorem XI.12].

3 Estimates for $\mathcal{N}_\pm$

In this section H_0 is a selfadjoint operator in $\mathcal{H}$ and G is an operator which acts from $\mathcal{H}$ into $\mathcal{K}$ and satisfies the conditions (1.3).

3.1 Monotonicity estimates

For $\lambda \in \mathbf{R}$ let

$$E(\lambda) := E_{H_0}(-\infty, \lambda), \quad G_-(\lambda) := GE(\lambda), \quad G_+(\lambda) := GE_{H_0}(\lambda, +\infty). \tag{3.1}$$

Lemma 3.1 *Let numbers* λ, $\lambda_\pm$ *be such that* $\pm(\lambda - \lambda_\pm) > 0$. *Then Condition 2.1 is satisfied (or not satisfied) for the pairs* H_0, $G_\pm(\lambda_\pm)$ *and* H_0, G *simultaneously and the following estimate holds:*

$$\mathcal{N}_\pm(\lambda; H_0, G) \le \mathcal{N}_\pm(\lambda; H_0, G_\pm(\lambda_\pm)). \tag{3.2$\pm$}$$

Proof The first assertion follows from Lemma 2.2. To be definite, let us prove (3.2$-$). It is easy to see that

$$A(\lambda + i0; H_0, G) \ge A(\lambda + i0; H_0, G_-(\lambda_-)),$$
$$K(\lambda + i0; H_0, G) = K(\lambda + i0; H_0, G_-(\lambda_-)).$$

This gives (3.2$-$). $\square$

Corollary 3.2 *For any $\lambda \in \mathbf{R}$ the following estimate*[1] *holds:*

$$\int_{\pm(t-\lambda)\geq 0} \mathcal{N}_\pm(t; H_0, G)dt \leq \|G_\pm(\lambda)\|^2_{\mathbf{S}_2}. \tag{3.3$\pm$}$$

Proof Using Lemma 3.1, Proposition 2.6 and inequality (0.4), we obtain the estimates:

$$\int_{\pm(t-\lambda)\geq 0} \mathcal{N}_\pm(t; H_0, G)dt \leq \int_{\pm(t-\lambda)\geq 0} \mathcal{N}_\pm(t; H_0, G_\pm(\lambda))dt$$
$$= \int_{\pm(t-\lambda)\geq 0} |\xi(t; H_0 \mp G^*_\pm(\lambda)G_\pm(\lambda), H_0)|dt \leq \int_{-\infty}^{\infty} |\xi(t; H_0 \mp G^*_\pm(\lambda)G_\pm(\lambda), H_0)|dt$$
$$\leq \|G^*_\pm(\lambda)G_\pm(\lambda)\|_{\mathbf{S}_1} = \|G_\pm(\lambda)\|^2_{\mathbf{S}_2}. \quad \square$$

Lemma 3.3 *Let the function $f : \mathbf{R} \to [0, \infty)$ be such that $\pm f$ is nondecreasing. Then the following inequality holds:*

$$\int_{-\infty}^{\infty} \mathcal{N}_\pm(\lambda; H_0, G)f(\lambda)d\lambda \leq \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}. \tag{3.4$\pm$}$$

Proof To be definite, let us consider the case of the lower signs. First assume that $f(\lambda) \to 0$ as $\lambda \to +\infty$. Denote

$$\lambda_{max} := \sup \operatorname{supp} f \leq +\infty \tag{3.5}$$

Since $G\sqrt{f(H_0)} \in \mathbf{S}_2$, one has $G_-(\lambda) \in \mathbf{S}_2$ for $\lambda < \lambda_{max}$. Fix some numbers R_1, R_2 such that $R_1 < R_2 < \lambda_{max}$ (from what follows it is easy to see that for a bounded f, one can take $R_1 = -\infty$). For $\lambda \in (R_1, R_2)$ put $S(\lambda) := \int_{R_1}^{\lambda} \mathcal{N}_-(t; H_0, G)dt$ (here the integral converges due to the inclusion $G_-(\lambda) \in \mathbf{S}_2$ and the inequality (3.3$-$)). Integrating by parts, we get:

$$\int_{R_1}^{R_2} \mathcal{N}_-(\lambda; H_0, G)(f(\lambda) - f(R_2))d\lambda = \int_{R_1}^{R_2} (f(\lambda) - f(R_2))dS(\lambda)$$
$$= -\int_{R_1}^{R_2} S(\lambda)df(\lambda) \leq -\int_{R_1}^{R_2} \|G_-(\lambda)\|^2_{\mathbf{S}_2} df(\lambda), \tag{3.6}$$

where on the last step we have used the inequality (3.3$-$). Next, integrating by parts again, we find:

$$-\int_{R_1}^{R_2} \|G_-(\lambda)\|^2_{\mathbf{S}_2} df(\lambda) = -\mathrm{Tr}\,(G_-(R_2)\int_{R_1}^{R_2} E(\lambda)d(f(\lambda) - f(R_2))G^*_-(R_2))$$
$$= \mathrm{Tr}\,(G_-(R_2)(f(R_1) - f(R_2))E(R_1)G^*_-(R_2)$$
$$+\mathrm{Tr}\,(G_-(R_2)(f(H_0) - f(R_2)I)E_{H_0}(R_1, R_2)G^*_-(R_2)) \leq \mathrm{Tr}\,(G_-(R_2)f(H_0)E(R_1)G^*_-(R_2))$$
$$+\mathrm{Tr}\,(G_-(R_2)f(H_0)G^*_-(R_2)) \leq \|G\sqrt{f(H_0)}E(R_1)\|^2_{\mathbf{S}_2} + \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}. \tag{3.7}$$

[1] We remind that in the estimates from above all the quantities in the right hand side are supposed to be finite.

Combining (3.6) and (3.7), we see that

$$\int_{R_1}^{R_2} \mathcal{N}_-(\lambda; H_0, G)(f(\lambda) - f(R_2))d\lambda \le \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2} + \|G\sqrt{f(H_0)}E(R_1)\|^2_{\mathbf{S}_2}.$$

Taking $R_1 \to -\infty$ and $R_2 \to \lambda_{max}$ in the last inequality, we arrive at (3.4−).

Now suppose $\lim_{\lambda\to+\infty} f(\lambda) =: f(\infty) \ne 0$. Then $G\sqrt{f(H_0)} \in \mathbf{S}_2$ implies $G \in \mathbf{S}_2$. Hence, by Proposition 2.6 and inequality (0.4),

$$\int_{-\infty}^{\infty} \mathcal{N}_-(\lambda; H_0, G)d\lambda \le \|G^*G\|_{\mathbf{S}_1} = \|G\|^2_{\mathbf{S}_2}. \tag{3.8}$$

Next, from the first part of the proof it follows that

$$\int_{-\infty}^{\infty} \mathcal{N}_-(\lambda; H_0, G)(f(\lambda) - f(\infty))d\lambda \le \|G\sqrt{f(H_0) - f(\infty)I}\|^2_{\mathbf{S}_2}$$
$$= \mathrm{Tr}\,(G(f(H_0) - f(\infty)I)G^*) = \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2} - f(\infty)\|G\|^2_{\mathbf{S}_2}.$$

Combining the last inequality with (3.8), we arrive at (3.4−). □

The following result of [4] is very close to Lemma 3.3.

Proposition 3.4 *Let $H_0 = H_0^*$ in $\mathcal{H}$, $V = V^* \in \mathbf{S}_1(\mathcal{H})$ and $H = H_0 + V$. Let $f(\lambda)$, $\lambda \in \mathbf{R}$ be a continuous*[2] *nonnegative nonincreasing function. Then the following inequalities are valid:*

$$\mathrm{Tr}\,(f(H)V) \le \int_{-\infty}^{\infty} \xi(\lambda; H, H_0)f(\lambda)d\lambda \le \mathrm{Tr}\,(f(H_0)V).$$

It is easy to see that for trace-class perturbations of a definite sign and bounded functions f Proposition 3.4 and Lemma 3.3 are equivalent.

In some important special case the estimates of the type (3.4−) for the SSF can be proved straightforwardly, without using $\mathcal{N}_-$.

Proposition 3.5 *Let $H_0 \ge 0$ be a selfadjoint operator in $\mathcal{H}$, G satisfy the condition* (1.3) *and $GR_0(-1) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$. Put $H_+ = H_+(H_0, G)$. Then for $k = 1$ and any $\lambda_0 < 0$ the inclusion* (2.10+) *holds; thus, the SSF for the pair H_0, H_+ is correctly defined. The following estimate holds:*

$$\int_{-\infty}^{\infty} (\lambda - \lambda_0)^{-2}\xi(\lambda; H_+, H_0)d\lambda \le \|GR_0(\lambda_0)\|^2_{\mathbf{S}_2}, \quad \forall \lambda_0 < 0. \tag{3.9}$$

Proof The inclusion (2.10+) (for $k = 1$) follows from (1.6). The SSF $\xi(\lambda; H_+, H_0)$ is defined on the basis of (2.10+) via (0.8) (with $H = H_+$ and $h(\lambda) = (\lambda - \lambda_0)^{-1}$). Applying (0.4) and taking into account the fact that $T(\lambda_0; H_0, G) \ge 0$, we find

$$\int_{-\infty}^{\infty} (\lambda - \lambda_0)^{-2}\xi(\lambda; H_+, H_0)d\lambda \le \|(H_+ - \lambda_0 I)^{-1} - (H_0 - \lambda_0 I)^{-1}\|_{\mathbf{S}_1}$$
$$= \|(GR_0(\lambda_0))^*(I + T(\lambda_0; H_0, G))^{-1}(GR_0(\lambda_0))\|_{\mathbf{S}_1} \le \|GR_0(\lambda_0)\|^2_{\mathbf{S}_2}. \quad \square$$

[2]It is easy to see that the condition of continuity of f can be dropped

3.2 Partition of spectrum

Let the Borel sets $\delta_1, \delta_2 \subset \mathbf{R}$, be such that $\sigma(H_0) \subset \delta_1 \cup \delta_2$. Introduce the notations $G_j = GE_{H_0}(\delta_j)$, $j = 1, 2$.

Lemma 3.6 *Let for some $\lambda \in \mathbf{R}$ Condition 2.1 be satisfied for the pair of operators H_0, G_j, $j = 1, 2$. Then it is satisfied for the pair H_0, G and the following estimates hold:*

$$\mathcal{N}_\pm(\lambda; H_0, G) \le \mathcal{N}_\pm(\lambda; H_0, (1-\theta)^{-1/2} G_1) + \mathcal{N}_\pm(\lambda; H_0, \theta^{-1/2} G_2), \quad \forall \theta \in (0,1); \quad (3.10\pm)$$

$$\mathcal{N}_\pm(\lambda; H_0, G) \ge \mathcal{N}_\pm(\lambda; H_0, (1+\theta)^{-1/2} G_1) - \mathcal{N}_\mp(\lambda; H_0, \theta^{-1/2} G_2), \quad \forall \theta > 0. \quad (3.11\pm)$$

Proof 1. First suppose $\delta_1 \cap \delta_2 = \emptyset$. Clearly, for all $z \in \rho(H_0)$,

$$\begin{aligned} T(z; H_0, G) &= T(z; H_0, G_1) + T(z; H_0, G_2), \\ A(z; H_0, G) &= A(z; H_0, G_1) + A(z; H_0, G_2), \\ K(z; H_0, G) &= K(z; H_0, G_1) + K(z; H_0, G_2). \end{aligned} \qquad (3.12)$$

From here follows the first assertion of the lemma. Using (3.12) and (1.1), we obtain

$$\begin{aligned} n_\pm(1; A(\lambda+i0; H_0, G) + tK(\lambda+i0; H_0, G)) &\le n_\pm(\theta; A(\lambda+i0; H_0, G_1) + tK(\lambda+i0; H_0, G_1)) \\ &+ n_\pm(1-\theta; A(\lambda+i0; H_0, G_2) + tK(\lambda+i0; H_0, G_2)). \end{aligned}$$

Integrating the last inequality with respect to t with the weight $\pi^{-1}(1+t^2)^{-1}$, we arrive at (3.10). Similarly (using (1.2) instead of (1.1)) one proves (3.11).

2. The general case ($\delta_1 \cap \delta_2 \ne \emptyset$) can be reduced to the one considered above by using Lemma 2.4. Indeed, writing the estimate (3.10) for the pair of sets δ_1, $\delta_2 \setminus \delta_1$ and observing that $0 \le (GE_{H_0}(\delta_2 \setminus \delta_1))^*(GE_{H_0}(\delta_2 \setminus \delta_1)) \le (GE_{H_0}(\delta_2))^*(GE_{H_0}(\delta_2))$, we find

$$\begin{aligned} \mathcal{N}_\pm(\lambda; H_0, G) &\le \mathcal{N}_\pm(\lambda; H_0, (1-\theta)^{-1/2} G_1) + \mathcal{N}_\pm(\lambda; H_0, \theta^{-1/2} GE_{H_0}(\delta_2 \setminus \delta_1)) \\ &\le \mathcal{N}_\pm(\lambda; H_0, (1-\theta)^{-1/2} G_1) + \mathcal{N}_\pm(\lambda; H_0, \theta^{-1/2} G_2). \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{N}_\pm(\lambda; H_0, G) &\ge \mathcal{N}_\pm(\lambda; H_0, (1+\theta)^{-1/2} G_1) - \mathcal{N}_\mp(\lambda; H_0, \theta^{-1/2} GE_{H_0}(\delta_2 \setminus \delta_1)) \\ &\ge \mathcal{N}_\pm(\lambda; H_0, (1+\theta)^{-1/2} G_1) - \mathcal{N}_\mp(\lambda; H_0, \theta^{-1/2} G_2). \quad \square \end{aligned}$$

Remark 3.7 Let us sketch another possible proof of Lemma 3.6, which clarifies the role of the fact that the perturbation has a definite sign. As above, suppose that $\delta_1 \cap \delta_2 = \emptyset$. In order to prove (3.10), we first observe that, as it is easy to check, for any $\theta \in (0,1)$

$$V = G^* G \le (1-\theta)^{-1} G_1^* G_1 + \theta^{-1} G_2^* G_2 = \tilde{G}^* \tilde{G},$$

where the operator $\tilde{G} = (1-\theta)^{-1/2} G_1 \oplus \theta^{-1/2} G_2$ acts from $\mathcal{H}$ into $\mathcal{K} \oplus \mathcal{K}$. Here the condition $V \ge 0$ is essential. Next, use Lemma 2.4 (for $V_1 = V$, $V_2 = \tilde{G}^* \tilde{G}$) and Lemma 2.3:

$$\begin{aligned} \mathcal{N}_\pm(\lambda; H_0, G) &\le \mathcal{N}_\pm(\lambda; H_0, (\tilde{G}^* \tilde{G})^{1/2}) = \mathcal{N}_\pm(\lambda; H_0, \tilde{G}) \\ &= \mathcal{N}_\pm(\lambda; H_0, (1-\theta)^{-1/2} G_1) + \mathcal{N}_\pm(\lambda; H_0, \theta^{-1/2} G_2). \end{aligned}$$

The proof of (3.11) is similar. Note also that Lemma 3.6 can be reformulated for the case of partition of the spectrum into an arbitrary finite number of sets δ_j.

Corollary 3.8 *Suppose that for some open interval $\delta \subset \mathbf{R}$ the following inclusion holds:*

$$GE_{H_0}(\delta) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}). \tag{3.13}$$

Then for a. e. $\lambda \in \delta$ for the pair H_0, G Condition 2.1 holds and

$$\mathcal{N}_\pm(\lambda; H_0, G) \in L_{1,\mathrm{loc}}(\delta). \tag{3.14}$$

In particular, if (3.13) *holds for any bounded interval $\delta \subset \mathbf{R}$, then*

$$\mathcal{N}_\pm(\lambda; H_0, G) \in L_{1,\mathrm{loc}}(\mathbf{R}).$$

Proof To be definite, consider the case of upper signs. In (3.10+) take $\theta = 1/2$, $\delta_1 = \delta$, $\delta_2 = \mathbf{R} \setminus \delta$. Due to (3.13), Proposition 2.6 and the inequality (0.4), one has

$$\mathcal{N}_+(\lambda; H_0, \sqrt{2}G_1) = -\xi(\lambda; H_0 - 2G_1^*G_1, H_0) \in L_1(\mathbf{R}).$$

At the same time, it is easy to see that $T(\lambda + i0; H_0, G_2)$ is selfadjoint and $\mathcal{N}_+(\lambda; H_0, \sqrt{2}G_2) = n_+(1; 2T(\lambda + i0; H_0, G_2))$ is a nonincreasing $\mathbf{Z}_+$–valued function of $\lambda \in \delta$, possibly with a singularity at $\sup \delta$. From here it follows that $\mathcal{N}_+(\lambda; H_0, \sqrt{2}G_2) \in L_{1,\mathrm{loc}}(\delta)$, and we arrive at (3.14). □

Note that the statement, analogous to Corollary 3.8, is false for the SSF. Namely, one can construct such operators H_0, G and an interval δ, that the hypothesis of Corollary 3.8 holds true, but for some $\phi \in C_0^\infty(\delta)$ the difference $\phi(H_\pm) - \phi(H_0) \notin \mathbf{S}_1$. Thus, the SSF for the pair $H_\pm$, H_0 cannot be defined on the interval δ in any reasonable sense.

4 Estimates for $\mathcal{N}_+ - N_+$

The contents of this section is aimed at the applications to the asymptotics of SSF in the large coupling constant limit. Operator H_0 is supposed to be nonnegative; we are interested in $\mathcal{N}_+(\lambda; H_0, G)$. For $\lambda < \inf \sigma(H_0)$ the question of asymptotics of $\mathcal{N}_+(\lambda; H_0, \sqrt{t}G) = N_+(\lambda; H_0, G, t)$ as $t \to \infty$ is well studied in applications. For this reason, we estimate the difference

$$Q(\lambda; \Lambda_-, \alpha) := \mathcal{N}_+(\lambda; H_0, G) - N_+(\Lambda_-; H_0, G, \alpha),$$

where $\Lambda_- < \inf \sigma(H_0)$ is fixed and $\alpha > 0$ is some auxiliary parameter. We define

$$Q_\pm(\lambda; \Lambda_-, \alpha) := (Q(\lambda; \Lambda_-, \alpha))_\pm.$$

We also use the notations (3.1) and set $\tilde{E}(\lambda) = I - E(\lambda)$.

4.1 Local estimates

Lemma 4.1 *Let the conditions* (1.3) *be satisfied for the operators* $H_0 \geq 0$ *and* G. *Then, for any* $\theta \in (0,1)$ *and any* $\Lambda_- \leq \lambda_- < \lambda_+ < \Lambda_+$, $\Lambda_- < \inf \sigma(H_0)$, *the following estimate holds:*

$$\int_{\lambda_-}^{\lambda_+} Q_+(\lambda, \Lambda_-, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)})d\lambda \leq \theta^{-1} \|GE_{H_0}(\lambda_-, \Lambda_+)\|^2_{\mathbf{S}_2}. \tag{4.1}$$

Proof We use the estimate (3.10+) for $\delta_1 = [\Lambda_+, +\infty)$, $\delta_2 = (-\infty, \Lambda_+)$, $\lambda \in (\lambda_-, \lambda_+) \subset \delta_2$:

$$\mathcal{N}_+(\lambda; H_0, G) \leq \mathcal{N}_+(\lambda; H_0, (1-\theta)^{-1/2} G\tilde{E}(\Lambda_+)) + \mathcal{N}_+(\lambda; H_0, \theta^{-1/2} G_-(\Lambda_+)). \tag{4.2}$$

Let us estimate from above the first summand in the right hand side of (4.2). In order to do this, we first note that for $\lambda \leq \lambda_+$ the following simple relation holds:

$$R_0(\lambda)\tilde{E}(\Lambda_+) \leq \frac{\Lambda_+ - \Lambda_-}{\Lambda_+ - \lambda_+} R_0(\Lambda_-),$$

which implies

$$T(\lambda + i0; H_0, G\tilde{E}(\Lambda_+)) \leq \frac{\Lambda_+ - \Lambda_-}{\Lambda_+ - \lambda_+} T(\Lambda_-; H_0, G). \tag{4.3}$$

The relation (4.3) yields the estimate (for any $\lambda \leq \lambda_+$):

$$\mathcal{N}_+(\lambda; H_0, (1-\theta)^{-1/2} G\tilde{E}(\Lambda_+)) = n_+(1; (1-\theta)^{-1} T(\lambda + i0; H_0, G\tilde{E}(\Lambda_+))$$
$$\leq n_+(1; \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)} T(\Lambda_-; H_0, G)) = N_+(\Lambda_-; H_0, G, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)}).$$

Substituting the result into (4.2), we find:

$$Q_+(\lambda; \Lambda_-, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)}) \leq \mathcal{N}_+(\lambda; H_0, \theta^{-1/2} G_-(\Lambda_+)).$$

Integrating the last inequality over λ and using (3.3+), we obtain:

$$\int_{\lambda_-}^{\lambda_+} Q_+(\lambda, \Lambda_-, \frac{\Lambda_+ - \Lambda_-}{(1-\theta)(\Lambda_+ - \lambda_+)})d\lambda \leq \int_{\lambda_-}^{\lambda_+} \mathcal{N}_+(\lambda; H_0, \theta^{-1/2} G_-(\Lambda_+))d\lambda \leq$$
$$\leq \int_{\lambda_-}^{\infty} \mathcal{N}_+(\lambda; H_0, \theta^{-1/2} G_-(\Lambda_+))d\lambda \leq \theta^{-1} \|GE_{H_0}(\lambda_-, \Lambda_+)\|^2_{\mathbf{S}_2}. \quad \square$$

Lemma 4.2 *Assume the hypothesis of Lemma 4.1. Then for any* θ_1, $\theta_2 > 0$ *and* $\Lambda_- \leq \lambda_- < \lambda_+$, $\Lambda_- < \inf \sigma(H_0)$, *the following estimate holds:*

$$\int_{\lambda_-}^{\lambda_+} Q_-(\lambda, \Lambda_-, (1 + \theta_1 + \theta_2)^{-1})d\lambda \leq \theta_1^{-1}(\lambda_+ - \lambda_-)\|G_-(\lambda_+)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2}$$
$$+\theta_2^{-1}\|G_-(\lambda_+)\|^2_{\mathbf{S}_2}. \tag{4.4}$$

Proof We use (3.11+) for $\delta_1 = [\lambda_+, +\infty)$, $\delta_2 = (-\infty, \lambda_+)$, $\lambda \in (\lambda_-, \lambda_+) \subset \delta_2$, $\theta = \theta_2$:

$$\mathcal{N}_+(\lambda; H_0, G) \geq \mathcal{N}_+(\lambda; H_0, (1+\theta_2)^{-1/2} G\tilde{E}(\lambda_+)) - \mathcal{N}_-(\lambda; H_0, \theta_2^{-1/2} G_-(\lambda_+)). \qquad (4.5)$$

Let us estimate from below the first summand in the right hand side of (4.5). In order to do this, first observe that for $\lambda \in (\lambda_-, \lambda_+)$ the following obvious relation holds:

$$\tilde{E}(\lambda_+) R_0(\lambda) \geq \tilde{E}(\lambda_+) R_0(\Lambda_-) = R_0(\Lambda_-) - E(\lambda_+) R_0(\Lambda_-),$$

which implies

$$T(\lambda + i0; H_0, G\tilde{E}(\lambda_+)) \geq T(\Lambda_-; H_0, G\tilde{E}(\lambda_+)) = T(\Lambda_-; H_0, G) - T(\Lambda_-; H_0, G_-(\lambda_+)).$$

From here, using (1.2), we derive

$$\begin{aligned} \mathcal{N}_+(\lambda; H_0, (1+\theta_2)^{-1/2} G\tilde{E}(\lambda_+)) &= n_+(1+\theta_2, T(\lambda+i0; H_0, G\tilde{E}(\lambda_+))) \\ &\geq n_+(1+\theta_2+\theta_1; T(\Lambda_-; H_0, G)) - n_+(\theta_1; T(\Lambda_-; H_0, G_-(\lambda_+))) \\ &= N_+(\Lambda_-; H_0, G, (1+\theta_1+\theta_2)^{-1}) - n_+(\theta_1; T(\Lambda_-; H_0, G_-(\lambda_+))). \end{aligned} \qquad (4.6)$$

Let us estimate the second summand in the right hand side of the last inequality in the following way:

$$\begin{aligned} n_+(\theta_1; T(\Lambda_-; H_0, G_-(\lambda_+))) &\leq \theta_1^{-1} \| T(\Lambda_-; H_0, G_-(\lambda_+)) \|_{\mathbf{S}_1} \\ &= \theta_1^{-1} \| G_-(\lambda_+) R_0^{1/2}(\Lambda_-) \|_{\mathbf{S}_2}^2. \end{aligned} \qquad (4.7)$$

Substituting (4.6) and (4.7) into (4.5), we find:

$$Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1}) \leq \theta_1^{-1} \| G_-(\lambda_+) R_0^{1/2}(\Lambda_-) \|_{\mathbf{S}_2}^2 + \mathcal{N}_-(\lambda; H_0, \theta_2^{-1/2} G_-(\lambda_+)).$$

Let us integrate the last inequality over λ and estimate the second summand in the right hand side according to (0.4):

$$\begin{aligned} \int_{\lambda_-}^{\lambda_+} Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1}) d\lambda &\leq \theta_1^{-1}(\lambda_+ - \lambda_-) \| G_-(\lambda_+) R_0^{1/2}(\Lambda_-) \|_{\mathbf{S}_2}^2 \\ + \int_{\lambda_-}^{\lambda_+} \mathcal{N}_-(\lambda; H_0, \theta_2^{-1/2} G_-(\lambda_+)) d\lambda &\leq \theta_1^{-1}(\lambda_+ - \lambda_-) \| G_-(\lambda_+) R_0^{1/2}(\Lambda_-) \|_{\mathbf{S}_2}^2 \\ &\quad + \theta_2^{-1} \| G_-(\lambda_+) \|_{\mathbf{S}_2}^2. \quad \square \end{aligned}$$

4.2 Weighted estimates

Lemma 4.3 *Assume the hypothesis of Lemma 4.1, let $\Lambda_- < \inf \sigma(H_0)$ and $f : (\Lambda_-, \infty) \to [0, \infty)$ be a nonincreasing function, $f(\lambda) \to 0$ as $\lambda \to \infty$. Then for any θ_1, $\theta_2 \in (0,1)$ the following estimate holds:*

$$\int_{\Lambda_-}^{\infty} Q_+(\lambda; \Lambda_-, (1-\theta_1)^{-1}(1-\theta_2)^{-1}) f(\lambda) d\lambda \leq \theta_1^{-1} \| G\sqrt{f(\theta_2 H_0 + \Lambda_-(1-\theta_2))} \|_{\mathbf{S}_2}^2. \qquad (4.8)$$

Proof Replacing $H_0 \mapsto H_0+\Lambda_-$, $f(x) \mapsto f(x-\Lambda_-)$, we see that it is sufficient to consider the case $\Lambda_- = 0$. Introduce the notation (3.5) The inclusion $G\sqrt{f(\theta_2 H_0)} \in \mathbf{S}_2$ implies that $G_-(\lambda/\theta_2) \in \mathbf{S}_2$ for $\lambda < \lambda_{max}$. Taking $\Lambda_- = \lambda_- = 0$, $\theta = \theta_1$ and $\lambda_+/\Lambda_+ = \theta_2$ in (4.1), we get:

$$\int_0^\lambda Q_+(t; 0, (1-\theta_1)^{-1}(1-\theta_2)^{-1})dt \leq \theta_1^{-1}\|G_-(\lambda/\theta_2)\|^2_{\mathbf{S}_2}, \quad \lambda < \lambda_{max}. \tag{4.9}$$

Denote the left hand side of (4.9) by $S(\lambda)$. Similarly to the proof of Lemma 3.3, integrating by parts and using (3.7), we obtain (for $R < \lambda_{max}$):

$$\begin{aligned}\int_0^R Q_+(\lambda; 0, (1-\theta_1)^{-1}(1-\theta_2)^{-1})(f(\lambda)-f(R))d\lambda &= \int_0^R (f(\lambda)-f(R))dS(\lambda)\\ = -\int_0^R S(\lambda)df(\lambda) \leq -\theta_1^{-1}\int_0^R \|G_-(\lambda/\theta_2)\|^2_{\mathbf{S}_2}df(\lambda) &\leq \theta_1^{-1}\|G\sqrt{f(\theta_2 H_0)}\|^2_{\mathbf{S}_2}.\end{aligned}$$

Taking $R \to \lambda_{max}$, we arrive at (4.8). □

Lemma 4.4 *Assume the hypothesis of Lemma 4.1, and let $f : (0,\infty) \to [0,\infty)$ be a nonincreasing integrable function and $F(\lambda) = \int_\lambda^\infty f(t)dt$. Then, for all $\Lambda_- < 0$, θ_1, $\theta_2 > 0$, the following estimate holds:*

$$\begin{aligned}\int_0^\infty Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1})f(\lambda)d\lambda \leq \theta_1^{-1}\|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|^2_{\mathbf{S}_2}&\\ +(\theta_1^{-1}+\theta_2^{-1})\|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}.&\end{aligned} \tag{4.10}$$

Proof First assume that f is bounded. Taking $\lambda_- = 0$ in (4.4), we obtain:

$$\int_0^\lambda Q_-(t, \Lambda_-, (1+\theta_1+\theta_2)^{-1})dt \leq \theta_1^{-1}\lambda\|G_-(\lambda)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2} + \theta_2^{-1}\|G_-(\lambda)\|^2_{\mathbf{S}_2}. \tag{4.11}$$

Introduce the notation (3.5) For $\lambda < \lambda_{max}$, due to the inclusion $G\sqrt{f(H_0)} \in \mathbf{S}_2$, the right hand side of (4.11) is finite; denote the left hand side of this inequality by $S(\lambda)$. Following the proof of Lemma 4.3, we integrate by parts, use (4.11) and find (for $R < \lambda_{max}$):

$$\begin{aligned}\int_0^R Q_-(\lambda, \Lambda_-, (1+\theta_1+\theta_2)^{-1})(f(\lambda)-f(R))d\lambda &= \int_0^R (f(\lambda)-f(R))dS(\lambda)\\ = -\int_0^R S(\lambda)df(\lambda) &\leq -\theta_1^{-1}\int_0^R \lambda\|G_-(\lambda)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2}df(\lambda)\\ &\quad -\theta_2^{-1}\int_0^R \|G_-(\lambda)\|^2_{\mathbf{S}_2}df(\lambda).\end{aligned} \tag{4.12}$$

Let us estimate the first summand in the right hand side of (4.12). Similarly to (3.7), we obtain:

$$-\int_0^R \lambda\|G_-(\lambda)R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2}df(\lambda) = -\mathrm{Tr}\,(G_-(R)R_0(\Lambda_-)\int_0^R \lambda E(\lambda)df(\lambda)G_-^*(R))$$

$$
\begin{aligned}
&= -\mathrm{Tr}\,(G_-(R)R_0(\Lambda_-)\int_0^R E(\lambda)d(F(\lambda)+\lambda f(\lambda))G_-^*(R)) \\
&= \mathrm{Tr}\,(G_-(R)R_0(\Lambda_-)(F(H_0)+H_0 f(H_0)-F(R)I-RF(R)I)G_-^*(R)) \\
&\le \|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|^2_{\mathbf{S}_2}+\|G\sqrt{f(H_0)}H_0^{1/2}R_0^{1/2}(\Lambda_-)\|^2_{\mathbf{S}_2} \le \|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|^2_{\mathbf{S}_2} \\
&\qquad + \|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}.
\end{aligned}
$$

Substituting the last estimate and (3.7) into (4.12), we obtain:

$$
\begin{aligned}
&\int_0^R Q_-(\lambda;\Lambda_-,(1+\theta_1+\theta_2)^{-1})(f(\lambda)-f(R))d\lambda \\
&\le \theta_1^{-1}\|GR_0^{1/2}(\Lambda_-)\sqrt{F(H_0)}\|^2_{\mathbf{S}_2}+(\theta_1^{-1}+\theta_2^{-1})\|G\sqrt{f(H_0)}\|^2_{\mathbf{S}_2}.
\end{aligned}
$$

Taking $R\to\lambda_{max}$, we arrive at (4.10). Now let f be unbounded (i.e., have a singularity at $x=0$). Pick a sequence of bounded integrable functions f_n such that $f_{n+1}(x)\ge f_n(x)$ and $f_n(x)\to f(x)$ $\forall x>0$. Using the inequality (4.10) for f_n and passing to the limit as $n\to\infty$, we obtain the desired result. □

5 Preliminary facts on the differential operators

5.1 Operators dominated by the "free Hamiltonian" In this subsection H_0 is such an operator that

$$H_0 = H_0^* \ge 0 \text{ in } \mathcal{H} = L_2(\mathbf{R}^d), \quad d\ge 1, \tag{5.1}$$

and for some constants $M>0$, $\beta>0$ the following pointwise estimate holds:

$$|e^{-H_0 t}\psi| \le M e^{\beta\triangle t}|\psi|, \quad \forall t>0, \quad \forall \psi\in\mathcal{H}. \tag{5.2}$$

Below we often use the following obvious observation. Let

$$\eta(\lambda)=\int_0^\infty e^{-t\lambda}d\mu(t), \quad \mu \text{ is a finite measure on } (0,\infty). \tag{5.3}$$

Then (5.2) implies

$$|\eta(H_0)\psi| \le M\eta(\beta(-\triangle))|\psi|, \quad \forall\psi\in\mathcal{H}. \tag{5.4}$$

In particular, taking $d\mu(t)=e^{-t\gamma}t^{m-1}dt$ for any $m>0$, $\gamma>0$, we find:

$$|(H_0+\gamma I)^{-m}\psi| \le M(\beta(-\triangle)+\gamma I)^{-m}|\psi|, \quad \forall\psi\in\mathcal{H}. \tag{5.5}$$

The following two propositions are obvious generalizations of the corresponding statements from [1].

Proposition 5.1 *Let $V\ge 0$ be a multiplication by a measurable function in $\mathcal{H}$ and suppose V is $(-\triangle)$-form compact. Then V is H_0-form compact.*

Proof follows the pattern of [1]. The inequality (5.5) implies the pointwise estimate

$$\left|\sqrt{V}(H_0+I)^{-1/2}\psi\right| \le M\sqrt{V}(\beta(-\triangle)+I)^{-1/2}|\psi|, \quad \forall\psi\in\mathcal{H}.$$

By hypothesis, the operator in the right hand side of the last relation is compact; hence (see [1, Theorem 2.2]), the operator in the left hand side is compact too. □

Sufficient conditions for $(-\triangle)$-form compactness of a multiplication operator in various terms can be found, for example, in [22, 5].

Proposition 5.2 (i) *Let F be a multiplication by a function $F \in L_2(\mathbf{R}^d)$ and η be a function of the type* (5.3). *Then:*

$$\|F\eta(H_0)\|^2_{\mathbf{S}_2} \leq (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-d/2} \int_0^\infty \lambda^{\frac{d}{2}-1}\eta^2(\lambda)d\lambda \int |F(x)|^2 dx; \tag{5.6}$$

in particular,

$$FR_0^m(-1) \in \mathbf{S}_2(\mathcal{H}), \quad m > d/4. \tag{5.7}$$

(ii) *Let F be a multiplication by a function $F \in l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d))$. Then, the following inclusion holds:*

$$FR_0^m(-1) \in \mathbf{S}_1(\mathcal{H}), \quad m > d/2. \tag{5.8}$$

Proof (i) By (5.4), one has:

$$\|F\eta(H_0)\|_{\mathbf{S}_2} \leq \|F\|_{L_2}\|\eta(H_0)\|_{2,\infty} \leq \|F\|_{L_2} M \|\eta(\beta(-\triangle))\|_{2,\infty},$$

where $\|\cdot\|_{2,\infty}$ is the norm of an operator acting from $L_2(\mathbf{R}^d)$ into $L_\infty(\mathbf{R}^d)$. Calculating the norm in the right hand side of the last inequality, we obtain (5.6).

(ii) was proved in [1, Theorem 2.12] for the case of a magnetic Schrödinger operator H_0 (see also [32, Theorem B.9.2]). The proof of [1] is carried over to the general case without changes. □

5.2 Schrödinger operator

As a basic example of operators of the form (5.1), (5.2), we consider the Schrödinger operator with a variable metric and electromagnetic field. In $\mathbf{R}^d$, $d \geq 1$, we fix a real $d \times d$-matrix valued function $g = g(x)$ such that

$$g_- \mathbf{1} \leq g(x) \leq g_+ \mathbf{1}, \quad 0 < g_- \leq g_+ < \infty \tag{5.9}$$

(here and in what follows $\mathbf{1}$ is a unit $d \times d$–matrix), a magnetic real vector-potential $\mathbf{A} = \mathbf{A}(x)$,

$$|\mathbf{A}| \in L_{2,\mathrm{loc}}(\mathbf{R}^d), \tag{5.10}$$

and a scalar electric potential $U = U(x)$,

$$U \in L_{1,\mathrm{loc}}(\mathbf{R}^d), \quad U \geq 0. \tag{5.11}$$

Define a quadratic form

$$h^{(0)}_{g,\mathbf{A},U}[u,u] = \sum_{k,j=1}^d \int g_{kj}(x) \left(-i\frac{\partial u}{\partial x_k} - \mathbf{A}_k u\right) \overline{\left(-i\frac{\partial u}{\partial x_j} - \mathbf{A}_j u\right)} dx + \int U|u|^2 dx \tag{5.12}$$

on the domain $d[h^{(0)}_{g,\mathbf{A},U}] = C_0^\infty(\mathbf{R}^d)$.

Proposition 5.3 *Under the conditions* (5.9)–(5.11) *the form* $h^{(0)}_{g,\mathbf{A},U}$ *is closable and its closure* $h_{g,\mathbf{A},U}$ *has the domain*

$$d[h_{g,\mathbf{A},U}] = \{u \in L_2(\mathbf{R}^d) \mid i\nabla u + \mathbf{A}u \in L_2(\mathbf{R}^d), \quad U^{1/2}u \in L_2(\mathbf{R}^2)\}, \tag{5.13}$$

where the expression ∇u *is understood in the distribution sense. The corresponding selfadjoint operator* $H_0 = H_0(g, \mathbf{A}, U)$ *satisfies the conditions* (5.1), (5.2) *with some constants* $M > 0$, $\beta > 0$.

For a flat metric ($g_{ij}(x) \equiv \delta_{ij}$) Proposition 5.3 is well-known (see, e.g., [34] and references therein); in this case $M = \beta = 1$. Essentially, in the above stated form Proposition 5.3 was proved in [11]. Some comments on this point can be found in Appendix.

Proposition 5.4 *Let* $d \geq 3$, $H_0 = H_0(g, \mathbf{A}, U)$ *be a selfadjoint operator corresponding to the form* $h_{g,\mathbf{A},U}$, *and* V *be a multiplication by the function* $V(x) \geq 0$, $V \in L_{d/2}(\mathbf{R}^d)$. *Then* V *is* H_0*-form compact and*

$$N_+(\lambda; H_0, \sqrt{V}, 1) \leq C_{5.14}(d) g_-^{-d/2} \int V^{d/2}(x)dx, \quad \forall \lambda < 0. \tag{5.14}$$

Proof Obviously, $H_0(g, \mathbf{A}, U) \geq g_- H_0(1, \mathbf{A}, 0)$. For the operator $H_0(1, \mathbf{A}, 0)$ the magnetic variant of the Cwikel–Lieb–Rozenblum estimate is known (see [20], and [31, p. 168]):

$$N_+(\lambda; H_0(1, \mathbf{A}, 0), \sqrt{V}, 1) \leq C(d) \int V^{d/2}(x)dx, \quad \forall \lambda < 0.$$

From here follows(5.14). □

Note that in [23] an estimate of the form (5.14) has been proved under much more general conditions on $\mathbf{A}$ and g, than (5.9), (5.10). As for the estimates $N_+(\lambda; H_0, \sqrt{V}, 1)$ for $d = 2$, see [27].

5.3 Polyharmonic operator

Let

$$H_0 = (-\triangle)^l, \quad l > 0 \text{ in } \mathcal{H} = L_2(\mathbf{R}^d), \quad d \geq 1; \tag{5.15}$$

here l is not necessarily an integer. Denote $\kappa := d/(2l)$.

Proposition 5.5 (i) *Let* F *be a multiplication by a function* $F \in L_2(\mathbf{R}^d)$ *and* $f : (0, \infty) \to [0, \infty)$ *be such a function that* $\int_0^\infty \lambda^{\kappa-1} f^2(\lambda) d\lambda < \infty$. *Then*

$$\|Ff(H_0)\|^2_{\mathbf{S}_2} \leq (2\pi)^{-d} \omega_d \kappa \int_0^\infty \lambda^{\kappa-1} f^2(\lambda) d\lambda \int |F(x)|^2 dx; \tag{5.16}$$

in particular,

$$FR_0^m(-1) \in \mathbf{S}_2(\mathcal{H}), \quad m > \kappa/2. \tag{5.17}$$

(ii) *Let* F *be a multiplication by a function* $F \in l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d))$. *Then the following inclusion holds:*

$$FR_0^m(-1) \in \mathbf{S}_1(\mathcal{H}), \quad m > \kappa. \tag{5.18}$$

Proof (i) a straightforward estimate of a Hilbert–Schmidt norm;
(ii) follows from Theorem 11.1 of [6]. □

Proposition 5.6 [5] *Let H_0 be the operator* (5.15), *$\kappa > 1$ and V be a multiplication by the function $V(x) \geq 0$, $V \in L_\kappa(\mathbf{R}^d)$. Then V is H_0-form compact and*

$$N_+(\lambda; H_0, \sqrt{V}, 1) \leq C_{5.19}(d,l) \int V^\kappa(x)dx, \quad \forall \lambda < 0. \tag{5.19}$$

Estimates of $N_+(\lambda; H_0, \sqrt{V}, 1)$ for $\kappa \leq 1$ can be also found in [5]; they are slightly more complicated than (5.19).

6 Estimates for the SSF: applications

Below we apply the abstract results of §2–4 to the differential operators H_0 introduced in §5. In subsection 1 concrete sufficient conditions for the equality (2.11) are given and in subsections 2, 3 estimates for $\mathcal{N}_\pm$ are obtained. Estimates for the SSF can be obtained directly by combining the results of subsections 1 and 2, 3; there is no need in recording all the statements of this kind explicitly.

To treat the Schrödinger operator $H_0(g, \mathbf{A}, U)$, we need only the properties (5.1) and (5.2). Therefore, we keep in mind that the Schrödinger operator is of the primary interest for the applications, but in the formulations of the theorems write that H_0 is an arbitrary operator, satisfying (5.1), (5.2).

6.1 Connection between $\mathcal{N}_\pm$ and SSF

Theorem 6.1 (i) *Let operator H_0 satisfy the conditions* (5.1), (5.2), *and the perturbation potential $V = V(x) \geq 0$ in $\mathbf{R}^d$ be $(-\triangle)$-form compact and $V \in L_1(\mathbf{R}^d)$. Then Condition 2.1 for the pair H_0, $\sqrt{V}$ is satisfied for a.e. $\lambda \in \mathbf{R}$ and $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V}) \in L_{1,\text{loc}}(\mathbf{R})$.*

(ii) *Let H_0 be the polyharmonic operator* (5.15) *and the perturbation potential $V = V(x) \geq 0$ in $\mathbf{R}^d$ be H_0-form compact and $V \in L_1(\mathbf{R}^d)$. Then Condition 2.1 for the pair H_0, $\sqrt{V}$ is satisfied for a.e. $\lambda \in \mathbf{R}$ and $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V}) \in L_{1,\text{loc}}(\mathbf{R})$.*

Proof Let us check the hypothesis of Corollary 3.8 (for $\mathcal{K} = \mathcal{H}$ and $G = \sqrt{V}$).

(i) Condition (1.3) (i.e., H_0-form compactness) follows from Proposition 5.1. The inclusion (3.13) is valid for any bounded interval $\delta \subset \mathbf{R}$ due to Proposition 5.2(i).

(ii) The inclusion (3.13) holds true for any bounded interval $\delta \subset \mathbf{R}$ due to Proposition 5.5(i). $\square$

In connection with the hypothesis of Theorem 6.1(i) we note that the inclusion $V \in L_1(\mathbf{R}^d)$ implies $(-\triangle)$-form compactness of V for $d = 1$, but, in general, does not imply even $(-\triangle)$-form boundedness of V for $d \geq 2$.

Theorem 6.2 (i) *Assume the hypothesis of Theorem 6.1(i) and suppose $d \leq 3$. Let $H_\pm = H_\pm(H_0, \sqrt{V})$; then, for $k = 1$ and any $\lambda_0 < \inf(\sigma(H_\pm) \cup \sigma(H_0))$, the inclusion* (2.10) *holds and thus the SSF for the pair $H_\pm$, H_0 is correctly defined. For a.e. $\lambda \in \mathbf{R}$ the equality* (2.11) *holds.*

(ii) *Assume the hypothesis of Theorem 6.1(ii) and suppose $\kappa < 2$. Let $H_\pm = H_\pm(H_0, \sqrt{V})$; then for $k = 1$ and any $\lambda_0 < \inf(\sigma(H_\pm) \cup \sigma(H_0))$ the inclusion* (2.10) *holds; thus, the SSF for the pair $H_\pm$, H_0 is correctly defined. For a.e. $\lambda \in \mathbf{R}$ the equality* (2.11) *holds.*

Proof It suffices to refer to the Propositions 2.7 and 2.8(i) (for $\mathcal{K} = \mathcal{H}$ and $G = \sqrt{V}$). The inclusion $\sqrt{V}R_0(-1) \in \mathbf{S}_2$ holds due to Proposition 5.2(i) in the case (i) and Proposition 5.5(i) in the case (ii) (compare with [1, Theorem 2.11]). □

We were unable to extend Theorem 6.2(i) to the case $d \geq 4$ and Theorem 6.2(ii) to the case $\kappa \geq 2$. However, the quantities $\mathcal{N}_\pm(\lambda; H_0, \sqrt{V})$ in these cases are correctly defined (see Theorem 6.1).

Theorem 6.3 (i) *Assume the hypothesis of Theorem 6.1(i) and suppose* $V \in l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d))$. *Let* $H_\pm = H_\pm(H_0, \sqrt{V})$. *Then, for all integers* $k > (d-1)/2$ *and all large enough* $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm))$ *the inclusion* (2.10) *holds; thus, the SSF for the pair* H_0, $H_\pm$ *is correctly defined. For a.e.* $\lambda \in \mathbf{R}$ *the equality* (2.11) *holds.*

(ii) *Assume the hypothesis of Theorem 6.1(i) and suppose* $V \in l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d))$. *Let* $H_\pm = H_\pm(H_0, \sqrt{V})$. *Then, for all integers* $k > \kappa - \frac{1}{2}$ *and all large enough* $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm))$ *the inclusion* (2.10) *holds; thus, the SSF for the pair* H_0, $H_\pm$ *is correctly defined. For a.e.* $\lambda \in \mathbf{R}$ *the equality* (2.11) *holds.*

Proof It suffices to refer to the Propositions 2.7 and 2.8(ii) (for $\mathcal{K} = \mathcal{H}$ and $G = \sqrt{V}$). The inclusion (2.9) follows from Proposition 5.2(i) in the case (i) and Proposition 5.5(i) in the case (ii). The inclusion

$$(\sqrt{V}R_0^{1/2}(-1))^*(\sqrt{V}R_0^{k+\frac{1}{2}}(-1)) = R_0^{1/2}(-1)VR_0^{k+\frac{1}{2}}(-1) \in \mathbf{S}_1(\mathcal{H})$$

is given by Proposition 5.2(ii) in the case (i) and by Proposition 5.5(ii) in the case (ii) (compare with [1, corollary 2.13]). □

In connection with the hypothesis of Theorem 6.3(i) we note that the inclusion $V \in l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d))$ implies $(-\triangle)$-form compactness of V for $d \leq 4$, but, in general, does not imply even $(-\triangle)$-form boundedness of V for $d \geq 5$.

6.2 Estimates for $\mathcal{N}_-$

Theorem 6.4 (i) *Assume the hypothesis of Theorem 6.1(i). Then, for any* $t > 0$ *the following estimate holds true:*

$$\int_0^\infty \mathcal{N}_-(\lambda; H_0, \sqrt{V})e^{-t\lambda}d\lambda \leq (2\pi)^{-d}\omega_d\frac{d}{2}M^2\beta^{-d/2}\int_0^\infty \lambda^{\frac{d}{2}-1}e^{-t\lambda}d\lambda \int V(x)dx =$$

$$= (2\pi)^{-d}\omega_d\frac{d}{2}M^2\beta^{-d/2}t^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)\int V(x)dx. \tag{6.1}$$

(ii) *Assume the hypothesis of Theorem 6.1(ii) and let* $f : (0, \infty) \to [0, \infty)$ *be a nonincreasing function. Then, the following estimate holds true:*

$$\int_0^\infty \mathcal{N}_-(\lambda; H_0, \sqrt{V})f(\lambda)d\lambda \leq (2\pi)^{-d}\omega_d\kappa\int_0^\infty \lambda^{\kappa-1}f(\lambda)d\lambda \int V(x)dx, \tag{6.2}$$

Proof (i) It suffices to apply Lemma 3.3 for $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $f(\lambda) = e^{-t\lambda}$ (the case of lower signs) and estimate the right hand side of (3.4−) according to (5.6).

(ii) Apply Lemma 3.3 for $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$ and estimate the right hand side of (3.4−) according to (5.16). □

Remark 6.5 1) Let a function $\eta = \eta(\lambda)$ have the form (5.3). Assuming the hypothesis of Theorem 6.4(i), (6.1) implies the estimate

$$\int_0^\infty \mathcal{N}_-(\lambda; H_0, \sqrt{V})\eta(\lambda)d\lambda \leq (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-d/2} \int_0^\infty \lambda^{\frac{d}{2}-1}\eta(\lambda)d\lambda \int V(x)dx. \quad (6.3)$$

2) Let the perturbation potential $V = V(x) \geq 0$ decay rapidly enough as $|x| \to \infty$. Recall the asymptotic formulae for the SSF for $\lambda \to \infty$:

$$\xi(\lambda; H_0(g, \mathbf{A}, U) + V, H_0(g, \mathbf{A}, U)) \sim (2\pi)^{-d}\omega_d \frac{d}{2}\lambda^{\frac{d}{2}-1} \int V(x)(\det g(x))^{-1/2}dx, \quad (6.4)$$

$$\xi(\lambda; (-\triangle)^l + V, (-\triangle)^l) \sim (2\pi)^{-d}\omega_d \kappa \lambda^{\kappa-1} \int V(x)dx. \quad (6.5)$$

The relations (6.4), (6.5) can be derived formally by computing the corresponding phase space volume. Under certain conditions on g, $\mathbf{A}$, U, V, the formulae (6.4), (6.5) can be justified either as pointwise asymptotics or in some average sense; we do not touch this subject. It is clear that (6.4), (6.5) are in correspodence with (6.3), (6.2); moreover, the constant is sharp in (6.2), and for the flat metric — in (6.3) as well.

6.3 Estimates for $\mathcal{N}_+$

Theorem 6.6 (i) *Assume the conditions of Theorem 6.1(i). Then for any $t > 0$, $\Lambda_- < 0$ and $\theta \in (0,1)$ the following estimate holds:*

$$\int_{\Lambda_-}^\infty (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1-\theta)^{-2}))_+ e^{-t\lambda}d\lambda \leq$$

$$\leq (2\pi)^{-d}\omega_d \frac{d}{2} M^2 \beta^{-\frac{d}{2}} \theta^{-1-\frac{d}{2}} e^{-(1-\theta)\Lambda_- t} t^{-\frac{d}{2}} \Gamma(\frac{d}{2}) \int V(x)dx. \quad (6.6)$$

(ii) *Assume the conditions of Theorem 6.1(ii). Let $\Lambda_- < 0$ and $f : (\Lambda_-, \infty) \to [0, \infty)$ be a nonincreasing function. Then for any $\theta \in (0,1)$ the following estimate holds:*

$$\int_{\Lambda_-}^\infty (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1-\theta)^{-2}))_+ f(\lambda)d\lambda \leq$$

$$\leq (2\pi)^{-d}\omega_d \kappa \theta^{-1-\kappa} \int_{(1-\theta)\Lambda_-}^\infty (\lambda - (1-\theta)\Lambda_-)^{\kappa-1} f(\lambda)d\lambda \int V(x)dx. \quad (6.7)$$

Proof (i) It suffices to apply Lemma 4.3 for $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta$, $f(\lambda) = e^{-t\lambda}$ and estimate the right hand side of (4.8) according to (5.6).

(ii) It suffices to apply Lemma 4.3 for $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta$ and estimate the right hand side of (4.8) according to (5.16). □

Remark 6.7 Let the function $\eta = \eta(\lambda)$ have the form (5.3) with the measure μ such that the integral in (5.3) is convergent for any $\lambda > \Lambda_-$; for example, $\eta(\lambda) = (\lambda - \lambda_0)^{-\gamma}$, $\lambda_0 < \Lambda_-$, $\gamma > 0$. Then, assuming the hypothesis of Theorem 6.6, (6.6) implies the estimate

$$\int_{\Lambda_-}^\infty (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1-\theta)^{-2}))_+ \eta(\lambda)d\lambda \leq$$

$$\leq (2\pi)^{-d}\omega_d\frac{d}{2}M^2\beta^{-\frac{d}{2}}\theta^{-1-\frac{d}{2}}\int_{(1-\theta)\Lambda_-}^{\infty}(\lambda-(1-\theta)\Lambda_-)^{\frac{d}{2}-1}\eta(\lambda)d\lambda\int V(x)dx. \tag{6.8}$$

Corollary 6.8 (i) *Assume the hypothesis of Theorem 6.1(i), let $H_0 = H_0(g, \mathbf{A}, U)$ and $d \geq 3$. Then, for any function η of the form (5.3) and any $\theta \in (0,1)$ the following estimate holds true:*

$$\int_0^\infty \mathcal{N}_+(\lambda; H_0, \sqrt{V})\eta(\lambda)d\lambda \leq C_{5.14}(d)g_-^{-d/2}(1-\theta)^{-d}\int_0^\infty \eta(\lambda)d\lambda\int V^{d/2}(x)dx+$$

$$+(2\pi)^{-d}\omega_d\frac{d}{2}M^2\beta^{-\frac{d}{2}}\theta^{-1-\frac{d}{2}}\int_0^\infty \lambda^{(d/2)-1}\eta(\lambda)d\lambda\int V(x)dx. \tag{6.9}$$

(ii) *Assume the hypothesis of Theorem 6.1(ii) and let $\kappa > 1$. Then, for any $\theta \in (0,1)$ and any nonincreasing function $f : (0,\infty) \to [0,\infty)$ the following estimate holds true:*

$$\int_0^\infty \mathcal{N}_+(\lambda; H_0, \sqrt{V})f(\lambda)d\lambda \leq C_{5.19}(d,l)(1-\theta)^{-2\kappa}\int_0^\infty f(\lambda)d\lambda\int V^{\kappa}(x)dx+$$

$$+(2\pi)^{-d}\omega_d\kappa\theta^{-1-\lambda}\int_0^\infty \lambda^{\kappa-1}f(\lambda)d\lambda\int V(x)dx. \tag{6.10}$$

Proof It suffices to relax the inequalities (6.7), (6.8), transferring the terms with N_+ into the right hand side, then use the estimates (5.14), (5.19) and let $\Lambda_- = 0$. □

Theorem 6.9 (i) *Assume the hypothesis of Theorem 6.1(i). Then for any $\Lambda_- < 0$, $p > \max\{1,\kappa\}$ and $\theta \in (0,1)$ the following estimate holds:*

$$\int_0^\infty (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1+\theta)^{-1}))_-(\lambda-\Lambda_-)^{-p}d\lambda \leq$$

$$\leq (2\pi)^{-d}\omega_d\frac{d}{2}M^2\beta^{-\frac{d}{2}}\theta^{-1}|\Lambda_-|^{\kappa-p}B(\frac{d}{2}, p-\frac{d}{2})\frac{4p-2}{p-1}\int V(x)dx. \tag{6.11}$$

(ii) *Assume the hypothesis of Theorem 6.1(ii) and let $\Lambda_- < 0$ and $f : (0,\infty) \to [0,\infty)$ be a nonincreasing integrable function. Then, for any $\theta \in (0,1)$ the following estimate holds:*

$$\int_0^\infty (\mathcal{N}_+(\lambda; H_0, \sqrt{V}) - N_+(\Lambda_-; H_0, \sqrt{V}, (1+\theta)^{-1}))_- f(\lambda)d\lambda \leq$$

$$\leq (2\pi)^{-d}\omega_d\kappa\theta^{-1}\int_0^\infty\left(4\lambda^{\kappa-1}+2\int_0^\lambda \frac{t^{\kappa-1}}{t-\Lambda_-}dt\right)f(\lambda)d\lambda\int V(x)dx. \tag{6.12}$$

Proof (i) It suffices to apply Lemma 4.4 for $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta/2$, $f(\lambda) = (\lambda-\Lambda_-)^{-p}$ and estimate the right hand side of (4.10) according to (5.6).

(ii) Apply Lemma 4.4 for $\mathcal{K} = \mathcal{H}$, $G = \sqrt{V}$, $\theta_1 = \theta_2 = \theta/2$ and estimate the right hand side of (4.10) according to (5.16). □

Remark 6.10 For $l \leq 1$ the semigroup $e^{-t(-\triangle)^l}$ in $L_2(\mathbf{R}^d)$ is positivity preserving. Using this fact, one can apply the construction of §5–6 to differential operators H_0 of the order $l < 1$, which satisfy the condition

$$\left|e^{-tH_0}\psi\right| \leq Me^{-t\beta(-\triangle)^l}|\psi|, \quad \forall\psi \in L_2(\mathbf{R}^d).$$

A typical example is a relativistic magnetic Schrödinger operator

$$(H_0(\mathbf{1}, \mathbf{A}, 0) + I)^{1/2} + U(x)$$

— see, e.g., [26].

7 Appendix: Remarks on the proof of Proposition 5.3

1. In [11], Proposition 5.3 has been proved under the additional assumption $U \in L_\infty(\mathbf{R}^d)$. In fact, it is easy to carry over the desired statement to the general case. Indeed, first note that

$$g_- h^{(0)}_{\mathbf{1},\mathbf{A},U} \le h^{(0)}_{g,\mathbf{A},U} \le g_+ h^{(0)}_{\mathbf{1},\mathbf{A},U}. \tag{7.1}$$

Under the conditions (5.10), (5.11), the form $h^{(0)}_{\mathbf{1},\mathbf{A},U}$ is closable and its domain is given by the expression in the right hand side of (5.13) — see [34]. From here and (7.1) follows the first statement of Proposition 5.3.

Next, the results of [11] imply that under the conditions (5.9), (5.10) and

$$\tilde U \ge 0, \quad \tilde U \in L_\infty(\mathbf{R}^d), \tag{7.2}$$

the following estimate holds:

$$\left| e^{-tH_0(g,\mathbf{A},\tilde U)}\psi \right| \le e^{-tH_0(g,0,0)}|\psi|, \quad \forall t>0, \quad \psi \in L_2(\mathbf{R}^d). \tag{7.3}$$

Take $U_n(x) = \min\{U(x), n\}$; similarly to [34, Theorem 2.1], one checks that $H_0(g,\mathbf{A},U_n) \to H_0(g,\mathbf{A},U)$ in the strong resolvent sense. Then, writing (7.3) for $\tilde U = U_n$ and passing to the limit as $n \to \infty$ over a subsequence which converges almost everywhere in $\mathbf{R}^d$, one gets:

$$\left| e^{-tH_0(g,\mathbf{A},U)}\psi \right| \le e^{-tH_0(g,0,0)}|\psi|, \quad \forall t>0, \quad \psi \in L_2(\mathbf{R}^d) \tag{7.4}$$

(compare to the proof of Theorem 3.1 of [16]). It is well-known (see, e.g., [10]), that the semigroup $e^{-tH_0(g,0,0)}$ is positivity preserving and for some constants $M>0$, $\beta>0$ the following estimate holds:

$$e^{-tH_0(g,0,0)}|\psi| \le M e^{-t\beta(-\triangle)}|\psi|, \quad \forall t>0, \quad \psi \in L_2(\mathbf{R}^d). \tag{7.5}$$

Combining (7.4) and (7.5), we get (5.2).

2. The proof of (7.3) in [11] is based on a certain new domination criterion for semigroups [24]. Below we show that (7.3) can be proved by using the "classical" semigroup theory technique: Kato's inequality [15], domination criterion [33, 13] and approximation argument [16, 34].

For $g \in C^1(\mathbf{R}^d)$ of the form (5.9), $\mathbf{A} \in C^1(\mathbf{R}^d)$, and $\tilde U$ of the form (7.2), Lemma A of [15] gives the pointwise estimate

$$\mathrm{Re}\,(\overline{\mathrm{sign}\, u} H_0(g,\mathbf{A},\tilde U)u) \ge H_0(g,0,0)|u|,$$

in the sense of distributions. From here, using the domination criterion [33, 13], we get (7.3) for the above specified class of functions g, $\mathbf{A}$, $\tilde U$. Now let g, $\mathbf{A}$ be of the form (5.9), (5.10), a sequence $\mathbf{A}_n \in C^1(\mathbf{R}^d)$ be such that $\mathbf{A}_n \to \mathbf{A}$ in $L_{2,\mathrm{loc}}$, and a sequence of matrix-valued functions $g_n(x)$, satisfying $g_-\mathbf{1} \le g_n(x) \le g_+\mathbf{1}$ (with the same g_-, g_+ as in (5.9)), be such that $g_n \to g$ almost everywhere in $\mathbf{R}^d$. Following the proof of Theorem 4.1 of [34] (see also Proposition 2.5 of [23]), one checks that $H(g_n, \mathbf{A}_n, \tilde{U}) \to H(g, \mathbf{A}, \tilde{U})$ in the strong resolvent sense. Here the key point of the proof is the fact that for $\phi \in C_0^\infty(\mathbf{R}^d)$ one has $g_n(\nabla - i\mathbf{A}_n)\phi \to g(\nabla - i\mathbf{A})\phi$ strongly in L_2. Next, in the inequality

$$\left| e^{-tH_0(g_n, \mathbf{A}_n, \tilde{U})}\psi \right| \le e^{-tH_0(g_n, 0, 0)}|\psi|$$

we pass to the limit as $n \to \infty$ over a subsequence converging almost everywhere in $\mathbf{R}^d$. This gives (7.3).

3. Note that in [11] the matrix $g(x)$ was subjected to a much weaker condition than (5.9), which allowed to consider also the case of nonselfadjoint $H_0(g, \mathbf{A}, U)$. Of course, the alternative scheme of proof, sketched above, does not include this case.

References

[1] J. Avron, I. Herbst, B. Simon, *Schrödinger operators with magnetic fields.* I. *General interactions*, Duke Math. J. **45** (1978), 847–883.

[2] M. Sh. Birman, S. B. Entina, *Stationary approach in abstract scattering theory.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **31**, no. 2 (1967), 401–430. English translation in: Math. USSR Izvestija **1**, no. 1 (1967), 391–420.

[3] M. Sh. Birman, M. Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert space.* Translated from Russian. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.

[4] M. Sh. Birman, M. Z. Solomyak, *Remarks on the spectral shift function.* (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions, 6. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27** (1972), 33–46. English transl. in J. Soviet Math. **3** no. 4 (1975), 408-419.

[5] M. Sh. Birman, M. Z. Solomyak, *Schrödinger operator. Estimates for the number of bound states as function-theoretical problem.* Amer. Math. Soc. Transl. (2) **150** (1992), 1–54.

[6] Birman, M. Sh.; Solomyak, M. Z. *Estimates for the singular numbers of integral operators.* (Russian) Uspehi Mat. Nauk **32**, no. 1(193) (1977), 17–84. English translation in: Russ. Math. Surveys **32**, no. 1 (1977), 15–89.

[7] M. Sh. Birman, M. Z. Solomyak, *Estimates for the number of negative eigenvalues of the Schrodinger operator and its generalizations.* Estimates and asymptotics for discrete spectra of integral and differential equations (Leningrad, 1989–90), 1–55, Adv. Soviet Math., 7, Amer. Math. Soc., Providence, RI, 1991.

[8] M. Sh. Birman, D. R. Yafaev, *The spectral shift function. The papers of M. G.. Kreĭn and their further development.* (Russian) Algebra i Analiz **4**, no. 5 (1992), 1–44. English translation in: St. Petersburg Math. J. **4**, no. 5 (1993), 833–870.

[9] V. S. Buslaev, *Trace formulas for the Schrödinger operator in a three-dimensional space.* (Russian) Dokl. Akad. Nauk SSSR, **143**, no. 5 (1962), 1067–1070.

[10] E. B. Davies, *Heat kernels and spectral theory,* Cambridge Univ. Press, 1989.

[11] M. Demuth, E. M. Ouhabaz, *Scattering theory for Schrödinger operators with magnetic fields,* Math. Nachr. **185** (1997), 49–58.

[12] R. Geisler, V. Kostrykin, R. Schrader, *Concavity properties of Krein's spectral shift function,* Rev. Math. Phys., **7**, no. 2 (1995), 161–181.

[13] H. Hess, R. Schrader, D. A. Uhlenbrock, *Domination of semigroups and generalization of Kato's inequality,* Duke Math. J. **44** (1977), 893–904.

[14] T. Kato, *Wave operators and similarity for some non-selfadjoint operators,* Math. Annal., **162** (1965), 259–279.

[15] T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1973), 135–148.

[16] T. Kato, *Remarks on Schrödinger operators with vector potentials*, Int. Eq. Oper. Theory **1** (1979), 103–113.

[17] R. Konno, S. T. Kuroda, *On the finiteness of perturbed eigenvalues,* J. Fac. Sci. Univ., Tokyo, Sect. I, **13** (1966), 55–63.

[18] L. S. Koplienko, *On the theory of the spectral shift function*, Problemy Mat. Fiz. **5** (1971), 62-79. (Russian)

[19] M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sb. **33 (75)**, no. 3 (1953), 597-626. (Russian)

[20] E. Lieb, *The number of bound states of one-body Schrödinger operators and the Weyl problem,* Proc. Am. Math. Soc. Symposia Pure Math. **36** (1980), 241–252.

[21] I. M. Lifshits, *On a problem in perturbation theory*, Uspekhi Mat. Nauk **7** (1952), no. 1 (47), 171-180. (Russian)

[22] V. G. Mazya, T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Monographs and Studies in Math. **23**, Pitman, Boston–London, 1985.

[23] M. Melgaard, G. Rozenblum, *Spectral estimates for magnetic operators*, Math. Scand. **79** (1996), 237–254.

[24] E.-M. Ouhabaz, *Invariance of closed convex sets and domination criteria for semigroups*, Potential Anal. **5** (1996), 611–625.

[25] A. B. Pushnitski, *Representation for the spectral shift function for perturbations of a definite sign.* (Russian) Algebra i analiz, **9**, no. 6 (1997), 197–213. English translation to appear in St. Petersburg Math. J.

[26] G. V. Rozenblum, M. Z. Solomyak, *CLR-estimate for the generators of positivity preserving and positively dominated semigroups*, Algebra i analiz **9**, no. 6 (1997), 214–236. English translation to appear in St. Petersburg Math. J.

[27] G. Rozenblum, M. Solomyak, *On the number of negative Eigenvalues for the two-dimensional magnetic Schrödinger operator*, “Differential Operators and Spectral Theory. Collection of papers, dedicated to the 70-th birthday of M. Sh. Birman” (to appear).

[28] M. Reed, B. Simon, *Methods of modern mathematical physics*, vol. 3, Academic Press, London, 1979.

[29] M. Reed, B. Simon, *Methods of modern mathematical physics*, vol. 2, Academic Press, 1975.

[30] D. Robert, *Semi-classical asymptotics of the spectral shift function*, “Differential Operators and Spectral Theory. Collection of papers, dedicated to the 70-th birthday of M. Sh. Birman” (to appear).

[31] B. Simon, *Functional integration and quantum physics*, Academic Press, NY, 1979.

[32] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982), 447–526.

[33] B. Simon, *Kato's inequality and comparison of semigroups*, J. Funct. Anal. **32** (1979), 97–101.

[34] B. Simon, *Maximal and minimal Schrödinger forms,* J. Oper. Theory **1** (1979), 37–47.

[35] A. V. Sobolev, *Efficient bounds for the spectral shift function*, Ann. Inst. H. Poincare Phys. Theor. **58**, no. 1 (1993), 55–83.

[36] D. R. Yafaev, *Mathematical scattering theory. General theory*, Amer. Math. Soc., Providence, RI, 1992.

Department of Mathematical Physics
Faculty of Physics
St-Petersburg State University
198904, St.Petersburg, Ul'yanovskaya, 1
RUSSIA
E-mail: pushnit@mph.phys.spbu.ru