A Criterion for Dynamical Localization in Random Schrödinger Models

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Abstract

We study dynamical properties of random Schrödinger operators $H^{(\omega)}$ defined on the Hilbert space $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$. We give sufficient conditions on the decay of the Green's function to obtain the dynamical localization property

$$\mathbb{E}\left(\sup_{T>1}\langle\langle |X|^2\rangle\rangle_{T,f_I(H^{(\omega)})\psi}\right) < \infty\ ,$$

where $\mathbb{E}$ is the expectation over randomness, f_I is any smooth characteristic function of a bounded energy-interval I and ψ is a state vector in the domain of $H^{(\omega)}$ with compact spatial support. The quantity $\langle\langle |X|^2\rangle\rangle_{T,\varphi}$ denotes the Cesaro mean up to time T of the second moment of position $\langle |X|^2\rangle_{t,\varphi}$ at times $0 \leq t \leq T$ of an initial state vector φ. Under weaker assumptions, we also prove a theorem on the absence of diffusion. The results are applied to a simple Anderson-type model in the lattice case and to a model with a correlated random potential in the continuous case.

Key-Words: Random Schrödinger operators, dynamical localization, diffusion exponents, correlated random potentials.

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1 Introduction

The study of random Schrödinger operators has a long history going back to the fundamental work of Anderson [3] in 1958. Random Schrödinger operators occur in probabilistic single-particle models which are commonly accepted [27, 20] to provide a minimal description for electronic properties of disordered materials such as doped semiconductors or metals with impurities. Anderson argued that the presence of disorder induces the so-called phenomenon of localization: an electron initially located in a bounded region will essentially remain there for all times. This, in turn, should imply a vanishing conductivity – a fact which is experimentally verified.

The first rigorous works on random Schrödinger operators are due to Pastur [22] who concentrated on their spectral properties. Later on, these results were extended to establish the almost-sure decomposition of the spectrum into pure-point and continuous parts, to show the existence and regularity properties of the integrated density of states, as well as exponential localization [16, 7, 23]. Here, exponential localization means that the spectrum is almost surely pure point in some set of energies, with exponentially decaying eigenfunctions. For energies at the bottom of the spectrum or near band edges, this property is known for multi-dimensional Anderson models on the lattice and for certain random Schrödinger operators in multi-dimensional continuous space, see e.g. [19, 15, 29, 2, 8, 18, 5, 17].

Early rigorous studies of transport coefficients can be found in [19, 13, 15]. A relevant observable describing transport properties of a probabilistic single-particle model is the energy-resolved mean diffusion constant. For a random Schrödinger operator $H^{(\omega)}$ on the lattice and an initial state localized at $y \in \mathbb{Z}^d$, it is defined as [15, 26]

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\varepsilon^2 \sum_{x \in \ell^2(\mathbb{Z}^d)} (x-y)^2 \int_0^\infty \mathrm{e}^{\varepsilon t} |\langle \delta_x, \mathrm{e}^{-\imath t H^{(\omega)}} \chi_I(H^{(\omega)}) \delta_y \rangle|^2 dt \right) . \tag{1}$$

Here $\mathbb{E}$ is the expectation on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and χ_I is the characteristic function of the set $I \subset \mathbb{R}$ of allowed energies. Another quantity of great interest (see e.g. [26, 1]), and closely connected to the previous one, is

$$\mathbb{E}\Big(\langle\langle |X|^2 \rangle\rangle_{T, \chi_I(H^{(\omega)})\psi} \Big) = \mathbb{E}\Big(T^{-1} \int_0^T \langle |X|^2 \rangle_{t, \chi_I(H^{(\omega)})\psi} \, dt \Big) , \tag{2}$$

the expectation of the Cesaro mean up to time T of the second moment of position $\langle |X|^2 \rangle_{t,\chi_I(H^{(\omega)})\psi} := \| \, |X| \mathrm{e}^{-\imath t H} \chi_I(H^{(\omega)}) \psi \|^2$ at times $0 \le t \le T$ of an initial state ψ with energy in $I \subset \mathbb{R}$ (see also Definition 2.2 below). It is used to calculate the diffusion exponents and the localization length. Until the beginning of the 90's, it was generally believed that the occurrence of pure-point spectrum was a sufficient criterion for the dynamical localization property

$$\mathbb{E}\left(\sup_{T>1}\langle\langle|X|^2\rangle\rangle_{T,\chi_I(H^{(\omega)})\psi}\right) < \infty\ . \tag{3}$$

However, we now know [11, 4] that pure-point spectrum and even exponential localization are not sufficient conditions for dynamical localization or the vanishing of the conductivity. The study of dynamical properties requires additional investigations. This was done recently by Aizenmann and Graf [1] who proved the dynamical localization property (3) for a large class of discrete random models. More recently, De Bievre and Germinet [10] showed $\sup_{T>0}\langle|X|^2\rangle_{T,\chi_I(H^{(\omega)})\psi} < \infty$, $\mathbb{P}$-almost surely, for both discrete and continuous models exhibiting exponential localization in I. This is quite a strong result, since it gives a bound for a quantity which has not been averaged over time. However, this bound provides no information about the expectation $\mathbb{E}$. Our main goal here is to show that the somewhat different property (3) does also hold for a large class of random Schrödinger operators on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$ if we replace the characteristic function χ_I by any smooth function f_I with compact support in I (Theorem 2.1). By relaxing some of the assumptions, a result on the absence of diffusion is also established (Theorem 2.2).

Similar to the strategy of Fröhlich and Spencer [13], who related the conductivity at some fixed energy E to the behavior of the Green's function at this energy, it is the strategy of the present paper to derive the dynamical localization property (3) from appropriate decay estimates of the Green's function. However, in order to calculate $\mathbb{E}\left(\sup_{T>1}\langle\langle|X|^2\rangle\rangle_{T,f_I(H^{(\omega)})\psi}\right)$ we need a more refined analysis that takes into account the dependence of the Green's function on the energy E and on the realization $\omega\in\Omega$ of the random potential. For proving dynamical localization, this can be done with the help of Von Dreifus and Klein's estimates [29] on the decay of the Green's function which are uniform in energy (see (8) below). For proving absence of diffusion, it suffices to have Combes-Hislop type decay estimates [8] at fixed energies (see (9) below). Although the second result on the vanishing of the diffusion constant is weaker than dynamical localization, it is worth mentioning, since the required Combes-Hislop estimates are proven for a very large class of random operators. On the contrary, decay estimates which are uniform in energy are only known for few cases (e.g. [29, 18, 10]).

The paper is organized as follows: In Section 2, we present the assumptions needed and we state our main results. In Section 3, we first apply Theorem 2.2 to a periodic continuous Schrödinger operator perturbed by an alloy-type random potential, thus proving absence of diffusion at energies near band-edges. Examples 2 and 3 are concerned with an application of Theorem 2.1 and establish dynamical localization for a multi-dimensional Anderson model on the lattice and for a continuous Schrödinger operator with a correlated alloy-type random potential, respectively. For the latter example we need a "variable-energy" multi-scale analysis that gives estimates which are uniform in energy. Details of these calculations are deferred to the Appendix. Section 4 is devoted to the proofs of the theorems.

2 Main results

We first describe precisely the quantities of interest. For the physical relevance of these quantities, one can refer for example to [28, 6, 1, 26].

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random operator H, that is a random variable $\Omega \ni \omega \mapsto H^{(\omega)}$ which takes on values in the space of linear operators on a Hilbert space $\mathcal{H}$. Here $\mathcal{H}$ will either be the space $\mathrm{L}^2(\mathbb{R}^d)$ of square-integrable functions or the space $\ell^2(\mathbb{Z}^d)$ of square-summable sequences. By $X = (X_1, \dots, X_d)$ we denote the self-adjoint multiplication operator on $\mathcal{H}$, and $|X| := \left(\sum_{j=1}^d X_j^2\right)^{1/2}$ is its modulus. For the random operator H we consider the following

Assumptions 2.1

i) H is measurable and $\mathbb{P}$-almost surely self-adjoint with constant domain $\mathcal{D}(H)$.

ii) There is $\mathcal{C} \subseteq \mathcal{H}$ such that $\mathcal{C}$ is $\mathbb{P}$-almost surely a core for H and $(\imath + \varepsilon X_j)^{-1}\mathcal{C} \subseteq \mathcal{C}$ for all $\varepsilon \in]0,1[$ and $j = 1, \dots, d$.

iii) The commutators $[H, X_j] := HX_j - X_jH$, $j = 1, \dots, d$, are well-defined and relatively operator bounded with respect to H, i.e. for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists $b(\omega) < \infty$ such that $\|[H^{(\omega)}, X_j](H^{(\omega)} + \imath)^{-1}\| \leq b(\omega)$. We further assume that $\mathbb{E}\big(b^2(\omega)\big) < \infty$.

iv) $\mathcal{D}(H) \cap \mathcal{D}(|X|)$ is dense in $\mathcal{H}$.

v) For $L > 1$, $z \in \mathbb{C} \setminus \mathbb{R}$ and $q, q' \in \mathbb{Z}^d$ with $\|q - q'\|_\infty > 2L$ there is an operator $D^{(\omega)}_{L,q'}(z)$ defined on $\mathcal{D}(H^{(\omega)})$ such that

$$\mathbf{1}_{q'}(H^{(\omega)} - z)^{-1}\,\mathbf{1}_q = \mathbf{1}_{q'}\,D^{(\omega)}_{L,q'}(z)(H^{(\omega)} - z)^{-1}\,\mathbf{1}_q \tag{4}$$

holds for $\mathbb{P}$-almost every ω. Here $\mathbf{1}_q$ acts as the multiplication operator corresponding to the function

$$\mathbf{1}_q(x) := \begin{cases} 1 & \text{if } \|x - q\|_\infty \leq 1\,, \\ 0 & \text{else}\ . \end{cases} \tag{5}$$

Remark 2.1 For conditions on the random operator H to fulfill i), see [16, 7, 23]. Among others, the measurability of H assures that functions of the type $(E, \omega) \mapsto \|(H^{(\omega)} - E - \imath\varepsilon)^{-1}\varphi\|$, which are continuous in E, are jointly measurable in E and ω. Assumption ii) is true for a large class of operators with $\mathcal{C} = C_0^\infty(\mathbb{R}^d)$, see e.g. [14]. Assumption iii) naturally occurs in the proof of Lemma 4.3 and helps to guarantee the finiteness, for all strictly positive T, of the quantity $\langle\langle |X|^2 \rangle\rangle_{T,\psi}$ given below in Definition 2.2. Assumption v) assures the existence of a geometric resolvent equation, which is written down in an abstract form in Eq. (4) in order to suit both the discrete and the continuous case. Concrete examples for the operators $D_{L,q'}^{(\omega)}(z)$, which in the sequel will be supposed to exhibit some nice decay properties, can be found in Section 3 (see also [8] and [29]).

Definition 2.1 Let $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-negative function. The operator $H^{(\omega)}$ is said to be (ϱ, E, L, q)-regular, if

$$\sup_{\varepsilon \neq 0} \| \mathbf{1}_q \, D_{L,q}^{(\omega)}(E + \imath\varepsilon) \| \leq \varrho(L)^{1/2} \, . \tag{6}$$

(M1) For H satisfying 2.1 i) and v), we suppose that there exists an increasing sequence $\{L_k\}_{k\in\mathbb{N}}$, an interval $I \subseteq \mathbb{R}$, a number $p > 0$ and a non-negative function $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{k=0}^{\infty} L_{k+1}^{d+2} (\varrho(L_k) + L_k^{-p}) < \infty \tag{7}$$

and

$$\begin{aligned} \mathbb{P}\{\omega \mid H^{(\omega)} \text{ is } (\varrho, E, L_k, q'') - \text{regular for all } E \in I \\ \text{and for either } q'' = q \text{ or } q'' = q'\} \geq 1 - L_k^{-p} \end{aligned} \tag{8}$$

for all $k \in \mathbb{N}$ and all $q, q' \in \mathbb{Z}^d$ with $\|q - q'\|_\infty > 2L_k$.

Losely speaking, (M1) assumes a decay with good probability of the localized resolvents (see the examples in Section 3). Such a decay can be shown by performing a "variable-energy" multi-scale analysis in the spirit of Von Dreifus and Klein (see e.g. [29, 18, 10]). For the weaker Theorem 2.2 to hold, we only need a weaker form of (M1), in that the set of events giving rise to decay need not be uniform in energy.

Multi-Scale Assumption 2

(M2) For H satisfying 2.1 i) and v), we assume that there exist $L_0 > 1$, $\alpha > 1$, $\beta > 0$, $m > (3+d+\beta)$, $p > 3+d+\beta$ such that for $\rho(x) := x^{-m}$, $L_k := {L_0}^{\alpha^k}$ and for all $E \in I$ and all $q \in \mathbb{Z}^d$, $\|q\|_\infty > 2L_k$, we have

$$\mathbb{P}\left\{\omega \mid H^{(\omega)} \text{ is } (\varrho, E, L_k, q) - \text{regular}\right\} \geq 1 - L_k^{-p} \, . \tag{9}$$

Definition 2.2 Take $\psi \in \mathcal{H}$ such that for all $t > 0$, $\mathrm{e}^{-\imath t H}\psi \in \mathcal{D}(|X|)$. The Cesaro mean at time $T > 0$ of the second moment of ψ is defined as

$$\langle\langle |X|^2 \rangle\rangle_{T,\psi} := \frac{1}{T} \int_0^T \| \, |X| \mathrm{e}^{-\imath t H} \psi \|^2 dt \, , \tag{10}$$

where $\imath$ is the imaginary unit. In analogy to [1] and [26] we say that ψ is dynamically localized if

$$\mathbb{E}\left(\sup_{T>1} \langle\langle |X|^2 \rangle\rangle_{T,\psi} \right) < \infty \, , \tag{11}$$

where $\mathbb{E}$ is the expectation associated with $\mathbb{P}$. The diffusion exponents are defined as in [26]

$$\sigma_{\text{diff}}^{+}(\psi) \quad := \quad \limsup_{T\to\infty} \frac{\log \mathbb{E}\left(\langle\langle |X|^2\rangle\rangle_{T,\psi}\right)}{\log T} \ , \tag{12}$$

$$\sigma_{\text{diff}}^{-}(\psi) \quad := \quad \liminf_{T\to\infty} \frac{\log \mathbb{E}\left(\langle\langle |X|^2\rangle\rangle_{T,\psi}\right)}{\log T} \ . \tag{13}$$

When the limit $\varepsilon \to 0$ exists, the diffusion constant is given by

$$D(\psi) := \lim_{\varepsilon\to 0} \mathbb{E}\left(\varepsilon^2 \int_{\mathbb{R}} \| \, |X| (H^{(\omega)} - E - \imath\varepsilon)^{-1}\psi\|^2 \, dE\right) \ . \tag{14}$$

Remark 2.2 If the upper diffusion exponent obeys $\sigma_{\text{diff}}^{+}(\psi) < 1$, then the diffusion constant vanishes, $D(\psi) = 0$.

The main results of this paper are summarized in the following two theorems.

Theorem 2.1 *(Dynamical localization)*
Consider a random operator H satisfying Assumptions 2.1 and the Multi-Scale Assumption (M1). Let I' be any compact subset of I such that $\text{dist}(\partial I, \partial I') > 0$. *Then we have dynamical localization in the sense that for all compactly supported $\varphi \in \mathcal{D}(H)$ and all $f_{I'} \in C_0^\infty(I')$*

$$\mathbb{E}\Big(\sup_{T>1} \langle\langle |X|^2\rangle\rangle_{T, f_{I'}(H)\varphi}\Big) < \infty \ . \tag{15}$$

This implies in particular that the diffusion exponents and the diffusion constant vanish, $\sigma_{\text{diff}}^{\pm}\big(f_{I'}(H)\varphi\big) = D\big(f_{I'}(H)\varphi\big) = 0$.

In case one can only establish the weaker Multi-Scale Assumption (M2), a weaker dynamical property can still be shown.

Theorem 2.2 *(Absence of diffusion)*
Let H be a random operator satisfying Assumptions 2.1 and the Multi-Scale-Assumption (M2), and let I' be as above. Then for all compactly supported $\varphi \in \mathcal{D}(H)$ and all $f_{I'} \in C_0^\infty(I')$, we have

$$\sigma_{\text{diff}}^{+}\big(f_{I'}(H)\varphi\big) < 1 \ . \tag{16}$$

This implies in particular that the diffusion constant vanishes, $D\big(f_{I'}(H)\varphi\big) = 0$.

3 Applications

We present here some examples of random Schrödinger operators for which the assumptions of Theorem 2.1 or 2.2 are fulfilled. The first example concerns absence of diffusion and the two following ones dynamical localization. Our aim is to use, as much as possible, the spectral results already known for these models in order to prove either (M1) or (M2), and to show that our theorems can easily be applied to a very large class of random Schrödinger operators. We will adopt the following notations: For $L > 0$ and $x \in \mathbb{R}^d$, $\Lambda_L(x)$ is the cube $\{y \in \mathbb{R}^d \mid \|x-y\|_\infty \leq L\}$ and $\chi_{\Lambda_L(x)}$ denotes a smooth characteristic function of $\Lambda_L(x)$, i.e. $\chi_{\Lambda_L(x)} \in C^2(\mathbb{R}^d)$ and for some fixed $\delta > 0$,

$$\chi_{\Lambda_L(x)} = \begin{cases} 1 & \text{if } \|x-y\|_\infty \leq L - \delta \ , \\ 0 & \text{if } \|x-y\|_\infty \geq L \ , \\ \in [0,1] & \text{elsewhere} \ . \end{cases}$$

Example 1: Random perturbations of periodic continuous Schrödinger operators

We consider a specific case of the random Schrödinger operators studied in [5],

$$H^{(\omega)} = -\Delta + V_{\mathrm{per}} + V^{(\omega)} \quad \text{on } L^2(\mathbb{R}^d) \ ,$$

where V_{per} is a bounded periodic potential such that $-\Delta + V_{\mathrm{per}}$ has a gap (B_-, B_+) in its spectrum and $V^{(\omega)} = g \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u(x-i)$ with

(H1) The potential u is positive, with compact support, such that $u(x) \geq 1$ on $[-1/2, 1/2]$, $\|u\|_\infty < \infty$.

(H2) $(\lambda_i)_{i \in \mathbb{Z}^d}$ is a stationary process of independent and identically distributed random variables. We assume that the probability distribution of λ_i has a Lebesgue density $h \in C^0(\mathbb{R})$, with support $[-M, M]$, and for some $\nu > 0$, $\mathbb{P}\{\mid \lambda \pm M \mid < \varepsilon\} \leq \varepsilon^{d+2+\nu}$

(H3) The coupling constant g satisfies $g < (B_+ - B_-)/(2M u_{\max})$, where $u_{\max} = \| \sum u(x-i)\|_\infty$.

It is proven in [5] that there exist non-empty compact subsets of $\mathbb{R}$, $I_+ \neq \emptyset$ and $I_- \neq \emptyset$, at the edge of the almost-sure spectrum of $H^{(\omega)}$, such that for $\alpha = (d+4)/(d+1)$, $m = p = 4 + d$, and for some $L_0 < \infty$, if $L_k = L_0^{\alpha^k}$, $E \in I_+ \cup I_-$, $q' \in \mathbb{Z}^d$,

$$\mathbb{P}\{\omega \mid \sup_{\varepsilon \neq 0} \|[-\Delta, \chi_{\Lambda_{L_k}(q')}](H^{(\omega)}_{\Lambda_{L_k}(q')} - E - \imath\varepsilon)^{-1} \mathbf{1}_{q'}\| \leq L_k^{-\frac{m}{2}}\} \geq 1 - L_k^{-p} \ , \quad (17)$$

where $H^{(\omega)}_{\Lambda_{L_k}(q')} = -\Delta + V_{\mathrm{per}} + \sum_{i \in \mathbb{Z}^d \cap \Lambda_{L_k}(q')} \lambda_i(\omega) u(x-i)$ and $\mathbf{1}_{q'}$ is defined in (5). According to the geometric resolvent equation, for $L > 0$ and $q' \in \mathbb{Z}^d$,

$$\begin{aligned} \chi_{\Lambda_L(q')}(H^{(\omega)} - E - \imath\varepsilon)^{-1} &= (H^{(\omega)}_{\Lambda_L(q')} - E - \imath\varepsilon)^{-1} \chi_{\Lambda_L(q')} \\ &+ (H^{(\omega)}_{\Lambda_L(q')} - E - \imath\varepsilon)^{-1} [-\Delta, \chi_{\Lambda_L(q')}](H^{(\omega)} - E - \imath\varepsilon)^{-1} \ . \end{aligned}$$

This implies (4) and (M2) with $D^{(\omega)}_{L,q'}(z) = (H_{\Lambda_L(q')} - z)^{-1}[-\Delta, \chi_{\Lambda_L(q')}]$. Assumptions 2.1 i)-v) are easily verified with $\mathcal{C} = C_0^\infty(\mathbb{R}^d)$, and $b(\omega) = 3(1 +$

$\|V_{\rm per}\|_\infty + (B_+ - B_-)/2)$; thus the conclusion of Theorem 2.2 holds for this model with $I = I_+ \cup I_-$.

Example 2: Discrete Anderson model

We consider the random family [3, 13, 29, 7, 23]

$$H^{(\omega)} = -\Delta_d + V^{(\omega)} \quad \text{in } \ell^2(\mathbb{Z}^d) ,$$

where $(-\Delta_d\psi)(n) := \sum_{i\in\mathbb{Z}^d,\, |i-n|=1} \psi(i)$, is the discrete Laplacian and $V^{(\omega)}(n)$, $n \in \mathbb{Z}^d$, are independent and identically distibuted random variables with absolutely continuous density $g(\lambda) := d\mu/d\lambda$ satisfying $\int \lambda^2 g(\lambda)d\lambda < \infty$. Although dynamical localization has already been proved in [1] for this model, we consider this issue to demonstrate the applicability of our general Statement 2.1.

For the discrete Anderson model, Assumptions 2.1 i), ii) and iv) are true with $\mathcal{C} = \mathcal{D}(H^{(\omega)}) = \mathcal{H}$. Since $[H^{(\omega)}, X_j]$ is bounded, iii) is fulfilled with $b(\omega) = 2d$ for $\mathbb{P}$-a.e. ω. Now, the usual geometric resolvent equation in $\ell^2(\mathbb{Z}^d)$ gives, for $\mathrm{Im}\, z \neq 0$, and for all $q, q' \in \mathbb{Z}^d$ such that $\|q - q'\|_\infty > 2L$,

$$\begin{aligned}&\langle \mathbf{1}_{q'} ,\, (H^{(\omega)} - z)^{-1}\mathbf{1}_q\rangle \\ &= \textstyle\sum_{u\in\Lambda, u'\notin\Lambda, \|u-u'\|_\infty=1} \langle \mathbf{1}_{q'} ,\, (H^{(\omega)}_{\Lambda_L(q')} - z)^{-1}\mathbf{1}_u\rangle\, \langle \mathbf{1}_{u'} ,\, (H^{(\omega)} - z)^{-1}\mathbf{1}_q\rangle \ ,\end{aligned} \tag{18}$$

where $H^{(\omega)}_{\Lambda_L(q')}$ is the operator $H^{(\omega)}$ restricted to the box $\Lambda_L(q')$ with Dirichlet boundary conditions. Thus (18) gives us the property v) for H if we define $D^{(\omega)}_{L,q'}(z)$ by its action on $\varphi \in \mathcal{H}$ according to

$$\left(D^{(\omega)}_{L,q'}(z)\varphi\right)(x) = \sum_{u\in\Lambda, u'\notin\Lambda, \|u-u'\|_\infty=1} \langle \mathbf{1}_{u'} ,\, \varphi\rangle (H^{(\omega)}_{\Lambda_L(q')} - z)^{-1}\mathbf{1}_u(x) \ .$$

Moreover, thanks to [29, Theorem 2.2 and Proposition A.11], there exists $0 < E_0 < \infty$ such that for $I = (-\infty, -E_0] \cup [E_0, +\infty)$ and for all q, q', $\|q - q'\|_\infty > 2L$, we have

$$\begin{aligned}\mathbb{P}\Big\{\omega|\ \forall E \in I,\ H^{(\omega)}\text{is } (\rho, E, L_k, q'')\text{−regular for either } q'' = q \text{ or } q'' = q'\Big\}& \\ \geq 1 - L_k^{-p}& \ ,\end{aligned}$$

with $p > 4d + 2$, $\rho(x) = \mathrm{e}^{-mx/2}$ for some fixed $m > 0$, $L_k = L_0^{(3/2)^k}$ for L_0 finite depending only on m and p. This gives exactly the Multi-Scale Assumption (M1). One can thus apply Theorem 2.1 with any compact subset I' of I.

Example 3: Anderson model with correlated potentials

We consider the Hamiltonian on $L^2(\mathbb{R}^d)$

$$H^{(\omega)} = -\Delta + \sum_{i\in\mathbb{Z}^d} \lambda_i(\omega)u(x - i) \ , \tag{19}$$

for which the random potentials $\lambda_i(\omega)u(x-i)$ at each site i are correlated. This model has been studied e.g. by [30, 9]. Correlated potentials means here that the coupling constant λ_i are not independently distributed. In this case, it has been proven that the bottom of the spectrum is dense pure point. Our goal here is to prove, with the help of a "variable-energy" multi-scale analysis, dynamical localization at the bottom of the spectrum. Let us first present the hypothesis for this model:

(H1) The site potential $u(x)$ is non-negative, not identically zero, compactly supported and $\|u\|_\infty < \infty$.

(H2) $\{\lambda_i\}_{i\in\mathbb{Z}^d}$ forms a stationary stochastic process of identically distributed random variables.

(H3) The conditional probability distribution of λ_0, given $\lambda_0^\perp = \{\lambda_i,\ i \neq 0\}$, is absolutely continuous with respect to Lebesgue measure with density h_0 which is compactly supported, $\|h_0\|_\infty < \infty$ and $\mathbb{E}(|\lambda_0|^{\max(2,d)}) < \infty$.

(H4) Let A be any given event on $\Lambda_L(0)$ (i.e. depending only on $\lambda_i(\omega)$, $i \in \Lambda_L(0) \cap \mathbb{Z}^d$). We denote by $A(x)$ the event A shifted to $\Lambda_L(x)$. For any given $\alpha > 1$ and $\beta > 1$, we assume that there exist $K_0(\alpha)$ even and $C(K_0) < \infty$ such that for all integer $K \geq K_0$, one can find $\theta(K, \alpha) > \beta$ having the property that for $L \gg 1$, $\forall x_1, x_2, \cdots, x_K \in \mathbb{Z}^d$ with $\|x_i - x_j\|_\infty \geq (1/2)L^\alpha$, $(i \neq j)$, and for all events A on $\Lambda_L(0)$,

$$\mathbb{P}\Big\{ \cap_{i=1}^K A(x_i) \Big\} \leq C\, \mathbb{P}(A)^\theta .$$

Existence of random processes satisfying (H2)-(H4) are given for example in [30]. Under these hypotheses, it is known that H_ω is an ergodic family, almost surely essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. This implies i), ii) and iv) of Assumption 2.1. Furthermore, for $\mathbb{P}$-a.e. ω, the potential $V^{(\omega)}$ is bounded uniformly in ω, and thus $[H^{(\omega)}, X_j]$ is relatively $H^{(\omega)}$-bounded, with relative bound $2(1 + \|V\|_\infty^2)^{1/2}$.Assumption 2.1 v) is simply the geometric resolvent equation, i.e., for $1 \leq L < \infty$ and q, q', such that $\|q - q'\|_\infty > 2L$ one has,

$$\mathbf{1}_{q'}(H^{(\omega)} - z)^{-1}\mathbf{1}_q \quad = \quad \mathbf{1}_{q'}(H^{(\omega)}_{\Lambda_L(q')} - z)^{-1}[H^{(\omega)}, \chi_{\Lambda_L(q')}](H_\omega - z)^{-1}\mathbf{1}_q \ , \tag{20}$$

where $\chi_{\Lambda_L(q')}$ is precisely defined in (59). Let $E_- \equiv \inf(\Sigma)$ denote the bottom of the almost sure spectrum of $H^{(\omega)}$. Following [8, 30], we know that for all $p > 0$, $m > 0$, S, N, and $L_0 < \infty$ there exists $E(L_0) > E_-$ such that with the notations of the Appendix and $I = [E_-, E(L_0)]$, we have the initial decay:

$$\mathbb{P}\left\{\forall E \in I,\ \exists (n_i')_{i=1,\ldots S} \text{ s.t. } \mathfrak{H}_{L_1}(E, y_1, m, (n_i')) \ \text{ or } \mathfrak{H}_{L_1}(E, y_2, m, (n_i'))\ \right\} \geq 1 - L_0^{-p}\ . \tag{21}$$

Thus, according to the estimate (65) and Lemmas A.1 and A.2 we take $\alpha = 3/2$ and $K_0 > 0$ such that $\theta(K_0, 3/2) > 3/2 + d - 1$; We also fix $S = 4$, $N = 4 + K_0$, $p = (K_0 + 1)/2$, $w = 2d + K_0/2 + 1$ and $m > 4w + 2(d - 1)$. Then we obtain, with the notations of the Appendix, for $L_k = L_0^{(\frac{3}{2})^k}$, and q, q' such that $\|q-q'\|_\infty > 2L_k$,

$$\mathbb{P}\Big\{\forall E \in I \text{ either for } \tilde{q}=q \text{ or } \tilde{q}=q',$$
$$\sup_{\varepsilon>0} \|[-\Delta,\, \chi_N^{L_k}(\tilde{q})](H_{\Lambda_{L_k(\tilde{q})}} - E - \imath\varepsilon)^{-1}\chi_0^{L_k}(\tilde{q})\| \le L_k^{-m}\Big\} \ge 1 - L_k^{-p} \; . \quad (22)$$

This inequality and (20) imply the Multi-Scale Assumption (8). Thus we can apply Theorem 2.1 to obtain dynamical localization at low energies for this model.

4 Proof of the Main results

The first two lemmas are the key results of this section. Together with Lemma 4.3 and Lemma 4.4, respectively, they imply Theorem 2.1 on dynamical localization and Theorem 2.2 on the absence of diffusion.

Lemma 4.1 *Let H satisfy Assumptions 2.1 i), v) and the Multi-Scale Assumption (M1). Then there is a constant $c_1 < \infty$ such that*

$$\mathbb{E}\left\{\sup_{\varepsilon\neq 0} \int_I \varepsilon \|\, |X|(H^{(\omega)} - E - \imath\varepsilon)^{-1}\varphi\|^2 dE\right\} \le c_1 \quad (23)$$

for all normalized $\varphi \in \mathcal{H}$ with compact support suppφ. *The constant c_1 depends only on the size of* suppφ.

Proof. Without loss of generality we assume that supp$\varphi \subseteq \{x \mid \|x\|_\infty < L_0\}$. The proof is done in three steps.

First we consider $\omega \in \Omega_k(q,q')$ with $\|q-q'\| > 2L_k$, where

$$\Omega_k(q,q') := \{\omega \mid H^{(\omega)} \text{ is } (\varrho, E, L_k, q'') - \text{regular for all } E \in I$$
$$\text{and either } q''=q \text{ or } q''=q'\}\,. \quad (24)$$

If $H^{(\omega)}$ is (ϱ, E, L_k, q')-regular, we apply (4) to get

$$\|\,\mathbf{1}_{q'}(H^{(\omega)} - E - \imath\varepsilon)^{-1}\,\mathbf{1}_q\,\|^2 \le \varrho(L_k)\|(H^{(\omega)} - E - \imath\varepsilon)^{-1}\,\mathbf{1}_q\,\|^2. \quad (25)$$

In case $H^{(\omega)}$ is (ϱ, E, L_k, q)-regular, we apply the adjoint of (4) to obtain the bound (25) with q replaced by q' and ε replaced by $-\varepsilon$ on the right-hand side of (25). In any case one has for all $\omega \in \Omega_k(q,q')$, with $\|q-q'\| > 2L_k$, the estimate

$$\|\,\mathbf{1}_{q'}(H^{(\omega)} - E - \imath\varepsilon)^{-1}\,\mathbf{1}_q\,\|^2$$
$$\le \varrho(L_k)\left(\|(H^{(\omega)} - E - \imath\varepsilon)^{-1}\,\mathbf{1}_q\,\|^2 + \|(H^{(\omega)} - E + \imath\varepsilon)^{-1}\,\mathbf{1}_{q'}\,\|^2\right)\,. \quad (26)$$

Second we derive the upper bound

$$\mathbb{E}\left\{\sup_{\varepsilon\neq 0}\int_I \varepsilon\|\,\mathbf{1}_{q'}(H^{(\omega)}-E-\imath\varepsilon)^{-1}\varphi\|^2 dE\right\}\leq 2\pi(2L_0+1)^d\varrho(L_k)+\pi L_k^{-p} \tag{27}$$

for all $q'\in\mathbb{Z}^d$ with $L_0+2L_k<\|q'\|_\infty\leq L_0+2L_{k+1}$. To this end we note that

$$\begin{aligned}\|\,\mathbf{1}_{q'}(H^{(\omega)}-E-\imath\varepsilon)^{-1}\varphi\|^2 &\leq \\ (2L_0+1)^d &\sum_{q\in\Lambda_{L_0}(0)\cap\mathbb{Z}^d}\|\,\mathbf{1}_{q'}(H^{(\omega)}-E-\imath\varepsilon)^{-1}\,\mathbf{1}_q\,\|^2\,\|\,\mathbf{1}_q\,\varphi\|^2\end{aligned} \tag{28}$$

where $\Lambda_{L_0}(0)$ is defined as in Section 3, and

$$\int_I \|\,(H^{(\omega)}-E-\imath\varepsilon)^{-1}\psi\|^2 dE\leq\frac{\pi}{\varepsilon} \tag{29}$$

for all $\psi\in\mathcal{H}$ with $\|\psi\|=1$. For $\omega\in\Omega_k(q,q')$ we use (26), (28) and (29) to bound the supremum on the left-hand side of (27). If $\omega\notin\Omega_k(q,q')$ we use (29) and the Multi-Scale Assumption (M1) for this purpose. This yields (27).

Finally we derive (23). The norm on the left-hand side can be estimated according to

$$\begin{aligned}\|\,|X|(H-E-\imath\varepsilon)^{-1}\varphi\|^2 &\leq \sum_{q'\in\mathbb{Z}^d}\|\,|X|\,\mathbf{1}_{q'}\,\|^2\|\,\mathbf{1}_{q'}(H^{(\omega)}-E-\imath\varepsilon)^{-1}\varphi\|^2\\ &\leq 4d(2L_0+1)^2\|(H^{(\omega)}-E-\imath\varepsilon)^{-1}\varphi\|^2+\\ \sum_{k=0}^{\infty}\sum_{q'\in\mathbb{Z}^d,L_0+2L_k<\|q'\|_\infty\leq L_0+2L_{k+1}} & d\,(L_0+2L_{k+1}+1)^2\|\,\mathbf{1}_{q'}(H^{(\omega)}-E-\imath\varepsilon)^{-1}\varphi\|^2\,.\end{aligned} \tag{30}$$

Monotone convergence, (27) and (29) yield

$$\begin{aligned}&\mathbb{E}\left\{\sup_{\varepsilon\neq 0}\int_I \varepsilon\|\,|X|(H-E-\imath\varepsilon)^{-1}\varphi\|^2 dE\right\}\leq 4\pi d\,(2L_0+1)^2\\ &+2^d d^2\pi\sum_{k=0}^{\infty}(2L_{k+1}-2L_k+2)(L_0+2L_{k+1}+1)^{d+1}\Big(2(2L_0+1)^d\varrho(L_k)+L_k^{-p}\Big)\,.\end{aligned} \tag{31}$$

Thanks to (7) the above sum is finite. ▮

Lemma 4.2 *Let H satisfy Assumptions 2.1 i), v) and (M2), then for $\varphi\in\mathcal{H}$, normalized and compactly supported, there exist $c_2<\infty$ and $\upsilon>0$ such that for all $0<\varepsilon<1$*

$$\mathbb{E}\left\{\varepsilon^2\int_I \|\,|X|(H^{(\omega)}-E-\imath\varepsilon)^{-1}\varphi\|^2 dE\right\}<\varepsilon^{\upsilon}\ .$$

Proof. We assume here, without loss of generality, that $\mathrm{supp}(\varphi) \subset \{x \mid \|x\|_\infty \le L_0\}$; then we obtain, for $E \in I$, $\varepsilon > 0$ and $J_0 \in \mathbb{N}$

$$\begin{aligned} & \| |X| (H^{(\omega)} - E - \imath\varepsilon)^{-1} \varphi \|^2 \\ \le \ & (2L_0+1)^2 \sum_{k \ge J_0} \sum_{q \in \mathrm{supp}(\varphi) \cap \mathbb{Z}^d} \sum_{q' \in \mathbb{Z}^d, L_0 + 2L_k < \|q'\|_\infty \le L_0 + 2L_{k+1}} \\ & \left\{ \| |X| \mathbf{1}_{q'} \|_\infty^2 \| \mathbf{1}_{q'} (H^{(\omega)} - E - \imath\varepsilon)^{-1} \mathbf{1}_q \|^2 \| \mathbf{1}_q \varphi \|^2 \right\} \\ & + d(L_{J_0} + L_0 + 1)^2 \| (H^{(\omega)} - E - \imath\varepsilon)^{-1} \varphi \|^2 \ . \end{aligned} \tag{32}$$

For fixed E and q we define

$$\Omega(E,q) = \{\omega \mid H^{(\omega)} \text{ is } (\rho, E, L_k, q)\text{−regular}\} \ .$$

Thus, according to Assumption (M2), we get

$$\begin{aligned} & \varepsilon^2 \int \left(\int_I \| |X| (H^{(\omega)} - E - \imath\varepsilon)^{-1} \varphi \|^2 dE \right) d\mathbb{P}(\omega) \\ \le \ & (2L_0+1)^2 \sum_{k \ge J_0} \sum_{q \in \mathrm{supp}(\varphi) \cap \mathbb{Z}^d} \sum_{q',\, L_0 + 2L_k \le \|q'\|_\infty \le L_0 + 2L_{k+1}} \\ & \left\{ \int_I \left(\int_{\Omega(E,q)} \frac{d(L_0 + 2L_{k+1} + 1)^2}{L_k^m} \varepsilon^2 \| (H^{(\omega)} - E - \imath\varepsilon)^{-1} \|^2 d\mathbb{P}(\omega) \right. \right. \\ & \left. \left. + \int_{\Omega^c(E,q)} d(L_0 + 2L_{k+1} + 1)^2 \varepsilon^2 \| \mathbf{1}_{q'} (H^{(\omega)} - E - \imath\varepsilon)^{-1} \|^2 d\mathbb{P}(\omega) \right) dE \right\} \end{aligned} \tag{33}$$

$$+ d(L_0 + 2L_{J_0} + 1)^2 \int \left(\varepsilon^2 \int_I \| (H^{(\omega)} - E - \imath\varepsilon)^{-1} \varphi \|^2 dE \right) d\mathbb{P}(\omega) \ . \tag{34}$$

By using the inequality $\| (H^{(\omega)} - E - \imath\varepsilon)^{-1} \| \le \varepsilon^{-1}$ in (33) and inequality (29) in (34), we derive the upper bound

$$\varepsilon^2 \int \left(\int_I \| |X| (H^{(\omega)} - E - \imath\varepsilon)^{-1} \varphi \|^2 dE \right) d\mathbb{P}(\omega) \le C(\varepsilon L_{J_0}^2 + |I| L_{J_0}^{-\beta}) \ .$$

Here C is a constant depending only on L_0 and d. This gives the expected result for an appropriate dependence of J_0 on ε: for $K \in \mathbb{N}$ such that $2\alpha^{-K} < 1$ and if $L_J < \varepsilon^{-1} \le L_{J+1}$, we take $J_0 = J - K$. We thus obtain $\upsilon = \min\{\beta\alpha^{-K-1}, 1 - 2\alpha^{-K}\}$. ∎

The next lemma guarantees that $\langle\langle |X|^2 \rangle\rangle_{T,\varphi}$ is well-defined for all $T > 1$ and for φ as in Theorem 2.1. It extends the results of Radin and Simon [24] to a larger class of operators H, although the set S we consider is slightly different.

Lemma 4.3 *Let $H^{(\omega)}$ be a random operator satisfying Assumptions 2.1 i)-iv) and let*

$$S \equiv \{\varphi \in \mathcal{H},\ \| |X| \varphi \| < \infty,\ \| H^{(\omega)} \varphi \| < \infty\} \ . \tag{35}$$

Then for $\mathbb{P}$-a.e. ω, $e^{-\imath tH^{(\omega)}}$ maps S into S, and there exists a finite constant $c^{(\omega)}$ such that for all $t\in\mathbb{R}$, $d=1,\ldots,j$, $\varphi\in S$ and $\psi\in\mathcal{H}$, one has

$$\langle\psi,\,[\mathrm{e}^{-\imath tH^{(\omega)}},X_j]\varphi\rangle=-\imath\int_0^t\langle\psi,\,\mathrm{e}^{-\imath(t-s)H^{(\omega)}}[H^{(\omega)},X_j]\mathrm{e}^{-\imath sH^{(\omega)}}\varphi\rangle\,ds\ , \tag{36}$$

and

$$\|\,|X|\mathrm{e}^{-\imath tH^{(\omega)}}\varphi\|\leq c^{(\omega)}|t|\left(\|\varphi\|+\|H^{(\omega)}\varphi\|\right)+\|\,|X|\varphi\|\ . \tag{37}$$

Remark 4.1 Strictly speaking, the set S depends on ω, but due to Assumption 2.1 i), S is $\mathbb{P}$-almost surely constant.

Proof. Since there is no possible confusion, we drop the superscript ω. Fix $j\in\{1,\ldots,d\}$ and let

$$F_\varepsilon=(H+\imath)\frac{1}{\imath+\varepsilon X_j}(H+\imath)^{-1}\ .$$

By Assumption 2.1 ii) and since $\mathcal{C}$ is a core for H, the domain of F_ε contains the dense subset $\mathcal{B}\equiv(H+\imath)\mathcal{C}$. Now on B, one can write

$$F_\varepsilon\equiv-\frac{\varepsilon}{\imath+\varepsilon X_j}[H,X_j](H+\imath)^{-1}F_\varepsilon+\frac{1}{\imath+\varepsilon X_j}\ . \tag{38}$$

Thus for $\varepsilon<\varepsilon_1$ small enough, one has

$$F_\varepsilon=\left(1+\frac{\varepsilon}{\imath+\varepsilon X_j}[H,X_j](H+\imath)^{-1}\right)^{-1}\frac{1}{\imath+\varepsilon X_j}\ ,$$

which implies that F_ε is uniformly bounded in ε. From (38) one obtains

$$(\imath+\varepsilon X_j)F_\varepsilon=1-\varepsilon[H,X_j](H+\imath)^{-1}F_\varepsilon\ ,$$

which is an operator uniformly bounded in ε for all $\varepsilon<\varepsilon_2$ small enough. Hence, the operator

$$(H+\imath)\frac{X_j}{\imath+\varepsilon X_j}(H+\imath)^{-1}\quad=\quad[H,X_j](H+\imath)^{-1}F_\varepsilon+X_jF_\varepsilon \tag{39}$$

is bounded. Now one has

$$\frac{X_j}{\imath+\varepsilon X_j}\mathrm{e}^{-\imath tH}=\mathrm{e}^{-\imath tH}\left(\mathrm{e}^{\imath tH}\frac{X_j}{\imath+\varepsilon X_j}\mathrm{e}^{-\imath tH}-\frac{X_j}{\imath+\varepsilon X_j}\right)+\mathrm{e}^{-\imath tH}\frac{X_j}{\imath+\varepsilon X_j}\ . \tag{40}$$

From (39), since $X_j/(\imath+\varepsilon X_j)$ maps $\mathcal{D}(H)$ into $\mathcal{D}(H)$, one obtains for the term in parenthesis in the right hand side of (40), in the weak sense

$$\mathrm{e}^{\imath tH}\frac{X_j}{\imath+\varepsilon X_j}\mathrm{e}^{-\imath tH}-\frac{X_j}{\imath+\varepsilon X_j}=\imath\int_0^t\mathrm{e}^{\imath sH}[H,\frac{X_j}{\imath+\varepsilon X_j}]\mathrm{e}^{-\imath sH}\,ds\ . \tag{41}$$

Furthermore, by writing

$$[H, \frac{X_j}{\imath + \varepsilon X_j}] = [H, X_j]\frac{1}{\imath + \varepsilon X_j} + X_j[H, \frac{1}{\imath + \varepsilon X_j}] \ ,$$

and

$$\begin{aligned} & X_j[H, \frac{1}{\imath + \varepsilon X_j}] \\ = \ & X_j \left(\frac{1}{\imath + \varepsilon X_j}(\imath + \varepsilon X_j)H\frac{1}{\imath + \varepsilon X_j} - \frac{1}{\imath + \varepsilon X_j}H(\imath + \varepsilon X_j)\frac{1}{\imath + \varepsilon X_j} \right) \\ = \ & X_j \frac{1}{\imath + \varepsilon X_j}[\varepsilon X_j, H]\frac{1}{\imath + \varepsilon X_j} \ , \end{aligned}$$

one obtains, together with (40) and (41)

$$\begin{aligned} \frac{X_j}{\imath + \varepsilon X_j}\mathrm{e}^{-\imath tH} \ = \ & \imath\, \mathrm{e}^{-\imath tH} \int_0^t \mathrm{e}^{\imath sH}[H, X_j](H+\imath)^{-1}(H+\imath)\frac{1}{\imath + \varepsilon X_j}\mathrm{e}^{-\imath sH}\, ds \\ & + \imath \int_0^t e^{\imath sH} X_j \frac{\varepsilon}{\imath + \varepsilon X_j}[X_j, H](H+\imath)^{-1}(H+\imath)\frac{1}{\imath + \varepsilon X_j}e^{-\imath sH} ds \\ & + \mathrm{e}^{-\imath tH}\frac{X_j}{\imath + \varepsilon X_j} \ . \end{aligned}$$

Then for all $\varphi \in \mathcal{D}(H)$,

$$\begin{aligned} \|\frac{X_j}{\imath + \varepsilon X_j}\mathrm{e}^{-\imath tH}\varphi\| \ \leq \ & |t|\, \|[H, X_j](H+\imath)^{-1}\|\, \|F_\varepsilon\|\, \|(H+\imath)\varphi\| \\ & + |t|\, \|\frac{\varepsilon X_j}{\imath + \varepsilon X_j}\|\, \|[H, X_j](H+\imath)^{-1}\|\, \|F_\varepsilon\|\, \|(H+\imath)\varphi\| \\ & + \|\frac{X_j}{\imath + \varepsilon X_j}\varphi\| \\ \leq \ & c|t| \left(\|H\varphi\| + \|\varphi\|\right) + \|\frac{X_j}{\imath + \varepsilon X_j}\varphi\| \ . \end{aligned}$$

The constant c is independent of ε. Taking the limit $\varepsilon \to 0$, one gets for all $\varphi \in S$

$$\|X_j \mathrm{e}^{-\imath tH}\varphi\| \leq c|t| \left(\|H\varphi\| + \|\varphi\|\right) + \|X_j\varphi\| \ .$$

This proves (37). Now, since we know that $\mathrm{e}^{-\imath tH}$ maps S into S, one has for all φ and ψ in S:

$$\langle \mathrm{e}^{\imath tH} X_j \mathrm{e}^{-\imath tH}\varphi, \psi\rangle \ = \ \langle X_j\varphi, \psi\rangle + \imath \int_0^t \langle \mathrm{e}^{\imath sH}[H, X_j]\mathrm{e}^{-\imath sH}\varphi, \psi\rangle\, ds \ . \quad (42)$$

Since the integrand in (42) is uniformly bounded in $s \in [0, t]$ and since S is dense in $\mathcal{H}$, one can extend (42) to all $\psi \in \mathcal{H}$, which gives (36). ▮

The following lemma exhibits the relation between the asymptotic behavior in T of the second moment of position $\langle\langle |X|^2\rangle\rangle_{T,\varphi}$, for φ localized in energy in a compact set I, and the behavior of the resolvents $(H-E-\imath\varepsilon)^{-1}$ for energies $E+\imath\varepsilon$ approaching the real axis. In particular, it shows that the main contribution to the second moment of position is due to energies E in I. It is an easy generalization of a result of Montcho [21] stated for operators $H=-\Delta+V$, where V is bounded below (See also [15] for related results in the case of random potentials which are piecewise constant).

Lemma 4.4 *Let H be a random operator satisfying Assumptions 2.1 i)-iv). Let $f\in C_0^\infty(\mathbb{R})$ be a non-negative function and $I\supset \mathrm{supp}\ f$ any compact interval such that $\delta\equiv \mathrm{dist}(\partial I, \mathrm{supp}\ f)>0$; then for all $\varphi\in\mathcal{D}(H)\cap\mathcal{D}(|X|)$ there exist constants c_1,c_2 and c_3 depending only on φ, δ and $\|f\|_{L^\infty(\mathbb{R})}$ such that for all $\varepsilon\equiv 1/T>0$ we have, for $\mathbb{P}$ a.e. ω:*

$$\begin{aligned}&\frac{1}{T}\int_0^T \|\,|X|\mathrm{e}^{-\imath tH^{(\omega)}}f(H^{(\omega)})\varphi\|^2dt\\ &\qquad\le c_1+c_2b^2(\omega)+c_3\varepsilon\int_I \|\,|X|(H^{(\omega)}-E+\imath\frac{\varepsilon}{2})^{-1}\varphi\|^2dE\quad . \qquad (43)\end{aligned}$$

The random variable $b(\omega)$ was defined in Assumption 2.1 iii).

Proof. We will drop the label ω in the proof. Since H is self-adjoint and $|X|$ is closed, we have for $\varepsilon=1/T$ [25, Lemma 1 on p. 412]

$$\begin{aligned}\frac{1}{T}\int_0^T \|\,|X|\mathrm{e}^{-\imath tH}f(H)\varphi\|^2dt &\le \mathrm{e}\,\varepsilon\int_{\mathbb{R}}\|\,|X|\mathrm{e}^{-\varepsilon\frac{t}{2}}\Theta(t)\mathrm{e}^{-\imath tH}f(H)\varphi\|^2dt\\ &= \frac{\mathrm{e}\,\varepsilon}{2\pi}\int_{\mathbb{R}}\|\,|X|R_\varepsilon(E)f(H)\varphi\|^2dE\ , \qquad (44)\end{aligned}$$

where $R_\varepsilon(E)=(H-E+\imath\varepsilon/2)^{-1}$ and $\Theta(t)$ is the Heaviside function. In order to bound the right-hand side of (44), we split the range of integration over E into two parts and fix $j\in\{1,\ldots,d\}$

$$\begin{aligned}&\int_{\mathbb{R}\setminus I}\|X_jR_\varepsilon(E)f(H)\varphi\|^2dE\\ &\le\ 4\Big(\int_{\mathbb{R}\setminus I}\|R_\varepsilon(E)f(H)X_j\varphi\|^2dE \qquad (45)\\ &\quad+\int_{\mathbb{R}\setminus I}\|[X_j,f^{\frac12}(H)]f^{\frac12}(H)R_\varepsilon(E)\varphi\|^2dE \qquad (46)\\ &\quad+\int_{\mathbb{R}\setminus I}\|f^{\frac12}(H)R_\varepsilon(E)[H,X_j]R_\varepsilon(E)f^{\frac12}(H)\varphi\|^2dE \qquad (47)\\ &\quad+\int_{\mathbb{R}\setminus I}\|f^{\frac14}(H)R_\varepsilon(E)f^{\frac14}(H)[X_j,f^{\frac12}(H)]\varphi\|^2dE\Big)\ . \qquad (48)\end{aligned}$$

Since $\mathrm{dist}(\partial I, \mathrm{supp}\ f) > 0$, one can bound the term (45) from above by

$$\int_{\mathbb{R}\setminus I} \|R_\varepsilon(E) f(H)\|^2 \|X_j\varphi\|^2 dE \quad \leq \quad \frac{2\|f\|_\infty^2}{\delta} \|X_j\varphi\|^2 \ . \tag{49}$$

We now treat the term (46). Let $\tilde{\varphi}_E \equiv R_\varepsilon(E)\varphi$; then obviously, since φ is in $\mathcal{D}(H)$, $\|\tilde{\varphi}_E\|$ and $\|H\tilde{\varphi}_E\|$ are finite and from Lemma 4.3 one has, by using the equality $R_\varepsilon(E) = -\imath \int_0^\infty \mathrm{e}^{-\varepsilon t/2} \mathrm{e}^{\imath t(H-E)} dt$

$$\begin{aligned} \|\,|X|\tilde{\varphi}_E\| \quad &\leq \quad \int_0^{+\infty} \mathrm{e}^{-\frac{\varepsilon t}{2}} \|\,|X| \mathrm{e}^{\imath t(H-E)} \varphi\| dt \\ &\leq \quad c \int_0^{+\infty} \mathrm{e}^{-\frac{\varepsilon t}{2}} \Big(t(\|\varphi\| + \|H\varphi\|) + \|\,|X|\varphi\|\Big) \\ &< \quad \infty \ . \end{aligned} \tag{50}$$

Therefore $\tilde{\varphi}_E$ is in the set S defined in Lemma 4.3. By writing $f^{1/2}(H) = \int_{\mathbb{R}} \widehat{f^{1/2}}(t) \mathrm{e}^{\imath t H} dt$, one can prove with the same arguments as above that $f^{1/2}(H) R_\varepsilon(E)\varphi \in S$. Then we get from Lemma 4.3

$$\begin{aligned} &\|[f^{\frac{1}{2}}(H), X_j] f^{\frac{1}{2}} R_\varepsilon(E)\varphi\| \\ &\leq \quad \sup_{\xi\in\mathcal{H}, \|\xi\|=1} \int_{\mathbb{R}} \int_0^t |\widehat{f^{\frac{1}{2}}}(t)|\, |\langle \xi, \mathrm{e}^{-\imath(s-t)H} [H, X_j] \mathrm{e}^{\imath s H} f^{\frac{1}{2}}(H) R_\varepsilon(E)\varphi\rangle| ds\, dt \\ &\leq \quad \int_{\mathbb{R}} t|\widehat{f^{\frac{1}{2}}}(t)|\, \|[H, X_j](H+\imath)^{-1}\|\, \|(H+\imath) f^{\frac{1}{4}}(H)\|\, \|R_\varepsilon(E) f^{\frac{1}{4}}(H)\varphi\| dt \ , \end{aligned}$$

and, using Assumption 2.1 iii), (46) is bounded above by

$$\begin{aligned} &\int_{\mathbb{R}\setminus I} \|[f^{\frac{1}{2}}(H), X_j] f^{\frac{1}{2}} R_\varepsilon(E)\varphi\|^2 dE \\ &\quad \leq \quad b^2 (\max_{E\in I} |E| + 1)^2 \Big(\int_{\mathbb{R}} t|\widehat{f^{\frac{1}{2}}}(t)| dt \Big)^2 \ \frac{2\|f\|_\infty}{\delta} \ . \end{aligned} \tag{51}$$

To bound the term (47) one writes

$$\begin{aligned} &\int_{\mathbb{R}\setminus I} \|f^{\frac{1}{2}}(H) R_\varepsilon(E) [H, X_j] R_\varepsilon(E) f^{\frac{1}{2}}(H)\varphi\|^2 dE \\ &\quad \leq \quad \int_{\mathbb{R}\setminus I} \Big(\|f^{\frac{1}{2}}(H) R_\varepsilon(E)\|^2 \|[H, X_j](H+\imath)^{-1}\|^2 \\ &\qquad\qquad \times \|(H+\imath) f^{\frac{1}{4}}(H)\|^2 \|f^{\frac{1}{4}}(H) R_\varepsilon(E)\|^2 \|\varphi\|^2 dE \Big) \\ &\quad \leq \quad \frac{2b^2}{3\delta^3} \|f\|_\infty^2 (\max_{E\in I} |E| + 1)^2 \|\varphi\|^2 \ . \end{aligned} \tag{52}$$

Finally, for (48) one has

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus I} \|f^{\frac14}(H)R_\varepsilon(E)f^{\frac14}(H)[X_j, f^{\frac12}(H)]\varphi\|^2 dE \\
&\quad\leq\quad \Big(\int_{\mathbb{R}\setminus I} f^{\frac14}(H)R_\varepsilon(E)\|^2 dE\Big)\Big(\sup_{\xi\in\mathcal{H},\|\xi\|=1} |\langle f^{\frac14}(H)\xi,\, [f^{\frac12}(H), X_j]\|\rangle|\Big)^2 \\
&\quad\leq\quad \frac{2\|f\|_\infty^{\frac12}}{\delta} \sup_{\xi\in\mathcal{H}} \Big(|\int_{\mathbb{R}} \widehat{f^{\frac12}}(t)\langle f^{\frac14}(H)\xi,\, [\mathrm{e}^{\imath tH}, X_j]\varphi\rangle dt|\Big)^2 \\
&\quad\leq\quad \frac{2\|f\|_\infty^{\frac12}}{\delta}\Big(\sup_{\xi\in\mathcal{H}} \int_{\mathbb{R}} |\widehat{f^{\frac12}}(t)| \int_0^t |\langle f^{\frac14}(H)\xi,\, \mathrm{e}^{-\imath(s-t)H}[H, X_j]\mathrm{e}^{\imath sH}\varphi\rangle| ds\, dt\Big)^2 \\
&\quad\leq\quad \frac{2b^2\|f\|_\infty}{\delta} \max_{E\in I}(|E|+1)^2\|\varphi\|^2\Big(\int_{\mathbb{R}} t|\widehat{f^{\frac12}}(t)|\Big)^2 . \qquad (53)
\end{aligned}
$$

Inequalities (49)-(53) imply that there exists $c<\infty$ independent of ω such that

$$
\int_{\mathbb{R}\setminus I} \|\,|X|\mathrm{e}^{-\imath tH} f(H)\varphi\|^2 dE \quad\leq\quad c\, b^2 . \qquad (54)
$$

As to the remaining contribution to (44), we write

$$
\begin{aligned}
&\varepsilon\int_I \|X_j R_\varepsilon(E) f(H)\varphi\|^2 dE \\
&\quad\leq\quad 2\varepsilon\int_I \|[X_j, f(H)]R_\varepsilon(E)\varphi\|^2 dE + 2\varepsilon\int_I \|X_j R_\varepsilon(E)\varphi\|^2 dE . \qquad (55)
\end{aligned}
$$

The first term in (55) is bounded from above by

$$
\begin{aligned}
&4\varepsilon\int_I \|[X_j, f^{\frac12}(H)]f^{\frac12}(H)R_\varepsilon(E)\varphi\|^2 dE \\
&\qquad\qquad + 4\varepsilon\int_I \|f^{\frac12}(H)[X_j, f^{\frac12}(H)]R_\varepsilon(E)\varphi\|^2 dE \\
&\qquad \leq 4\varepsilon\int_I\int_{\mathbb{R}} |\widehat{f^{\frac12}}(t)|\, \|[X_j, \mathrm{e}^{\imath tH}]f^{\frac12}(H)R_\varepsilon(E)\varphi\|^2 dt\, dE \\
&\qquad\qquad + 4\varepsilon\int_I\int_{\mathbb{R}} |\widehat{f^{\frac12}}(t)|\, \|f^{\frac12}(H)[X_j, \mathrm{e}^{\imath tH}]R_\varepsilon(E)\varphi\|^2 dt\, dE . \qquad (56)
\end{aligned}
$$

According to the fact that $R_\varepsilon(E)\varphi$ and $f^{\frac12}(H)R_\varepsilon(E)\varphi$ are both in the set S defined in (35), one can prove as above with the help of Lemma 4.3 that each of the two terms in (56) are bounded above by

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}} t|\widehat{f^{\frac12}}(t)|dt\Big)^2 \|[H, X_j](H+\imath)^{-1}\|^2 \\
&\qquad\qquad \times\, \|(H+\imath)f^{\frac12}(H)\|^2 \varepsilon\int_I \|R_\varepsilon(E)\varphi\|^2 dE \qquad (57) \\
&\quad\leq\quad \pi\Big(\int_{\mathbb{R}} t|\widehat{f^{\frac12}}(t)|dt\Big)^2 b^2(\max_{E\in I}|E|+1)^2\|f\|_\infty . \qquad (58)
\end{aligned}
$$

This inequality together with (54) and (55) implies (43). ∎

Remark 4.2 Although stated for random operators, those last two lemmas are purely deterministic, in the sense that their statements remain true if we consider a non-random H satisfying Assumptions 2.1 i)-iv).

A Appendix

We present here the two lemmas necessary for the "variable-energy" multi-scale analysis needed in Example 3. For the rest of this section, we fix $\alpha > 1$, $\ell < \infty$, $N > 4$ and S even, $2 < S < N-1$. For $L = N\ell^\alpha$ we define for all $n \in \{0, 1, \cdots, N\}$

$$\chi_n^L(x) = \begin{cases} J_{\Lambda_{L/(4N)}(x),0} & n = 0 \ , \\ J_{\Lambda_{nL/N}(x), \frac{(L/N)^{1/\alpha}}{4N}} & 1 \leq n \leq N-1 \ , \\ J_{\Lambda_L(x), L/(4N)} & n = N \ , \end{cases} \tag{59}$$

where, for $\delta > 0$, $J_{\Lambda,\delta}$ is a smooth characteristic function such that, for $\partial\Lambda$ being the boundary of the box Λ,

$$J_{\Lambda,\delta}(y) = \begin{cases} 1 & \text{if } y \in \Lambda \text{ and } \mathrm{dist}(y, \partial\Lambda) < \delta \ , \\ 0 & \text{if } y \notin \Lambda \ , \end{cases} \tag{60}$$

and for $\delta = 0$, $J_{\Lambda,\delta}$ is the characteristic function of Λ. One also defines the frame $\mathfrak{F}_L(x,n)$ of the box $\Lambda_{\frac{nL}{N}}(x)$ as the set of $y \in \Lambda_{\frac{nL}{N}}(x) \cap (\frac{\ell}{4N}\mathbb{Z})^d$ such that $nL/N - 2\ell/(4N) < \|x-y\|_\infty \leq nL/N - \ell/(4N)$. Thus $\mathrm{supp}\nabla\chi_n^L(x) \subset \cup_{y \in \mathfrak{F}_L(x,n)} \Lambda_{\ell/(4N)}(y)$. Note that $\mathrm{Card}(\mathfrak{F}) \leq c_{d,N} L^{(1-1/\alpha)(d-1)}$, $c_{d,N}$ being a constant depending only on d and N. Now, for $w, m > 0$ and $n_i \in \{0, 1, \ldots, N\}$, $i \in \{1, \ldots, S\}$, we define $\mathfrak{W}_L(E, x, w, (n_i)_{i=1,\ldots,S})$ to be the property: $\forall i = 1, \ldots, S$,

$$\sup_{\varepsilon > 0} \|(H_{\Lambda_{\frac{n_i L}{N}}(x)} - E - \imath\varepsilon)^{-1}\| \leq L^w \ ,$$

and $\mathfrak{H}_L(E, x, m, (n_i)_{i=1,\ldots,S})$ as the property: $\forall i = 1, \ldots, S$, $\forall y \in \mathfrak{F}_L(x, n_i)$,

$$\sup_{\varepsilon > 0} \|[-\Delta, \chi_N^\ell(y)](H_{\Lambda_\ell(y)} - E - \imath\varepsilon)^{-1}\chi_0^\ell(y)\| \leq \ell^{-m} \ .$$

Here H_Λ refers to H restricted to Λ with Dirichlet boundary conditions. We call a box $\Lambda_\ell(y)$ having this last property an (m, E)-good box.

Lemma A.1 *Deterministic estimates*
Given $x \in \mathbb{Z}^d$, $L < \infty$ and $E \in \mathbb{R}$, if $(S-\alpha)m > \alpha(S+1)w + S(d-1)(\alpha-1)$ and if there are S integers $(n_i)_{i=1,\ldots,S}$, $1 \leq n_1 < n_2 < \cdots < n_S \leq N-1$ such that $\mathfrak{W}_L(E, x, w, (n_i))$ and $\mathfrak{H}_L(E, x, m, (n_i))$ hold, then

$$\sup_{\varepsilon > 0} \|[-\Delta, \chi_N^L(x)](H_{\Lambda_L(x)} - E - \imath\varepsilon)^{-1}\chi_0^L(x)\| \leq L^{-m} \ .$$

The proof of the deterministic estimates is now well-known (see e.g. [30] for discrete models and [12] for continuous models). For self-consistency, we give here the most important steps of the proof of Lemma A.1.

Proof. Let $W_{L,n,x} = [-\Delta, \chi_n^L(x)]$, $(n \neq 0)$ and $R_\Lambda = (H_\Lambda - E - \imath\varepsilon)^{-1}$. We first apply S times the geometric resolvent equation to $W_{L,N,x} R_{\Lambda_L(x)} \chi_0^L(x)$ with the increasing sequence of boxes $\Lambda_{n_i L/N}(x)$, $i = 1, \ldots, S$,

$$\begin{aligned} & W_{L,N,x} R_{\Lambda_L(x)} \chi_0^L(x) \\ & \quad = W_{L,N,x} R_{\Lambda_{\frac{n_S L}{N}}(x)} \chi_{n_S}^{\partial}(x) W_{L,n_S,x} R_{\Lambda_{\frac{n_{S-1} L}{N}}(x)} \chi_{n_{S-1}}^{\delta} \\ & \qquad \ldots W_{L,n_2,x} R_{\Lambda_{\frac{n_1 L}{N}}(x)} \chi_{n_1}^{\partial}(x) W_{L,n_1,x} R_{\Lambda_{\frac{n_1 L}{N}}(x)} \chi_0^L(x) \ , \end{aligned} \tag{61}$$

where $\chi_{n_i}^{\partial}(x)$ is the characteristic function of the support of $\nabla \chi_{n_i}^L(x)$. Now, again with the help of the geometric resolvent equation, observe that for $j = 1, \ldots S$

$$\begin{aligned} & \| W_{L,n_j,x} R_{\Lambda_{\frac{n_j L}{N}}(x)} \chi_{n_{j-1}}^{\partial} \| \\ & \quad \leq \sum_{y \in \mathfrak{F}_L(x, n_{j-1})} \| W_{L,n_j,x} R_{\Lambda_{\frac{n_j L}{N}}(x)} \chi_0^{\ell}(y) \| \\ & \quad \leq \sum_{y \in \mathfrak{F}_L(x, n_{j-1})} \| W_{L,n_j,x} R_{\Lambda_{\frac{n_j L}{N}}(x)} \chi_N^{\ell}(y) \| \, \| W_{\ell,N,y} R_{\Lambda_\ell}(y) \chi_0^{\ell}(y) \| \ . \end{aligned} \tag{62}$$

In each term of this last sum, the first factor is estimated by using $\mathfrak{W}_L(E, x, w, (n_i))$, and the second factor by using $\mathfrak{H}_L(E, x, m, (n_i))$. Thus, (61) together with (62) gives

$$\begin{aligned} \sup_{\varepsilon > 0} \| W_{L,N,x} R_{\Lambda_L(x)} \chi_0^L \| & \leq (\tilde{c}\, c_{d,N})^S \tilde{c} \big(L^{(1-\frac{1}{\alpha})(d-1)} L^w \ell^{-m} \big)^s L^w \\ & \leq L^{-m} \end{aligned}$$

for L large enough. Remark that here we have used the following inequality (see e.g. [8, Appendix 1] or [12]), for all L', z and $n \in \{1, \ldots, N-1\}$:

$$\| W_{L',n,z} R_{\Lambda_{L'}}(z) \chi_0^{L'} \| \leq \tilde{c} \| R_{\Lambda_{L'}}(z) \| \ .$$

∎

We now state the probabilistic part of this analysis. Since (8) requires a result which is uniform in energy, we must do a "variable-energy" multi-scale analysis like in [29], adapted to correlated potentials. One of the key estimates we use here is a correlated Wegner estimate proven in [9]:

Proposition A.1 *Combes-Hislop-Mourre correlated Wegner estimate*
We assume (H1) - (H4) for the model (19)*. Then there exist* $I = [\alpha, \beta]$*, a compact interval in the almost-sure spectrum of* $H^{(\omega)}$*,* $\ell_0 < \infty$ *and* $C_W > 0$ *such that for* $E \in I$*, for any bounded open cubes* Λ_1*,* Λ_2 *with* $\mathrm{dist}(\Lambda_1, \Lambda_2) \geq \ell_0$ *and for all* $0 < \eta < 1$

$$\mathbb{E}\big(\,\mathrm{Tr}\,(\mathbf{E}_1[E-\eta, E+\eta])\,\mathrm{Tr}\,(\mathbf{E}_2[E-\eta, E+\eta])\,\big) \leq C_W \eta^2 |\Lambda_1||\Lambda_2| \;, \tag{63}$$

where $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *are, respectively, the spectral families associated to* H_{Λ_1} *and* H_{Λ_2}*.*

We can then establish the following lemma:

Lemma A.2 *Probabilistic estimates*
With the same notations as before, if there exist $L_0 < \infty$ *and* $p > 0$ *such that* $(\alpha-1)(d-1)(N-S)/(\theta(N-S,\alpha)-\alpha) < p < w - 2d$*, and for all* x_1*,* x_2*,* $\|x_1 - x_2\|_\infty > 2L_0$*,*

$$\mathbb{P}\Big\{\forall E \in I,\ \exists (n_i)_{i=1,\dots S}\ \text{s.t.}\ \big(\mathfrak{W}_{L_0}(E, x_1, w, (n_i)) \text{ and } \mathfrak{H}_{L_0}(E, x_1, m, (n_i))\ \big)$$
$$\text{or}\ \big(\mathfrak{W}_{L_0}(E, x_2, w, (n_i)) \text{ and } \mathfrak{H}_{L_0}(E, x_2, m, (n_i))\ \big)\Big\} \geq 1 - L_0^{-p}\ ,$$

then for $L_1 = N L_0^\alpha$ *and for all* y_1*,* y_2*,* $\|y_1 - y_2\|_\infty > 2L_1$*, we have:*

$$\mathbb{P}\Big\{\forall E \in I,\ \exists (n_i')_{i=1,\dots S}\ \text{s.t.}\ \big(\mathfrak{W}_{L_1}(E, y_1, w, (n_i')) \text{ and } \mathfrak{H}_{L_1}(E, y_1, m, (n_i'))\ \big)$$
$$\text{or}\ \big(\mathfrak{W}_{L_1}(E, y_2, w, (n_i')) \text{ and } \mathfrak{H}_{L_1}(E, y_2, m, (n_i'))\ \big)\Big\} \geq 1 - L_1^{-p}\ . \tag{64}$$

Proof. Let y_1 and y_2 be such that $\|y_1 - y_2\|_\infty > 2L_1$. We denote by $\overline{\mathfrak{W}}$ (resp. $\overline{\mathfrak{H}}$) the complementary events of $\mathfrak{W}$ (resp. $\mathfrak{H}$). We start by estimating the probability of the complement of the event that appears in (64)

$$\begin{aligned}
&\mathbb{P}\Big\{\exists E \in I,\ \forall (n_i)_{i=1,\dots,S}\ \big(\overline{\mathfrak{W}}_{L_1}(E, y_1, w, (n_i)) \text{ or } \overline{\mathfrak{H}}_{L_1}(E, y_1, m, (n_i))\big)\\
&\qquad \text{and } \big(\overline{\mathfrak{W}}_{L_1}(E, y_2, w, (n_i)) \text{ or } \overline{\mathfrak{H}}_{L_1}(E, y_2, m, (n_i))\big)\Big\}\\
&\leq\ \mathbb{P}\{\exists E \in I,\ \forall (n_i)\ \overline{\mathfrak{W}}_{L_1}(E, y_1, m, (n_i)) \text{ and } \overline{\mathfrak{W}}_{L_1}(E, y_2, m, (n_i))\}\\
&\qquad +\mathbb{P}\{\exists E \in I,\ \forall (n_i)\ \overline{\mathfrak{H}}_{L_1}(E, y_1, w, (n_i)) \text{ and } \overline{\mathfrak{H}}_{L_1}(E, y_2, m, (n_i))\}\\
&\qquad +\mathbb{P}\{\exists E \in I,\ \forall (n_i)\ \overline{\mathfrak{W}}_{L_1}(E, y_1, w, (n_i)) \text{ and } \overline{\mathfrak{H}}_{L_1}(E, y_2, m, (n_i))\}\\
&\qquad +\mathbb{P}\{\exists E \in I,\ \forall (n_i)\ \overline{\mathfrak{H}}_{L_1}(E, y_1, m, (n_i)) \text{ and } \overline{\mathfrak{W}}_{L_1}(E, y_2, w, (n_i))\}\\
&\leq\ \sum_{(n_i)\in\mathcal{V}} \mathbb{P}\Big\{\exists E \in I,\ \forall j,k \in \{1,\dots,N-S\},\\
&\qquad\qquad \mathrm{dist}(\sigma(H_{\Lambda_{\frac{n_j L_1}{N}}(y_1)}), E) < L_1^{-w} \text{and } \mathrm{dist}(\sigma(H_{\Lambda_{\frac{n_k L_1}{N}}(y_2)}), E) < L_1^{-w}\Big\}\\
&\qquad +3\mathbb{P}\{\exists E \in I,\ \forall (n_i),\ \overline{\mathfrak{H}}_{L_1}(E, y_1, m, (n_i))\}\ ,
\end{aligned} \tag{65}$$

where $\mathcal{V} = \{(n_i)\,|\,1 \leq n_1 < n_2 < \cdots < n_{N-S} \leq N-1\}$. The probability in the first term in (65) is bounded from above by

$$\mathbb{P}\{\exists E \in I,\ \mathrm{Tr}(\mathbf{E}_1(J_w))(\mathrm{Tr}(\mathbf{E}_2(J_w))) \geq 1\}\ , \tag{66}$$

where $J_w = (E - L_1^{-w}, E + L_1^{-w})$ and $\mathbf{E}_1$, $\mathbf{E}_2$ are, respectively, the spectral family of $H_{\Lambda_{n_1 L_1/N}(y_1)}$ and $H_{\Lambda_{n_2 L_1/N}(y_2)}$. Now (66) is bounded from above by

$$\begin{aligned}
&\mathbb{P}\left\{\int_I \mathrm{Tr}(\mathbf{E}_1(J_w))\mathrm{Tr}(\mathbf{E}_2(\mathrm{J_w}))\,\mathrm{dE} \geq \frac{\mathrm{L}_1^{-\mathrm{w}}}{2}\right\} \\
&\leq \int d\mathbb{P}(\omega)\mathbf{1}_{\{\int_I \mathrm{Tr}(\mathbf{E}_1(J_w))\,\mathrm{Tr}(\mathbf{E}_2(J_w))\,dE \geq \frac{L_1^{-w}}{2}\}} \\
&\leq \int d\mathbb{P}(\omega) 2L_1^w \int_I \mathrm{Tr}(\mathbf{E}_1(J_w))\,\mathrm{Tr}(\mathbf{E}_2(J_w)) \\
&\leq 2L_1^w \int_I \mathbb{E}\{\mathrm{Tr}(\mathbf{E}_1(J_w))\,\mathrm{Tr}(\mathbf{E}_2(J_w))\} dE \\
&\leq 2C_W |I| L_1^w L_1^{-2w} |\Lambda_{\frac{n_1 L_1}{N}}(y_1)|\,|\Lambda_{\frac{n_2 L_1}{N}}(y_2)| \qquad (67)\\
&\leq 2^{2d+1} C_W |I| L_1^{-w+2d}\ . \qquad (68)
\end{aligned}$$

In inequality (67) we have used Proposition A.1. The second term in (65) is estimated as follows

$$\begin{aligned}
&\mathbb{P}\{\exists E \in I,\ \forall (n_i),\ \overline{\mathfrak{H}}_{L_1}(E, y_1, m, (n_i))\} \\
&\quad\leq \mathbb{P}\big\{\exists E \in I, \exists 1 \leq n_1 < \cdots < n_{N-S} \leq N-1,\ \exists (z_i)_{i=1,\ldots,N-S}, \\
&\qquad\quad z_i \in \mathfrak{F}_{L_1}(y_1, n_i),\ \text{s.t. } \Lambda_{L_0}(z_i) \text{ are not } (m,E)\text{−good boxes}\big\} \\
&\quad\leq \binom{N-1}{N-S} (\mathrm{Card}(\mathfrak{F}_{L_1}))^{N-S} \\
&\qquad \times \mathbb{P}\{\exists E \in I,\ \forall i \in \{1, \ldots, N-S\},\ \Lambda_{L_0}(z_i) \text{ are not } (m,E)\text{−good boxes}\} \\
&\quad\leq c(N,S,d) L_1^{(1-\frac{1}{\alpha})(d-1)(N-S)} L_0^{-p\theta(N-S,\alpha)}\ , \qquad (69)
\end{aligned}$$

for some uniform constant $c(N,S,d)$ depending only on N, S and d. In the last inequality we have used Assumption (H4). Now, (68) and (69) give

$$\begin{aligned}
&\mathbb{P}\Bigg\{\forall E \in I,\ \exists (n_i')_{i=1,\ldots S} \text{ s.t. } \Big(\mathfrak{W}_{L_1}(E, y_1, w, (n_i')) \text{ and } \mathfrak{H}_{L_1}(E, y_1, m, (n_i'))\Big) \\
&\qquad \text{or } \Big(\mathfrak{W}_{L_1}(E, y_2, w, (n_i')) \text{ and } \mathfrak{H}_{L_1}(E, y_2, m, (n_i'))\Big)\Bigg\} \\
&\qquad\geq 1 - 2^{2d+1}\binom{N-1}{N-S} C_W |I| L_1^{-w+2d} \\
&\qquad\quad -3c(N,S,d) N^{\frac{p\theta(N-S,\alpha)}{\alpha}} L_1^{(1-\frac{1}{\alpha})(d-1)(N-S) - \frac{p\theta(N-S,\alpha)}{\alpha}} \\
&\qquad\geq 1 - L_1^{-p}\ . \qquad (70)
\end{aligned}$$

∎

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