Singular Reduction of Axially Symmetric Perturbations of the Isotropic Harmonic Oscillator

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Abstract

The normal form of an axially symmetric perturbation of the isotropic harmonic oscillator is invariant under a 2-torus action and thus integrable in three degrees of freedom. The reduction of this symmetry is performed in detail, showing how the singularities of the reduced phase space determine the distribution of periodic orbits and invariant 2-tori in the original perturbation. To illustrate these results a particular quartic perturbation is analysed.

§1 Introduction

Consider a Hamilton function H in 3-space that is invariant under rotation about the vertical axis, *i.e.* axially symmetric. We assume that the origin is an elliptic equilibrium and that the three (normal) frequencies of this equilibrium are equal. This means that the linearization $DX_H(0)$ is the isotropic harmonic oscillator, or in 1:1:1 resonance. Thus, near the origin the Hamiltonian system is a small axially symmetric perturbation of a (maximally) superintegrable system — all orbits of the isotropic harmonic oscillator are periodic.

The situation where H is an arbitrary (small) axially symmetric perturbation of the isotropic harmonic oscillator is only slightly more general. After a translation the origin is again an equilibrium, and the only further effect is that the perturbation may also detune the frequencies of the equilibrium.

In astrophysics one models the motion in the core of an (axially symmetric) galaxy as perturbations of the isotropic harmonic oscillator. The dominant part in 1:1:1 resonance accounts for the central part of the galaxy being spherically symmetric. Furthermore one assumes an additional reflectional symmetry, cf. [Saaf;68], [Verhulst;79], [de Zeeuw;85,87] and references therein.

Only few general statements can be made about perturbations of superintegrable systems in three degrees of freedom. In [Arnol'd;63] KAM-theory is adapted to show that, if the perturbation "removes the degeneracy", a measure-theoretically large part of the phase space becomes filled by invariant 3-tori. Unlike perturbations of non-degenerate integrable systems, where the geometry of the *Cantor-fibration* into 3-tori is imposed by the unperturbed system, the distribution of invariant 3-tori in the perturbation of a superintegrable system is determined by the perturbation itself.

The axial symmetry implies that, next to the Hamilton function H , the third component N of the angular momentum is an integral of motion. Thus we are one integral short an integrable system. The energy H_0 of the isotropic harmonic oscillator is a "formal integral" : it becomes an integral only after a normalizing co-ordinate transformation and the truncation of the higher order terms. In this way one obtains a normal form of order n that is ε^{n+1}-close to H (if $\|H - H_0\| \sim \varepsilon$). Often it is sufficient to work with the normal form of order 1 which may be computed as the average $\bar{H}$ of H along the periodic orbits of the isotropic harmonic oscillator.

The integrable system $\bar{H}$ (with integrals $\bar{H}, N$ and H_0) makes the phase space a *ramified 3-torus bundle*, consisting of 3-parameter families of invariant 3-tori, 2-parameter families of invariant 2-tori, 1-parameter families of periodic orbits and the equilibrium at the origin. It is exactly this distribution of 3-tori, 2-tori and periodic orbits that determines how the Cantor families of invariant 3-tori and 2-tori and the periodic orbits of X_H are distributed in phase space. Therefore, our first goal is to understand the integrable approximation $\bar{H}$ of H . The 2-torus symmetry generated by X_N and X_{H_0} allows to reduce to a one-degree-of-freedom problem. We require these systems to be *structurally stable* as a family, parametrized by the two actions N and H_0. If they were not we would try the normal form of order 2 *etc.* The next step is to reconstruct the dynamics of the normal form $\bar{H}$ from the family of one-degree-of-freedom systems, and the final step has to relate the dynamics defined by $X_{\bar{H}}$ to the original system X_H .

The main object of the present paper is to explain the reduction of the 2-torus symmetry, which lies at the basis of the program sketched above. Since there are nontrivial isotropy groups, regular reduction does not apply. Indeed, all reduced phase spaces turn out to have singular points. We also keep in mind that we want to use the reduction to understand the dynamics of X_H . Both H and $\bar{H}$ are axially symmetric, so first reducing this symmetry allows to perform the perturbation analysis in two degrees of freedom. It turns out that the reduction of the axial symmetry transforms the unperturbed system X_{H_0} into the 2:1 resonance, which is then reduced in a second step.

This paper is organized as follows. In the remaining part of this introduction we recall the dynamics of the isotropic harmonic oscillator. In Section 2 the reduction procedure is carried out. We study the singularities that occur and give a physical interpretation of the corresponding motion. This reduction procedure shall be applied to the normal form. The necessary theorem is generalized in Section 3 to the present case of *Poisson spaces*. The final two sections explain how the reduction is used to study a perturbation of the isotropic harmonic oscillator. We exemplify our results with the perturbing galactic-type potential $\varepsilon(x_1^2+x_2^2)x_3^2$. Section 4 contains the study of the one-degree-of-freedom problem defined by the normal form and the reconstruction of the corresponding two-degree-of-freedom system (with the axial symmetry still reduced). In Section 5 we see how the flow of the normal form is perturbed when the higher order terms are added. Furthermore the dynamics of the full system is reconstructed.

1.1 The isotropic harmonic oscillator

We are going to perturb the harmonic oscillator with three equal frequencies. Therefore, we now recapitulate the main facts about this system. The presentation of these well-known results also allows us to fix some notations. The reader may find more details in [Jauch,Hill;40], [Fradkin;65], [Marmo,Saletan,Simoni,Vitale;85].

In one degree of freedom the harmonic oscillator has the equations of motion

$$\dot{x} = \frac{\partial H_0}{\partial p} = p \quad , \quad \dot{p} = -\frac{\partial H_0}{\partial x} = -\omega^2 x \qquad ;$$

the energy being $H_0 = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2$. The superposition of three such systems is called an *isotropic harmonic oscillator* (in three degrees of freedom) if the frequency $\omega > 0$ is the same for all three subsystems. As a result the corresponding flow

$$\varphi_t(x,p) = \begin{pmatrix} (\cos\omega t)I_3 & \frac{1}{\omega}(\sin\omega t)I_3 \\ -\omega(\sin\omega t)I_3 & (\cos\omega t)I_3 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

is periodic with common period $\frac{2\pi}{\omega}$, here I_3 denotes the 3×3 identity matrix. In particular the isotropic harmonic oscillator is a *maximally superintegrable* Hamiltonian system. This can also be deduced from the symmetries of the system. Indeed, H_0 is invariant under an action of $U(3)$, the group of unitary complex 3×3-matrices. To show this, and also for other purposes, it is convenient to introduce the following complex notation.

In complex variables $z_j = \omega x_j + \mathrm{i} p_j$ $(j{=}1{,}2{,}3)$ the symplectic structure becomes

$$\sum_{j=1}^{3} \mathrm{d}x_j \wedge \mathrm{d}p_j \quad = \quad \frac{\mathrm{i}}{2\omega} \sum_{j=1}^{3} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$$

and the energy reads

$$H_0 \quad = \quad \frac{1}{2}(p\,|\,p) + \frac{\omega^2}{2}(x\,|\,x) \quad = \quad \frac{1}{2}(z\,|\,z) \qquad .$$

Here and in the following $(..|..)$ denotes the standard inner product (in the hermitian case of complex entries : linear in the first argument). In z-variables it is immediate that the Hamilton function H_0 is invariant under the standard action $U(3) \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$. From Noether's theorem, or by direct verification, one concludes that real and imaginary part of all components of the hermitian 3×3-matrix $(z_j \bar{z}_k)_{j,k=1,2,3}$ are invariant under the flow $\varphi_t(z) = \mathrm{e}^{-\mathrm{i}\omega t} z$ of our equations of motion $\dot{z} = -\mathrm{i}\omega\, z$. On the other hand, a short computation shows that the common level sets of the $z_j \bar{z}_k$ coincide with the trajectories of this flow.

Remark : The trace of the matrix $(z_j \bar{z}_k)_{j,k=1,2,3}$ is $2H_0$. Therefore, many authors prefer to restrict the above action to the group $SU(3)$ of unitary 3×3-matrices with determinant 1 . The non-diagonal components of the imaginary part $\operatorname{Im} z_j \bar{z}_k = \omega(x_j p_k - x_k p_j)$ yield the angular momentum, induced by the subgroup $SO(3) < SU(3)$ of orthogonal real 3×3-matrices with determinant 1 . The real part $(\operatorname{Re} z_j \bar{z}_k)_{j,k=1,2,3}$ determines the ellipse in configuration space along which the flow proceeds, cf. [Holas,March;90]. The trajectory in phase space is completed by a similar ellipse in momentum space, with a phase shift of $\frac{\pi}{2}$.

1.2 Axially symmetric perturbations

Our main interest is not so much the isotropic harmonic oscillator itself, but a (small) perturbation H_ε of H_0 . We assume H_ε to be *axially symmetric* ; by this we mean that H_ε is invariant under the *axial S^1-action*

$$\begin{array}{rccc} \varrho : & S^1 \times T^*\mathbb{R}^3 & \longrightarrow & T^*\mathbb{R}^3 \\ & (\nu, (x,p)) & \mapsto & (\exp_\nu x, \exp_\nu p) \end{array} \tag{1}$$

where $\exp_\nu$ stands for $\exp\begin{pmatrix} 0 & -\nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Note that ϱ is the lift of the action $(\nu, x) \mapsto \exp_\nu x$ of S^1 on the configuration space $\mathbb{R}^3$ to the cotangent bundle. The corresponding geometric description is a rotation by an angle ν about the vertical axis (the axis along x_3). In z-variables the axial S^1-action becomes

$$\begin{array}{rccc} \varrho : & S^1 \times \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ & (\nu, z) & \mapsto & \exp_\nu z \end{array} \quad . \tag{2}$$

Because of the axial symmetry H_ε induces a Hamilton function, again denoted H_ε , on the quotient space $T^*\mathbb{R}^3/_\varrho \;=\; \mathbb{C}^3/_{S^1}$. In this way the number of degrees of freedom is lowered from three to two.

Remark : As already mentioned, perturbations of the isotropic harmonic oscillator H_0 are encountered in various situations. Given an equilibrium of a Hamiltonian system H with linearization $DX_H(0) = X_{H_0}$, we may perform the scaling $x \mapsto \varepsilon x$, $p \mapsto \varepsilon p$, $t \mapsto \varepsilon^2 t$ to obtain $H_\varepsilon = H_0 + \epsilon H_1 + \varepsilon^2 H_2 + \ldots$. Here H_i contains the terms of order $i+2$ in (x,p) . If H_ε is invariant under an S^1-action, then the co-ordinates (x,p) can often be chosen in such a way that this S^1-action turns into the axial S^1-action.

§2 Symmetry reduction

We are given an axially symmetric Hamilton function H_ε on $\mathbb{C}^3$. To study H_ε we first want to understand its average $\bar{H}_\varepsilon$, which in addition is invariant under the periodic flow φ of the isotropic harmonic oscillator H_0 . Using these symmetries, we may reduce $\bar{H}_\varepsilon$ to a one-degree-of-freedom system.

The axial S^1-action ϱ leaves H_0 invariant, so φ and ϱ commute and induce a 2-torus action on $\mathbb{C}^3$. But rather than directly reducing this 2-torus action, we proceed step by step and first reduce the axial S^1-action. Indeed, also the original Hamilton function H_ε is axially symmetric, and we may later consider the axially reduced H_ε as a perturbation of $\bar{H}_\varepsilon$ in *two* degrees of freedom, considerably facilitating our task.

2.1 Reduction of the axial symmetry

The axial S^1-action ϱ on $\mathbb{C}^3$ is not a free action since the subspace $\{\,(0,0,z_3) \mid z_3 \in \mathbb{C}\,\}$ is left fixed under all rotations. Therefore, the regular reduction theorem in [Abraham,Marsden;78] does not apply. Instead, we have to resort to singular reduction, as described in [Arms,Cushman,Gotay;91], [Sjamaar,Lerman;91], [Cushman,Bates;97]. Correspondingly, to obtain the equations of motion from the Hamilton function, we work with Poisson brackets rather than symplectic structures.

The key idea is to use a set $\pi_1, \ldots, \pi_n$ of *generators* of the ring of ϱ-invariant functions as co-ordinates on the reduced space $\mathbb{C}^3/_{S^1}$. To be precise, the π_i allow an embedding of $\mathbb{C}^3/_{S^1}$ in $\mathbb{R}^n$, where the relations between the π_i determine the image of this embedding (which we identify with $\mathbb{C}^3/_{S^1}$). Restricting to a subset of $\mathbb{C}^3$ where ϱ acts (properly and) freely yields an open smooth submanifold in $\mathbb{C}^3/_{S^1}$, while points with nontrivial isotropy group lead to singularities of $\mathbb{C}^3/_{S^1}$.

Proposition 2.1

Let ϱ denote the axial S^1-action (2) *on $\mathbb{C}^3$. The functions*

$$\begin{aligned} \sigma_1 &= z_1^2 + z_2^2 \\ \sigma_2 &= z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 + \bar{z}_1 z_2 \\ \sigma_3 &= z_3 \end{aligned} \tag{3}$$

generate the Poisson algebra of (smooth) ϱ-invariant functions on $\mathbb{C}^3$. They are subject to the relations $\sigma_1\bar{\sigma}_1 = (\mathrm{Re}\,\sigma_2)^2 - (\mathrm{Im}\,\sigma_2)^2$, $\mathrm{Re}\,\sigma_2 \geq 0$.

Proof :

Using $z_j = \omega x_j + \mathrm{i} p_j$ we express the invariant polynomials

$$\begin{array}{lll} \pi_1 = x_1^2 + x_2^2 & \pi_3 = x_1 p_1 + x_2 p_2 & \pi_5 = x_3 \\ \pi_2 = p_1^2 + p_2^2 & \pi_4 = x_1 p_2 - x_2 p_1 & \pi_6 = p_3 \end{array}$$

as

$$\begin{array}{lll} \pi_1 = \frac{1}{2\omega^2}(\mathrm{Re}\,\sigma_1 + \mathrm{Re}\,\sigma_2) & \pi_3 = \frac{1}{2\omega}\mathrm{Im}\,\sigma_1 & \pi_5 = \frac{1}{\omega}\mathrm{Re}\,\sigma_3 \\ \pi_2 = \frac{1}{2}(\mathrm{Re}\,\sigma_2 - \mathrm{Re}\,\sigma_1) & \pi_4 = \frac{1}{2\omega}\mathrm{Im}\,\sigma_2 & \pi_6 = \mathrm{Im}\,\sigma_3 \end{array} ,$$

leading the situation back to Chapter IV, Claim 2.1 of [Cushman,Bates;97] ; **q.e.d.**

An elementary computation yields the Poisson structure, see *table 2.1.* In particular $N(\sigma) = \frac{1}{2\omega}\mathrm{Im}\,\sigma_2$ is a *Casimir element* of the Poisson algebra. This does not come as a surprise, $N = \frac{\mathrm{i}}{2\omega}(z_1\bar{z}_2 - \bar{z}_1 z_2) = x_1 p_2 - x_2 p_1$ is the third component of the angular momentum and generates the axial S^1-action ϱ (*i.e.* ϱ is the flow of the Hamiltonian vector field X_N).

$\{A,B\}$	$\mathrm{Re}\,\sigma_1$	$\mathrm{Im}\,\sigma_1$	$\mathrm{Re}\,\sigma_2$	$\mathrm{Im}\,\sigma_2$	$\mathrm{Re}\,\sigma_3$	$\mathrm{Im}\,\sigma_3$	B
$\mathrm{Re}\,\sigma_1$	0	$4\omega\mathrm{Re}\,\sigma_2$	$4\omega\mathrm{Im}\,\sigma_1$	0	0	0	
$\mathrm{Im}\,\sigma_1$	$-4\omega\mathrm{Re}\,\sigma_2$	0	$-4\omega\mathrm{Re}\,\sigma_1$	0	0	0	
$\mathrm{Re}\,\sigma_2$	$-4\omega\mathrm{Im}\,\sigma_1$	$4\omega\mathrm{Re}\,\sigma_1$	0	0	0	0	
$\mathrm{Im}\,\sigma_2$	0	0	0	0	0	0	
$\mathrm{Re}\,\sigma_3$	0	0	0	0	0	ω	
$\mathrm{Im}\,\sigma_3$	0	0	0	0	$-\omega$	0	
A							

Table 2.1: The Poisson bracket on the reduced phase space.

In fact *table 2.1* defines a Poisson structure on $\mathbb{C}^3$. The function defined by

$$P(\sigma) \;=\; (\mathrm{Re}\,\sigma_1)^2 + (\mathrm{Im}\,\sigma_1)^2 - (\mathrm{Re}\,\sigma_2)^2 + (\mathrm{Im}\,\sigma_2)^2 \qquad (4)$$

is the second Casimir element, so $P^{-1}(0) \subseteq \mathbb{C}^3$ is invariant under every Hamiltonian system on "the σ-space" $\mathbb{C}^3$. Note that this also holds for the reduced phase space $\mathbb{C}^3/_{S^1} = \{\sigma \in \mathbb{C}^3 \mid P(\sigma) = 0\,,\ \mathrm{Re}\,\sigma_2 \geq 0\}$, cf. the remark below. All other Casimir elements are functions of N and P as the Poisson bracket in *table 2.1* has rank 4 .

The relations do not involve σ_3 — the axial symmetry only concerns the planar subsystem. To understand the singularity of the reduced system it is therefore sufficient to study it after projecting $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$ to the first two co-ordinates. In $\mathbb{C}^2 \cong \mathbb{R}^4$ the relations

$$(\mathrm{Re}\,\sigma_1)^2 + (\mathrm{Im}\,\sigma_1)^2 + (\mathrm{Im}\,\sigma_2)^2 \;=\; (\mathrm{Re}\,\sigma_2)^2 \quad , \qquad \mathrm{Re}\,\sigma_2 \geq 0$$

describe a cone — the upper half of the standard double cone in $\mathbb{R}^4$. If we fix the value a of N , the level set is (the upper half of) a hyperboloid for $a \neq 0$, and (half of) a standard cone in $\mathbb{R}^3$ for $a = 0$. To return to the full reduced phase space we have to attach a plane $\mathbb{C}$ to every point. Hence, the level sets $\{\sigma \in \mathbb{C}^3 \mid N(\sigma) = a\,,\ P(\sigma) = 0\}$ are diffeomorphic to $\mathbb{C}^2$ for $a \neq 0$, while for $a = 0$ this set is only homeomorphic to $\mathbb{C}^2$, having a singular stratum $\{(0,0,\sigma_3) \mid \sigma_3 \in \mathbb{C}\}$.

Remark : The Poisson space $\mathbb{C}^3/_{S^1}$ is stratified into symplectic strata, in accordance with [Sjamaar,Lerman;91]. In particular, the singular stratum $\{\sigma \in \mathbb{C}^3 \mid P(\sigma) = 0\,,\ \mathrm{Re}\,\sigma_2 = 0\}$ of $P^{-1}(0)$ is invariant under every Hamiltonian system on the σ-space $\mathbb{C}^3$.

Remark : It is not always the case that the Poisson structure on a singular reduced space is the restriction of a Poisson structure on the "ambient space" $\mathbb{R}^n$ where n denotes the number of generators of the ring of invariant functions. A counter example is the 1:1:2 resonance, see [Egilsson;95]. It works here because the singular reduced space is a level set of a Casimir element.

2.2 Reduction of the oscillator symmetry

The flow φ of the isotropic harmonic oscillator commutes with the axial S^1-action (2) and induces an action on $\mathbb{C}^3/_{S^1} \subseteq \mathbb{C}^3$ which we denote by the same letter. We extend φ to the whole σ-space $\mathbb{C}^3$ in the natural way and obtain the flow

$$\varphi_t(\sigma) \;=\; (\mathrm{e}^{-2\mathrm{i}\omega t}\sigma_1, \sigma_2, \mathrm{e}^{-\mathrm{i}\omega t}\sigma_3)$$

of the Hamiltonian vector field defined by (the natural extension of) H_0 on $\mathbb{C}^3$. Indeed, $H_0(\sigma) = \frac{1}{2}\mathrm{Re}\;\sigma_2 + \frac{1}{2}(\mathrm{Re}\;\sigma_3^2 + \mathrm{Im}\;\sigma_3^2)$, and using *table 2.1* we obtain the equations of motion $\dot{\sigma}_1 = -2\mathrm{i}\omega\sigma_1$, $\dot{\sigma}_2 = 0$, $\dot{\sigma}_3 = -\mathrm{i}\omega\sigma_3$. Normalizing the period to 2π , we obtain the reduced *oscillator symmetry*

$$\begin{array}{rccc} \psi : & S^1 \times \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ & (\ell, \sigma) & \mapsto & \varphi_{\frac{\ell}{\omega}}(\sigma) \end{array} \tag{5}$$

which is generated by $L : \sigma \mapsto \frac{1}{2\omega}(\mathrm{Re}\sigma_2 + \sigma_3\bar{\sigma}_3)$. The subspace $\{\,(0,\sigma_2,0) \bigm| \sigma_2 \in \mathbb{C}\,\}$ is left fixed under all rotations ψ_ℓ , and the π-rotation ψ_π fixes the whole subspace $\{\,\sigma_3 = 0\,\}$ of $\mathbb{C}^3$, so ψ is not a free action. In fact we have reduced H_0 to the 2:1 resonance, cf. [Kummer;86], and our next goal is to further reduce to one degree of freedom. Again we use invariant theory to perform the reduction.

Proposition 2.2

Let ψ denote the S^1-action $(\ell, \sigma) \mapsto (\mathrm{e}^{-2\mathrm{i}\ell}\sigma_1, \sigma_2, \mathrm{e}^{-\mathrm{i}\ell}\sigma_3)$ on $\mathbb{C}^3$. The complex-valued function

$$\tau_1 \;=\; \sigma_1\bar{\sigma}_3^2 \tag{6}$$

together with the real-valued functions

$$\begin{aligned} \tau_2 &= \mathrm{Re}\,\sigma_2 \\ \tau_3 &= \sigma_3\bar{\sigma}_3 \\ \tau_4 &= \mathrm{Im}\,\sigma_2 \\ \tau_5 &= \sigma_1\bar{\sigma}_1 \end{aligned} \tag{6}$$

generates the Poisson algebra of (smooth) ψ-invariant functions on $\mathbb{C}^3$. The generators satisfy the relations $\tau_1\bar{\tau}_1 = \tau_3^2\tau_5$, $\tau_3 \geq 0$, $\tau_5 \geq 0$.

Proof :

The invariants $\tau_3, \tau_5, \mathrm{Re}\,\tau_1, \mathrm{Im}\,\tau_1$ perform the standard reduction of the 2:1 resonance, see Appendix B, Claim 4.6 of [Cushman,Bates;97], and this process is parametrized by $\sigma_2 = \tau_2 + \mathrm{i}\tau_4$; **q.e.d.**

Like after the axial reduction, the Poisson structure, given by *table 2.2*, extends to the whole space $\mathbb{C} \times \mathbb{R}^4$. The Casimir elements $N : \tau \mapsto \frac{1}{2\omega}\tau_4$ and $P : \tau \mapsto \tau_5 - \tau_2^2 + \tau_4^2$ already present before the reduction are completed by the Casimir elements $L : \tau \mapsto \frac{1}{2\omega}(\tau_2 + \tau_3)$ and $R : \tau \mapsto \tau_1\bar{\tau}_1 - \tau_3^2\tau_5$. In particular $R^{-1}(0) \subseteq \mathbb{C} \times \mathbb{R}^4$ is invariant under any Hamiltonian system on “the τ-space” $\mathbb{C} \times \mathbb{R}^4$, and again this also holds for the reduced phase space $\mathbb{C}^3/_{S^1} = \{ \tau \in \mathbb{C} \times \mathbb{R}^4 \mid R(\tau) = 0, \tau_3 \geq 0, \tau_5 \geq 0 \}$. From the axial reduction we inherit the inequality $\tau_2 \geq 0$.

$\{A, B\}$	$\mathrm{Re}\,\tau_1$	$\mathrm{Im}\,\tau_1$	τ_2	τ_3	τ_4	τ_5	B
$\mathrm{Re}\,\tau_1$	0	$4\omega(\tau_2\tau_3^2 - \tau_3\tau_5)$	$4\omega\mathrm{Im}\,\tau_1$	$-4\omega\mathrm{Im}\,\tau_1$	0	$8\omega\tau_2\mathrm{Im}\,\tau_1$	
$\mathrm{Im}\,\tau_1$	$-4\omega(\tau_2\tau_3^2 - \tau_3\tau_5)$	0	$-4\omega\mathrm{Re}\,\tau_1$	$4\omega\mathrm{Re}\,\tau_1$	0	$-8\omega\tau_2\mathrm{Re}\,\tau_1$	
τ_2	$-4\omega\mathrm{Im}\,\tau_1$	$4\omega\mathrm{Re}\,\tau_1$	0	0	0	0	
τ_3	$4\omega\mathrm{Im}\,\tau_1$	$-4\omega\mathrm{Re}\,\tau_1$	0	0	0	0	
τ_4	0	0	0	0	0	0	
τ_5	$-8\omega\tau_2\mathrm{Im}\,\tau_1$	$8\omega\tau_2\mathrm{Re}\,\tau_1$	0	0	0	0	
A							

Table 2.2: The Poisson bracket on the second reduced phase space.

The zeros of the real polynomial $u^2 - v^2w$ in three variables (u, v, w) form the so-called Whitney umbrella, homeomorphic to a cone on a figure eight. Restricting to $v \geq 0$ yields “a 2-dimensional cone on a pinched S^1”. For fixed τ_2, τ_4 the zeros of $R(\tau) = (\mathrm{Re}\,\tau_1)^2 + (\mathrm{Im}\,\tau_1)^2 - \tau_3^2\tau_5$ also form a cone : on a figure eight, rotated about the long axis. The restriction $\tau_3 \geq 0$ makes this surface look like a turnip, that is a pinched 2-sphere. The singularity ‘at the pinch’ is a standard cone $u^2 + v^2 - w^2 = 0$. For future reference we call this 3-dimensional semi-algebraic variety a *turnip cone*, so $\mathbb{C}^3/_{S^1} = \{ \tau \in \mathbb{C} \times \mathbb{R}^4 \mid R(\tau) = 0, \tau_3 \geq 0, \tau_5 \geq 0 \}$ is the product of a plane with a turnip cone.

To obtain the double reduced phase space we have to impose $\tau_2 \geq 0$ and intersect $\mathbb{C}^3/_{S^1}$

with $P^{-1}(0)$. This allows us to eliminate $\tau_5 = \tau_2^2 - \tau_4^2$. The projection

$$\pi \, : \, \mathbb{C} \times \mathbb{R}^4 \longrightarrow \mathbb{C} \times \mathbb{R}^3 \tag{7}$$

is a global chart of the submanifold $P^{-1}(0) \subseteq \mathbb{C} \times \mathbb{R}^4$. Thus we are led to a semi-algebraic subvariety V of $\mathbb{C} \times \mathbb{R}^3$, given by the relations

$$|\tau_1|^2 \;=\; \tau_3^2\,(\tau_2^2 - \tau_4^2) \quad , \quad \tau_3 \geq 0 \quad , \quad \tau_2 \geq |\tau_4| \qquad . \tag{8}$$

Here $\tau_1 \in \mathbb{C}$ and $\tau_2, \tau_3, \tau_4 \in \mathbb{R}$ are co-ordinates on $\mathbb{C} \times \mathbb{R}^3$.

Proposition 2.3

Let $V \subseteq \mathbb{C} \times \mathbb{R}^3$ be defined by (8) and let S be the union of

$$\begin{aligned} S_1 &= \{\, (0, \tau_2, 0, \tau_4) \in \mathbb{C} \times \mathbb{R}^3 \;\big|\; |\tau_4| < \tau_2 \,\} \\ S_2 &= \{\, (0, 0, \tau_3, 0) \in \mathbb{C} \times \mathbb{R}^3 \;\big|\; \tau_3 > 0 \,\} \\ S_3 &= \{\, (0, \tau_2, 0, \tau_4) \in \mathbb{C} \times \mathbb{R}^3 \;\big|\; |\tau_4| = \tau_2 \,\} \qquad . \end{aligned}$$

Then $V \backslash S$ is a smooth manifold and every point in S is a singularity of V . The local structure of V along S_1 is the product of a 2-dimensional standard cone with a plane, while along S_2 it is the product of a 3-dimensional standard cone with a line. In nonzero points of S_3 the local structure of V is the product of a line with a turnip cone. All families of singularities extend to the origin.

Proof :

From the implicit mapping theorem we may conclude that V is smooth in all points where $DR = (2\,\mathrm{Re}\,\tau_1\,,\, 2\,\mathrm{Im}\,\tau_1\,,\, -2\,\tau_2\tau_3^2\,,\, -2\,\tau_3\,(\tau_2^2 - \tau_4^2)\,,\, 2\,\tau_3^2\tau_4)$ does not vanish. This holds if $\tau_1 \neq 0$ or if $\tau_3\tau_4 \neq 0$. Note that $\tau_4 = 0$ implies that $\tau_2 = 0$ under the condition $\tau_1 = 0 \neq \tau_3$. Thus S contains the singularities of V .

For $\tau_2^2 - \tau_4^2 > 0$, as well as for $\tau_3 > 0$, the cone structure of the singularity can be read directly off from

$$(\mathrm{Re}\,\tau_1)^2 \;+\; (\mathrm{Im}\,\tau_1)^2 \;-\; \tau_3^2\,(\tau_2^2 - \tau_4^2) \;=\; 0 \qquad .$$

For the singularities along $S_3\backslash\{0\}$ we can write the above equation as

$$(\mathrm{Re}\,\tau_1)^2 \;+\; (\mathrm{Im}\,\tau_1)^2 \;-\; \tau_3^2\,(\tau_2 + \tau_4)\,(\tau_2 - \tau_4) \;=\; 0 \qquad .$$

Since $\tau_2 > 0$ we find that either $\tau_2 + \tau_4 > 0$ or $\tau_2 - \tau_4 > 0$, furthermore recall that $\tau_3 \geq 0$. Thus the singularities of V along $S_3 \backslash \{0\}$ are two 1-parameter families of turnip cones ; **q.e.d.**

The two remaining Casimir elements N and L slice V into subvarieties which are invariant under every Hamiltonian flow. Because the projection $\mathbb{C} \times \mathbb{R}^3 \longrightarrow \mathbb{C} \times \mathbb{R}$ is a global chart of $N^{-1}(a) \cap L^{-1}(b)$, we can eliminate $\tau_4 = 2\omega a$ and $\tau_3 = 2\omega b - \tau_2$. Thus $V \cap N^{-1}(a) \cap L^{-1}(b)$ becomes the variety

$$V_{a,b} \quad = \quad \{ \, \tau \in \mathbb{C} \times \mathbb{R} \;\; \big| \;\; |\tau_1| = \sqrt{F_{a,b}(\tau_2)} \, , \; 2\omega |a| \leq \tau_2 \leq 2\omega b \, \} \qquad ,$$

where

$$F_{a,b}(\tau_2) \quad = \quad (\tau_2 - 2\omega b)^2 \, (\tau_2^2 - 4\omega^2 a^2) \qquad .$$

For $|a| = b$ this set reduces to the point $\tau = (0, 2\omega b)$, all these point sets coming from the singularities in $S_3 \subseteq V$, and $|a| > b$ implies $V_{a,b} = \emptyset$. In the remaining cases $V_{a,b}$ is a 2-dimensional surface of revolution, generated by $\sqrt{F_{a,b}}$ between the two roots $\tau_2 = 2\omega|a|$ and $\tau_2 = 2\omega b$. For $0 < |a| < b$ this yields a pinched sphere that looks like a turnip, see *figure 2.1a*, while for $a = 0 < b$ one obtains a topological sphere with two singularities, see *figure 2.1b*.

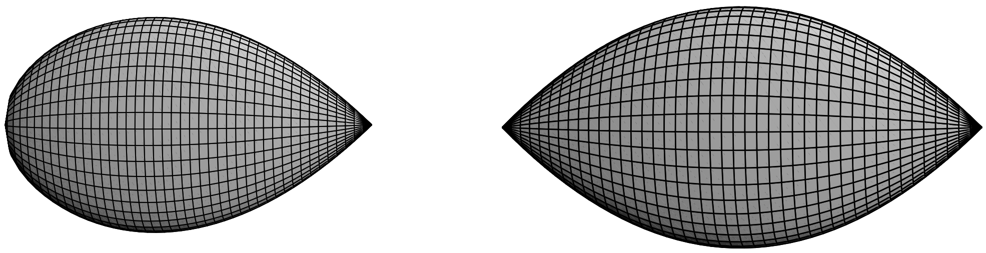

Figure 2.1: The twice reduced spaces $V_{a,b} \subseteq \mathbb{C} \times \mathbb{R}$, with the horizontal axis along τ_2 .
a). For $0 < |a| < b$ this surface has the form of a turnip.
b). For $a = 0 < b$ this surface has the form of a lemon.

Note that the Poisson structure on $V_{a,b}$ induced by *table 2.2* can also be written as the vector triple product

$$\{G, H\} \quad = \quad 2\omega \left(\nabla G \times \nabla H \,\middle|\, \nabla ((\mathrm{Re}\,\tau_1)^2 + (\mathrm{Im}\,\tau_1)^2 - F_{a,b}) \right) \qquad .$$

As implied by singular reduction, the singular strata are preserved by all Hamiltonian flows. The singularities at $\tau = (0, 2\omega b)$ come from $S_1 \subseteq V$, while $S_2 \subseteq V$ induces for $a = 0$ a singularity at the origin.

cases $b > 0$	Topological structure of $N^{-1}(a) \cap L^{-1}(b)$
$a > b$	empty set
$a = b$	diffeomorphic to S^1
$0 < a < b$	diffeomorphic to $S^3 \times S^1$
$a = 0$	stratified into $\mathbb{T}^2 \times \mathbb{R}^2$ and S^1
$-b < a < 0$	diffeomorphic to $S^3 \times S^1$
$a = -b$	diffeomorphic to S^1
$a < -b$	empty set

Table 2.3: The subsets of the original phase space $T^*\mathbb{R}^3$ with third component N of the angular momentum equal to a and energy $H_0 = \omega L$ of the isotropic harmonic oscillator equal to $\omega b > 0$.

We end this section with a description of the subsets

$$N^{-1}(a) \cap L^{-1}(b) \quad = \quad \{ (x,p) \in T^*\mathbb{R}^3 \;\; \big| \;\; N(x,p) = a \, , \, L(x,p) = b \}$$

of the original phase space, see *table 2.3*. In the co-ordinates

$$\begin{array}{lll} r = \omega x_1 + p_2 & t = \omega x_2 + p_1 & v = \omega x_3 \\ s = \omega x_1 - p_2 & u = \omega x_2 - p_1 & w = p_3 \end{array}$$

the defining equations of $N^{-1}(a) \cap L^{-1}(b) \subseteq T^*\mathbb{R}^3$ become

$$\begin{array}{rcl} r^2 + u^2 + v^2 + w^2 & = & 2\omega(b+a) \\ s^2 + t^2 + v^2 + w^2 & = & 2\omega(b-a) \end{array} \quad .$$

This shows again that $N^{-1}(a) \cap L^{-1}(b)$ is empty for $|a| > b$, that $N^{-1}(0) \cap L^{-1}(0)$ is the origin and that $N^{-1}(\pm b) \cap L^{-1}(b)$, $b > 0$ is an S^1 . Furthermore we conclude that $N^{-1}(a) \cap L^{-1}(b)$ is diffeomorphic to the product $S^3 \times S^1$ for $0 < |a| < b$.

The axial S^1-action simultaneously rotates the (r,u)-plane and the (s,t)-plane. Under the axial reduction, the 2-torus

$$\{ \, r^2 + u^2 = 2\omega(b+a) - v^2 - w^2 \, , \; s^2 + t^2 = 2\omega(b-a) - v^2 - w^2 \, , \; v, w \text{ fixed} \, \}$$

gets reduced to an S^1 , and the circle $\{ \min(r^2+u^2, s^2+t^2) = 0 \, , \, v,w \text{ fixed} \}$ becomes a point. Thus the image of $S^3 \times S^1$ under the axial reduction mapping is a smooth 3-sphere. In case $a = 0 < b$ the axial reduction map is singular at $r^2 + u^2 = s^2 + t^2$ and the image of $N^{-1}(0) \cap L^{-1}(b)$ is

$$\{ \, \sigma \in \mathbb{C}^3 \;\; \big| \;\; \sigma_1 \bar{\sigma}_1 = \mathrm{Re}\, \sigma_2 \, , \; \mathrm{Im}\, \sigma_2 = 0 \, , \; \mathrm{Re}\, \sigma_2 + \sigma_3 \bar{\sigma}_3 = 2\omega b \, \}$$

which is only a topological 3-sphere, because it has a cone-like singularity at the singular circle $\{ (0,0,\sigma_3) \in \mathbb{C}^3 \; \big| \; \sigma_3 \bar{\sigma}_3 = 2\omega b \}$. From this, one reconstructs the subset $\{ (x,p) \in$

$T^*\mathbb{R}^3 \mid N(x,p) = 0\,,\, L(x,p) = b\}$ which is not a submanifold, since the derivative of N vanishes along the circle $C = \{(0,0,x_3,0,0,p_3) \in T^*\mathbb{R}^3 \mid \omega^2 x_3^2 + p_3^2 = 2\omega b\}$. Here the local structure is the product of a line (corresponding to the circle) with the 2-torus cone $\{(r,s,t,u) \in \mathbb{R}^4 \mid r^2 + u^2 - s^2 - t^2 = 0\}$. The complement of the singular circle C in $N^{-1}(0) \cap L^{-1}(b)$ is a 1-parameter family of 3-tori which shrink to a 2-torus as $\omega^2 x_3^2 + p_3^2 \to 0$; in other words : the regular part of $N^{-1}(0) \cap L^{-1}(b)$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}^2$.

Remark : Note that the structure of $N^{-1}(0) \cap L^{-1}(b)\,,\, b > 0$ which we have found above is topologically necessary since all subsets $N^{-1}(a) \cap L^{-1}(b)\,,\, -b \leq a \leq b$ have to fit together to form the 5-sphere $\{(x,p) \in T^*\mathbb{R}^3 \mid H_0(x,p) = \omega b\}$.

2.3 Physical interpretation

The two S^1-actions $\varrho : (\nu, z) \mapsto \exp_\nu z$ and $\psi : (\ell, z) \mapsto \mathrm{e}^{-\mathrm{i}\ell} z$ on the original phase space $\mathbb{C}^3$ commute and give rise to a 2-torus action

$$\begin{array}{rccc} \psi\circ\varrho : & (S^1 \times S^1) \times \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ & ((\nu,\ell), z) & \mapsto & \varrho_\nu(\psi_\ell(z)) \end{array}$$

with momentum mapping $(N,L) : \mathbb{C}^3 \longrightarrow \mathbb{R}^2$. The composite

$$\mathbb{C}^3 \xrightarrow{\sigma} \mathbb{C}^3 \xrightarrow{\tau} \mathbb{C} \times \mathbb{R}^4 \xrightarrow{\pi} \mathbb{C} \times \mathbb{R}^3$$

of the mappings defined by (3), (6) and (7) makes $\mathbb{C}^3$ a ramified 2-torus bundle over $V = \mathrm{im}(\pi\circ\tau\circ\sigma)$. Indeed, restricted to $(\pi\circ\tau\circ\sigma)^{-1}(V \setminus S)$ the 2-torus symmetry is free and proper and all values of the momentum mapping are regular. Here the regular reduction applies, see Section 4.3 of [Abraham,Marsden;78].

Let us show that all fibres of $\pi\circ\tau\circ\sigma : \mathbb{C}^3 \longrightarrow V$ can be reconstructed from their projection onto configuration space. The orbits of the S^1-action (5) coincide with the trajectories of the isotropic harmonic oscillator in phase space. Scaling this ellipse in configuration space by ω yields the corresponding ellipse in the space of momenta. Recall that the oscillator flow along these two ellipses is synchronised with a phase shift of $\frac{\pi}{2}$. The reconstructed fibre of the ramified 2-torus bundle is the orbit swept out by these ellipses under the axial S^1-action (2). The effect of ϱ_ν is to rotate in both the configuration space and the space of momenta about the third co-ordinate axis.

This explains immediately why there is a 2:1 resonance in the second reduction. The condition $\tau_3 = 0$ implies that the S^1-family of ellipses corresponding to a singular point in $S_1 \subseteq V$ is contained in the (x_1, x_2)-plane. Because of the geometric symmetry of an ellipse, the axial π-rotation ϱ_π and the π-rotation ψ_π along the ellipse coincide. This S^1-family of ellipses has the non-trivial isotropy group $\mathbf{Z}_2 = \{(0,0),(\pi,\pi)\} < S^1 \times S^1$. While the fibre still is a 2-torus, the bundle is not locally trivial over points of S_1 . At points of S_3 , *i.e.* when moreover $\tau_4 = \pm\tau_2$, the ellipse degenerates into a circle $\mathcal{C}$ and the "S^1-family of ellipses" consists of exactly this circle $\mathcal{C}$ in the (x_1, x_2)-plane. Here the isotropy group is $S^1 = \{ (\nu, \ell) \in S^1 \times S^1 \mid \nu + \ell = 2\pi \}$; the only exception is the equilibrium in the origin which is invariant under the whole 2-torus action. The isotropy group of a point in a fibre above $S_2 \subseteq V$ is $S^1 = \{ (\nu, \ell) \in S^1 \times S^1 \mid \ell = 0 \}$. The corresponding ellipse is a rectilinear one along the x_3-axis.

It is instructive to further identify the fibres over smooth points of V . We first consider points in $V \backslash S$ with $\tau_1 = 0$. Since $\tau_3 \neq 0$, this implies $\tau_2 = |\tau_4| \neq 0$; the whole energy $\frac{1}{2}\tau_2$ of the (z_1, z_2)-subsystem of the isotropic harmonic oscillator is contained in the third component $N = \frac{1}{2\omega}\tau_4$ of the angular momentum. Hence, the corresponding S^1-family of ellipses projects into a circle in the (x_1, x_2)-plane.

The other 2-tori are parametrised by $\tau_2 > 0$, $\tau_3 > 0$, $\tau_4 \in]-\tau_2, \tau_2[$ and the argument $\arg \tau_1 \in S^1$ of τ_1 . While $N = \frac{1}{2\omega}\tau_4$ and $\frac{1}{2}\tau_2$ describe the geometry of the projection of the ellipse into the (x_1, x_2)-plane, the z_3-subsystem has the energy $\frac{1}{2}\tau_3$. The angle $\arg \tau_1 = \arg(z_1^2 + z_2^2) - 2 \arg z_3$ determines the synchronisation of (the S^1-families of) the two subsystems. Let us illustrate this with two significant examples.

For $\tau_4 = 0$ the (z_1, z_2)-subsystem has no angular momentum, this leads to (S^1-families of) rectilinear ellipses in the (x_1, x_2)-plane. Proper synchronisation yields (S^1-families of) rectilinear ellipses in the configuration space. These are characterised by $G = 0$, where $G := \|x \times p\|$ denotes the total angular momentum. Using the relation

$$2 \,\mathrm{Re}\, \tau_1 + \tau_2^2 + \tau_3^2 - \tau_4^2 \;=\; \sum_{j,k=1}^{3} (z_j \bar{z}_k)^2 \;=\; 4\omega^2 L^2 - 4\omega^2 G^2 \tag{9}$$

we obtain $\mathrm{Re}\, \tau_1 = |\tau_1|$. The whole upper arc in the plane $\mathrm{Im}\, \tau_1 = 0$ connecting the two singularities of $V_{0,b}$ (cf. *figure 2.1b*) corresponds to rectilinear ellipses.

The circular ellipses are characterised by $G = L$. It follows from (9) that

$$\cos(\arg \tau_1) \;=\; \frac{-1}{2|\tau_1|}(\tau_2^2 + \tau_3^2 - \tau_4^2)$$

and, hence, $\mathrm{Im}\, \tau_1 = 0$, $\mathrm{Re}\, \tau_1 = -\tau_3^2 = -(\tau_2^2 - \tau_4^2)$. For fixed $|a| \leq b$ the sole solution in $V_{a,b}$ is $(\tau_1, \tau_2) = (-\frac{\omega^2}{b^2}(b^2 - a^2)^2, \frac{\omega}{b}(a^2 + b^2))$. In particular the point $(\tau_1, \tau_2) = (-\omega^2 b^2, \omega b)$ on $V_{0,b}$ corresponds to the S^1-family of circular ellipses with radius $\sqrt{b/\omega}$ that are perpendicular to the (x_1, x_2)-plane.

Remark : Completing the angle ν conjugate to N and the angle ℓ conjugate to L by an angle g conjugate to G , one obtains a special set of action-angle variables for the isotropic harmonic oscillator, the *nodal-Lissajous variables*, cf. [Ferrer,Garate;96]. For a thorough discussion of the relation between the invariants $\tau_1, \tau_2, \tau_3, \tau_4$ and the nodal-Lissajous variables (ℓ, g, ν, L, G, N) the reader is referred to [Ferrer,Hanßmann,Palacián,Yanguas;98].

§3 The normal form

As remarked in Subsection 1.2 we may reduce a given axially symmetric Hamilton function H_ε to two degrees of freedom. To obtain a further reduction to one degree of freedom we use the fact that H_ε is a small perturbation of the isotropic harmonic oscillator H_0 . Indeed, the normal form $\bar{H}_\varepsilon$ of order 1 is by construction invariant under the oscillator symmetry ψ and inherits from H_ε the invariance under the axial S^1-action ϱ .

Our next aim is to formulate and prove a basic theorem in the present context of reduced phase spaces with singularities. Then we specialize to a particular quartic perturbation of the superintegrable oscillator H_0 .

3.1 Normalization in Poisson spaces

To compute the normal form of order 1 , we have to average H_ε along the periodic flow of H_0 . Since this process preserves symmetries, cf. [Broer;79,81], [Churchill,Kummer,Rod;83], [Cushman;92], reducing $\bar{H}_\varepsilon$ from $T^*\mathbb{R}^3$ to $T^*\mathbb{R}^3/_{S^1}$ and averaging (the reduced) H_ε on $T^*\mathbb{R}^3/_{S^1}$ leads to the same Hamilton function $\bar{H}_\varepsilon$ on $T^*\mathbb{R}^3/_{S^1}$. While H_ε is ε-close to H_0 , it will be ε^2-close to $\bar{H}_\varepsilon$. We formulate this using the C^2-norm $\|f\|_{C^2(B)} = \max\{\|f\|_B, \|Df\|_B, \|D^2 f\|_B\}$ on some compact domain B of a Poisson space P . Note that this norm is also well-defined on the Poisson algebra $\mathcal{A} \subseteq C(T^*\mathbb{R}^3/_{S^1})$, in spite of the singularities of the Poisson space $T^*\mathbb{R}^3/_{S^1}$. Indeed, $\mathcal{A}$ can be identified with the subalgebra $C^\infty_\varrho(T^*\mathbb{R}^3)$ of smooth functions that are invariant under (1).

Theorem 3.1

On the Poisson space P let H_0 be a Hamilton function with periodic flow of constant frequency $\omega > 0$ that has the origin $0 \in P$ as sole equilibrium. If H_ε satisfies the estimate $\|H_\varepsilon - H_0\|_{C^2(B)} < \gamma\varepsilon$ on the closed ball B_R of radius R (with some constant γ), *then there is a canonical transformation ψ such that $\|H_\varepsilon \circ \psi - H_0\|_{B_r} < \Gamma\varepsilon^2$ on the closed ball B_r of radius $r := R - \gamma\varepsilon$ (with some constant Γ).*

Proof :

We construct our canonical transformation as the time one flow ψ_1^ε of a Hamiltonian vector field with Hamilton function

$$K \;:=\; \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} t\cdot(H_\varepsilon - H_0)\circ\varphi_t \,\mathrm{d}t$$

where φ_t denotes the flow of H_0 . The bound $\|X_K\|_{B_R} \le \frac{\pi}{\omega}\|D(H_\varepsilon - H_0)\|_{B_R} < \frac{\pi}{\omega}\gamma\varepsilon$ ensures that ψ_1^ε maps the ball B_r into B_R . From $\{K, H_0\} = H_\varepsilon - \bar{H}_\varepsilon$ we derive

$$H_\varepsilon\circ\psi_1^\varepsilon - \bar{H}_\varepsilon \;=\; H_0\circ\psi_1^\varepsilon \;-\; \bar{H}_0 \;-\; \{H_0, K\} \;+\; (H_\varepsilon - H_0)\circ\psi_1^\varepsilon \;-\; (H_\varepsilon - H_0) \quad . \tag{10}$$

Since $\|D\{f,g\}\| \le \mathit{const}\,\|f\|_{C^2}\|g\|_{C^2}$, using the inequalities

$$\|(H_\varepsilon - H_0)\circ\psi_1^\varepsilon - (H_\varepsilon - H_0)\|_{B_r} \;\le\; \|D(H_\varepsilon - H_0)\|_{B_R}\,\|\psi_1^\varepsilon - \mathrm{id}\|_{B_r}$$

$$\|\psi_1^\varepsilon - \mathrm{id}\|_{B_r} \;\le\; \int_0^1 \|X_K\circ\psi_t^\varepsilon\|_{B_r}\,\mathrm{d}t \;\le\; \|X_K\|_{B_R}$$

and

$$\begin{aligned}\|H_0\circ\psi_1^\varepsilon - \bar{H}_0 - \{H_0,K\}\|_{B_r} \;&\le\; \int_0^1 \|\{H_0,K\}\circ\psi_1^\varepsilon - \{H_0,K\}\|_{B_r}\,\mathrm{d}t \\ &\le\; \|D\{H_0,K\}\|_{B_R}\,\|X_K\|_{B_R}\end{aligned}$$

we obtain the desired conclusion from (10) ; **q.e.d.**

The normal form $\bar{H}_\varepsilon$ can be reduced to one degree of freedom, see Section 2. This defines a family $\bar{H}_\varepsilon^{a,b}$ on the twice reduced spaces $V_{a,b}$, depending on the values (a,b) of the momentum mapping (N,L) . Under variation of the (distinguished) parameters a and b the family may undergo bifurcations. For instance, a centre and a saddle can meet in a *centre-saddle bifurcation* and vanish, or two saddles can have the same energy and get connected by heteroclinic orbits. Next to these codimension one bifurcations there may also occur codimension two bifurcations at isolated points (a,b) in the parameter plane.

3.2 An astrophysical example

The general case of axially symmetric cubic and quartic perturbations of the isotropic harmonic oscillator will be studied in a subsequent paper. Here we illustrate our results with the particular example

$$H_\varepsilon(x,p) \;=\; \frac{1}{2}(p\,|\,p) \;+\; \frac{\omega^2}{2}(x\,|\,x) \;+\; \varepsilon\,(x_1^2+x_2^2)\,x_3^2 \tag{11}$$

which has also been considered in [Caranicolas;94] and [Ferrer,Gárate;96]. The normal form (of order one) of H_ε is

$$\begin{aligned}\bar{H}_\varepsilon(x,p) \;&=\; H_0(x,p) \;+\; \frac{3\varepsilon}{8\omega^4}(\omega^2x_1^2+p_1^2+\omega^2x_2^2+p_2^2)(\omega^2x_3^2+p_3^2)\\ &\quad-\;\frac{\varepsilon}{4\omega^2}(x_1p_3-x_3p_1)^2 \;-\; \frac{\varepsilon}{4\omega^2}(x_2p_3-x_3p_2)^2 \qquad .\end{aligned}$$

Using the invariants (3) of the axial symmetry the Hamilton function H_0 of the isotropic harmonic oscillator reads

$$H_0(\sigma) \;=\; \frac{1}{2}\mathrm{Re}\,\sigma_2 \;+\; \frac{1}{2}\sigma_3\bar{\sigma}_3$$

while the perturbations H_ε and $\bar{H}_\varepsilon$ become

$$H_\varepsilon(\sigma) \;=\; H_0(\sigma) \;+\; \frac{\varepsilon}{2\omega^4}(\mathrm{Re}\,\sigma_1+\mathrm{Re}\,\sigma_2)(\mathrm{Re}\,\sigma_3)^2$$

and

$$\bar{H}_\varepsilon(\sigma) \;=\; H_0(\sigma) \;+\; \frac{\varepsilon}{8\omega^4}\mathrm{Re}\,(\sigma_1\bar{\sigma}_3^2) \;+\; \frac{\varepsilon}{4\omega^4}\sigma_3\bar{\sigma}_3\mathrm{Re}\,\sigma_2 \qquad .$$

The Hamilton function H_ε is not invariant under the induced harmonic oscillator symmetry (5) and therefore does not induce a function on the doubly reduced phase space. The normal form $\bar{H}_\varepsilon$ is by construction invariant under (5) and thus induces

$$\bar{H}_\varepsilon(\tau) \;=\; \frac{1}{2}(\tau_2+\tau_3) \;+\; \frac{\varepsilon}{8\omega^4}\mathrm{Re}\,\tau_1 \;+\; \frac{\varepsilon}{4\omega^4}\tau_2\tau_3 \qquad .$$

on the doubly reduced phase space. For fixed values a and b of N and L the normal form $\bar{H}_\varepsilon$ of the perturbation H_ε induces a Hamilton function

$$\bar{H}_\varepsilon^{a,b}(\tau) \;=\; \omega b \;+\; \frac{\varepsilon}{8\omega^4}\mathrm{Re}\,\tau_1 \;-\; \frac{\varepsilon}{4\omega^4}\tau_2^2 \;+\; \frac{\varepsilon b}{2\omega^3}\tau_2$$

on $\mathbb{C}\times\mathbb{R}$. In the following two sections we analyse this example. On every step we first discuss the general statements that can be made, before exploiting features special to this system.

Remark : In cylindrical co-ordinates (r,ν,ζ) , with canonically conjugate momenta (p_r,N,p_ζ) , the perturbed Hamilton function H_ε reads

$$H_\varepsilon(r,\nu,\zeta,p_r,N,p_\zeta) \;=\; \frac{1}{2}(p_r^2+p_\zeta^2) \;+\; \frac{N^2}{2r^2} \;+\; \frac{\omega^2}{2}(r^2+\zeta^2) \;+\; \varepsilon r^2\zeta^2 \qquad .$$

For $N=0$ this is a perturbed harmonic oscillator in 1:1 resonance, cf. *e.g.* [Hénon,Heiles;64]. Unlike the set of invariants (3) of the axial symmetry, the radius $r=\sqrt{\frac{1}{2\omega^2}(\mathrm{Re}\,\sigma_1+\mathrm{Re}\,\sigma_2)}$

is not smooth at the origin. This square root also accounts for the difference to the 2:1 resonance. As we have seen in Subsection 2.3, the interaction of the axially reduced planar subsystem $(\zeta, p_\zeta) = 0$ with the subsystem $(r, p_r) = 0$ is better described by the 2:1 resonance.

§4 The averaged flow

We are given an axially symmetric perturbation H_ε of the isotropic harmonic oscillator H_0 . In Section 2 we could reduce the normal form $\bar{H}_\varepsilon$ to a one-degree-of-freedom problem. In fact we have a whole family of one-degree-of-freedom systems, depending on the values a and b of N and L . In this section we study the normal form in its own right. For fixed a, b we give the phase portrait of the one-degree-of-freedom system $X_{\bar{H}_\varepsilon^{a,b}}$, and we describe how these systems fit together.

4.1 The one-degree-of-freedom system

Let us start with a rough description of the expected phase portraits of $X_{\bar{H}_\varepsilon^{a,b}}$ on the twice reduced spaces $V_{a,b}$. For an open and dense part of the parameter plane the normal form is a *Bott-Morse function*. This leads to finitely many saddles and centres on $V_{a,b}$, furthermore the singular points of $V_{a,b}$ are always equilibria.

To be more explicit, the orbits of the Hamiltonian vector field $X_{\bar{H}_\varepsilon^{a,b}}$ are the intersections of the level sets $\{ \bar{H}_\varepsilon^{a,b}(\tau_1, \tau_2) = h \}$ with $V_{a,b}$ in $\mathbb{C} \times \mathbb{R}$. From this the phase portraits can be obtained by geometric considerations.

In our particular example the level sets are the products of the $(\operatorname{Im} \tau_1)$-axis with the parabola

$$\operatorname{Re} \tau_1 \quad = \quad 2\,\tau_2^2 \; - \; 4\omega b\,\tau_2 \; + \; \frac{8\omega^4 h}{\varepsilon} \; - \; \frac{8\omega^5 b}{\varepsilon} \qquad .$$

Since $V_{a,b}$ is a surface of revolution, the points where a level set is tangent (*i.e.* the equilibria of $X_{\bar{H}_\varepsilon^{a,b}}$) lie in the plane $\operatorname{Im} \tau_1 = 0$. Therefore, we merely have to study the intersections of the above parabola with

$$(\operatorname{Re} \tau_1)^2 \quad = \quad F_{a,b}(\tau_2) \quad = \quad (\tau_2 - 2\omega b)^2\,(\tau_2^2 - 4\omega^2 a^2) \qquad .$$

The possible positions of the parabola with respect to $\mathrm{Re}\,\tau_1 = \pm\sqrt{F_{a,b}(\tau_2)}$ are depicted in *figure 4.1.* From these one immediately obtains the phase portraits of *figure 4.2.* For instance, when $a = 0$ there is a centre at $\tau = (\omega^2 b, \omega b)$ and a saddle at $\tau = (-\omega^2 b, \omega b)$, while the two singular points are both centres.

Note that the sole bifurcation, a centre-saddle bifurcation, is of codimension one. Indeed, the scaling

$$\tau_1 \mapsto 4\omega^2 b^2\, \tau_1 \qquad \tau_2 \mapsto 2\omega b\, \tau_2 \qquad a \mapsto b\, a \tag{12}$$

shows that only the ratio $\mu = \frac{a}{b}$ is relevant for the qualitative dynamics of the reduced system. Replacing furthermore h by $\frac{\varepsilon b^2}{2\omega^2} h + \omega b$ and rescaling time we are led to the Hamilton function

$$\mathcal{H}(\tau) \;=\; \mathrm{Re}\,\tau_1 \;+\; 2\,\tau_2\,(1 - \tau_2) \tag{13}$$

on the phase space

$$V_\mu \;=\; \{\, \tau \in \mathbb{C} \times [\mu, 1] \;\big|\; R_\mu(\tau) = 0 \,\} \qquad , \tag{14}$$

where $R_\mu(\tau) = \tau_1 \bar{\tau}_1 - (\tau_2 - 1)^2(\tau_2^2 - \mu^2)$, with Poisson bracket

$$\{\, f \,,\, g \,\} \;=\; (\, \nabla f \times \nabla g \,|\, \nabla R_\mu \,) \qquad . \tag{15}$$

The dependence of the phase space on μ makes this a one-parameter family of Hamiltonian systems.

Proposition 4.1

The family of Hamiltonian systems defined by (13) – (15) *is structurally stable.*

Proof :

To show structural stability we study the behaviour of the equilibria of $X_{\mathcal{H}}$ on $R_\mu^{-1}(0)$. These are the points on V_μ where $\nabla\mathcal{H}$ is parallel to ∇R_μ . Introducing

$$F_\mu(\tau_2) \;=\; (\tau_2 - 1)^2\,(\tau_2^2 - \mu^2)$$

(whence $R_\mu(\tau) = (\mathrm{Re}\,\tau_1)^2 + (\mathrm{Im}\,\tau_1)^2 - F_\mu(\tau_2)$) we are led to the equation

$$\lambda \cdot \begin{pmatrix} 1 \\ 0 \\ 2 - 4\tau_2 \end{pmatrix} \;=\; \begin{pmatrix} 2\,\mathrm{Re}\,\tau_1 \\ 2\,\mathrm{Im}\,\tau_1 \\ -F'_\mu(\tau_2) \end{pmatrix}$$

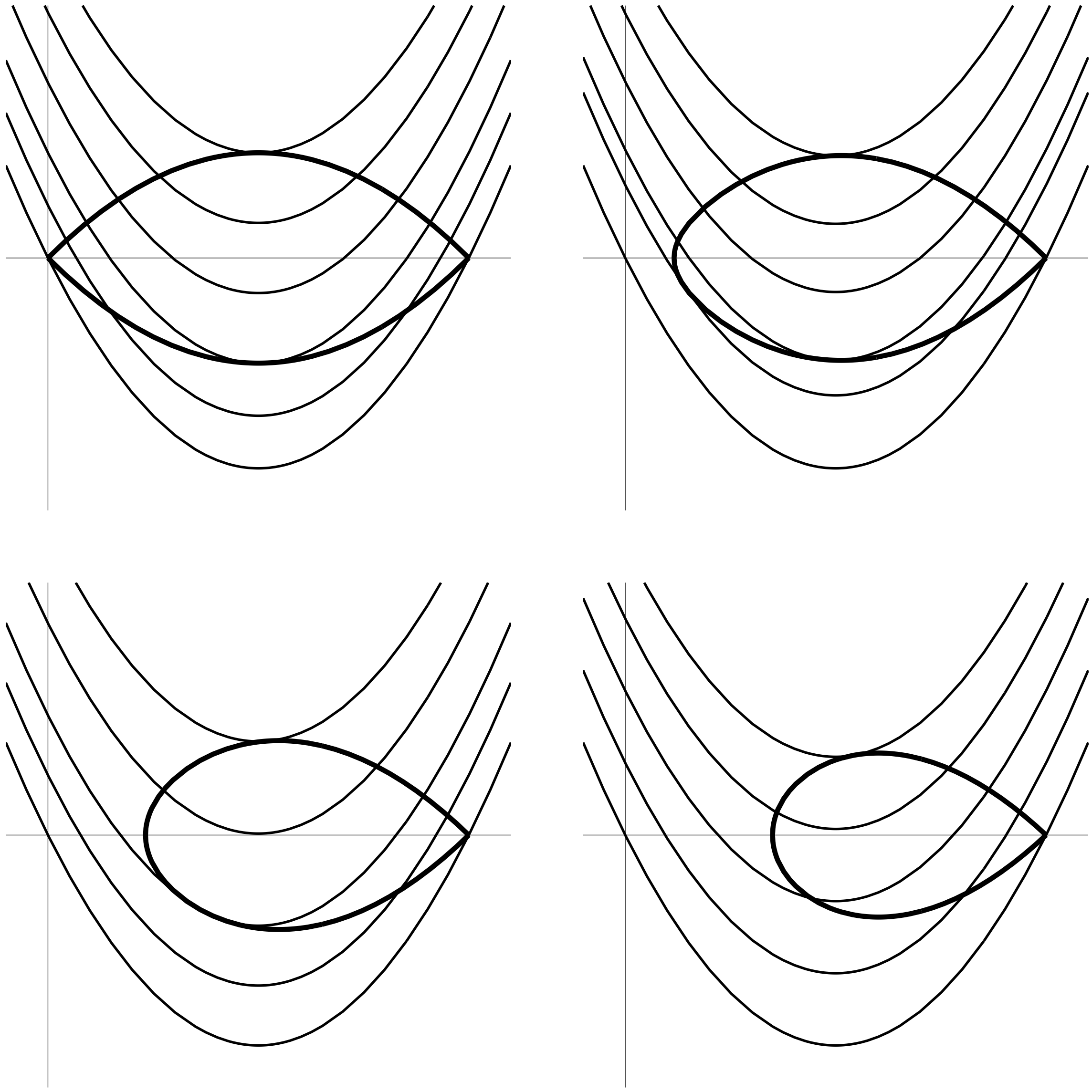

Figure 4.1: Intersections of $\bar{H}_\varepsilon^{a,b} = h$ with the phase space $V_{a,b}$ in the plane $\operatorname{Im}\tau_1 = 0$ for significant values h . The four ratios $\mu = \frac{a}{b}$ are $\mu = 0$, $\mu = 0.116$, $\mu = 0.231839$ and $\mu = 0.35$. The axes are τ_2 (horizontal) and $\operatorname{Re}\tau_1$ (vertical).

under the condition $(\operatorname{Re}\tau_1)^2 + (\operatorname{Im}\tau_1)^2 = F_\mu(\tau_2)$. The Lagrange multiplier λ must be equal to $2\operatorname{Re}\tau_1 = 2\epsilon\sqrt{F_\mu(\tau_2)}$, $\epsilon = \pm 1$, and the solutions $\hat{\tau}_2 = \hat{\tau}_2(\mu)$ of the third equation yield the equilibria $(\hat{\epsilon}\sqrt{F_\mu(\hat{\tau}_2)} + \mathrm{i}0, \hat{\tau}_2)$. Note that the sign ϵ distinguishes between the upper and lower arc of $V_\mu \cap \{\operatorname{Im}\tau_1 = 0\}$, see *figure 4.1*.

The case $F_\mu(\hat{\tau}_2) = 0$ is readily analysed : these equilibria are necessarily the singular

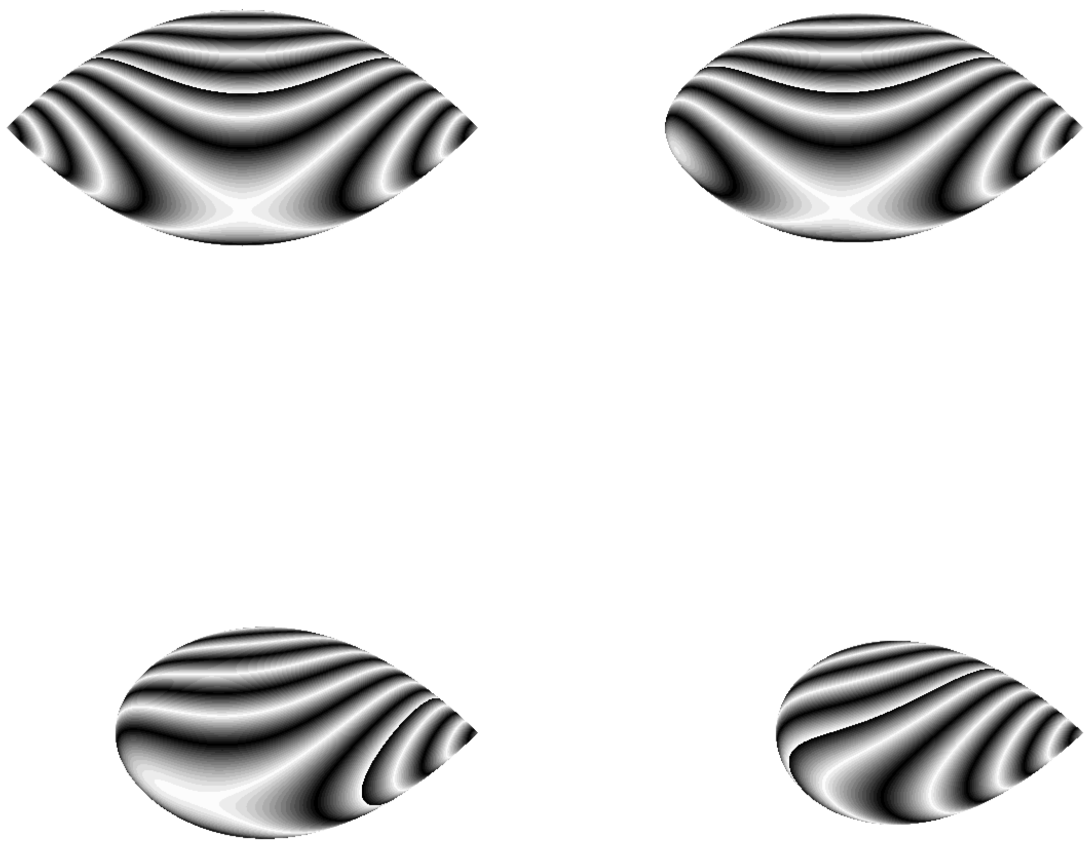

Figure 4.2: Possible phase portraits of the twice reduced system. The four ratios $\mu = \frac{a}{b}$ are the same as in *figure 4.1*. Corresponding to the maximal energy value there is in all cases a further centre, hidden at the other side.

points of the phase space $R_\mu^{-1}(0)$, and since

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}\,\epsilon\sqrt{F_\mu(\tau_2)}\;\Big|_{\tau_2\,=\,1} \;=\; -\epsilon\sqrt{1-\mu^2} \;<\; 2 \;=\; \frac{\mathrm{d}}{\mathrm{d}\tau_2}\Big(2\,\tau_2^2 - 2\,\tau_2 + h\Big)\Big|_{\tau_2\,=\,1}$$

and, for $\mu = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}\,\epsilon\sqrt{F_0(\tau_2)}\;\Big|_{\tau_2\,=\,0} \;=\; \epsilon \;>\; -2 \;=\; \frac{\mathrm{d}}{\mathrm{d}\tau_2}\Big(2\,\tau_2^2 - 2\,\tau_2 + h\Big)\Big|_{\tau_2\,=\,0}$$

the equilibrium at $\tau = (0 + \mathrm{i}0, 1)$ is a centre, as well as, for $\mu = 0$, the equilibrium at the origin.

Thus we concentrate on $F_\mu(\hat{\tau}_2) \neq 0$. In the neighbourhood of an equilibrium we choose local co-ordinates q and p on the phase space V_μ according to

$$\begin{pmatrix} \operatorname{Re}\tau_1 \\ \operatorname{Im}\tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} \hat{\epsilon}\sqrt{F_\mu(\hat{\tau}_2 + q) - p^2} \\ p \\ \hat{\tau}_2 + q \end{pmatrix} .$$

Geometrically this corresponds to a projection onto the $(\operatorname{Im}\tau_1, \tau_2)$-plane. In particular q and p are not symplectic co-ordinates, having Poisson bracket $\{q, p\} = -2\hat{\epsilon}\sqrt{F_\mu(\hat{\tau}_2 + q) - p^2}$. In such co-ordinates the Hamilton function reads

$$\mathcal{H}_\mu(q, p) = \hat{\epsilon}\sqrt{F_\mu(\hat{\tau}_2 + q) - p^2} - 2q^2 + 2q - 4\hat{\tau}_2 q + 2\hat{\tau}_2 - 2\hat{\tau}_2^2 .$$

As $\frac{\partial^2 \mathcal{H}_\mu(0,0)}{\partial p^2} = \frac{-\hat{\epsilon}}{\sqrt{F_\mu(\hat{\tau}_2)}} \neq 0$ and $\frac{\partial^2 \mathcal{H}_\mu(0,0)}{\partial q \partial p} = 0$ the equilibrium at the origin is, respectively, a centre, a saddle or a parabolic equilibrium for negative, positive and vanishing $\hat{\epsilon}\frac{\partial^2 \mathcal{H}_\mu(0,0)}{\partial q^2}$. Let us show that the family undergoes a centre-saddle bifurcation at ocurring parabolic equilibria. We have to check that the three determining coefficients

$$\begin{aligned} A &:= \frac{1}{2}\frac{\partial^2 \mathcal{H}_{\hat{\mu}}(0,0)}{\partial p^2} = \frac{-\hat{\epsilon}}{2\sqrt{F_{\hat{\mu}}(\hat{\tau}_2)}} \\ B &:= \frac{1}{2}\frac{\partial^3 \mathcal{H}_{\hat{\mu}}(0,0)}{\partial q^3} = \frac{3\hat{\epsilon}\hat{\mu}^2\,(\hat{\tau}_2 - \hat{\mu}^2)}{2\sqrt{\hat{\tau}_2^2 - \hat{\mu}^2}^{\,5}} \\ C &:= -\frac{\partial^2 \mathcal{H}_{\hat{\mu}}(0,0)}{\partial \mu \partial q} = \frac{-\hat{\epsilon}\hat{\mu}\,(\hat{\tau}_2 - \hat{\mu}^2)}{\sqrt{\hat{\tau}_2^2 - \hat{\mu}^2}^{\,3}} \end{aligned}$$

do not vanish at the parameter value $\hat{\mu}$ where the bifurcation takes place, cf. [Meyer;70,75], [Broer,Chow,Kim,Vegter;93], [Hanßmann;95]. It is not necessary to calculate the actual value $\hat{\mu}$ (and $-\hat{\mu}$) by equating

$$\frac{\partial^2 \mathcal{H}_{\hat{\mu}}(0,0)}{\partial q^2} = -\hat{\epsilon}\,\frac{2\hat{\tau}_2(\hat{\tau}_2^2 - \hat{\mu}^2) + \hat{\mu}^2(1 - \hat{\tau}_2)}{\sqrt{\hat{\tau}_2^2 - \hat{\mu}^2}^{\,3}} - 4 \tag{16}$$

to zero. Indeed, $\hat{\mu} \neq 0$ because $\frac{\partial^2 \mathcal{H}_0(0,0)}{\partial q^2} = -2\hat{\epsilon} - 4 < 0$, and $F_{\hat{\mu}}(\hat{\tau}_2) \neq 0$ implies $\hat{\tau}_2^2 - \hat{\mu}^2 > 0$. From (16) we conclude that the equilibria on the upper arc, *i.e.* with $\hat{\epsilon} = +1$ are centres. For $\hat{\epsilon} = -1$ the signs of the determining coefficients are $\operatorname{sgn} A = +1$, $\operatorname{sgn} B = -1$ and $\operatorname{sgn} C = \operatorname{sgn}\hat{\mu}$. Hence, as $|\mu|$ increases, a saddle and a centre meet in the parabolic equilibrium and vanish in a centre-saddle bifurcation. In particular, $|\hat{\mu}|$ is uniquely determined. As shown above we have $|\hat{\mu}| \in \,]0, \hat{\tau}_2[\, \subseteq \,]0, 1[$.

Finally no global bifurcations (between stable and unstable manifolds of different saddles) occur since there is always at most one saddle **;** **q.e.d.**

Remark : Whenever $\bar{H}_\varepsilon - H_0$ is homogeneous in the original variables ωx and p , the scaling (12) reveals the reduced system $\bar{H}_\varepsilon^{a,b}$ to depend on the distinguished parameters only via $\mu = \frac{a}{b}$. The most important case is that $\bar{H}_\varepsilon - H_0$ contains only fourth order terms, *i.e.* the original Hamilton function H_ε is a perturbation of H_0 by terms of order three and four. To unfold occurring bifurcations of codimension two within the single system defined by $\bar{H}_\varepsilon$, one would either have to allow for a detuning of the frequencies of H_0 , or also include terms of order (five and) six. Note that near an equilibrium (with linearization governed by H_0) these latter terms are of the same magnitude as the additional terms in the normal form of order 2 .

Remark : It is instructive to compare the phase portraits of *figure 4.2* with *figure 5* in [Ferrer,Gárate;96]. While corresponding flows are conjugate for $a \neq 0$, those with $a = 0$ are not. Indeed, the south pole of the sphere $S^2_{L,0}$ in [Ferrer,Gárate;96] corresponds to all rectilinear ellipses. Hence, to pass from $V_{0,b}$ to $S^2_{L,0}$ one has to identify all points on the arc $\operatorname{Im}\tau_1 = 0 \, , \operatorname{Re}\tau_1 \geq 0$. While the saddle at the north pole of $S^2_{L,0}$ corresponds to the saddle $\tau = (-\omega^2 b^2, \omega b)$ of $V_{0,b}$, the degenerate equilibrium at the south pole gets blown up to three centres. See also [Ferrer,Hanßmann,Palacián,Yanguas;98].

4.2 Reconstruction of the averaged flow

Our next goal is to reconstruct the dynamics of $X_{\bar{H}_\varepsilon}$ on the subset $\{ \sigma \in \mathbb{C}^3 \mid P(\sigma) = 0 \}$ defined by the relation (4). Note that we refrain from reconstructing the complete flow of $X_{\bar{H}_\varepsilon}$ on $T^*\mathbb{R}^3$. Indeed, since the original Hamilton function H_ε is axially symmetric, we can perform the perturbation from $\bar{H}_\varepsilon$ to H_ε on the (first) reduced phase space $\{ \sigma \in \mathbb{C}^3 \mid P(\sigma) = 0 \, , \operatorname{Re}\sigma_2 \geq 0 \}$. The momentum mapping N of the reduced axial symmetry being an integral of motion for both $\bar{H}_\varepsilon$ and H_ε , this perturbation analysis takes place in two degrees of freedom, allowing for better results.

For the reconstruction we have to attach a 1-torus (*i.e.* an S^1) to every point on $V_{a,b}$. In this way the periodic orbits of $X_{\bar{H}_\varepsilon^{a,b}}$ give rise to invariant 2-tori, and the equilibria lead to periodic orbits of $X_{\bar{H}_\varepsilon}$. Thus the 3-dimensional invariant subvarieties $N^{-1}(a) \cap L^{-1}(b)$ of $P^{-1}(0)$ become ramified 2-torus bundles. The regular fibres form families of 2-tori which are separated by the stable and unstable manifolds of hyperbolic periodic orbits (or, occasionally, of parabolic periodic orbits) and shrink down to elliptic periodic orbits with $\sigma \neq 0$. The situation near the periodic orbit $\sigma(t) = (\sigma_1(t), 2\omega(b + \mathrm{i}a), 0)$ (with $|\sigma_1(t)| = 2\omega\sqrt{b^2 - a^2}$) of $X_{\bar{H}_\varepsilon}$ is more involved, reflecting the 2:1 resonance. Indeed, the frequency of the periodic orbit is $2\omega + O(\varepsilon)$, while the surrounding tori have the frequency $\omega + O(\varepsilon)$ next to a frequency of order ε accounting for the slow motion on the twice reduced phase space. As $a \to \pm b$ this periodic orbit shrinks to an equilibrium, here the whole "ramified torus bundle" $\{ \sigma \in \mathbb{C}^3 \mid P(\sigma) = 0 \, , \, L(\sigma) = b \, , \, N(\sigma) = \epsilon b \}$, $\epsilon = \pm 1$, $b \geq 0$ consists of this single point.

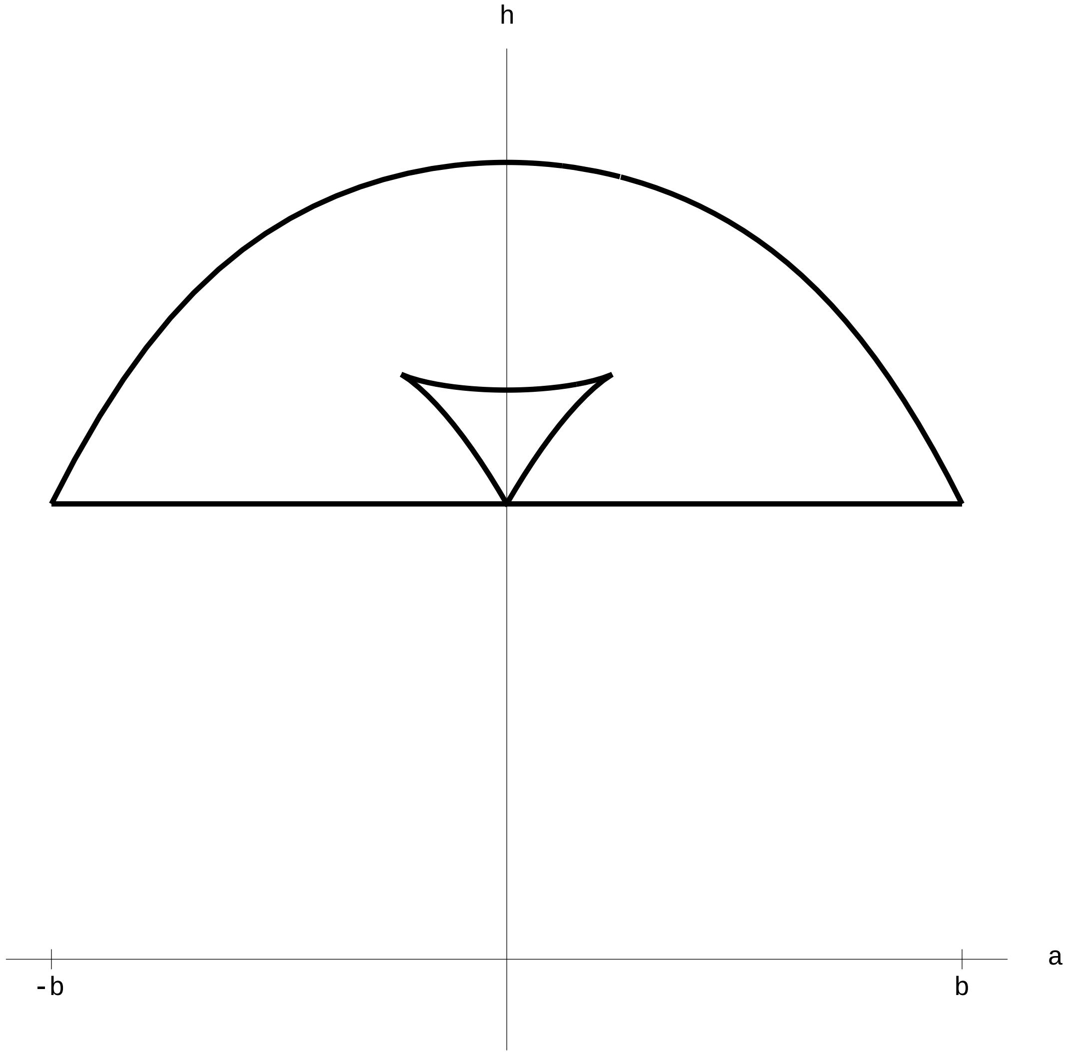

Figure 4.3: The b-slice Σ_b of the set Σ of critical values (h, a, b) of $(\bar{H}_\varepsilon, N, L)$.

We also have to understand how the invariant level sets fit together to form the (reduced) phase space $\{ \sigma \in \mathbb{C}^3 \mid P(\sigma) = 0 \, , \, \mathrm{Re}\, \sigma_2 \geq 0 \}$. We can collect the relevant information in one picture, the set Σ of critical values of

$$(\bar{H}_\varepsilon, N, L): \quad \{ \sigma \in \mathbb{C}^3 \mid P(\sigma) = 0\,,\ \mathrm{Re}\,\sigma_2 \geq 0 \} \quad \longrightarrow \quad \mathbb{R}^3 \qquad .$$

Note that the lift of this mapping to $T^*\mathbb{R}^3$ is the energy-momentum mapping of $\bar{H}_\varepsilon$ on $T^*\mathbb{R}^3$.

In our particular example the set Σ can be obtained from the proof of Proposition 4.1. The critical values are symmetric with respect to the reflection $a \mapsto -a$. For a fixed value b of L the slice Σ_b of critical values is depicted in *figure 4.3*. The scaling (12) shows that the whole set Σ of critical values is a cone over Σ_b , distances in the a-direction increase linearly with b as does the minimal energy on Σ_b, while the maximal energy on Σ_b behaves like $b + \varepsilon b^2$. In particular also a level set $\bar{H}_\varepsilon = h$ of Σ is diffeomorphic to the depicted b-slice.

§5 Implications for the original system

In the previous section we could give a quite detailed description of the dynamics of $\bar{H}_\varepsilon$ on the (axially) reduced phase space $\{ \sigma \in \mathbb{C}^3 \mid P(\sigma) = 0\,, \mathrm{Re}\,\sigma_2 \geq 0 \}$. This is due to the oscillator symmetry. The behaviour is governed by the reduced system with one degree of freedom. But unlike these one-degree-of-freedom systems the vector field $X_{\bar{H}_\varepsilon}$ is not structurally stable (in the class of all Hamiltonian vector fields on $P^{-1}(0)$). Under any small perturbation it can (and usually does) lose its integrability.

We have to keep track of several perturbation problems here. First, our "original Hamilton function" H_ε is a perturbation (of order ε) of the isotropic harmonic oscillator H_0 . Using Theorem 3.1 we consider H_ε as a perturbation (of order ε^2) of its normal form $\bar{H}_\varepsilon$. Furthermore we know that perturbations of the family $\bar{H}_\varepsilon^{a,b}$ of one-degree-of-freedom systems defined by $\bar{H}$ do not lead to qualitative changes. This is what we want to use to gain information about the perturbation from $\bar{H}_\varepsilon$ to H_ε on $P^{-1}(0)$.

Note that we did not reconstruct the complete flow of $X_{\bar{H}_\varepsilon}$ on $T^*\mathbb{R}^3$. In this way the perturbation analysis takes place in two degrees of freedom, considerably facilitating our task. For instance, the surviving maximal tori provided by KAM-theory now divide the energy shells, which leads to global stability results (in the sense of Lyapunov). In a final step we reconstruct the complete flow of X_{H_ε} on $T^*\mathbb{R}^3$.

5.1 Persistence in two degrees of freedom

Our aim is to understand the dynamics of a generic axially symmetric perturbation of the isotropic harmonic oscillator. In Subsection 2.1 we used the S^1-symmetry of the Hamilton function H_ε to reduce the phase space to $\{\sigma \in \mathbb{C}^3 \mid P(\sigma) = 0, \operatorname{Re}\sigma_2 \geq 0\}$ and in Subsections 4.2 we described the dynamics of the integrable approximation $\bar{H}_\varepsilon$ of H_ε . We now extract from this information a (partial) description of the flow defined by X_{H_ε} .

First we use that the set $\{(0,0,\sigma_3) \mid \sigma_3 \in \mathbb{C}^3\}$ of singularities of $P^{-1}(0)$ is necessarily X_{H_ε}-invariant. The intersections with the energy levels $\{H_\varepsilon = h\}$ correspond to the solutions of the one-degree-of-freedom subsystem

$$\dot{\sigma}_3 \;=\; -2\omega\mathrm{i}\,\frac{\partial H_\varepsilon}{\partial \bar{\sigma}_3} \qquad .$$

In particular there is an equilibrium in an ε-neighbourhood of the origin. In applications this will usually be the origin itself.

We now turn to the regular subset $\{\sigma \in \mathbb{C}^3 \mid P(\sigma) = 0, \operatorname{Re}\sigma_2 \geq 0\}$. Theorem 3.1 yields the inequality

$$\|H_\varepsilon \circ \psi - \bar{H}_\varepsilon\|_{A(h)} \;\leq\; \Gamma(h) \cdot \varepsilon^2$$

for sufficiently small ε . Here $\|\,..\,\|_{A(h)}$ denotes the supremum norm on the union $A(h)$ of the energy shells $\{H = h'\}$ with $h' \leq h$, and $\Gamma(h)$ is a constant that only depends on h (and not on ε).

We look for persistence of the dynamics of $X_{\bar{H}_\varepsilon}$ on the energy shells. To apply the implicit mapping theorem on equilibria and periodic orbits (including parabolic periodic orbits that undergo a (periodic) centre-saddle bifurcation) we need estimates $|H_\varepsilon \circ \psi - \bar{H}_\varepsilon| < c\varepsilon$ on some neighbourhood, with a constant c independent of ε . Such estimates are true for $\varepsilon < \frac{c}{\Gamma(h)}$.

The persistence of most invariant 2-tori follows from [Arnol'd;63], [Broer,Huitema;91]. The crucial point is the asymptotic behaviour of the frequency mapping as $\varepsilon \to 0$. With respect to suitably chosen actions (L, I) the frequency vector is of the form

$$D\bar{H}_\varepsilon(L, I) \;=\; \begin{pmatrix} \omega \\ 0 \end{pmatrix} + O(\varepsilon) \qquad .$$

To ensure the persistence of a large Cantor set of 2-tori on each energy shell we need that the determinant

$$\det \begin{pmatrix} D^2\bar{H}_\varepsilon & D\bar{H}_\varepsilon \\ D\bar{H}_\varepsilon & 0 \end{pmatrix} \;=\; -\omega^2 \frac{\partial^2 \bar{H}_\varepsilon}{\partial I^2} + O(\varepsilon^2) \tag{17}$$

is of order ε . Hence the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\partial^2 \bar{H}_\varepsilon}{\partial I^2}$ has to be nonzero at $\varepsilon = 0$, which is a generic condition. The momentum mapping N of the reduced axial S^1-action is an integral of motion not only for $\bar{H}_\varepsilon$, but also for H_ε . The 2-dimensional invariant tori divide the 3-dimensional invariant submanifolds of constant N and H_ε , they are nested within these 3-spheres. Therefore, the centres and the elliptic periodic orbits are stable in the sense of Lyapunov.

While the stable manifold of *e.g.* a hyperbolic periodic orbit coincides with the unstable manifold in the averaged system $\bar{H}_\varepsilon$, these manifolds split for generic H_ε . This gives rise to transverse homoclinic orbits and, as we vary the energy, to tangency bifurcations ; cf. [Robinson;70a,b], [Broer,Tangerman;86], [Duarte;94]. This applies *mutatis mutandis* to heteroclinic orbits.

In our particular example the polar subsystem $\sigma_1 = \sigma_2 = 0$ is the harmonic oscillator $\dot{\sigma_3} = -\mathrm{i}\omega\,\sigma_3$, in particular the origin remains fixed. The planar subsystem $\sigma_3 = 0$ is also invariant : again the perturbation vanishes and the equations of motion $\dot{\sigma}_1 = -2\mathrm{i}\omega\,\sigma_1\,,\ \dot{\sigma}_2 = 0$ are those of the axially reduced 1:1 resonance. This accounts for the centres at $\sigma = (0, |a| + \mathrm{i}a, 0)\,,\ a \in \mathbb{R}$ and the elliptic periodic orbits $(\mathrm{e}^{-2\mathrm{i}\omega t}\sigma_1, b + \mathrm{i}a, 0)\,,\ |a| < b$. For sufficiently small ε also the second family of elliptic periodic orbits that originates from the centres $(0, |a| + \mathrm{i}a, 0)$ survives the perturbation from $\bar{H}_\varepsilon$ to H_ε . This also holds true for the continuation of the elliptic periodic orbits $(0, 0, \mathrm{e}^{-\mathrm{i}\omega t}\sigma_3)$ and the 2-parameter family of hyperbolic periodic orbits. Furthermore the (periodic) centre-saddle bifurcation survives where both families merge to a 1-parameter family of parabolic periodic orbits and vanish.

To show the persistence of large Cantor sets of invariant 2-tori on each energy shell we remark that the reduced system $X_{\bar{H}_\varepsilon^{a,b}}$ on $V_{a,b}$ depends on ε only through a global multiplication. Furthermore, the frequency $\frac{\partial \bar{H}_\varepsilon}{\partial I}$ is an analytic function, so it suffices to show that this function is not constant. Now the system has hyperbolic periodic orbits, and upon approaching a separatrix, the frequency $\frac{\partial \bar{H}_\varepsilon}{\partial I}$ converges to zero. Extracting some neighbourhood of the isolated zeros of $\frac{\partial^2 \bar{H}_\varepsilon}{\partial I^2}$ we get the necessary lower bound $\varepsilon\kappa$ of the determinant (17).

5.2 Reconstruction of the full system

We finally return to three degrees of freedom and reconstruct the dynamics of the full system X_{H_ε} on $T^*\mathbb{R}^3$. With the exception of $\{\,(0,0,\sigma_3) \mid \sigma_3 \in \mathbb{C}^3\,\}$, every point of the reduced phase space $\{\,\sigma \in \mathbb{C}^3 \mid P(\sigma) = 0\,,\, \operatorname{Re}\sigma_2 \geq 0\,\}$ gets a 1-torus (*i.e.* an S^1) attached. Elliptic and hypoelliptic equilibria thereby turn into periodic orbits of the same type. Elliptic, hyperbolic and parabolic periodic orbits in $P^{-1}(0)$ become, respectively, normally elliptic, hyperbolic and parabolic invariant 2-tori on $T^*\mathbb{R}^3$. From invariant 2-tori with quasi-periodic motion we get invariant 3-tori that may be resonant, but do not foliate into periodic orbits. The set $\{(0,0,\sigma_3) \mid \sigma_3 \in \mathbb{C}^3\}$ of singularities simply becomes $T_2 := \{(x,p) \in T^*\mathbb{R}^3 \mid x_1 = x_2 = 0 = p_1 = p_2\}$. These periodic orbits are oscillator "ellipses" that survive the perturbation from H_0 to H_ε . The singular points of the second reduction also influence the dynamics. Recall that $T^*\mathbb{R}^3$ is a ramified 2-torus bundle over V , and that the singular fibres are exactly the ones attached to points in the singular subset $S = S_1 \cup S_2 \cup S_3$ of V . By T_i $(i=1,2,3)$ we denote the union of the fibres of points in S_i . The above considerations prove the following result.

Theorem 5.1

Let $H_\varepsilon \in C^\infty(T^*\mathbb{R}^3)$ *be a generic axially symmetric Hamilton function that has an equilibrium at the origin. Moreover, suppose that the linearization of* H_ε *at* 0 *is in 1:1:1 resonance. In a small neighbourhood of the origin the flow has the following properties.*

The subset $T_2 \subseteq T^*\mathbb{R}^3$ *consists of periodic orbits of* H_ε *which shrink down to the origin. From this equilibrium also originate two 1-parameter families of periodic orbits of* H_ε *which are close to* T_3 . *For* $N > 0$ *these are (almost) direct planar circular orbits, while* $N < 0$ *corresponds to the retrograde case. These latter periodic orbits and the equilibrium are stable in the sense of Lyapunov. Near* T_1 *the system has a 2-parameter family of invariant 2-tori.*

The existence of three periodic orbits on each energy shell $H_\varepsilon = const$ follows already from index-theoretical considerations, cf. [Moser;70]. As we have seen, the axial symmetry not only completely determines the polar periodic orbit in T_2 , but also forces the other two periodic orbits to be almost planar.

Some remarks on the topological structure of the invariant subsets are in order. The energy levels $H_\varepsilon = const$ are 5-spheres, indeed they are perturbed from $H_0 = const$. Intersecting with $N = a$ leads to 4-dimensional invariant submanifolds close to $N^{-1}(a) \cap L^{-1}(b)$. For $a \neq 0$ the latter are diffeomorphic to $S^3 \times S^1$, while $N^{-1}(0) \cap L^{-1}(b)$ has a more intricate structure described at the end of Subsection 2.2. Since $\{(x,p) \mid N(x,p) = a \neq 0,\, H_\varepsilon(x,p) = const\}$ is reconstructed from the smooth 3-sphere $\{\sigma \mid P(\sigma) = 0,\, N(\sigma) = a \neq 0,\, H_\varepsilon(\sigma) = const\}$, this set is again diffeomorphic to $S^3 \times S^1$. For topological reasons $\{(x,p) \mid N(x,p) = 0,\, H_\varepsilon(x,p) = const\}$ must therefore be homeomorphic to $N^{-1}(0) \cap L^{-1}(b)$. The singular circle is exactly the periodic orbit $T_2 \cap \{H_\varepsilon = const\}$.

The dynamics on the "regular part" $(\pi \circ \tau \circ \sigma)^{-1}(V \backslash S)$ of $T^*\mathbb{R}^3$ depend heavily on the actual form of the perturbation H_ε . Already the normal behaviour of the periodic orbits in $T_2 \cup \{(0,0)\}$ or the invariant 2-tori near T_1 can only be determined when H_ε is specified.

Let us collect what we found out about our particular example.

Theorem 5.2

Consider the Hamilton function (11) *on* $T^*\mathbb{R}^3$. *For sufficiently small* ε *the corresponding flow has the following properties.*

A measure-theoretically large part of the phase space is filled by 3*-dimensional Cantor families of invariant* 3*-tori. On each energy shell the measure of their complement goes to zero as* $\varepsilon \to 0$. *The motion on these* 3*-tori is quasi-periodic.*

Different Cantor families of invariant 3*-tori are separated by the stable and unstable manifolds of* 2*-parameter families of normally hyperbolic invariant* 2*-tori. The latter arise and vanish in (quasi-)periodic centre-saddle bifurcations. The Cantor families of invariant* 3*-tori shrink down to* 2*-parameter families of normally elliptic* 2*-tori.*

The periodic orbits in T_2 *are elliptic. The two families of invariant* 2*-tori that originate from these vanish in the (quasi-)periodic centre-saddle bifurcations. From the elliptic periodic orbits at* T_3 *originate two families of normally elliptic invariant* 2*-tori, the ones at* T_1 *being in 1:1 resonance. The origin and all elliptic periodic orbits and normally elliptic* 2*-tori are stable in the sense of Lyapunov.*

We can be more specific about the motion in configuration space corresponding to the normally hyperbolic 2-tori when $N = 0$, *i.e.* only parametrised by the value h of the energy. Recall that for the normal form $\bar{H}_\varepsilon$ such 2-tori are foliated by periodic orbits along circular ellipses with radius $\frac{\sqrt{h}}{\omega}$ which are perpendicular to the (x_1, x_2)-plane. Now note that for $N = 0$ also the motion of H_ε in configuration space is planar as $\frac{x_2p_3 - x_3p_2}{x_3p_1 - x_1p_3}$ becomes a conserved quantity. In this way we recover the (almost circular) periodic orbits found with a semi-numerical method in [Caranicolas;94]. Similarly, the normally elliptic 2-tori at $N = 0$ give rise to periodic motions very close to straight lines through the origin of the configuration space, passing the (x_1, x_2)-plane with slope 1 .

Remark : The validity of the normal form is restricted to a small neighbourhood of the origin. For instance, the two invariant circles $x_1^2 + x_2^2 = -\frac{\omega^2}{2\varepsilon}$, $x_3^2 = -\frac{\omega^2}{2\varepsilon}$, $p_i = 0$ are not accessible to this perturbation analysis.

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