Variational Methods and Nonlinear Quasigeostrophic Waves *

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Abstract. In this paper, the authors discuss the zonally periodic steady quasigeostrophic waves in a $\beta-$plane channel, by using variational methods. A class of steady quasigeostrophic waves are determined by the potential vorticity field profile, $g(\cdot)$, which is a function of the stream function. They show that zonally periodic steady quasigeostrophic waves exist when the bottom topography and the potential vorticity field are bounded. They also show that these waves are unique if, in addition, the potential vorticity field profile is increasing and passes through the origin. Finally, they show that the zonal periodic wave in the case with $g(\psi) = \arctan(\psi)$ is nonlinearly stable in the sense of Liapunov, under a boundedness condition for the potential vorticity field for this zonal periodic wave, or equivalently, under suitable conditions on the bottom topography, β parameter, and zonal period T.

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1 Introduction

Geophysical fluid dynamicists often use simplified partial differential equation models which are intended to capture the key features of large scale phenomena and filter out undesired fast (high frequency) oscillations. An important example of such a partial differential equation is the quasigeostrophic model [16],[6]:

$$q_t + J(\psi, q) = 0, \tag{1.1}$$

where q is the potential vorticity

$$q = \Delta\psi - \frac{1}{R^2}\psi + \beta y + h(x, y),$$

$\psi(x, y, t)$ is the stream function, R is the Rossby deformation radius, $\beta > 0$ is the meridional gradient of the Coriolis parameter, and $h(x, y)$ is the bottom topography. Moreover, Δ is the Laplacian operator in the x, y plane and $J(f, g) = f_x g_y - f_y g_x$ is the Jacobian operator. This same equation also models plasma drift waves [8].

The quasigeostrophic equation may be derived as an approximation of the rotating shallow water equations by the conventional asymptotic expansion in small Rossby number [16]. Schochet [18] has recently shown that the shallow water flows converge to the quasigeostrophic flows in Sobolev norms in the limit of zero Rossby number, i.e., at asymptotically high rotation rate, for appropriate initial data.

Steady patterns correspond to permanent geophysical regimes and they have been studied in, e.g., [6], p221, [16], p93, Meacham and Flierl [12], [15]. There has also been much research on stability of steady states of the quasigeostrophic equation; see, for example, Holm et al. [10], Benzi et al. [3], McIntyre and Shepherd [11], Mu [13],[14], and Ripa [17].

The methods of functional analysis have often been applied effectively in geophysical fluid dynamics. See, e.g., Bourgoise and Beale [5], and Babin et al. [1], [2] for recent interesting results using this approach. This paper shows existence, uniqueness and nonlinear stability of zonally periodic steady quasigeostrophic waves under suitable conditions on the bottom topography and the potential vorticity field, by applying well-known variational methods in direct calculations, together with results from nonlinear partial differential equations, that illustrate these methods.

2 Steady Quasigeostrophic Waves

Steady quasigeostrophic waves satisfy

$$J(\psi, q) = 0. \tag{2.2}$$

Since $J(\psi, -\frac{1}{R^2}\psi) = 0$, this equation leads to

$$J(\psi, \Delta\psi + \beta y + h(x,y)) = 0, \tag{2.3}$$

which implies that ψ and $\Delta\psi + \beta y + h(x,y)$ are functionally dependent. We assume this dependence may be expressed as a semilinear elliptic partial differential equation

$$\psi_{xx} + \psi_{yy} + \beta y + h(x,y) = g(\psi), \tag{2.4}$$

where $g(\psi)$ is an arbitrary smooth function. Once $g(\psi)$ has been determined by specifying the potential vorticity at one point on each streamline, (2.4) determines the steady flow ([16], p.93). The quantity $\psi_{xx}+\psi_{yy}+\beta y+h(x,y)$ is called the potential vorticity. Due to the potential vorticity conservation, the boundedness of $g(\psi)$ means boundedness of the potential vorticity field. In the following we assume that $g(\psi)$ may be chosen and study zonally periodic steady flows.

3 Variational Methods and Topographic Effect

We consider equation (2.4) on the zonal domain ($\beta-$channel)

$$-\infty < x < \infty, \quad 0 < y < 1.$$

This problem has a variational principle, $\delta\mathcal{L} = 0$, with Lagrangian function

$$\mathcal{L}(\psi) = \int [\frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2 - (\beta y + h(x,y))\psi + G(\psi)]dxdy, \tag{3.5}$$

where $G(\psi) = \int_0^\psi g(s)ds$. We seek a solution ψ that is periodic in zonal direction x and satisfies zero Dirichlet boundary conditions in meridional y direction, namely, $\psi = 0$ at $y = 0$ and $y = 1$.

We denote as $H_T^1 = H_T^1((0,T)\times(0,1))$ the subspace of the usual Sobolev space $H^1((-\infty,\infty)\times(0,1))$ consisting of functions which are periodic in x with period $T > 0$ and satisfy zero Dirichlet boundary conditions at $y = 0$ and $y = 1$. Similarly we can define $L_T^2 = L_T^2((0,T)\times(0,1))$, and H_T^p, L_T^p for positive integers p. We also denote as $\|\cdot\|_1, \|\cdot\|$ the usual norms in H_T^1, L_T^2, respectively. Note that since the Poincaré inequality holds [4], [7], the expression $(\int \nabla\psi\cdot\nabla\psi dxdy)^{\frac{1}{2}}$ is an equivalent norm in H_T^1. Here and hereafter, $\int dxdy = \int_0^T\int_0^1 dxdy$. Moreover,

$$(w_1, w_2) = \int w_1 w_2 \; dxdy$$

is the usual scalar product in L^2_T.

A weak zonally periodic solution $\psi(x, y)$ in H^1_T for (2.4) is defined to satisfy

$$-(\nabla\psi, \nabla\phi) + (\beta y + h(x, y), \phi) - (g(\psi), \phi) = 0, \ \forall\phi \in H^1_T \tag{3.6}$$

or

$$\int [-\nabla\psi \cdot \nabla\phi + (\beta y + h(x, y))\phi - g(\psi)\phi] dxdy = 0, \ \forall\phi \in H^1_T, \tag{3.7}$$

where the left hand side defines a semi-bilinear form.

We use standard variational methods (as discussed in [19] and references therein) to prove existence of zonally periodic waves. To this end we show that $\mathcal{L}(\psi)$ is weakly lower semi-continuous and coercive, which implies that $\mathcal{L}(\psi)$ attains its infimum. The minimizer ψ^* for this infimum is a weak zonally periodic solution of (2.4). Recall that $\mathcal{L}(\psi)$ is called weakly lower semi-continuous, if for any weakly convergent sequence $\{\psi_n\}$ in H^1_T, there holds

$$\mathcal{L}(\psi) \leq \lim \inf_{n\to\infty} \mathcal{L}(\psi_n).$$

If

$$\mathcal{L}(\psi) \to \infty \ \ as \ \ \|\psi\|_1 \to \infty,$$

then $\mathcal{L}(\psi)$ is called coercive.

Bona et al. [4] considered steady periodic waves in stratified flows by variational methods. However, their results are valid for a different class of nonlinearity and do not apply to the present problem.

In showing that the Lagrangian $\mathcal{L}(\psi)$ is weakly lower semi-continuous, we first rewrite it as

$$\mathcal{L}(\psi) = \frac{1}{2}(\nabla\psi, \nabla\psi) - (\beta y + h(x, y), \psi) + \int [\int_0^\psi g(s)ds] dxdy. \tag{3.8}$$

Let $\{\psi_n\}$ be a weakly convergent sequence in H^1_T. In view of the inequalities,

$$\begin{aligned} 0 &\leq \frac{1}{2}(\nabla\psi - \nabla\psi_n, \nabla\psi - \nabla\psi_n) \\ &= \frac{1}{2}(\nabla\psi, \nabla\psi) - (\nabla\psi, \nabla\psi_n) + \frac{1}{2}(\nabla\psi_n, \nabla\psi_n) \end{aligned} \tag{3.9}$$

and

$$\lim_{n\to\infty} (\nabla\psi, \nabla\psi_n) = (\nabla\psi, \nabla\psi), \tag{3.10}$$

we have

$$
\begin{aligned}
0 &\leq \liminf_{n\to\infty} \frac{1}{2}(\nabla\psi - \nabla\psi_n, \nabla\psi - \nabla\psi_n) \\
&= -\frac{1}{2}(\nabla\psi, \nabla\psi) + \liminf_{n\to\infty} \frac{1}{2}(\nabla\psi_n, \nabla\psi_n)\,. \qquad (3.11)
\end{aligned}
$$

Thus,

$$
\liminf_{n\to\infty} \frac{1}{2}(\nabla\psi_n, \nabla\psi_n) \;\geq\; \frac{1}{2}(\nabla\psi, \nabla\psi). \qquad (3.12)
$$

We also have

$$
\lim_{n\to\infty}(\beta y + h(x,y), \psi_n) = (\beta y + h(x,y), \psi), \qquad (3.13)
$$

as long as $h(x,y)$ is $T-$periodic in x, and $h(x,0) = \beta + h(x,1) = 0$. (These conditions ensure that $\beta y + h(x,y)$ is in L_T^2.) Hence, the first and the second term in $\mathcal{L}(\psi)$ are weakly lower semi-continuous. Now consider the third term. We assume that $g(\psi)$ is bounded: $|g(\psi)| \leq M_1$. Note that

$$
\begin{aligned}
&\liminf_{n\to\infty}\{\int[\int_0^{\psi_n} g(s)ds]dxdy - \int[\int_0^{\psi} g(s)ds]dxdy\} \\
=\;& \liminf_{n\to\infty} \int[\int_{\psi}^{\psi_n} g(s)ds]dxdy \\
=\;& \liminf_{n\to\infty} \int g(\psi + \theta(\psi_n - \psi))(\psi_n - \psi)dxdy \\
\geq\;& -\liminf_{n\to\infty} M_1 \int |\psi_n - \psi| dxdy \\
\geq\;& 0, \qquad (3.14)
\end{aligned}
$$

where we have used the mean value theorem of integral calculus to write $\int_{\psi}^{\psi_n} g(s)ds = g(\psi + \theta(\psi_n - \psi))(\psi_n - \psi)$ for some θ in $(0,1)$.

Equations (3.12), (3.13), (3.14) together imply

$$
\liminf_{n\to\infty} \mathcal{L}(\psi_n) \geq \mathcal{L}(\psi). \qquad (3.15)
$$

This calculation proves the following.

Lemma 1 *If $g(\psi)$ is bounded, $h(x,y)$ is $T-$periodic in x, and $h(x,0) = \beta + h(x,1) = 0$, then the Lagrangian function $\mathcal{L}(\psi)$ is weakly lower semi-continuous in H_T^1.*

Next we show coercivity.

Lemma 2 *If the bottom topography $h(x,y)$ and $g(\psi)$ are bounded, then the Lagrangian function $\mathcal{L}(\psi)$ is coercive in H_T^1.*

Proof. Since the bottom topography is bounded: $|h(x,y)| \le M_2$, we have the estimate

$$\begin{aligned}\int[(\beta y + h(x,y))\psi]dxdy &\le (\beta + M_2)\int|\psi|dxdy \\ &\le C_1/\epsilon + C_2\epsilon\|\psi\|_1^2, \end{aligned} \tag{3.16}$$

by the Cauchy-Schwarz, Youngs and Poincaré inequalities, where $C_1, C_2 > 0$ are constants depending on β, M_2, T, and $\epsilon > 0$ is to be determined. Similarly,

$$\begin{aligned}\int[\int_0^\psi g(s)ds]dxdy &\le M_1\int|\psi|dxdy \\ &\le C_3/\epsilon + C_4\epsilon\|\psi\|_1^2. \end{aligned} \tag{3.17}$$

Thus we have

$$\begin{aligned}\mathcal{L}(\psi) &= \frac{1}{2}(\nabla\psi, \nabla\psi) - (\beta y + h(x,y), \psi) + \int[\int_0^\psi g(s)ds]dxdy \\ &\ge (\frac{1}{2} - C_2\epsilon - C_4\epsilon)\|\psi\|_1^2 - C_1/\epsilon - C_3/\epsilon, \end{aligned} \tag{3.18}$$

where we can take $\epsilon > 0$ such that $\frac{1}{2} - C_2\epsilon - C_4\epsilon > 0$. Therefore, the estimate (3.18) implies that

$$\mathcal{L}(\psi) \to \infty \ \text{ as } \ \|\psi\|_1 \to \infty,$$

i.e., $\mathcal{L}(\psi)$ is coercive. This completes the proof of Lemma 2.

By Theorem 1.2 in [19], we conclude there exists a weak solution $\psi(x,y)$ in H_T^1 of (2.4), i.e., there exists a weak zonally periodic steady quasigeostrophic wave.

Actually this weak solution $\psi(x,y)$ is smooth (with continuous second order derivatives) as along as $h(x,y)$ and $g(\psi)$ are smooth. In fact, it follows from the L^p-elliptic theory [7] that ψ is in H^p for any $p < \infty$. The Sobolev embedding theorem [7] implies that ψ is also in the Hölder space $C^{1,\alpha}$, and, furthermore, the Schauder theory [7] yields that ψ is in fact in $C^{2,\alpha}$.

In conclusion, we obtain the following theorem.

Theorem 1 *Consider the steady quasigeostrophic waves satisfying*

$$\psi_{xx} + \psi_{yy} + \beta y + h(x,y) = g(\psi), \tag{3.19}$$

on the zonal domain ($\beta-$channel)

$$-\infty < x < \infty, \ \ 0 < y < 1,$$

where: $\psi(x,y)$ is the streamfunction satisfying zero Dirichlet boundary condition at $y = 0$ and $y = 1$; $\beta > 0$ is the meridional gradient of the Coriolis *parameter; $h(x,y)$ is the smooth bottom topography; and the potential vorticity field $g(\psi)$ is a smooth function.*

Assume boundedness of the bottom topography $h(x,y)$ and the potential vorticity field $g(\psi)$.

(i) If $h(x,y)$ is $T-$periodic in x, and $h(x,0)=0, \beta+h(x,1)=0$, then there exists a smooth zonally periodic steady quasigeostrophic wave of period T; in particular,

(ii) If $h=h(y)$ is independent of x and $h(0)=h(1)=0$, then, for any $T>0$, there exists a smooth zonally periodic steady quasigeostrophic wave of period T.

Examples of bottom topography $h(x,y)$ and potential vorticity field $g(\psi)$ satisfying the conditions in Theorem 1 are: $h(x,y)=y(y-1)\sin(\frac{2\pi}{T}x)-\beta$ and $g(\psi)=\sin(\psi)$, $g(\psi)=\frac{1}{1+\psi^2}$, $g(\psi)=\tanh(\psi)=\frac{e^{\psi}-e^{-\psi}}{e^{\psi}+e^{-\psi}}$, or $g(\psi)=\arctan(\psi)$. Note that $-1<\tanh(\psi)<1$ and $-\frac{\pi}{2}<\arctan(\psi)<\frac{\pi}{2}$.

4 Uniqueness and Nonlinear Stability of Zonal Periodic Patterns

In this section, we discuss the uniqueness and nonlinear stability of zonally periodic steady quasigeostrophic waves. We assume that $g(\psi)$ is increasing and passes through the origin, i.e., $g(0)=0$. This implies that

$$g(s)s \geq 0, \tag{4.20}$$

$$(g(s_1)-g(s_2))(s_1-s_2) \geq 0, \tag{4.21}$$

for all s. We can now use a result in Zeidler [20], Corollary 26.13, p.572, to conclude that the zonally periodic steady quasigeostrophic waves claimed in Theorem 1 are unique. Thus, we have the following result.

Theorem 2 *Assume the conditions in Theorem 1. Moreover, assume that the potential vorticity field $g(\psi)$ is increasing and passes through the origin ($g(0)=0$), as well as being bounded.*

Then, the zonally periodic steady quasigeostrophic waves whose existence is shown in the proof of Theorem 1 are unique.

For example, potential vorticity fields $g(\psi)$ satisfying the conditions of Theorem 2 are: $g(\psi)=\tanh(\psi)$ and $g(\psi)=\arctan(\psi)$. Note that $\tanh(s)$ is increasing, $\tanh(0)=0$, and $\arctan(\psi)$ is increasing, $\arctan(0)=0$.

Under a slight additional condition, the corresponding unique zonally periodic steady quasigeostrophic waves can be shown to be nonlinearly stable (in the sense of Liapunov). For existence (Theorem 1) and uniqueness (Theorem 2) of zonally periodic steady quasigeostrophic waves, we have assumed that $g(\cdot)$ is bounded and increasing. The nonlinear stability result (in the spirit of Arnold's first theorem) in Holm et al. [10] requires that the derivative of the inverse of $g(\cdot)$ to be bounded between two positive constants. This imples that the derivative of $g(\cdot)$ itself is also bounded between two positive constants, which further implies that the potential vorticity field $g(\cdot)$ is unbounded. Thus the nonlinear stability result in [10] does not directly apply in our case. Note also that some other nonlinear stability results for steady quasigeostrophic waves are in the spirit of Arnold second theorem, which requires that the inverse of g (and thus g itself) to be decreasing (e.g., [11], [13],[14], and [17]), do not apply to our case either.

However, by following Holm et. al [9], we modify the inverse of g (or g itself) so that it satisfies the nonlinear stability conditions in Holm et al. [10], and thus we can still show nonlinear stability, under suitable conditions on bottom topography $h(x, y)$, β parameter and period T.

We illustrate this idea in an example with the potential vorticity field $g(\psi) = \arctan(\psi)$. As we remarked above, in this case, we have a unique zonally periodic steady quasigeostrophic wave of period T, $\psi_0(x, y)$, when topography $h(x, y)$ and the β parameter satisfy the conditions in Theorem 1. In the following we find sufficient conditions so that $\psi_0(x, y)$ is nonlinearly stable.

Note that

$$\Delta\psi_0 - \frac{1}{R^2}\psi_0 + \beta y + h(x, y) = \arctan(\psi_0), \tag{4.22}$$

or

$$\psi_0 = \tan(\Delta\psi_0 - \frac{1}{R^2}\psi_0 + \beta y + h(x, y)). \tag{4.23}$$

Note that $\frac{d}{ds}\tan(s) = \frac{1}{cos^s(s)} > 0$ (for $-\frac{\pi}{2} < s < \frac{\pi}{2}$), but this derivative is not bounded between two positive constants.

As in [9] we consider the conserved functional

$$H_C(\psi) = \int [\frac{1}{2}(\nabla\psi)^2 + \frac{1}{2R^2}\psi^2 + C(q)]dxdy, \tag{4.24}$$

where the potential vorticity $q(x, y) = \Delta\psi - \frac{1}{R^2}\psi + \beta y + h(x, y)$, and $C(q)$ is the Casimir function. To complete the nonlinear stability argument, we need to choose $C(q)$ so that ψ_0 is a critical point of $H_C(\psi)$, which requires that $C(q) = -\int_0^q g^{-1}(s)ds$, and that $-C''(q) = (g^{-1})'$ is bounded between two positive constants.

In the case $g(s) = \arctan(s)$ we have

$$
\begin{aligned}
C(s) &= -\int_0^s \tan(s)ds = log(|cos(s)|), & (4.25)\\
C'(s) &= -\tan(s), & (4.26)\\
C''(s) &= -\frac{1}{cos^2(s)} < 0, & (4.27)
\end{aligned}
$$

for $-\frac{\pi}{2} < s < \frac{\pi}{2}$. So $C(s)$ is convex downward, but $C''(s)$ is unbounded below.

We modify $C(s)$ to $\tilde{C}(s)$ so that $-\tilde{C}''(s)$ is bounded between two positive constants and ψ_0 is a critical point of $H_{\tilde{C}}(\psi)$.

To this end, we assume that

$$
\begin{aligned}
-\frac{\pi}{2} + c_1 = \min q_0(x,y) \le q_0(x,y) &= \Delta\psi_0 - \frac{1}{R^2}\psi_0 + \beta y + h(x,y) \\
&\le \max q_0(x,y) = \frac{\pi}{2} - c_2, \qquad (4.28)
\end{aligned}
$$

for some postive constants $0 < c_1, c_2 < \frac{\pi}{2}$. Here $q_0(x,y)$ is the potential vorticity corresponding to the zonally periodic steady state $\psi_0(x,y)$, and the minimum and maximum are taken on $D = [0,T] \times [0,1]$. This assumption means that the potential vorticity corresponding to the zonally periodic steady state takes value within $(-\frac{\pi}{2}, \frac{\pi}{2})$, the domain of definition of $\tan(s)$.

To modify $C(s)$, we cut it off outside $-\frac{\pi}{2} + c_1 \le s \le \frac{\pi}{2} - c_2$, and patch two quadratic polynomials smoothly (matching the first and second order derivatives) at end points $s = -\frac{\pi}{2} + c_1, \frac{\pi}{2} - c_2$. Namely, we define

$$
\tilde{C}(s) = \begin{cases} -\frac{1}{2\cos^2(-\frac{\pi}{2}+c_1)}s^2 + a_1 s + b_1, & s \le -\frac{\pi}{2} + c_1 \\ C(s) = \log|\cos(s)|, & -\frac{\pi}{2} + c_1 \le s \le \frac{\pi}{2} - c_2 \\ -\frac{1}{2\cos^2(\frac{\pi}{2}-c_2)}s^2 + a_2 s + b_2, & s \ge \frac{\pi}{2} - c_2 \end{cases}
$$

where

$$
\begin{aligned}
a_1 &= \frac{-\frac{\pi}{2} + c_1}{\cos^2(-\frac{\pi}{2} + c_1)} - \tan(-\frac{\pi}{2} + c_1), \\
b_1 &= \log|\cos(-\frac{\pi}{2} + c_1)| + \frac{(-\frac{\pi}{2} + c_1)^2}{2\cos^2(-\frac{\pi}{2} + c_1)} - a_1(-\frac{\pi}{2} + c_1), \\
a_2 &= \frac{\frac{\pi}{2} - c_2}{\cos^2(\frac{\pi}{2} - c_2)} - \tan(\frac{\pi}{2} - c_2), \\
b_2 &= \log|\cos(\frac{\pi}{2} - c_2)| + \frac{(\frac{\pi}{2} - c_2)^2}{2\cos^2(\frac{\pi}{2} - c_2)} - a_2(\frac{\pi}{2} - c_2).
\end{aligned}
$$

Since $\tilde{C}$ and C coincide on the interval $[\min q_0, \max q_0]$, it follows that ψ_0 is a critical point of $H_{\tilde{C}}(\psi)$. Moreover, $-C''$ is bounded between two positive constants

$$0 < \frac{1}{\cos^2(\frac{\pi}{2} - \max(c_1, c_2))} \leq -\tilde{C}''(s) \leq \frac{1}{\cos^2(\frac{\pi}{2} - \min(c_1, c_2))}, \qquad (4.29)$$

for all s.

Then we can use the nonlinear stability result in Holm et. al [10] for (multi-layer) quasigeostrophic steady flows, or follow the argument in Holm et. al [9], to conclude nonlinear stability for the unique zonally periodic steady quasigeostrophic waves claimed in Theorem 2. Note that for the periodic zonal channel here, the argument in [10] still goes through in our case. See Benzi et al. [3] for discussions on zonal periodic boundary conditions. Therefore we have the following result.

Theorem 3 *Assume that the bottom topography $h(x, y)$ satisfies the conditions in Theorem 1 and consider the case with the potential vorticity field $g(\psi) = \arctan(\psi)$. By Theorem 1 and Theorem 2, there exists a unique zonally periodic (with period T) steady quasigeostrophic wave $\psi_0(x, y)$ with the corresponding potential vorticity $q_0(x, y)$.*

If the potential vorticity field $q_0(x, y)$ takes values in a closed interval within $(-\frac{\pi}{2}, \frac{\pi}{2})$, the domain of definition for $g^{-1}(s) = \tan(s)$, then the unique zonally periodic quasigeostrophic wave is nonlinearly stable (in the sense of Liapunov) in the norm $\int[(\nabla\psi)^2 + \frac{1}{R^2}\psi^2 + (\Delta^2\psi)^2]dxdy$.

We remark that, although the nonlinear stability conditions are in terms of potential vorticity, they equivalently put constraints on the bottom topography, β parameter, and zonal period T.

5 Discussions

We have found conditions for the existence, uniqueness and nonlinear stability of zonally periodic steady quasigeostrophic waves in a $\beta-$plane channel. Namely, these waves exist and are unique, provided: (i) the bottom topography $h(x, y)$ is bounded, and (ii) the potential vorticity field $g(\psi)$ is bounded, and (for uniqueness) the function g is increasing and passes through the origin. Moreover, the zonal periodic wave in the case with $g(\psi) = \arctan(\psi)$ is shown to be nonlinearly stable in the sense of Liapunov, under a boundedness condition for the potential vorticity field for this zonal periodic wave, or equivalently, under suitable conditions on the bottom topography, β parameter, and zonal period T. This stability analysis can also be done for the case with $g(\psi) = \tanh(\psi)$ or other cases.

We also remark that if the bottom topography is independent of zonal direction: $h = h(y)$, then (2.4) becomes

$$\psi_{xx} + \psi_{yy} + \beta y + h(y) = g(\psi). \tag{5.30}$$

In this case, there exist $x-$*independent* nonlinear quasigeostrophic waves satisfying

$$\psi_{yy} + \beta y + h(y) = g(\psi(y)), \tag{5.31}$$

which may be regarded as zonally periodic with *any* period.

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