On the Asymptotic Behaviour of QMFs with Single Factor Loops

Carl Fredrik Stein[*†]
Department of Mathematics,
Chalmers University of Technology
S-412 96 Göteborg, Sweden

Matthias Holschneider[†]
CNRS, CPT Luminy
F-132 88 Marseille, France

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Abstract

Recently a parametrization of biorthogonal QMF by means of elementary loops in $SL(2)$ has been proposed generalizing the factorization of orthogonal QMF through loops in $U(2)$. These factorizations can easily be realized by some network of elementary propagators. In this paper we discuss the asymptotic shape of the QMFs that are obtained by iterating one elementary factor in this parametrization. It turns out that in the case of orthogonal QMFs the shape is asymptotically described by some Airy function as the supports of the filters increase. In the bi-orthogonal case instead we obtain asymptotically a quadratic chirp. Similar results are to be expected when more factors are involved.

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1 Introduction

Let L be some complex valued 2 by 2 matrix. We then associate with it the following linear operator $\mathcal{L}_0[L]$ acting in $L^2(\mathbf{Z})$. Its restriction to the linear space spanned by δ_0 and δ_1 is just given by L. We then extend it to all of $L^2(\mathbf{Z})$ by requiring that $\mathcal{L}_0[L]$ commutes with the translations by 2.

$$[\mathcal{L}_0, T_2] = 0.$$

We denote by $\mathcal{L}_1[L]$ the operator $T_1\mathcal{L}_0[L]T_{-1}$. Its action is given by letting act L on the space spanned by δ_{-1} and δ_0 and extending it by requiring

$$[\mathcal{L}_1, T_2] = 0.$$

See figure 1 for an illustration.

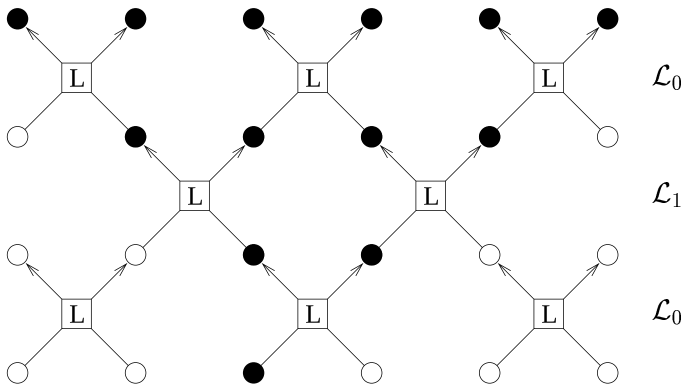

Figure 1: A portion of the structure of the operators considered in this paper. The boxes represent the matrices L and the circles illustrate sequence entries. The filled circles show the non-zero sequence entries resulting from applying $\mathcal{L}_0\mathcal{L}_1\mathcal{L}_0$ to a sequence with the filled circle in the bottom row as its only non-zero component.

Clearly we have that $\mathcal{L}_i\mathcal{L}'_i = \mathcal{L}_i[LL']$ but note that in general the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ do not commute even if both are associated to the same element $L \in M(2)$.

Consider now the special case when $L \in U(2)$. Then the operators $\mathcal{L}_i$ are unitary. As it was shown in [5], any unitary operator that commutes with the translations by two and leaves invariant the subspace of compactly supported sequences, can be factorized into operators of the above type. An immediate corollary of this is that the operators $\mathcal{L}_i$ can be used to generate all compactly supported QMFs. More precisely, if we consider

$$\alpha = \mathcal{L}_0[L_m]\mathcal{L}_1[L_{m-1}]\mathcal{L}_0[L_{m-2}]\ldots\mathcal{L}_0[L_1]\delta_0, \quad (1)$$

$$\beta = \mathcal{L}_0[L_m]\mathcal{L}_1[L_{m-1}]\mathcal{L}_0[L_{m-2}]\ldots\mathcal{L}_0[L_1]\delta_1, \quad (2)$$

then the pair (α, β) is a QMF if the all the matrices L_m are in $U(2)$. Vice versa for a compactly supported QMF, there is a sequence of matrices and some overall translation such that $(T_k\alpha, T_k\beta)$ can be written in the above form. Recently, this result has been extended to bi-orthogonal filter pairs, in which case the matrices L_k are in $SL(2)$, [3]. These factorizations can also be understood in terms of loop-group factorizations.

In this paper we are interested in the asymptotic behaviour of these QMF if the number of factors gets large. This question obviously makes sense only if we make some restrictive hypothesis on the set of factors involved. In this paper we actually assume that L is the same for all factors. So we are interested in the asymptotic behaviour of say

$$(\mathcal{L})^{n/2}\delta_0, \quad \mathcal{L} = \mathcal{L}_0[L]\mathcal{L}_1[L].$$

Here we only will consider even n. There are clearly some trivial situations. For instance $L = 1$ will be the identity.

The behaviour is also simple if $L = \sigma_i,\ i = 1, 2, 3.$, where the σ_i's are the Pauli spin matrices given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3}$$

It is natural to associate a time with the number of iterations n. Repeated application of $\mathcal{L}$ then naturally becomes an evolution in space-time. The diagonals in the space-time plane will be called the mathematical light cone. In the case where L equals one of the Pauli matrices σ_1 or σ_2, the operator $\mathcal{L}$ is a propagation along the light cone. More precisely, δ_0 propagates to the right and δ_1 propagates to the left.

Let us now consider a more complicated situation. We consider the case when $L = i\sin\theta Id + \cos\theta\sigma_1$. With no immediate analytic solution at hand we are referred to computer simulation. In figure 2 we show the result of applying $\mathcal{L}$ 800 times to the initial sequence δ_0. Surprisingly, a rich structure develops. From figure 2 we clearly see that the 'mass' of the sequence moves at a speed distinctly slower than that of the mathematical support. Furthermore we see that a horizontal structure develops at the wavefront. In Section 3 we prove that this wavefront approaches an Airy functions in a sense to be defined in that section. We shall also compute the effective support. Note once more, that each horizontal line corresponds to a QMF. Since Airy functions naturally appear in the description of propagation phenomena, these QMF might play a role in the wavelet analysis of propagation data. Because of this numerical finding we say that the matrices L which asymptotically produce this kind of behaviour are in the propagative class (a more precise definition is given below).

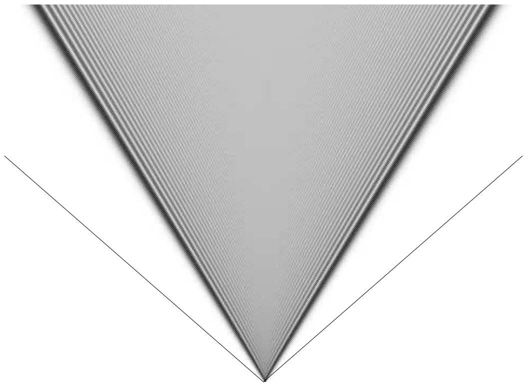

Figure 2: The result of iterating $L = i \sin\theta Id + \cos\theta\sigma_1$ 800 times, on the starting sequence δ_0. The solid lines indicate the mathematical light cone.

Consider now the case of a matrix L that is not unitary but only in $SL(2)$. As you can see in figure 3 its asymptotic behaviour is totally different. Instead of a propagative behaviour, the asymptotic behaviour looks rather like a diffusion and an over-all translation. Again this will be made precise in this paper.

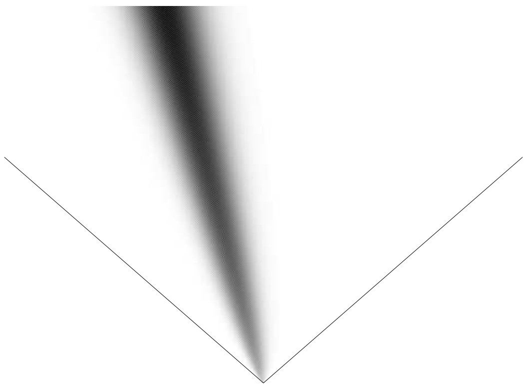

Figure 3: The result of iterating a non-unitary matrix L 400 times, on the starting sequence δ_0. The result has been renormalised to keep the maxima at each horizontal level the same. The solid lines indicate the mathematical light cone.

2 Classifications

Let us consider a 2×2-matrix $L \in GL(2)$, and let $\mathcal{L}$ be its associated operator just as in the introduction. Without loss of generality we may assume that

$$iL \in SL(2) \ . \tag{4}$$

This will be assumed throughout. By construction $\mathcal{L}_0$ and $\mathcal{L}_1$ commute with T_2. According to Bloch theory this implies that they leave the space spanned by

$$\left(e^{i\omega n}, (-1)^n e^{i\omega n}\right) \tag{5}$$

invariant. The action of $\mathcal{L}_0$ and $\mathcal{L}_1$ on the base functions of this space can be represented as 2×2 matrix-valued functions: $\widehat{L}_0(\omega)$ and $\widehat{L}_1(\omega)$. Consequently, the action of $\mathcal{L}^2$ on (5) is given by $\Gamma(\omega) = \widehat{L}_1(\omega)\widehat{L}_0(\omega)$. When we construct $\Gamma(\omega)$ in detail below, we shall see that it has a square root given by:

$$\widehat{L}(\omega) = \sigma_3 \widehat{L}_0(\omega) \ . \tag{6}$$

Since $\widehat{L}^n(\omega)$ represents the action of $\mathcal{L}^n$ on the space spanned by (5) for n even, we will focus much attention on the properties of $\widehat{L}(\omega)$. As can be verified by direct computation and as we will see in more detail below, for $iL \in SL(2)$ of the form

$$L = \begin{bmatrix} l_{00} & l_{01} \\ l_{10} & l_{11} \end{bmatrix} \tag{7}$$

the associated loop $\widehat{L}(\omega)$ reads

$$\widehat{L}(\omega) = \frac{1}{2}\begin{bmatrix} l_{00} + l_{11} + l_{10}e^{-i\omega} + l_{01}e^{i\omega} & l_{00} - l_{11} + l_{10}e^{-i\omega} - l_{01}e^{i\omega} \\ -l_{00} + l_{11} + l_{10}e^{-i\omega} - l_{01}e^{i\omega} & -l_{00} - l_{11} + l_{10}e^{-i\omega} + l_{01}e^{i\omega} \end{bmatrix} \tag{8}$$

In particular it follows that

$$\det \widehat{L}(\omega) = -\det L \ , \ \forall \omega \ , \tag{9}$$

which proves that the convention (4) implies that

$$\widehat{L}(\omega) \in SL(2) \, , \ \forall \ \omega \ . \tag{10}$$

We are now able to make the following definitions:

Definition 1 *We say a matrix L such that $iL \in SL(2)$ is admissible if $\widehat{L}(\omega)$ is not a constant matrix and if the eigenvalues of $\widehat{L}(\omega)$ are distinct for all ω and if they can be chosen to be C^∞-functions of ω. We say an admissible matrix L such that $iL \in SL(2)$ is propagative if both eigenvalues of $\widehat{L}(\omega)$ are on the unit circle for all ω. An admissible matrix L such that $iL \in SL(2)$ is called diffusive if the absolute values of the eigenvalues of $\widehat{L}(\omega)$ are not constant functions of ω.*

It is clear that there are matrices L such that $iL \in SL(2)$, which are neither propagative nor diffusive, for instance, $(-i)1$ is non-admissible.

Note that all admissible $L \in U(2)$ are propagative, but there are propagative matrices outside $U(2)$.

In terms of the components of L we have the following characterization.

Lemma 1 *A matrix L with $iL \in SL(2)$ is admissible if and only if not both l_{01} and l_{10} vanish and in addition*

$$\left\{ l_{01}e^{i\omega} + l_{10}e^{-i\omega} : \omega \in [0, 2\pi) \right\} \cap \{-2, 2\} = \emptyset \tag{11}$$

An admissible matrix is propagative if and only if $l_{10} = \overline{l_{01}}$.

There is no simple characterization of diffusive L in terms of the components of L. **Proof** We have $\widehat{L}(\omega)$ is constant if and only if $l_{01} = l_{10} = 0$. In addition, for $A \in SL(2)$ its eigenvalues are given by

$$\lambda_{1,2} = \frac{trA}{2} \pm \sqrt{\left(\frac{trA}{2}\right)^2 - 1}. \tag{12}$$

Applied to $\widehat{L}(\omega)$ this shows the first part of the statement, since

$$tr\widehat{L}(\omega) = l_{01}e^{i\omega} + l_{10}e^{-i\omega}. \tag{13}$$

Since the eigenvalues of $\widehat{L}(\omega)$ are distinct, it is diagonalizable for each ω. It is well known that the eigenvalues of a matrix in SL(2) are on the unit circle if and only if the trace of the matrix is a real number between -2 and 2 (inclusively). Evaluating the trace at $\omega = 0$ shows us that $\Im l_{10} = -\Im l_{01}$, and the same requirement for $\omega = \pi/2$ yields the condition $\Re l_{10} = \Re l_{01}$, which shows the necessity part of the second statement. On the other hand if $l_{10} = \overline{l_{01}}$, then

$$\begin{aligned} \mathrm{tr}\widehat{L}^*(\omega) &= l_{01}e^{i\omega} + \overline{l_{01}e^{i\omega}} & (14) \\ &= 2\Re\left(l_{01}e^{i\omega}\right), & (15) \end{aligned}$$

and the sufficiency part follows. **qed**

3 The asymptotic analysis of propagative case

For any sequence s_m with $m \in \mathbf{Z}$ we can define a measure η on $\mathbf{R}$ by

$$\eta(y) = \sum_m s_m \delta(y - m). \tag{16}$$

Specifically, the iterated action of $\mathcal{L}$ associated to a matrix L on an initial sequence s_0 with compact support gives rise to a sequence of measures η_n on $\mathbf{R}$ according to

$$\eta_n(y) = \sum_m \mathcal{L}^n(s)_m \, \delta(y-m) \tag{17}$$

In terms of these measures we can formulate the main result of this section.

Theorem 1 *Assume that L is such that $iL \in SL(2)$ is propagative. Let $\mathcal{L}$ be its associated operator and define the measures $\eta_n^{\pm}$ according to (17). Then there exists a unique number $\gamma_c \in (0,1)$, and phases φ, $\psi \in [0, 2\pi)$ such that for all initial conditions s_0 the renormalised measures*

$$\chi_n(y) = n^{\frac{1}{3}} e^{-in\varphi} e^{-in^{1/3}y\psi} \eta_n\left(n^{\frac{1}{3}}y + \gamma_c n\right) \tag{18}$$

converge as $n \to \infty$

$$\chi_n \to \chi = mAi(a\cdot), \text{ in S}', \tag{19}$$

where a is a real number and m is some complex number depending linearly on s. Moreover the linear form $s \to m$ is not identically 0. An analogue statement holds for $\eta_n\left(n^{\frac{1}{3}}y - \gamma_c n\right)$

Here Ai is the airy function as defined e.g. in [6]
Remark 1 : For a particular initial sequence, m may well be zero. Higher order terms will then dominate, but we do not treat this case.
Remark 2 : There are precisely two choices of the phases (φ, ψ).
Remark 3 : All constants in the theorem can be calculated explicitly using the ideas of the proof. In particular, we have that $\gamma_c = |l_{01}|$, see (67) below.

Let us now introduce the following characteristic asymptotic features:

Definition 2 *Let $CS(\mathbf{Z})$ denote space of all sequences of the integers with compact support. We then define*
(a) The asymptotic growth factor, F

$$F = \sup_{s \in CS(\mathbf{Z})} \inf \left\{ \Phi \in \mathbf{R} \mid \lim_{n\to\infty} \Phi^{-n} \mathcal{L}^n(s) = \mathbf{0} \right\} . \tag{20}$$

(b) The asymptotic exponent of algebraic decay, a

$$a = \sup_{s \in CS(\mathbf{Z})} \sup \left\{ \alpha \in \mathbf{R} \mid \lim_{n\to\infty} n^{\alpha} F^{-n} \mathcal{L}^n(s) = \mathbf{0} \right\} . \tag{21}$$

(c) The asymptotic opening angles, $\beta^{\pm}$

$$\beta^+ = \sup_{s \in CS(\mathbf{Z})} \lim_{\varepsilon \to 0} \left(\inf \left\{ \beta_\varepsilon \in \left(0, \frac{\pi}{2}\right) \mid \lim_{m,n\to\infty} n^{a+\varepsilon} F^{-n} \mathcal{L}^n(s)_m = 0, \right. \right.$$

$$
\begin{aligned}
& \quad \text{whenever } \frac{m}{n} > \tan\beta_\varepsilon\Big\}\Big) & (22)\\
\beta^- &= \sup_{s\in CS(\mathbf{Z})}\lim_{\varepsilon\to 0}\left(\sup\left\{\beta_\varepsilon \in \left(-\frac{\pi}{2},0\right) \mid \lim_{m,n\to\infty} n^{a+\varepsilon}F^{-n}\mathcal{L}^n(s)_m = 0,\right.\right.\\
& \quad \left.\left.\text{whenever } \frac{m}{n} < \tan\beta_\varepsilon\right\}\right) \ , & (23)
\end{aligned}
$$

where $\mathbf{0}$ denotes a sequence $s = \{s_x\}$, $x \in \mathbf{R}$ consisting only of zeros. The definition (c) is based on a definition in [4], but since our definition takes renormalisation into account its range of application is much larger. It is evident that Theorem 1 contains information about all these asymptotic features. In fact, for a propagative L they are:

$$
\begin{aligned}
F &= \sqrt{|\det L|} & (24)\\
a &= \frac{1}{3} & (25)\\
\tan\beta^\pm &= \pm\gamma_c = \pm|l_{01}| \ . & (26)
\end{aligned}
$$

In [4] it is shown that the 1-dimensional Dirac equation can be approximated by an operator associated with

$$
L = \begin{bmatrix} i\sin\delta & \cos\delta \\ \cos\delta & i\sin\delta \end{bmatrix} \ . \tag{27}
$$

We recognize this as the case illustrated in figure 2, and if we apply (26) to it we have that the asymptotic opening angle satisfies the expression:

$$
\tan\beta = |\cos\delta| \ . \tag{28}
$$

We will devote this entire section to the proof of Theorem 1. In the first subsection we will introduce $\widehat{L}(\omega)$ in more detail. In the next subsection we will show that iterates of $\mathcal{L}$ can be calculated from the inverse Fourier transform. In the final subsection we will complete the proof of Theorem 1.

3.1 On operators on $L^2(\mathbf{Z})$ which commute with translation by two

Since the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ commute with T^2, translations by two, the general theory for operators which commute with T^2 will be of great use for us in this paper. This theory is presented in [4], and here we will follow that presentation closely. We define $s = \{s_i\}$ to be the element in $L^2(\mathbf{Z})$ with i^{th} entry s_i. We will also make frequent use of the sequences $\delta_k = \{\delta_{ki}\}$.

Our analysis will be based on Fourier analysis, so we begin by recalling the definition of the Fourier transform of a sequence:

$$\hat{s}(\omega) = \sum_{k=-\infty}^{\infty} s_k e^{-i\omega k} \tag{29}$$

$$s_k = \frac{1}{2\pi}\int_0^{2\pi} \hat{s}(\omega)\, e^{i\omega k} d\omega \tag{30}$$

It is clear that any linear operator A which commutes with translations by two, T_2, is completely determined by its action on the two sequences δ_0 and δ_1. We introduce the projections onto the even and odd terms:

$$\{\Pi_e(s)\}_i = \frac{1}{2}\left((-1)^i s_i + s_i\right) = \begin{cases} s_i \text{ if } i \text{ is even} \\ 0 \text{ otherwise} \end{cases} \tag{31}$$

and similarly for the odd projection Π_o. We can now express A as:

$$A(s) = A(\delta_0) * \Pi_e(s) + (T_{-1}A(\delta_1)) * \Pi_o(s) \tag{32}$$

If we let $F_i(\omega) = \widehat{A(\delta_i)}(\omega)$, and take the discrete Fourier transform of (32) we obtain:

$$\widehat{A(s)}(\omega) = F_0(\omega)\,\widehat{\Pi_e(s)}(\omega) + e^{i\omega} F_1(\omega)\,\widehat{\Pi_o(s)}(\omega) \tag{33}$$

From the first equality in (31) we have that

$$\begin{aligned} \widehat{\Pi_e(s)}(\omega) &= \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(e^{-ik(\omega+\pi)} s_k + e^{-ik\omega} s_k\right) \\ &= \frac{1}{2}\left(\hat{s}(\omega+\pi) + \hat{s}(\omega)\right) \end{aligned} \tag{34}$$

The same calculation for Π_o yields:

$$\widehat{\Pi_o(s)}(\omega) = \frac{1}{2}\left(-\hat{s}(\omega+\pi) + \hat{s}(\omega)\right) \tag{35}$$

If we substitute (34) and (35) into (33) we finally obtain:

$$\widehat{A(s)}(\omega) = \frac{1}{2}\Big((\hat{s}(\omega) + \hat{s}(\omega+\pi))F_0(\omega) + (\hat{s}(\omega) - \hat{s}(\omega+\pi))\, e^{i\omega} F_1(\omega)\Big)\ . \tag{36}$$

If we substitute $\omega + \pi$ for ω in the last equation we get in addition:

$$\begin{aligned} A\widehat{(s+\pi)}(\omega) &= \frac{1}{2}\big((\hat{s}(\omega) + \hat{s}(\omega+\pi))\, F_0(\omega+\pi) + \\ &+ \ (\hat{s}(\omega) - \hat{s}(\omega+\pi))\, e^{i\omega} F_1(\omega+\pi)\Big)\ . \end{aligned} \tag{37}$$

A combination of the last two results enables us to express the action of A in Fourier space as a linear operator $\widehat{A}(\omega)$:

$$\begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega+\pi) \end{bmatrix} \mapsto \widehat{A}(\omega) \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega+\pi) \end{bmatrix} \tag{38}$$

where $\widehat{A}(\omega)$ is a 2×2-matrix valued function of $\omega \in [0, 2\pi)$ given by

$$\widehat{A}(\omega) = \frac{1}{2} \begin{bmatrix} F_0(\omega) + e^{i\omega} F_1(\omega) & F_0(\omega) - e^{i\omega} F_1(\omega) \\ F_0(\omega+\pi) + e^{i\omega} F_1(\omega+\pi) & F_0(\omega+\pi) - e^{i\omega} F_1(\omega+\pi) \end{bmatrix} \tag{39}$$

Henceforth, $\widehat{A}(\omega)$ will stand for this matrix-valued function. We also introduce the projection operator P defined by $P(a, b) = a$, projection onto the first component. Now the iterated application of the operator may be studied with the aid of

$$\widehat{A^n(s)}(\omega) = P\left(\widehat{A}(\omega)^n \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega+\pi) \end{bmatrix}\right) \tag{40}$$

Since $\mathcal{L}_0$ and $\mathcal{L}_1$ commute with translation by two in x, we can apply the previous section to express their action in Fourier space as the linear operators $\widehat{L}_i(\omega, t)$, where $i = 0, 1$. However, $\mathcal{L}_i(\delta_0)$ and $\mathcal{L}_i(\delta_1)$ for $i = 0, 1$ are determined entirely by L (in fact if $i = 0$ they are the first and second column of L respectively), and thus the same holds for $\widehat{L}_i(\omega, t)$. Consequently, we need only consider $\widehat{L}_0(\omega)$ and $\widehat{L}_1(\omega)$. However, a look at figure 1 convinces us that $\widehat{L}_1(\omega) = \widehat{T_1}(\omega)\widehat{L}_0(\omega)\widehat{T_1^*}(\omega)$, therefore we have that $\mathcal{L}^2$ is described in Fourier space as

$$\widehat{\Gamma}(\omega) \equiv \widehat{L}_1\widehat{L}_0 = \widehat{T_1}(\omega)\widehat{L}_0(\omega)\widehat{T_1^*}(\omega)\widehat{L}_0(\omega)\ . \tag{41}$$

If we calculate

$$\widehat{T_1}(\omega) = \begin{bmatrix} e^{-i\omega} & 0 \\ 0 & -e^{-i\omega} \end{bmatrix} = e^{-i\omega}\sigma_3 \tag{42}$$

we can simplify (41) according to

$$\widehat{\Gamma}(\omega) = \sigma_3\widehat{L}_0(\omega)\sigma_3\widehat{L}_0(\omega) = \widehat{L}^2(\omega) \tag{43}$$

where $\widehat{L}(\omega) = \sigma_3\widehat{L}_0(\omega)$. This proves the claim we made in Section 2. The explicit formula for $\widehat{L}(\omega)$ follows immediately from our general computations.

3.2 Computing iterates of L by Inverse Fourier Transform

Our main aim in this subsection we show how iterates of L associated to an admissible L can be computed by the Inverse Fourier Transform. For each admissible L, such that $iL \in SL(2)$, the eigenvalues can be written in the following form

$$\lambda_{1,2}(\omega) = e^{\pm i\zeta(\omega)}. \tag{44}$$

We choose, as we may, $\zeta(\omega)$ to be a C^∞ function. Then in fact we have the following lemma:

Lemma 2 *The following holds:*
(a) the eigenvalues $\lambda_{1,2}(\omega)$ of $\widehat{L}(\omega)$ are 2π-periodic.
(b) $\widehat{L}(\omega)$ is diagonalizable for all ω.
(c) There exists a C^∞ function $E : \mathbf{R} \to GL(2)$, with $E(\omega + 2\pi) = E(\omega)$ such that

$$\widehat{L}^*(\omega) = E(\omega)\, D(\omega)\, E^{-1}(\omega) \ , \tag{45}$$

where

$$D(\omega) = \begin{bmatrix} e^{i\zeta(\omega)} & 0 \\ 0 & e^{-i\zeta(\omega)} \end{bmatrix} . \tag{46}$$

Proof (a) follows by direct computation. (b) holds since by hypothesis on L the eigenvalues are distinct for all ω. Now $\widehat{L}(\omega)$ is 2π-periodic, and from (a) so is $D(\omega)$. Clearly, since we are working with complex valued matrices, the eigenvectors $e_i(\omega)$, $i = 1, 2$, associated to the first and second eigenvalue may be chosen to be periodic functions. Since as usual $E(\omega)$ can be constructed from these eigenvectors the periodicity of E follows. **qed**

Putting all together, we have shown that the iterates of $\mathcal{L}$ associated to an admissible L can be computed by Inverse Fourier Transform as follows

$$\mathcal{L}^n(s)_m = \frac{1}{2\pi}\int_0^{2\pi} P\left(E(\omega)\, D^n(\omega)\, E^{-1}(\omega)\, e^{i\omega m} \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega+\pi) \end{bmatrix}\right) d\omega \ . \tag{47}$$

Carrying out the computations more explicitly we have shown the following lemma which we state for further reference.

Lemma 3 *If $\mathcal{L}$ is associated to an admissible L, and if $s(x)$ is a sequence with compact support then there exist C^∞ vector valued functions $G_0(\omega)$ and $G_1(\omega)$, that are 2π-periodic and independent of n, such that*

$$\begin{aligned} \mathcal{L}^n(s)_m &= \frac{1}{2\pi} e^{n|\nu|_{max}} \int_0^{2\pi} G_0(\omega)\, e^{i\left(n\xi(\omega)+m\omega+in\left(\nu(\omega)+|\nu|_{max}\right)\right)} d\omega \\ &+ \frac{1}{2\pi} e^{n|\nu|_{max}} \int_0^{2\pi} G_1(\omega)\, e^{i\left(-n\xi(\omega)+m\omega+in\left(|\nu|_{max}-\nu(\omega)\right)\right)} d\omega \end{aligned} \tag{48}$$

where $|\nu|_{max}$ denotes the maximum of the absolute value of the function $\nu(\omega) = \Im(\zeta(\omega))$.

3.3 End of the Proof of Theorem 1

In this case, the eigenvalues are on the unit circle and we may write

$$\mathcal{L}^n (s)_m = \frac{1}{2\pi} \int_0^{2\pi} G_0 (\omega) e^{in\phi^+(\omega,\gamma)} d\omega + \frac{1}{2\pi} \int_0^{2\pi} G_1 (\omega) e^{in\phi^-(\omega,\gamma)} d\omega \quad (49)$$

with $\gamma = m/n$ and $\phi^{\pm}(\omega, \gamma) = \gamma\omega \pm \xi(\omega)$. Here $\xi(\omega)$ is real valued and satisfies

$$\xi(\omega + 2\pi) = \xi(\omega) + 2\pi k \quad (50)$$

for some $k \in \mathbf{Z}$ not depending on ω. Actually ξ is periodic as we will see below (see Eq. (59) below).

The main step in proving Theorem 1 is to estimate these integrals with the aid of a theorem which originates from Airy. We have used the form which appear in [6] and [2]:

Theorem 2 *(Airy) Let f be a real valued C^∞ function near 0 in $\mathbf{R}^2$ such that $\frac{\partial f}{\partial \omega} = \frac{\partial^2 f}{\partial \omega^2} = 0$, but $\frac{\partial^3 f}{\partial \omega^3} \neq 0$ at 0. Then there exists C^∞ real valued functions $a(g)$, $b(g)$ near 0 such that $a(0) = 0$, $b(0) = f(0,0)$ and*

$$\left| \int u(\omega, g) e^{ivf(\omega,g)} d\omega - w(g) e^{ivb(g)} \mathrm{Ai}\left(a(g) v^{\frac{2}{3}}\right) v^{-\frac{1}{3}} \right| \leq C v^{-\frac{2}{3}} , \quad (51)$$

provided that $u \in C_0^\infty$ and supp *u is sufficiently close to 0. Here $w \in C_0^\infty$, and C is independent of γ in some neighbourhood of 0. Moreover, there exists a real-valued C^∞ function $T(\omega, g)$ with $T(0,0) = 0$ and $\frac{\partial T(\omega,0)}{\partial \omega} > 0$ such that*

$$f(\omega, g) = \frac{T^3}{3} + a(g) T + b(g) \ . \quad (52)$$

This theorem can not be applied directly, since our functions are on the circle. Indeed, we have integrals of the type

$$\int_0^{2\pi} f(\omega) e^{i(-n\xi(\omega)+m\omega)} d\omega \quad (53)$$

with ξ satisfying (50). However, let $h(\omega)$ be a smooth real valued function, taking values between 0 and 1 such that $\sum_l h(\omega + 2\pi l) \equiv 1$. Then the above integral may be rewritten as

$$\int h(\omega) f(\omega) e^{i(-n\xi(\omega)+m\omega)} d\omega = \int h(\omega) f(\omega) e^{in(-\xi(\omega)+\omega(m/n))} d\omega \quad (54)$$

Here h and ξ are the uniquely defined smooth extensions to the whole real line of the previous functions defined a priori only on the circle.

From this we see that we only have to look for critical points of the total phase $\phi^{\pm}(\omega, \gamma) = \gamma\omega \pm \xi(\omega)$, $\gamma = m/n$, for values $\omega \in [0, 2\pi)$.

Lemma 4 *The following holds for $\xi(\omega)$. There are exactly two inflection points. More precisely, there exists a unique $\omega_1 \in [0, \pi)$ such that*

$$\left.\frac{d^2\xi}{d\omega^2}\right|_{\omega=\omega_1} = 0. \tag{55}$$

Furthermore,

$$\left.\frac{d^2\xi}{d\omega^2}\right|_{\omega=\omega_1+\pi} = 0. \tag{56}$$

For all other values of ω, we have $d^2\xi(\omega)/d\omega^2 \neq 0$. In addition, we have that

$$\left.\frac{d\xi}{d\omega}\right|_{\omega=\omega_1} = -\left.\frac{d\xi}{d\omega}\right|_{\omega=\omega_1+\pi} \tag{57}$$

and that

$$\left.\frac{d^3\xi}{d\omega^3}\right|_{\omega=\omega_1} = \left.\frac{d^3\xi}{d\omega^3}\right|_{\omega=\omega_1+\pi} \tag{58}$$

Moreover the following holds

$$\xi(\pi+\omega) = \pi - \xi(\omega) \tag{59}$$

Proof Since $\widehat{L}(\omega) \in SL(2)$, $\forall\omega$ we have that

$$\begin{aligned} \cos\xi(\omega) &= \mathrm{tr}\widehat{L}(\omega)/2 & (60)\\ &= \Re\left(l_{01}e^{i\omega}\right) & (61)\\ &= |l_{01}|\cos(\omega+\upsilon). & (62) \end{aligned}$$

where $\upsilon = \arg l_{01}$, and where we have used (15). It follows that

$$\begin{aligned} \frac{d\xi}{d\omega} &= \epsilon\frac{|l_{01}|\sin(\omega+\upsilon)}{\sqrt{1-|l_{01}|^2\cos^2(\omega+\upsilon)}}, & (63)\\ \frac{d^2\xi}{d\omega^2} &= \epsilon\frac{|l_{01}|\left(1-|l_{01}|^2\right)\cos(\omega+\upsilon)}{\left(1-|l_{01}|^2\cos^2(\omega+\upsilon)\right)^{3/2}} \text{ and} & (64)\\ \frac{d^3\xi}{d\omega^3} &= -\epsilon\frac{|l_{01}|\left(1-|l_{01}|^2\right)\sin(\omega+\upsilon)}{\left(1-|l_{01}|^2\cos^2(\omega+\upsilon)\right)^{3/2}}\left\{1+3\frac{|l_{01}|^2\cos(\omega+\upsilon)}{1-|l_{01}|^2\cos^2(\omega+\upsilon)}\right\}. & (65) \end{aligned}$$

where ϵ is either $+1$ or -1. Obviously, (64) implies the first two statement of the lemma, but then (57) follows from (63). Furthermore, if we insert $\cos(\omega+\upsilon) = 0$ and $\sin(\omega+\upsilon) = \pm 1$ into (65) we get the penultimate statement.

Clearly $tr\widehat{L}(\pi+\omega) = -tr\widehat{L}(\omega)$. Therefore either $\xi(\pi+\omega) = \pi \pm \xi(\omega)$. Note that (63) implies that

$$\frac{\partial}{\partial\omega}\left(\xi\left(\omega\right)+\xi\left(\omega+\pi\right)\right)=0,\ \forall\omega \tag{66}$$

Consequently $\xi(\omega)+\xi(\omega+\pi)=c$ and hence the last statement. **qed**

Set

$$\gamma_c = \left|\frac{d\xi(\omega_1)}{d\omega}\right| = |l_{01}| \tag{67}$$

Consider again the two phase functions $\phi^{\pm}(\omega,\gamma) = \gamma\omega \pm \xi(\omega)$. We then have shown that for $|\gamma| \leq \gamma_c$ there are two critical points for each of them. As γ approaches γ_c they get closer to give a rise to a cubic critical point. For $|\gamma| > \gamma_c$ there are no critical points any more. This already explains the effective support inside a cone smaller than the mathematical light cone, since the integrals in (49) are rapidly decreasing oscillatory integrals. This is precisely the situation to which Theorem 2 applies. A priori, for γ close to γ_c there are two contributions. One coming from each of the integrals in (49). We might therefore expect a superposition of two different Airy functions in the asymptotic behaviour. However, because of the symmetries of ξ the phases look locally the same. We may therefore conclude by invoking the following result the proof of which we may leave to the reader.

Lemma 5 *Let $\alpha_{m,n}$ $n \in \mathbf{Z}$, $m = 1, 2, \ldots$, be a double sequence of complex numbers subject to the following*

$$|\alpha_{m,n}| \leq C(1+n^2)^{\rho/2} \tag{68}$$

with some ρ, $C > 0$. Let $h(x)$ be a monotone function, tending to 0 as $x \to \infty$. Suppose there is a smooth function f of tempered growth at ∞ such that

$$\alpha_{n,m} \to f(mh(n)), \quad (n \to \infty) \tag{69}$$

where the convergence is uniform for $|mh(n)| \leq K$ for any $K > 0$. Then consider the following sequences of distributions

$$\eta_n = \sum_m \alpha_{n,m} h(n)\delta(x - mh(n)) \tag{70}$$

$$\chi_n = \sum_m \alpha_{n,m} h(n)^2 (-1)^m \delta(x - mh(n)). \tag{71}$$

We have $\eta_n \to f$ and $\chi_n \to f'$ in the sense of distributions.

The theorem follows upon considering $\alpha_{m,n} = (\mathcal{L}^n s)_{\lfloor\gamma_c n\rfloor+m}$, where $\lfloor x\rfloor$ denotes the nearest integer $\leq x$. **qed**

4 The asymptotic analysis of the diffusive case

In this section our main result is

Theorem 3 *Assume that L is such that $iL \in SL(2)$ and that L is diffusive. Let $\mathcal{L}$ be its associated operator. Define the measures η_n according to (17). Then there exists a unique number $\gamma_c \in [-1,1]$, and phases $\varphi,\ \psi \in [0,2\pi)$ such that for the renormalised measures*

$$(\chi)_n(y) = e^{-n|\nu|_{max}} n^{\frac{1}{2}} e^{-in\varphi} e^{-in^{1/2}y\psi} \eta_n \left(n^{\frac{1}{2}}y + \gamma_c n\right) \tag{72}$$

we have that

$$(\chi)_n \to \chi = m e^{i\left\{\frac{\Delta(\cdot)^2}{2}\right\}} e^{-\left\{\frac{(\cdot)^2}{2D}\right\}}, \text{ in S}' \text{ as } n \to \infty \ , \tag{73}$$

where D is a positive number, Δ is a real number and m is a complex number depending linearly on s. Moreover, the linear form $s \to m$ is not identically zero.

Remark 1: Once again m may be zero for a particular initial sequence. Higher order terms will then dominate, but we do not treat this case.
Remark 2: There are exactly two choices of the phases $(\varphi,\ \psi)$.
Remark 3: In this case the constants are must easily calculated by a numerical procedure suggested by the proof. It is, however, possible to construct explicit formulae.

4.1 Proof of Theorem 3

All the results in Subsections 3.1 and 3.2 hold in the non-propagative case as well. However, before we estimate the integrals in (48), we must prove a proposition, analogous to Theorem 2.

Proposition 1 *Let $K \subset \mathbf{R}$ be a sufficiently small neighbourhood of $(0,0)$. Let $u(\omega,g) \in C_0^\infty(K)$, and $f(\omega,g)$ be a complex-valued C^∞ function in some neighbourhood of $(0,0)$ fulfilling*

$$\Im f \geq 0 \ , \ \Im f(0,0) = 0 \ , \ \left.\frac{\partial f}{\partial \omega}\right|_{(\omega,g)=(0,0)} = 0 \ , \ \left.\frac{\partial^2 f}{\partial \omega^2}\right|_{(\omega,g)=(0,0)} \neq 0 \ . \tag{74}$$

Let us furthermore assume that $\Im \left.\frac{\partial f}{\partial g}\right|_{(\omega,g)=(0,0)} = 0$, and that for some $p > \frac{1}{3}$ and some $b > 0$ we have that

$$g \in \left[-bv^{-p}, bv^{-p}\right] \ . \tag{75}$$

Then for some $c \in C_0^\infty$ and for some C which only depends on b and p we have

$$\left| \int u(\omega, g)\, e^{ivf(\omega,g)} d\omega - c(g)\, v^{-\frac{1}{2}} e^{iv\chi_1} e^{ivg\chi_2} e^{iv\frac{g^2}{2}\chi_3} e^{-vD\frac{g^2}{2}} \right| \leq$$

$$\leq Cv^{-\frac{1}{2}-(3p-1)}\ ,\ v > 0\ , \qquad (76)$$

where χ_1, χ_2, χ_3 and D are real constants depending only on f and its derivatives in the origin. Explicit expressions for these constants follow directly from the proof.

In this case $f(\omega)$ is a complex number, and therefore the main idea of the proof of this proposition is to use a theorem by Melin and Sjöstrand [7], which extends the method of stationary phase to the complex case. The version we state is a slightly simplified $\mathbf{R}^{1+1}$ version of the corresponding theorem in [6].

Theorem 4 *(Melin and Sjöstrand) Let $K \subset \mathbf{R}^{1+1}$ be a sufficiently small neighbourhood of $(0,0)$. Let $u(\omega, g) \in C_0^\infty(K)$, and $f(\omega, g)$ be a complex-valued C^∞ function in some neighbourhood of $(0,0)$ fulfilling (74). Then for some constant b we have*

$$\left| \int u(\omega)\, e^{ivf(\omega,g)} d\omega - b\left(\left(\frac{v\frac{\partial^2 f}{\partial \omega^2}}{2\pi i}\right)^0\right)^{-\frac{1}{2}} e^{ivf^0} u^0 \right|$$

$$\leq Cv^{-\frac{3}{2}}\ ,\ v > 0\ . \qquad (77)$$

where for functions $p(\omega, g)$ we have used the notation from Hörmander [6] and let $p^0(g)$ stand for a function of only g such that $p(\omega, g)$ and $p^0(g)$ are in the same residue class modulo the ideal generated by $\frac{\partial f}{\partial \omega}$. When $g = 0$ we have that

$$\left(\left(\frac{v\frac{\partial^2 f}{\partial \omega^2}}{2\pi i}\right)^0\right)^{-\frac{1}{2}} = e^{-i\pi \operatorname{sign}\left(\left.\frac{\partial^2 f}{\partial \omega^2}\right|_{(\omega,g)=(0,0)}\right)/4} \sqrt{\frac{2\pi}{v\left|\left.\frac{\partial^2 f}{\partial \omega^2}\right|_{(\omega,g)=(0,0)}\right.}} \qquad (78)$$

For small $g \neq 0$ we determine the square root by continuity.

Proof of Proposition 1.
The conditions in Theorem 4 are clearly met, so to prove Proposition 1 we must evaluate $f^0(g)$. Let $I = I\left(\frac{\partial f}{\partial \omega}\right)$ be the ideal generated by $\frac{\partial f}{\partial \omega}$. From [6] we have that there exists a C^∞ function $W(g)$ such that $W(0)$ and

$$\frac{\partial f}{\partial \omega} = c(\omega, g)\,(\omega - W(g)) \qquad (79)$$

for some C^∞ function c in some neighbourhood of $(0,0)$. In [6] it is also shown that we have

$$f(\omega,g) = f^0(g) + \sum_{2\le\alpha<N} f^\alpha(g)\frac{(\omega - W(g))^\alpha}{\alpha!} \mod I^N . \tag{80}$$

If we differentiate (79) with respect to ω and set $\omega = g = 0$ we obtain that

$$c(0,0) = \left.\frac{\partial^2 f}{\partial\omega^2}\right|_{(\omega,g)=(0,0)} . \tag{81}$$

If we in addition differentiate (79) with respect to g and set $\omega = g = 0$ we find that

$$\left.\frac{dW}{dg}\right|_{g=0} = -\frac{\left.\frac{\partial^2 f}{\partial\omega\partial g}\right|_{(\omega,g)=(0,0)}}{\left.\frac{\partial^2 f}{\partial\omega^2}\right|_{(\omega,g)=(0,0)}} \tag{82}$$

Since $W(0) = 0$ this implies that

$$W(g) = -\frac{\left.\frac{\partial^2 f}{\partial\omega\partial g}\right|_{(\omega,g)=(0,0)}}{\left.\frac{\partial^2 f}{\partial\omega^2}\right|_{(\omega,g)=(0,0)}} g + \mathcal{O}\left(g^2\right) . \tag{83}$$

Now differentiate (80) twice with respect to ω and set $\omega = g = 0$ to obtain

$$f^2(g) = \left.\frac{\partial^2 f}{\partial\omega^2}\right|_{(\omega,g)=(0,0)} + \mathcal{O}(g) . \tag{84}$$

Insertion of (83) and (84) into (80) with $\omega = 0$, now yields

$$\begin{aligned} f^0(g) &= f(0,0) + g\left.\frac{\partial f}{\partial g}\right|_{(\omega,g)=(0,0)} + \\ &+ \frac{g^2}{2}\left[\left.\frac{\partial^2 f}{\partial g^2}\right|_{(\omega,g)=(0,0)} - \frac{\left(\left.\frac{\partial^2 f}{\partial\omega\partial g}\right|_{(\omega,g)=(0,0)}\right)^2}{\left.\frac{\partial^2 f}{\partial\omega^2}\right|_{(\omega,g)=(0,0)}}\right] + \mathcal{O}\left(g^3\right) . \end{aligned} \tag{85}$$

Using condition (75) and remembering that $p > \frac{1}{3}$ we have that

$$\exp Cvg^3 \le \exp Cv^{-(3p-1)} = 1 + \mathcal{O}\left(v^{-(3p-1)}\right) . \tag{86}$$

To conclude the proof we need the following estimates, which are obtained analogously to those above:

$$u^0(g) = u(0,0) + \mathcal{O}(g) \text{ , and} \tag{87}$$

$$\left(\left(\frac{v\frac{\partial^2 f}{\partial\omega^2}}{2\pi i}\right)^0\right)^{-\frac{1}{2}} = v^{-\frac{1}{2}}\sqrt{\frac{2\pi i}{\left.\frac{\partial^2 f}{\partial\omega^2}\right|_{(\omega,g)=(0,0)}}} + v^{-\frac{1}{2}}\mathcal{O}(g) . \tag{88}$$

Because of the condition (75) this completes the proof of Proposition 1. **qed**

Proof of Theorem 3
Recall that according to Lemma 3 we have

$$\begin{aligned}\mathcal{L}^{*n}(s)_m &= \frac{1}{2\pi}e^{n|\nu|_{max}}\int_0^{2\pi} G_0(\omega)\, e^{in\phi^+(\omega,\gamma)}d\omega + \\ &+ \frac{1}{2\pi}e^{n|\nu|_{max}}\int_0^{2\pi} G_1(\omega)\, e^{in\phi^-(\omega,\gamma)}d\omega \end{aligned} \tag{89}$$

where we now let $\phi^{\pm}(\omega,\gamma)=\gamma\omega\pm\zeta(\omega)+i|\nu|_{max}$, and $\gamma=\frac{m}{n}$. An immediate consequence of Lemma 2 is that

$$\zeta(\omega+2\pi)=\zeta(\omega)+2\pi k \tag{90}$$

Once more we cannot apply Theorem 1 directly, since our functions are only defined on the circle. However, we proceed just like in the propagative case and take a smooth function $h(\omega)$ taking values between 0 and 1 and satisfying $\sum_l h(\omega+2\pi l)\equiv 1$. Now we can rewrite (89) as

$$\begin{aligned}\mathcal{L}^{*n}(s)_m &= \frac{1}{2\pi}e^{n|\nu|_{max}}\int h(\omega)\, G_0(\omega)\, e^{in\phi^+(\omega,\gamma)}d\omega \\ &+ \frac{1}{2\pi}e^{n|\nu|_{max}}\int h(\omega)\, G_1(\omega)\, e^{in\phi^-(\omega,\gamma)}d\omega \end{aligned} \tag{91}$$

This, and the following theorem from [6], now tells us that we only have to look for critical points of $\phi^{\pm}$ for values $\omega\in[0,2\pi)$.

Theorem 5 *Let $K\subset\mathbf{R}$ be a compact set, X an open neighbourhood of K and k a positive integer. If $u\in C_0^{2k}(K)$, $f\in C^{3k+1}(X)$ and $\Im f\geq 0$ in X. Then*

$$v^k\left|\int u(\omega)\, e^{ivf(\omega)}d\omega\right|\leq C\sum_{\alpha\leq k}|D^\alpha u|\left(\left|f'\right|^2+\Im f\right)^{|\alpha|/2-k},\ v>0\ . \tag{92}$$

Here C is bounded when f stays in a bounded set in $C^{k+1}(X)$.

The following proposition gives most of the information about the critical values of $\phi^{\pm}$ that we need:

Proposition 2 *Let L and $\widehat{L}(\omega)$ be as in Theorem 3, and let the eigenvalues of $\widehat{L}(\omega)$ be*

$$\lambda_{1,2}=e^{\pm i\zeta(\omega)}\ , \tag{93}$$

where $\zeta=\zeta(\omega)=\xi(\omega)+i\nu(\omega)$, then either $\nu(\omega)$ is constant or there exists a unique $\omega_1\in[0,\pi)$ such that

$$\begin{aligned}|\nu(\omega_1)| &= |\nu(\omega_1+\pi)|=|\nu|_{max}\ ,\ \text{and} && (94)\\ |\nu(\omega)| &< |\nu|_{max}\ \text{if}\ \omega\neq\omega_1,\ \omega_1+\pi\ , && (95)\end{aligned}$$

and where the maxima are of the first order.

We defer the proof of Proposition 2 until the next subsection.

By assumption $\nu(\omega)$ is not constant. According to Proposition 2 there thus exists a unique ω_1 such that

$$|\nu(\omega_1)| = |\nu(\omega_1 + \pi)| = |\nu|_{max} \ . \tag{96}$$

Now two different situations can occur. These are, as illustrated in figure 4, that we either have that $\nu(\omega_1) = \nu(\omega_1 + \pi)$ which gives us the situation in figure 4(a) or that we have that $\nu(\omega_1) = -\nu(\omega_1 + \pi)$ as shown in figure 4(b). In the first of these cases one of the integrals in (91) has two critical points for $\omega \in [0, 2\pi)$. To enable the application of Proposition 1 in this case as well we now take a C_0^∞ partition of unity $\phi_k(\omega)$ such that $\phi_1 = 1$ in a neighbourhood of ω_1 and $\phi_2 = 1$ in a neighbourhood of $\omega_1 + \pi$. If we insert this partition of unity into the integrands in (89) we obtain

$$\begin{aligned} \mathcal{L}^{*n}(s)_m &= \frac{1}{2\pi} e^{n|\nu|_{max}} \sum_{k=1}^{\infty} \int h(\omega)\, \phi_k(\omega)\, G_0(\omega)\, e^{in\left(\phi^+(\omega,\gamma)\right)} d\omega \\ &+ \frac{1}{2\pi} e^{n|\nu|_{max}} \sum_{k=1}^{\infty} \int h(\omega)\, \phi_k(\omega)\, G_1(\omega)\, e^{in\left(\phi^-(\omega,\gamma)\right)} d\omega \ . \end{aligned} \tag{97}$$

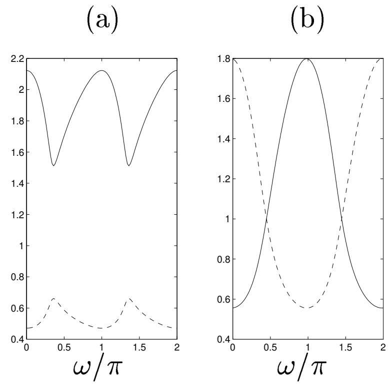

Figure 4: Typical variation of the absolute values of the two eigenvalues as ω goes from 0 to 2π, for (a) the case when $\nu(\omega_1) = \nu(\omega_1 + \pi)$, and (b) the case when $\nu(\omega_1) = -\nu(\omega_1 + \pi)$.

However, if we now let

$$\gamma_c = \operatorname{sign}(\nu(\omega_1)) \left.\frac{d\xi}{d\omega}\right|_{\omega=\omega_1} \tag{98}$$

both of the situations in figure 4 can be treated by the following lemma:

Lemma 6 *If we define*

$$
\begin{aligned}
r(\omega,\gamma) &= (\gamma+\gamma_c)(\omega+\omega_1) - \operatorname{sign}(\nu(\omega_1))\,\zeta(\omega+\omega_1) + i|\nu|_{max} && (99)\\
s(\omega,\gamma) &= (\gamma+\gamma_c)(\omega+\omega_1+\pi) - \\
&- \operatorname{sign}(\nu(\omega_1+\pi))\,\zeta(\omega+\omega_1+\pi) + i\,|\nu|_{max} && (100)
\end{aligned}
$$

we have that

$$(a)\ \Im r \geq 0,\ \Im s \geq 0\ , \tag{101}$$

$$(b)\ \Im r(0,0) = \Im s(0,0) = 0\ , \tag{102}$$

$$(c)\ \left.\frac{\partial r}{\partial \omega}\right|_{(\omega,\gamma)=(0,0)} = \left.\frac{\partial s}{\partial \omega}\right|_{(\omega,\gamma)=(0,0)} = 0\ , \tag{103}$$

$$(d)\ \left.\frac{\partial^2 r}{\partial \omega^2}\right|_{(\omega,\gamma)=(0,0)} = \left.\frac{\partial^2 s}{\partial \omega^2}\right|_{(\omega,\gamma)=(0,0)} \neq 0\ , \tag{104}$$

$$(e)\ \frac{\partial^2 r}{\partial\omega\partial\gamma} = \frac{\partial^2 s}{\partial\omega\partial\gamma} = 1\ . \tag{105}$$

Remark : For r we have either that

$$
\begin{aligned}
r(\omega,\gamma) &= \phi^{+}(\omega+\omega_1,\gamma+\gamma_c)\,, \text{ or that} && (106)\\
r(\omega,\gamma) &= \phi^{-}(\omega+\omega_1,\gamma+\gamma_c)\ . && (107)
\end{aligned}
$$

depending on the sign of $\nu(\omega_1)$. A similar statement holds for s.
Proof We have that

$$
\begin{aligned}
\Im r(\omega,\gamma) &= |\nu|_{max} - |\nu(\omega)| \geq 0 \text{ and} && (108)\\
\Im s(\omega,\gamma) &= |\nu|_{max} - |\nu(\omega+\pi)| \geq 0\ , && (109)
\end{aligned}
$$

which together with Proposition 2 prove (a) and (b). To prove the rest of the lemma recall that $\mathrm{tr}\widehat{L}(\pi+\omega) = -\mathrm{tr}\widehat{L}(\omega)$ which implies that

$$\zeta(\omega+\pi) = \pi \pm \zeta(\omega)\ . \tag{110}$$

If we take the imaginary part of this, we obtain that for ω in some neighbourhood around ω_1 we have

$$\zeta(\omega+\pi) = \pi + \frac{\operatorname{sign}(\nu(\omega_1))}{\operatorname{sign}(\nu(\omega_1+\pi))}\zeta(\omega)\ . \tag{111}$$

From this we have that

$$
\begin{aligned}
\left.\frac{\partial r}{\partial \omega}\right|_{(\omega,\gamma)=(0,0)} &= \gamma_c - \operatorname{sign}(\nu(\omega_1))\left.\frac{d\xi}{d\omega}\right|_{\omega=\omega_1} = 0\ , && (112)\\
\left.\frac{\partial s}{\partial \omega}\right|_{(\omega,\gamma)=(0,0)} &= \gamma_c - \operatorname{sign}(\nu(\omega_1+\pi))\left.\frac{d\xi}{d\omega}\right|_{\omega=\omega_1+\pi} && (113)\\
&= \gamma_c - \operatorname{sign}(\nu(\omega_1))\left.\frac{d\xi}{d\omega}\right|_{\omega=\omega_1} = 0\ , && (114)
\end{aligned}
$$

which proves (c). To prove (f) let us notice that

$$\left.\frac{\partial^2 r}{\partial\omega^2}\right|_{(\omega,\gamma)=(0,0)} = -\mathrm{sign}\left(\nu\left(\omega_1\right)\right)\left.\frac{d^2\zeta}{d\omega^2}\right|_{\omega=\omega_1}, \text{ and} \tag{115}$$

$$\left.\frac{\partial^2 s}{\partial\omega^2}\right|_{(\omega,\gamma)=(0,0)} = -\mathrm{sign}\left(\nu\left(\omega_1+\pi\right)\right)\left.\frac{d^2\zeta}{d\omega^2}\right|_{\omega=\omega_1+\pi}. \tag{116}$$

The equality of these expressions follows from (111), and since the minimum of the imaginary part of r at ω is of the first order according to Proposition 2, it follows that at least

$$\left.\frac{d^2\nu}{d\omega^2}\right|_{\omega=\omega_1} \neq 0\ , \tag{117}$$

which concludes the proof of (f). Finally (e) is a trivial consequence of the definitions of r and s. This concludes the proof of the lemma. **qed**

With Theorem 5 and Proposition 1 we have now all the tools needed to calculate $\alpha_{m,n} = \mathcal{L}^n\left(s\right)_{\lfloor\gamma_c n\rfloor+m}$, where $\lfloor x\rfloor$ denotes the nearest integer $\leq x$. Finally we use Lemma 5 to finish the proof. **qed**

4.2 Proof of Proposition 2

We begin the proof by noting that since $\widehat{L}\left(\omega\right)\in SL\left(2\right),\ \forall\omega$, we have that $\cos\zeta\left(\omega\right) = \mathrm{tr}\widehat{L}\left(\omega\right)/2$, and thus

$$\nu\left(\omega\right) \quad = \quad \Im\arccos\left(\frac{\mathrm{tr}\widehat{L}\left(\omega\right)}{2}\right). \tag{118}$$

However, the imaginary part of the arccos-function is well known (see for example [1]), and thus we have

$$\begin{aligned}
\nu\left(\omega\right) = \Im\arccos\left(\frac{\mathrm{tr}\widehat{L}\left(\omega\right)}{2}\right) \quad &= \quad \pm\ln\left[\alpha+\left(\alpha^2-1\right)^{1/2}\right] && (119)\\
&= \quad \pm\mathrm{arccosh}\alpha && (120)
\end{aligned}$$

where

$$\begin{aligned}
\alpha \quad &= \quad \frac{1}{2}\left[\left(x+1\right)^2+y^2\right]^{1/2}+\frac{1}{2}\left[\left(x-1\right)^2+y^2\right]^{1/2}\ , && (121)\\
x \quad &= \quad x\left(\omega\right) = \Re\left(\frac{\mathrm{tr}\widehat{L}\left(\omega\right)}{2}\right) \text{ and} && (122)\\
y \quad &= \quad y\left(\omega\right) = \Im\left(\frac{\mathrm{tr}\widehat{L}\left(\omega\right)}{2}\right)\ . && (123)
\end{aligned}$$

Choosing always the plus sign in (120) corresponds to studying $\left|\nu\left(\omega\right)\right|$. We see that the level sets $\left|\nu\left(x,y\right)\right| = \tau$ are given by the equations

$$\frac{x^2}{\cosh^2\tau}+\frac{y^2}{\sinh^2\tau} = 1, \tag{124}$$

These level sets are for $\tau > 0$ ellipses with centre at $(0,0)$ foci at $(\pm 1, 0)$ and with the length of the major and minor semi-axis given by $\cosh \tau$ and $\sinh \tau$. The level set for $\tau = 0$ is the part of the real axis between -1 and 1. Consequently, finding the maximum of $|\nu(\omega)|$ is equivalent to finding the maximum value of τ obtained in (124)as ω changes. From (124) we also see that

$$|\nu(-x,-y)| = |\nu(x,y)| \tag{125}$$

Note now that from (13) we have that

$$\begin{aligned} \mathrm{tr}\widehat{L}(\omega)/2 &= \left(l_{01}e^{i\omega} + l_{10}e^{-i\omega}\right)/2 && (126) \\ &= \left(\frac{l_{01}+l_{10}}{2}\right)\cos\omega + i\left(\frac{l_{01}-l_{10}}{2}\right)\sin\omega\ . && (127) \end{aligned}$$

From this it is clear that $\mathrm{tr}\widehat{L}(\omega+\pi)/2 = -\mathrm{tr}\widehat{L}(\omega)/2$, so (125) tells us that $|\nu(\omega+\pi)| = |\nu(\omega)|$. To prove the existence of a unique maximiser of $|\nu(\omega)|$ in $[0,\pi)$, note that the locus of the equation in (127) is also an ellipse (or a straight line) with centre $(0,0)$, as can trivially be seen if $\left(\frac{l_{01}+l_{10}}{2}\right)$ and $\left(\frac{l_{01}-l_{10}}{2}\right)$ are real, and by a change of variables in the general case as well.

We therefore have the situation depicted in figure 5. From the geometry of the situation it is clear that there exists a τ_{min} and a τ_{max} such that the ellipse followed by $\mathrm{tr}\widehat{L}(\omega)/2$ as ω goes from 0 to 2π crosses all the level sets $|\nu(x,y)| = \tau, \tau \in (\tau_{min}, \tau_{max})$ exactly four times, and such that the level sets $|\nu(x,y)| = \tau, \tau > \tau_{max}$ or $\tau \in [0, \tau_{min})$ are not intersected at all. If $\tau_{max} = \tau_{min}$ then it is clear that $\nu(\omega)$ is constant. From the degree of the trigonometric polynomials involved it is also clear that the intersection of $\mathrm{tr}\widehat{L}(\omega)/2$ with the level set $|\nu(x,y)| = \tau_{max}$ is of the second order.

We have therefore proved that $|\nu(\omega)|$ is either constant or attains a unique first order maximum at some $\omega_1 \in [0,\pi)$ as well as at $\omega_1 + \pi$. This concludes the proof of Proposition 2. **qed**

It is easy to see that the situation in figure 5 corresponds to that in figure 4 (a). With the absolute values of the eigenvalues as in figure 4 (b) the ellipses would cross the real line between -1 and $+1$ and consequently look like that in figure 6.

5 Numerical illustrations of the asymptotic behaviour

The purpose of this section is to illuminate the asymptotic analysis by numerical examples. These will also give some idea of the speed of convergence towards the asymptotic solution.

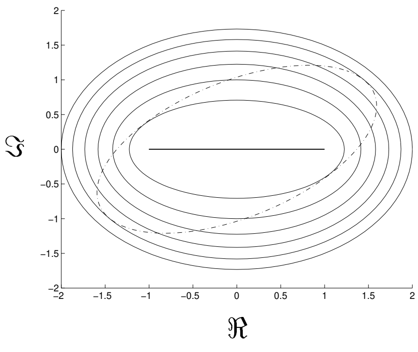

Figure 5: The solid lines show some level sets for the absolute value of the imaginary part of the arccos-function. The line segment between -1 and 1 corresponds to the imaginary part of the arccos-function being zero, and the absolute value of it grows with the size of the ellipse. The ellipse indicated by $- \cdot -$ shows the locus of $\cos\zeta(\omega) = \mathrm{tr}\widehat{L}(\omega)/2$ for a typical case. It is clear that there is a smallest and largest level set each intersecting the latter ellipse exactly twice.

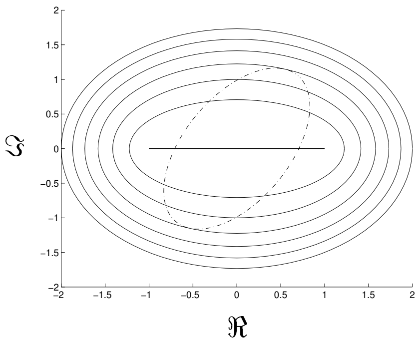

Figure 6: The solid lines show some level sets for the absolute value of the imaginary part of the arccos-function. The line segment between -1 and 1 corresponds to the imaginary part of the arccos-function being zero, and the absolute value of it grows with the size of the ellipse. The ellipse indicated by $- \cdot -$ shows the locus of $\cos\zeta(\omega) = \mathrm{tr}\widehat{L}(\omega)/2$. It is clear that there is a largest level set each intersecting the latter ellipse exactly twice, and that the only other level set intersecting it exactly twice is the level set corresponding to $\nu = 0$.

5.1 The convergence towards the Airy function

The primary difficulty when trying to illustrate the convergence towards the Airy functions is the very marked presence of a term which eventually will oscillate to zero in S'. One possibility around this problem is certainly to use an initial sequence which makes the amplitude of the oscillating term vanish. However, it is not an easy task to find such an initial sequence. Fortunately, one can find that if we restrict ourselves to even time-steps the oscillating term affects all the even sequence entries in one way and all the odd entries in another way. Therefore if we treat the odd and even sequence entries separately we see that each of them approaches an Airy function without any terms oscillating to zero. Needless to say, the amplitudes of the two limit Airy functions will be different, but apart from that they will be identical. This is illustrated in Figure 7 where we have applied $\mathcal{L}$ associated to the Dirac equation with $\delta = 0.5$ repeatedly. We have renormalised the result just as in Theorem 1 and finally we have linearly interpolated between the even and odd points of support to enhance the visual impact of the convergence.

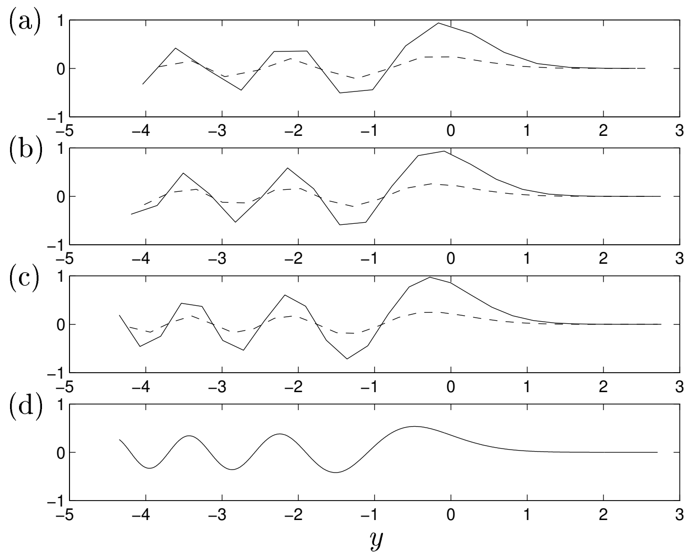

Figure 7: The renormalised measures from (17) for $\mathcal{L}$ associated to the L modelling the Dirac equation with $\delta = 0.5$ with a typical initial sequence are shown at (a) $n = 100$, (b) $n = 200$ and (c) $n = 400$. For the solid (dashed) lines we have linearly interpolated between the even (odd) points of the support to illustrate that these each converge towards an Airy function in S'. In (d) we show the limit Airy function obtained from Theorem 2.

5.2 Convergence towards the Gaussian functions

To illustrate the case when $\nu(\omega) \not\equiv 0$ we chose

$$L = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \tag{128}$$

By a calculation entirely analogous to that in the previous subsection we can show that even in this we will have to restrict ourselves to even or odd time-steps and treat odd and even terms separately to avoid the disturbing oscillations. In Figure 8 we show parts of the renormalised measures corresponding to (128) at different times.

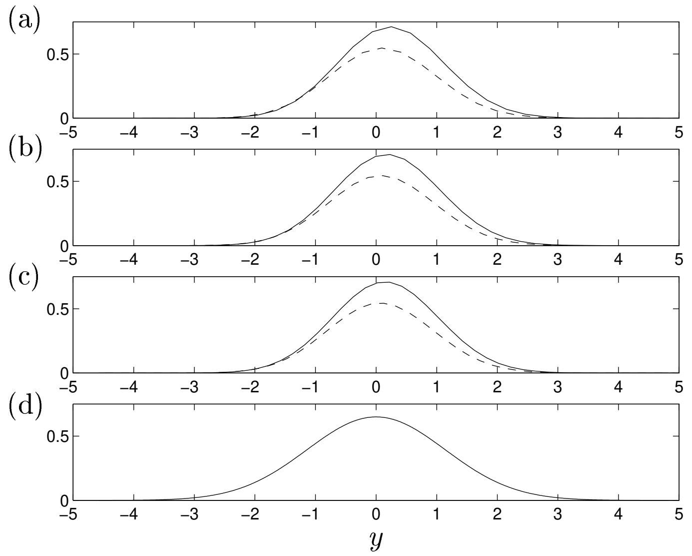

Figure 8: The renormalised measures from (17) for the $\mathcal{L}$ associated to the L given in (128), with a typical initial sequence are shown at (a) $n = 40$, (b) $n = 70$ and (c) $n = 100$. For the solid (dashed) lines we have linearly interpolated between the even (odd) points of the support to illustrate that these each converge towards a Gaussian function in S'. In (d) we show the limit Gaussian function obtained from Theorem 3.

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