Dissipative Quasigeostrophic Dynamics under Random Forcing *

April 6, 1998

James R. Brannan[1], Jinqiao Duan[1], and Thomas Wanner[2]

1. Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29634, USA.
E-mail: duan@math.clemson.edu
2. Institute of Mathematics, University of Augsburg D-86135 Augsburg, Germany.

Abstract

The quasigeostrophic model is a simplified geophysical fluid model at asymptotically high rotation rate or at small Rossby number. We consider the quasigeostrophic equation with dissipation under random forcing in bounded domains. We show that global unique solutions exist for appropriate initial data. Unlike the deterministic quasigeostrophic equation whose well-posedness is well-known, there seems no rigorous result on global existence and uniqueness of the randomly forced quasigeostrophic equation. Our work provides such a rigorous result on global existence and uniqueness, under very mild conditions.

Key words: Quasigeostrophic model, random forcing, dissipation, stochastic partial differential equation.

*This work was supported by the National Science Foundation Grant DMS-9704345.

Running head:
Random Quasigeostrophic Dynamics

Author for correspondence:
Dr. Jinqiao Duan
Department of Mathematical Sciences
Clemson University,
Clemson, South Carolina 29634, USA.
E-mail: duan@math.clemson.edu
Fax: (864) 656-5230
Tel: (864) 656-2730

1 Introduction

The models for geophysical flows are usually too complicated for analysis. Simplified partial differential equation models which are intended to capture the key features of large scale phenomena and filter out undesired high frequency oscillations in geophysical flows have been derived at asymptotically high rotation rate or small Rossby number. An important example of such partial differential equations is the quasigeostrophic model.

The *deterministic* quasigeostrophic equation is ([20],[6])

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi \ , \tag{1}$$

where $\psi(x, y, t)$ is the stream function, $\beta \geq 0$ is the meridional gradient of the Coriolis parameter, $\nu > 0$ is the viscous dissipation constant, $r > 0$ is the Ekman dissipation constant. Moreover, $J(f, g) = f_x g_y - f_y g_x$ denotes the Jacobian operator.

The deterministic quasigeostrophic equation (1) has been derived as an approximation of the rotating shallow water equations by the conventional asymptotic expansion in small Rossby number ([20]). Schochet ([22]) has recently shown that the shallow water flows converge to the quasigeostrophic flows in Sobolev norms in the limit of zero Rossby number (i.e., at asymptotically high rotation rate), for appropriate initial data. For related work about the three dimensional baroclinic quasigeostrophic model, see, for example, [4], [11], [17] and [2].

Recently, a few authors have considered the randomly forced quasigeostrophic equation, in order to incorporate the impact of uncertain geophysical forces ([21], [13], [15], [18], [10]). They studied statistical issues such as estimating correlation coefficients for the *linearized* quasigeostrophic equation with random forcing. There is also recent work about the impact of random ocean bottom topography on quasigeostrophic dynamics ([16]).

The randomly forced quasigeostrophic equation takes the form ([18])

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi + \frac{dW}{dt} \ , \tag{2}$$

where $W(x, y, t)$ is a space-time Wiener process to be defined below. There does not seem to exist a mathematically rigorous theory of quasigeostrophic dynamics under random forcing. In this paper, we consider existence and uniqueness of solutions for the nonlinear quasigeostrophic equation (2) subject to Dirichlet boundary conditions and appropriate initial data.

2 Local existence and uniqueness of solution processes

Introducing $\omega(x,y,t) = \Delta\psi(x,y,t)$, the equation (2) can be written as

$$\omega_t + J(\psi,\omega) + \beta\psi_x = \nu\Delta\omega - r\omega + \frac{dW}{dt} , \tag{3}$$

where $(x,y) \in D$ and $D \subset \mathbb{R}^2$ denotes a bounded domain with sufficiently regular boundary. This equation is supplemented by zero Dirichlet boundary conditions ([5]) for both ψ and $\omega = \Delta\psi$, together with an appropriate initial condition, i.e., we require

$$\begin{aligned} \psi(x,y,t) &= 0 \quad \text{on } \partial D , &(4)\\ \omega(x,y,t) &= 0 \quad \text{on } \partial D , &(5)\\ \omega(x,y,0) &= \omega_0(x,y) . &(6) \end{aligned}$$

We note that the Poincaré inequality holds with these boundary conditions.

As it stands, (3) still has to be given a mathematically precise meaning. This can be done using the framework of stochastic partial differential equations ([7]). For this we (formally) rewrite (3) in the form

$$d\omega = (\nu\Delta\omega - r\omega - \beta\psi_x - J(\psi,\omega))dt + dW . \tag{7}$$

In the following we use the abbreviations $L^2 = L^2(D)$, $L^\infty = L^\infty(D)$, $H_0^k = H_0^k(D)$, $H^k = H^k(D)$, $0 < k < \infty$, for the standard Sobolev spaces. Let $< \cdot,\cdot >$ and $\|\cdot\| \equiv \|\cdot\|_2$ denote the standard scalar product and norm in L^2, respectively. Moreover, the norms for H_0^k, L^∞ are denoted by $\|\cdot\|_{H^k}$, $\|\cdot\|_\infty$, respectively. Due to the Poincaré inequality ([12], p. 164), $\|\Delta\varphi\|$ is an equivalent norm for H_0^2. It is well-known that the operator $A = \nu\Delta : L^2 \to L^2$ with domain $D(A) = H^2 \cap H_0^1$ is self-adjoint. Note that A generates an analytic semigroup $S(t)$ on L^2 ([19]). The spectrum of A consists of eigenvalues $0 > \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots$ with corresponding normalized eigenfunctions φ_1, φ_2, ... The set of these eigenfunctions is complete in L^2. For example, for the square domain $D = (0,1) \times (0,1)$ the eigenvalues are given by $-\nu(m^2+n^2)\pi^2$ for $m,n \in \mathbb{N}$, and the associated eigenfunctions are suitable multiples of $\sin(m\pi x)\sin(n\pi y)$.

Now we can define an appropriate class of Wiener processes W. Let $\beta_k(t)$, $k \in \mathbb{N}$, denote a family of independent real-valued Brownian motions.

Furthermore, choose positive constants μ_k, $k \in \mathbb{N}$, such that

$$\sum_{k=1}^{\infty} \frac{\mu_k}{|\lambda_k|^{1-\gamma}} < \infty$$

for some $0 < \gamma < 1$. Then we consider the Wiener process W defined by

$$W(t) := \sum_{k=1}^{\infty} \sqrt{\mu_k} \beta_k(t) \varphi_k \ , \quad t \geq 0 \ . \tag{8}$$

We further assume that

$$\kappa(D) = \inf_{0<\rho<diam(D)} \inf_{(x,y)\in D} \frac{meas(D \cap B(x,y;\rho))}{\rho^2} > 0,$$

where $diam(D)$ is the diameter of D (the least upper bound of two-point distances in D), $meas(\cdot)$ denotes the Lebesgue measure, and $B(x,y;\rho)$ is the open disk centered at (x,y) and with radius ρ. We also assume that the eigenfunctions φ_k satisfy

$$\varphi_k \in C_0(\bar{D}), \quad |\varphi_k(x,y)| \leq C,$$

$$|\partial_x \varphi_k(x,y)|, \quad |\partial_y \varphi_k(x,y)| \leq C\sqrt{|\lambda_k|},$$

for $(x,y) \in D, k \in \mathbb{N}$, and some constant $C > 0$. For the square domain $D = (0,1) \times (0,1)$, these conditions are all satisfied. Then, according to Theorem 5.2.9 in [8], the stochastic convolution

$$W_A(t) = \int_0^t S(t-s)dW(s) \ , \quad t > 0 \ , \tag{9}$$

has a continuous version with values in $C_0(D)$, the Banach space of continuous functions satisfying zero Dirichlet boundary conditions on D.

If we define the nonlinear operator F by $F(\omega) = -r\omega - \beta\psi_x - J(\psi,\omega)$, then (7) can be rewritten as the abstract evolution equation together with initial condition

$$d\omega = (A\omega + F(\omega))dt + dW \ , \tag{10}$$

$$\omega(0) = \omega_0 \tag{11}$$

or in the mild (integral) form

$$\omega(t) = S(t)\omega_0 + \int_0^t S(t-s)F(\omega(s))ds + W_A(t) \ , \tag{12}$$

where the stochastic convolution $W_A(t)$ is defined in (9).

By defining $U(t) = \omega(t) - W_A(t)$, we obtain a deterministic mild (integral) equation

$$U(t) = S(t)\omega_0 + \int_0^t S(t-s)F(U(s) + W_A(s))ds \ , \tag{13}$$

or in its differential form

$$U'(t) = AU(t) + F(U(t) + W_A(t)) \ , \tag{14}$$

$$U(0) = \omega_0. \tag{15}$$

In the following we prove the local existence of $U(t)$. We follow the approach in [9] or [8], p. 261. We first show that the integral in (13) makes sense for $U \in C([0,T]; L^2)$. Then, we obtain local existence for (13) by the Banach contraction mapping principle in L^2.

Note that since A generates an analytic semigroup $S(t)$ on L^2 and has only negative eigenvalues, we have ([19], p. 74), for $a > 0$,

$$S(t)(-A)^a = (-A)^a S(t), \tag{16}$$

$$\|(-A)^a S(t)u\| \leq \frac{c}{t^a} \cdot \|u\|, \tag{17}$$

$$\|S(t)u\| \leq c \cdot \|u\|. \tag{18}$$

Here and hereafter we use c to denote various constants.

We first show that $\int_0^t S(t-s)F(U(s) + W_A(s))ds$ makes sense for $U(\cdot) + W_A(\cdot)$ (and thus $U(\cdot)$) in $C([0,T]; L^2)$. Recalling that $\omega = U + W_A$, this follows from the following lemma.

Lemma 1 *Define the mapping* $\mathcal{F} : C([0,T]; H_0^1) \to C([0,T]; L^2)$ *by*

$$(\mathcal{F}(\omega))(t) := \int_0^t S(t-s)F(\omega(s))ds \ , \quad t \in [0,T] \ , \quad \omega \in C([0,T]; L^2) \ .$$

Then $\mathcal{F}$ *is continuous, and it can be extended to a continuous mapping from the space* $C([0,T]; L^2)$ *to* $C([0,T]; L^2)$. *Furthermore, the image of the extended mapping* $\mathcal{F}$ *is contained in* $C([0,T], H^a(D))$ *for* $0 \leq a < \frac{1}{2}$.

Proof: The continuity of $\mathcal{F} : C([0,T]; H_0^1) \to C([0,T]; L^2)$ is obvious. As for extending the domain of $\mathcal{F}$ let $\omega, \bar{\omega} \in C([0,T]; L^2)$ be arbitrary. Using the abbreviations $\omega = \Delta\psi$ and $\bar{\omega} = \Delta\bar{\psi}$ we get

$$\begin{aligned} F(\omega) - F(\bar{\omega}) &= r(\bar{\omega} - \omega) + \beta(\bar{\psi} - \psi)_x + \psi_x(\bar{\omega} - \omega)_y + (\bar{\psi} - \psi)_x \bar{\omega}_y \\ &\quad - \psi_y(\bar{\omega} - \omega)_x + (\psi - \bar{\psi})_y \bar{\omega}_x. \end{aligned}$$

Let $a \in [0,1)$, and consider an arbitrary $\varphi \in D((-A)^a)$. Then the above identity implies

$$\begin{aligned} I &= \left\langle (-A)^a \varphi, \int_0^t S(t-s) F(\omega(s)) ds - \int_0^t S(t-s) F(\bar{\omega}(s)) ds \right\rangle \\ &= \int_0^t \langle S(t-s)(-A)^a \varphi, [r(\bar{\omega} - \omega) + \beta(\bar{\psi} - \psi)_x + \psi_x(\bar{\omega} - \omega)_y \\ &\qquad + (\bar{\psi} - \psi)_x \bar{\omega}_y - \psi_y(\bar{\omega} - \omega)_x + (\psi - \bar{\psi})_y \bar{\omega}_x](s) \rangle ds \\ &\equiv \int_0^t (I_1 + I_2 + I_3 + I_4 + I_5 + I_6) ds, \end{aligned}$$

where

$$\begin{aligned} I_1 &= < S(t-s)(-A)^a \varphi, [r(\bar{\omega} - \omega)](s) >, \\ I_2 &= < S(t-s)(-A)^a \varphi, [\beta(\bar{\psi} - \psi)_x](s) >, \\ I_3 &= < S(t-s)(-A)^a \varphi, [\psi_x(\bar{\omega} - \omega)_y](s) >, \\ I_4 &= < S(t-s)(-A)^a \varphi, [(\bar{\psi} - \psi)_x \bar{\omega}_y](s) >, \\ I_5 &= < S(t-s)(-A)^a \varphi, [-\psi_y(\bar{\omega} - \omega)_x](s) >, \\ I_6 &= < S(t-s)(-A)^a \varphi, [(\psi - \bar{\psi})_y \bar{\omega}_x](s) > . \end{aligned}$$

Now we estimate $|I_k|$, $k = 1, \ldots, 6$, one by one, thereby omitting the argument s.

$$\begin{aligned} |I_1| &= | < S(t-s)(-A)^a \varphi, r(\bar{\omega} - \omega) > | \\ &\le r \|S(t-s)(-A)^a \varphi\| \cdot \|\bar{\omega} - \omega\| \\ &\le rc(t-s)^{-a} \cdot \|\varphi\| \cdot \|\bar{\omega} - \omega\|, \end{aligned}$$

$$\begin{aligned} |I_2| &= | < S(t-s)(-A)^a \varphi, \beta(\bar{\psi} - \psi)_x > | \\ &\le \beta \|S(t-s)(-A)^a \varphi\| \cdot \|(\bar{\psi} - \psi)_x\| \\ &\le \beta c(t-s)^{-a} \cdot \|\varphi\| \cdot \|\bar{\omega} - \omega\|, \end{aligned}$$

where we have used the Poincaré inequality ([12], p. 164) on $(\bar{\psi}-\psi)_x$ which has zero mean. Using the Cauchy-Schwarz inequality $|<u,v>| \leq \|u\|\|v\|$ we also obtain

$$\begin{aligned}
|I_3| &= |< S(t-s)(-A)^a\varphi, \psi_x(\bar{\omega}-\omega)_y >| \\
&= |< D_y[(S(t-s)(-A)^a\varphi)\psi_x], \bar{\omega}-\omega >| \\
&\leq |< D_y(S(t-s)(-A)^a\varphi)\psi_x, \bar{\omega}-\omega >| \\
&\qquad +|< (S(t-s)(-A)^a\varphi)\psi_{xy}, \bar{\omega}-\omega >| \\
&\leq \|D_y(S(t-s)(-A)^a\varphi)\psi_x\| \cdot \|\bar{\omega}-\omega\| \\
&\qquad +\|S(t-s)(-A)^a\varphi\|_\infty \cdot \|\psi_{xy}\| \cdot \|\bar{\omega}-\omega\|.
\end{aligned}$$

As for estimating $\|D_y(S(t-s)(-A)^a\varphi)\psi_x\|$ we get

$$\begin{aligned}
\|D_y(S(t-s)(-A)^a\varphi)\psi_x\| &\leq \|D_y(S(t-s)(-A)^a\varphi)\|_{\frac{1}{4}} \cdot \|\psi_x\|_{\frac{1}{4}} \\
&\leq c\|D_y(S(t-s)(-A)^a\varphi)\|_{H^{\frac{1}{2}}} \cdot \|\psi_x\|_{H^1} \\
&\leq c\|S(t-s)(-A)^a\varphi\|_{H^{\frac{3}{2}}} \cdot \|\omega\| \\
&\leq c\|A^{\frac{3}{4}+\rho}S(t-s)(-A)^a\varphi\| \cdot \|\omega\| \\
&\leq c(t-s)^{-(\frac{3}{4}+\rho+a)} \cdot \|\varphi\| \cdot \|\omega\|,
\end{aligned}$$

where we have used the inequality $\|uv\| \leq \|u\|_{\frac{1}{4}}\|v\|_{\frac{1}{4}}$, the continuity of the mapping $D_y : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ ([23], p. 56), the embedding $D((-A)^{\frac{3}{4}+\rho}) \hookrightarrow H^{\frac{3}{2}}$ for arbitrary $\rho > 0$ ([19], p. 243), and the facts that $H^{\frac{1}{2}}$ and H^1 are embedded in L^4 ([1], p. 217). Furthermore,

$$\begin{aligned}
\|S(t-s)(-A)^a\varphi\|_\infty &\leq c\|S(t-s)(-A)^a\varphi\|_{H^{1+\rho}} \\
&\leq c\|(-A)^{\frac{1}{2}+\rho}S(t-s)(-A)^a\varphi\| \\
&\leq c(t-s)^{-(\frac{1}{2}+\rho+a)} \cdot \|\varphi\|,
\end{aligned}$$

due to the smoothing property of the semigroup S and the embeddings $D((-A)^{\frac{1}{2}+\rho}) \hookrightarrow H^{1+\rho} \hookrightarrow L^\infty$ for arbitrary $\rho > 0$ ([19], pp. 208, 243). Altogether we get

$$|I_3| \leq [c(t-s)^{-(\frac{3}{4}+\rho+a)} + c(t-s)^{-(\frac{1}{2}+\rho+a)}] \cdot \|\varphi\| \cdot \|\omega\| \cdot \|\omega-\bar{\omega}\|.$$

Similarly, the following estimates can be obtained:

$$\begin{aligned}
|I_4| &\leq [c(t-s)^{-(\frac{3}{4}+\rho+a)} + c(t-s)^{-(\frac{1}{2}+\rho+a)}] \cdot \|\varphi\| \cdot \|\bar{\omega}\| \cdot \|\omega-\bar{\omega}\| , \\
|I_5| &\leq [c(t-s)^{-(\frac{3}{4}+\rho+a)} + c(t-s)^{-(\frac{1}{2}+\rho+a)}] \cdot \|\varphi\| \cdot \|\omega\| \cdot \|\omega-\bar{\omega}\| , \\
|I_6| &\leq [c(t-s)^{-(\frac{3}{4}+\rho+a)} + c(t-s)^{-(\frac{1}{2}+\rho+a)}] \cdot \|\varphi\| \cdot \|\bar{\omega}\| \cdot \|\omega-\bar{\omega}\| .
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
|I| &\leq \int_0^t (|I_1| + |I_2| + |I_3| + |I_4| + |I_5| + |I_6|)ds \\
&\leq \frac{rc+\beta c}{1-a} \cdot t^{1-a} \cdot \|\varphi\| \cdot \sup_{0\leq s\leq t} \|\omega(s) - \bar{\omega}(s)\| \\
&\quad + \left(\frac{8c}{1-4\rho-4a} \cdot t^{\frac{1}{4}-\rho-a} + \frac{4c}{1-2\rho-2a} \cdot t^{\frac{1}{2}-\rho-a} \right) \\
&\qquad \cdot \|\varphi\| \cdot \sup_{0\leq s\leq t} (\|\omega(s)\| + \|\bar{\omega}(s)\|) \cdot \sup_{0\leq s\leq t} \|\omega(s) - \bar{\omega}(s)\|,
\end{aligned}
$$

provided the positive constants a and ρ satisfy $0 < \rho + a < \frac{1}{4}$. This finally implies

$$
\int_0^t S(t-s)F(\omega(s))ds - \int_0^t S(t-s)F(\bar{\omega}(s))ds \in D((-A)^a) \quad \text{for} \quad 0 \leq a < \frac{1}{4},
$$

and

$$
\begin{aligned}
&\left\| (-A)^a \left(\int_0^t S(t-s)F(\omega(s))ds - \int_0^t S(t-s)F(\bar{\omega}(s))ds \right) \right\| \\
\leq\ & \frac{rc+\beta c}{1-a} \cdot t^{1-a} \cdot \sup_{0\leq s\leq t} \|\omega(s) - \bar{\omega}(s)\| \\
& + \left(\frac{8c}{1-4\rho-4a} \cdot t^{\frac{1}{4}-\rho-a} + \frac{4c}{1-2\rho-2a} \cdot t^{\frac{1}{2}-\rho-a} \right) \\
& \quad \cdot \sup_{0\leq s\leq t} (\|\omega(s)\| + \|\bar{\omega}(s)\|) \cdot \sup_{0\leq s\leq t} \|\omega(s) - \bar{\omega}(s)\|.
\end{aligned}
$$

Especially for $a = 0$ we obtain

$$
\begin{aligned}
&\left\| \int_0^t S(t-s)F(\omega(s))ds - \int_0^t S(t-s)F(\bar{\omega}(s))ds \right\| \\
\leq\ & (rc+\beta c) \cdot t \cdot \sup_{0\leq s\leq t} \|\omega(s) - \bar{\omega}(s)\| \\
& + \left(\frac{8c}{1-4\rho} \cdot t^{\frac{1}{4}-\rho} + \frac{4c}{1-2\rho} \cdot t^{\frac{1}{2}-\rho} \right) \\
& \quad \cdot \sup_{0\leq s\leq t} (\|\omega(s)\| + \|\bar{\omega}(s)\|) \cdot \sup_{0\leq s\leq t} \|\omega(s) - \bar{\omega}(s)\|,
\end{aligned}
$$

for every $0 < \rho < \frac{1}{4}$. This completes the proof of the lemma. □

We conclude from the above lemma that $\int_0^t S(t-s)F(U(s)+W_A(s))ds$, considered as a mapping with argument $U(\cdot)$, can be extended to a bounded map from $C([0,T], L^2(D))$ into itself.

Now we can follow [19], p. 196 or [7], p. 201, to obtain that (13) has a unique local solution $U(t)$, or (10), (11) has a unique local solution $\omega(x,y,t)$, on $[0,\tau)$, by the Banach contraction mapping principle. The solution $\omega(x,y,t)$ is in $C([0,\tau]; L^2(D))$, as well as in $C((0,\tau]; H^a(D))$, for arbitrary $0 \le a < \frac{1}{2}$.

3 Global solution processes

In this section, we show that the solution $U(t)$ is a priori bounded, in $L^2(D)$-norm, on any finite interval $[0,T]$. This implies that the local solution $U(t)$, and thus $\omega(x,y,t)$ is actually global in time.

We consider (14), (15) with W_A replaced by a regular function V from the space $C([0,T]; H_0^3(D))$ and ω_0 in $D(A)$

$$\begin{aligned} U'(t) &= AU(t) + F(U(t)+V(t)) \ , \\ & U(0) = \omega_0, \end{aligned} \tag{19}$$

where, we denote, $U = \Delta u, V = \Delta v$. More specifically, (19) is

$$U' = \nu \Delta U - r(U+V) - \beta(u+v)_x - J(u+v, U+V). \tag{20}$$

Due to the smoothing effect of the sectorial operator A, and the fact that F is locally Lipschitz in U from $H_0^{m+1} \cap H^{2m+2}$ to $H_0^m \cap H^{2m}$ for $m = 0,1,2$, we conclude that the solution U of (19) is in $H_0^k \cap H^{2k}$ for $k = 0,1,2$, and hence U is a strong solution (if V is smoother, the solution U is also smoother); see [14], p. 73.

We now estimate the norm $\|U(t)\|$.

Multiplying (20) by U and integrating over D, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U\|^2 &= -\nu \int_D |\nabla U|^2 - r\int_D (U+V)U \\ & \quad -\beta \int_D (u_x + v_x)U - \int_D J(u+v, U+V)U \\ &= -\nu \int_D |\nabla U|^2 - r \int_D (U^2 + UV) \end{aligned}$$

$$
\begin{aligned}
&\quad -\beta\int_D (u_xU+v_xU) - \int_D J(u,V)U - \int_D J(v,V)U \\
&= -\nu\|\nabla U\|^2 - r\int_D (U^2+UV) - \beta\int_D (u_xU+v_xU) \\
&\quad + \int_D (-u_xV_yU + u_yV_xU - v_xV_yU + v_yV_xU), \qquad (21)
\end{aligned}
$$

where we have used the fact that $\int_D J(u,U)U = \int_D J(v,U)U = 0$ via integration by parts; see also [17]. We estimate the right hand side of (21) term by term.

$$
-r\int_D (U+V)U \;\leq\; r(1+c\|V\|_\infty)\|U\|^2, \qquad (22)
$$

$$
\begin{aligned}
-\beta\int_D (u_xU+v_xU) &\leq \beta\int_D \frac{1}{2}[u_x^2+U^2+v_x^2+U^2] \\
&\leq \beta c(\|U\|^2+\|V\|^2) \\
&\leq \beta c(\|U\|^2+\|V\|_\infty^2), \qquad (23)
\end{aligned}
$$

where we have used the Poincaré inequality on u_x, v_x, which have zero mean.

$$
\begin{aligned}
\int_D (-u_xV_yU) &= \int_D (u_xU)_yV \\
&= \int_D (u_{xy}UV + u_xU_yV) \\
&\leq \|V\|_\infty\int_D |u_{xy}U| + \int_D (|u_x|\cdot\|V\|_\infty)|U_y| \\
&\leq \|V\|_\infty\int_D \frac{1}{2}(u_{xy}^2+U^2) + \int_D \left(\frac{1}{2\epsilon}u_x^2\|V\|_\infty^2 + \frac{\epsilon}{2}U_y^2\right) \\
&\leq \frac{c}{2}\|V\|_\infty\left(1+\frac{1}{\epsilon}\|V\|_\infty\right)\|U\|^2 + \frac{\epsilon}{2}\|\nabla U\|^2, \qquad (24)
\end{aligned}
$$

since $\int_D u_{xy}^2$ is bounded by $c\int_D(\Delta u)^2 = c\int_D U^2$. We have also used the Young inequality ([23]) to get that $(|u_x|\cdot\|V\|_\infty)|U_y| \leq \frac{1}{2\epsilon}u_x^2\|V\|_\infty^2 + \frac{\epsilon}{2}U_y^2$, for any positive real number $\epsilon > 0$. Similarly, we have

$$
\int_D u_yV_xU \;\leq\; \frac{c}{2}\|V\|_\infty\left(1+\frac{1}{\epsilon}\|V\|_\infty\right)\|U\|^2 + \frac{\epsilon}{2}\|\nabla U\|^2,
$$

$$
\begin{aligned}
\int_D(-v_x V_y U) &= \int_D (v_x U)_y V \\
&= \int_D (v_{xy}UV + v_x U_y V) \\
&\le \|V\|_\infty \int_D \frac{1}{2}(v_{xy}^2 + U^2) + \int_D \left(\frac{1}{2\epsilon}v_x^2\|V\|_\infty^2 + \frac{\epsilon}{2}U_y^2\right) \\
&\le \frac{1}{2}\|V\|_\infty\|U\|^2 + c\|V\|_\infty\|V\|^2 + \frac{c}{\epsilon}\|V\|_\infty^2\|V\|^2 + \frac{\epsilon}{2}\|\nabla U\|^2 \\
&\le \frac{1}{2}\|V\|_\infty\|U\|^2 + c\|V\|_\infty^3 + \frac{c}{\epsilon}\|V\|_\infty^4 + \frac{\epsilon}{2}\|\nabla U\|^2, \\
\int_D v_y V_x U &\le \frac{1}{2}\|V\|_\infty\|U\|^2 + c\|V\|_\infty^3 + \frac{c}{\epsilon}\|V\|_\infty^4 + \frac{\epsilon}{2}\|\nabla U\|^2. \qquad (25)
\end{aligned}
$$

Putting (22) - (25) into (21), we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U\|^2 &\le (-\nu + 2\epsilon)\cdot\|\nabla U\|^2 \\
&\quad +[r(1+c\|V\|_\infty) + \beta c + c\|V\|_\infty(1+\frac{1}{\epsilon}\|V\|_\infty) + \|V\|_\infty]\cdot\|U\|^2 \\
&\quad +\beta c\|V\|_\infty^2 + 2c\|V\|_\infty^3 + \frac{2}{\epsilon}c\|V\|_\infty^4. \qquad (26)
\end{aligned}
$$

Taking $\epsilon = \frac{\nu}{2}$, we finally obtain

$$
\frac{d}{dt}\|U(t)\|^2 \le A(t)\cdot\|U(t)\|^2 + B(t), \qquad (27)
$$

where

$$
\begin{aligned}
A(t) &= 2\left[r(1+c\|V\|_\infty) + \beta c + c\|V\|_\infty\left(1+\frac{2}{\nu}\|V\|_\infty\right) + \|V\|_\infty\right] > 0, \\
B(t) &= 2\beta c\|V\|_\infty^2 + 4c\|V\|_\infty^3 + \frac{8}{\nu}c\|V\|_\infty^4 > 0. \qquad (28)
\end{aligned}
$$

Hence by the Gronwall inequality ([23]), we obtain

$$
\|U(t)\|^2 \le \|\omega_0\|^2 e^{\int_0^t A(s)ds} + \int_0^t B(s)e^{\int_s^t A(\tau)d\tau}ds, \; 0 < t < T. \qquad (29)
$$

Note that $H_0^3(D)$ is embedded in $C_0(D)$, the trajectories of $W_A(t)$ can be uniformly approximated, on any finite interval $[0,T]$, by functions V in

$C([0,T]; H_0^3(D))$, and $D(A)$ is dense in $L^2(D)$. Thus the boundedness estimate (29) is true for any local solution $U(t)$ of (14). This shows that the unique (local) solution does not blow up on any finite intervals.

We thus have the following theorem.

Theorem 1 *For every initial condition $\omega_0(x,y) \in L^2(D)$, there exists a unique global mild solution $\omega(x,y,t)$ of the quasigeostrophic model (3), (4), (5), and (6). This solution is contained in the space $C([0,T]; L^2(D))$ for every $T > 0$, as well as in $C((0,T]; H^a(D))$ for all $0 \leq a < \frac{1}{2}$ and $T > 0$.*

4 Discussions

There has been recent work on geophysical problems modeled by the randomly forced quasigeostrophic equation (e.g., [21], [13], [15], [18], [10]). Unlike the deterministic quasigeostrophic equation whose well-posedness is well-known (e.g., [3], [22], [4]), there seems no rigorous results on global existence and uniqueness of the randomly forced quasigeostrophic equation. Our work provides such a rigorous result on global existence and uniqueness, under very mild conditions.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Babin, A. Mahalov and B. Nicolaenko, Global splitting, integrability and regularity of 3D Euler and Navier-Stokes equations for uniformly rotating fluids. *European J. Mech. B Fluids* **15** (1996), 291-300.

[3] A. F. Bennett and P. E. Kloeden, The dissipative quasigeostrophic equations, *Mathematika* **28** (1981), 265-285.

[4] A. J. Bourgeois and J. T. Beale, Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean, *SIAM J. Math. Anal.* **25** (1994), 1023-1068.

[5] P. Cessi and G. R. Ierley, Symmetry-breaking multiple equilibria in quasigeostrophic, wind-driven flows, *J. Phys. Oceanography*, **25** (1995), 1196-1205.

[6] B. Cushman-Roisin, *Introduction to Geophysical Fluid Dynamics*, Prentice Hall, 1994.

[7] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.

[8] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, 1996.

[9] G. Da Prato, A. Debussche and R. Temam, Stochastic Burgers equation, *Nonlinear Diff. Eqns. Appl.*, 1994, 389-402.

[10] T. DelSole and B. F. Farrell, A stochastically excited linear system as a model for quasigeostrophic turbulence: Analytic results for one- and two-layer fluids, *J. Atmos. Sci.* **52** (1995) 2531-2547.

[11] P. F. Embid and A. J. Majda, Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity, *Comm. PDEs* **21** (1996), 619-658.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer-Verlag, 1983.

[13] A. Griffa and S. Castellari, Nonlinear general circulation of an ocean model driven by wind with a stochastic component, *J. Marine Research* **49** (1991), 53-73.

[14] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.

[15] G. Holloway, Ocean circulation: Flow in probability under statistical dynamical forcing, in *Stochastic Models in Geosystems*, S. Molchanov and W. Woyczynski (eds.), Springer, 1996.

[16] V. Klyatskin and D. Gurarie, Random topography in geophysical models, in *Stochastic Models in Geosystems*, S. Molchanov and W. Woyczynski (eds.), Springer, 1996.

[17] D. D. Holm, Hamiltonian formulation of the baroclinic quasigeostrophic fluid equations, *Phys. Fluids* **29** (1986), 7-8.

[18] P. Müller, Stochastic forcing of quasi-geostrophic eddies, in *Stochastic Modelling in Physical Oceanography*, R. J. Adler, P. Müller and B. Rozovskii (eds.), Birkhäuser, 1996.

[19] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.

[20] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, 2nd edition, 1987.

[21] R. M. Samelson, Stochastically forced current fluctuations in vertical shear and over topography, *J. Geophys. Res.* **94** (1989) 8207-8215.

[22] S. H. Schochet, Singular limits in bounded domains for quasilinear symmetric hyperbolic systems having a vorticity equation, *J. Diff. Eqns.* **68** (1987), 400-428.

[23] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* Springer-Verlag, New York, 1988.