Time-Periodic Quasigeostrophic Motion under Dissipation and Forcing *

Jinqiao Duan
Department of Mathematical Sciences,
Clemson University, Clemson, South Carolina 29634, USA.
E-mail: duan@math.clemson.edu, Fax: (864)656-5230.

Abstract

The quasigeostrophic equation is a prototypical geophysical fluid model. In this paper, we consider time-periodic motions of this model under dissipation and time-dependent wind forcing. We show that when the wind forcing is time-periodic and the spatial square-integral of the wind forcing is bounded in time, the full nonlinear quasigeostrophic model has time-periodic motions, under some conditions on β parameter, Ekman number, viscousity and the domain size.

Key words: quasigeostrophic fluid model, nonlinear dynamics, time-periodic motion, dissipative dynamics

Short running title:
Time-Periodic Quasigeostrophic Motion

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1 Introduction

The two dimensional barotropic quasigeostrophic (QG) equation has been derived as an approximation of the rotating shallow water equations by the conventional asymptotic expansion in small Rossby number ([11]). The lowest order approximation gives the barotropic QG equation, which is also the conservation law for the zero-th order potential vorticity. Warn et al. [16] and Vallis [15] emphasize that this asymptotic expansion is generally secular for all but the simplest flows and propose a modified asymptotic method, which involves expanding only the fast modes. The barotropic QG equation also emerges at the lowest order in this modified expansion.

Schochet ([13]) has recently shown that quasigeostrophy is a valid approximation of the rotating shallow water equations in the limit of zero Rossby number, i.e., at asymptotically high rotation rate. For related work in the three dimensional baroclinic QG model, see, for example, [3] , [6] , and [2] .

It is known that the *linearized* two dimensional barotropic QG equation without forcing and without dissipation has time-periodic solutions, and these time-periodic solutions are damped away by Ekman or viscous dissipation ([11], pages 147 and 236).

In this paper, we show that when the wind forcing is periodic and when its spatial square-integral is bounded in time, the full nonlinear QG equation with Ekman and viscous dissipation has time-periodic solutions. We use a topological technique from nonlinear global analysis ([10]).

2 Quasigeostrophy

The two dimensional barotropic QG equation is ([11], page 234)

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi + f(x, y, t), \quad (1)$$

where $\psi(x, y, t)$ is the stream function, $\beta > 0$ is the meridional gradient of the Coriolis parameter, $\nu > 0$ is the viscous dissipation constant and $r > 0$ is the Ekman dissipation constant and $f(x, y, t)$ is the wind forcing. Moreover, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplacian operator in the x, y plane and $J(f, g) = f_x g_y - f_y g_x$ is the Jacobi operator. This equation is written in the often-studied case of infinite Rossby deformation radius and with flat bottom, applicable for planetary-scale solutions. The situation with infinite Rossby deformation radius is equivalent to the rigid-lid approximation ([5], p223).

In this paper, we assume that the wind forcing $f(x, y, t)$ is periodic in time with period $T > 0$.

Introducing the relative vorticity $\omega(x, y, t) = \Delta\psi(x, y, t)$, the equation (1) can be written as

$$\omega_t + J(\psi, \omega) + \beta\psi_x = \nu\Delta\omega - r\omega + f(x, y, t) \ , \tag{2}$$

where $(x, y) \in D$, an arbitrary bounded planar domain with piecewise smooth boundary. This equation is supplemented by zero Dirichlet boundary conditions for both ψ and $\omega = \Delta\psi$, together with an appropriate initial condition, i.e., we require

$$\begin{aligned} \psi(x, y, t) &= 0 \quad \text{on } \partial D \ , & (3) \\ \omega(x, y, t) &= 0 \quad \text{on } \partial D \ , & (4) \\ \omega(x, y, 0) &= \omega_0(x, y) \ . & (5) \end{aligned}$$

These boundary conditions have been used in analytical and numerical study of this model in, e.g., [4], [12],[1].

We note that the Poincaré inequality ([7])

$$\int_D g^2(x, y)dxdy \leq \frac{|D|}{\pi} \int_D |\nabla g|^2 dxdy \tag{6}$$

holds with these boundary conditions, where $|D|$ is the area of the domain D. The global well-posedness (smooth solutions) of this dissipative model can be obtained similarly as in, for example, [1], [17] and [18].

It is well-known that the *linearized* two dimensional barotropic QG equation without forcing and without dissipation

$$\Delta\psi_t + \beta\psi_x = 0, \qquad x, y \in [0, 1], \tag{7}$$

has time-periodic solutions ([11], p.146-149). For example, assume the basin is the unit square on the $\beta-$plane, with boundary condition $\psi = 0$. The boundary is a streamline in this case. By separating variables, one can find that the equation (7) has time-periodic solutions ([11], p.146-149)

$$\psi_{mn}(x, y, t) = cos(\frac{\beta x}{2\sigma_{mn}} + \sigma_{mn}t)sin(m\pi x)sin(n\pi y), m, n = 1, 2, 3, \cdots, \tag{8}$$

which are basin-scale traveling-wave oscillations with dispersion relation,

$$\sigma_{mn} = \frac{-\beta}{2\pi\sqrt{m^2 + n^2}} \ . \tag{9}$$

These are basin-scale normal modes (planetary waves) for QG in the rigid-lid approximation. Each mode consists of a carrier wave $cos(\frac{\beta x}{2\sigma_{mn}}+\sigma_{mn}t)$ moving to the left (westward) and modulated by an envelope of sine functions which maintain the boundary conditions. These time-periodic linear solutions are damped away by Ekman or viscous dissipation ([11], p. 236).

We address the question of whether there are any basin-scale time-periodic solutions in the full nonlinear dissipative QG dynamics (1) with time-periodic wind forcing.

3 Dissipativity

In the following we use the abbreviations $L^2 = L^2(D)$, $H_0^1 = H_0^1(D), \ldots$ for the standard Sobolev spaces. Furthermore, let $< \cdot, \cdot >$, $\| \cdot \|$ denote the standard scalar product and norm in L^2, respectively. We need the following inequalities ([14]).

Young inequality:

$$AB \leq \frac{\epsilon}{2}A^2 + \frac{1}{2\epsilon}B^2, \tag{10}$$

where A, B are non-negative real numbers and $\epsilon > 0$.

Gronwall inequality: If an integrable function $y(t)$ satisfies that

$$\frac{dy}{dt} \leq Ay + B, \quad t \geq 0, \tag{11}$$

for some constants A, B with $A \neq 0$, then

$$y(t) \leq [y(0) + \frac{B}{A}]e^{At} - \frac{B}{A}, \quad t > 0. \tag{12}$$

We now show that the system (2), under boundary conditions (3), (4), is a dissipative system in the sense ([10] or [8]) that all solutions $\omega(x, y, t)$ approach a bounded set in $L^2(D)$ as time goes to infinity. A $T-$time-periodic dissipative system in a Banach space has at least one $T-$time-periodic solution. This result follows from a Leray-Schauder topological degree argument and the Browder's principle ([10], p.235).

Multipling (2) by ω and integrating over D, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \int_D J(\psi, \omega)\omega dxdy + \beta \int_D \psi_x \omega dxdy \\ = \; & -\nu \int_D |\nabla \omega|^2 - r\|\omega\|^2 + \int_D f(x, y, t)\omega dxdy. \end{aligned} \tag{13}$$

Note that

$$\int_D J(\psi,\omega)\omega dxdy = 0, \tag{14}$$

via integration by parts; see also [1] or [9]. Moreover, using the Young and Poincaré inequalities, we get

$$\begin{aligned}
|\beta \int_D \psi_x \omega dxdy| &\leq \frac{1}{2}\beta(\int_D \psi_x^2 + \int_D \omega^2) \\
&\leq \frac{1}{2}\beta(\frac{|D|}{\pi}\int_D \omega^2 dxdy + \int_D \omega^2 dxdy) \\
&= \frac{1}{2}\beta(\frac{|D|}{\pi}+1)\int_D \omega^2 dxdy, \qquad (15)\\
-\nu \int_D |\nabla\omega|^2 dxdy &\leq -\frac{\pi\nu}{|D|}\int_D \omega^2 dxdy. \qquad (16)
\end{aligned}$$

We further assume that the square-integral of the wind forcing $f(x,y,t)$ with respect to x,y is bounded in time.

$$\int_D f(x,y,t)\omega dxdy \leq \frac{1}{\epsilon}\int_D f^2(x,y,t)dxdy + \epsilon\int_D \omega^2 dxdy \leq M + \epsilon \int_D \omega^2 dxdy, \tag{17}$$

where $\epsilon > 0$ is to be determined, and $M > 0$ is a time-independent constant since we have assume that the square-integral of $f(x,y,t)$ with respect to x,y is bounded in time.

Putting (14), (15), (16), (17) into (13), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \alpha\int_D \omega^2 dxdy \leq M, \tag{18}$$

where

$$\alpha = [r + \frac{\pi\nu}{|D|} - \frac{1}{2}\beta(\frac{|D|}{\pi}+1) - \epsilon].$$

Assume that

$$r + \frac{\pi\nu}{|D|} > \frac{1}{2}\beta(\frac{|D|}{\pi}+1). \tag{19}$$

We can then take $\epsilon > 0$ small enough such that $\alpha > 0$. Thus, by the Gronwall inequality, we have

$$\|\omega\|^2 \leq (\|\omega_0\|^2 - \frac{M}{\alpha})e^{-2\alpha t} + \frac{M}{\alpha}. \tag{20}$$

Hence all solutions ω enter a bounded set $\{\omega : \quad \|\omega\| \le \sqrt{\frac{M}{\alpha}}\}$ as time goes to infinity. The system (2) is therefore a dissipative system and hence has at least one $T-$time-periodic solution.

We then have the following result.

Theorem 1 *Assume that*

(i) the wind forcing $f(x, y, t)$ is time-periodic with period $T > 0$, and its square-integral with respect to x, y is bounded in time; and

(ii) $r + \frac{\pi\nu}{|D|} > \frac{1}{2}\beta(\frac{|D|}{\pi} + 1)$, where $\beta > 0$ is the meridional gradient of the Coriolis parameter, $\nu > 0$ is the viscous dissipation constant, $r > 0$ is the Ekman dissipation constant, and $|D|$ is the area of the bounded domain D.

Then the dissipative quasigeostrophic model

$$
\begin{aligned}
\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x &= \nu\Delta^2\psi - r\Delta\psi + f(x, y, t) \,, && (21)\\
\omega &= \Delta\psi \,, && (22)\\
\psi(x, y, t) &= 0 \quad on\ \partial D \,, && (23)\\
\omega(x, y, t) &= 0 \quad on\ \partial D \,, && (24)\\
\omega(x, y, t + T) &= \omega(x, y, t) \,, && (25)
\end{aligned}
$$

has at least one time-periodic solution with period $T > 0$.

We remark that it is generally difficult to show existence of time-periodic motions for a spatially extended evolution system. Our result provides such a proof of existence, under some conditions on β parameter, Ekman number, viscousity and the domain size, for a prototypical geophysical fluid model.

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