LONG TIME SEMICLASSICAL APPROXIMATION OF QUANTUM FLOWS: A PROOF OF THE EHRENFEST TIME

Dario BAMBUSI[1], Sandro GRAFFI[2], Thierry PAUL[3]

Abstract. *Let $\mathcal{H}$ be a holomorphic Hamiltonian of quadratic growth on $\mathbb{R}^{2n}$, b a holomorphic exponentially localized observable, H, B the corresponding operators on $L^2(\mathbb{R}^n)$ generated by Weyl quantization, and $U(t) = \exp iHt/\hbar$. It is proved that the L^2 norm of the difference between the Heisenberg observable $B_t = U(t)BU(-t)$ and its semiclassical approximation of order $N-1$ is majorized by $KN^{(6n+1)N}(-\hbar\log\hbar)^N$ for $t \in [0, T_N(\hbar)]$ where $T_N(\hbar) = -\dfrac{2\log\hbar}{N-1}$. Choosing a suitable $N(\hbar)$ the error is majorized by $C\hbar^{\log|\log\hbar|}$, $0 \le t \le |\log\hbar|/\log|\log\hbar|$. (Here K, C are constants independent of $N, \hbar$).*

1. Introduction and statement of the results

Denote $\Omega := \mathbb{R}^{2n}$ with coordinates (x, ξ). Let $\mathcal{H}(x, \xi) \in C^\infty(\Omega; \mathbb{R})$, and $b_t(x, \xi) := b \circ \phi_t^{\mathcal{H}} \equiv b(\phi_t^{\mathcal{H}}(x, \xi))$ be the time evolution of any bounded observable $b(x, \xi) \in C^\infty(\Omega; \mathbb{R})$ under the the flow $\phi_t^{\mathcal{H}} : \Omega \leftrightarrow \Omega$ generated by the Hamiltonian $\mathcal{H}$. Denote $H := Op^W(\mathcal{H})$ and $B = Op^W(b)$ the self-adjoint operators in $L^2(\mathbb{R}^n)$ representing the (Weyl) quantization of the symbols $\mathcal{H}, b$ and let $B_t := e^{iHt/\hbar} B e^{-iHt/\hbar}$ be the Heisenberg observable, i.e. the quantum evolution of the observable B under the unitary group generated by H.

The question of estimating how long the classical and quantum evolutions stay "close" one another or, better, how long the evolution of the quantum observables is determined by the corresponding classical one up to a prescribed error vanishing with $\hbar$ is one of the oldest problems of semiclassical analysis. According to a well known conjecture going back to Chirikov and Zaslavski [Ch,Za], this approximation can be valid on a time interval of

[1] Dipartimento di Matematica, Università di Milano, Italy. (bambusi@mat.unimi.it)
[2] Dipartimento di Matematica, Università di Bologna, Italy. (graffi@dm.unibo.it)
[3] CEREMADE, Université de Paris-Dauphine, France. (paulth@ceremade.dauphine.fr)

maximum duration $T \equiv T(\hbar)$ of order $-\log\hbar$, called the Ehrenfest time, if the error is required to vanish faster than any power of $\hbar$.

The origin of this conjecture, formally verified in some instances[Za] can be understood in the correspondence between symbols $b(x,\xi)$ (classical observables) and operators in Hilbert space B (quantum observables) provided by the Weyl quantization procedure:

$$(Bu)(x) = \frac{1}{(2\pi\hbar^n)}\int_{\mathbb{R}^{2n}} b\left(\frac{x+y}{2},\xi\right) e^{i\langle(x-y),\xi\rangle/\hbar}\, u(y)\, dy d\xi, \quad u\in\mathcal{S}(\mathbb{R}^n) \tag{1.1}$$

In this framework the problem can be formulated as follows: B_t solves the Heisenberg equation of motion

$$\dot{B}_t = \frac{i}{\hbar}[H, B_t] \tag{1.2}$$

If B_t admits a symbol, denoted $b_t(x,\xi;\hbar)$, by (1.2) it fulfills the equation

$$\dot{b}_t = \{\mathcal{H}, b_t\}_M \tag{1.3}$$

with the initial condition $b_0(x,\xi;\hbar) = b(x,\xi)$. Here $\{f,g\}_M(x,\xi)$ is the Moyal bracket of the two observables $f,g\in C^\infty(\mathbb{R}^{2n})$

$$\{f,g\}_M(x,\xi) := f\#g - g\#f \tag{1.4}$$

where $f\#g$, the symbol of the operator product FG, is expressed by the composition of the symbols f and g:

$$(f\#g)(x,\xi) = \frac{1}{(2\pi\hbar)^{n/2}}\int_{\mathbb{R}^{4n}} e^{-i\langle r,\rho\rangle/\hbar + i\langle w,\tau\rangle/\hbar} f(x+w,\rho+\xi)g(x+r,\tau+\xi)\, d\rho d\tau\, dr dw \tag{1.5}$$

$\{f,g\}_M$ admits the following formal expansion in powers of $\hbar$ [Fo,Ro,Vo]:

$$\{f,g\}_M(x,\xi) \sim \{f,g\} + \frac{1}{2^j}\sum_{|\alpha+\beta|=j\geq 1}(-1)^{|\beta|}\hbar^j\left(\partial_\xi^\alpha g D_x^\beta g\right)\cdot\left(\partial_\xi^\beta g D_x^\alpha f\right) \tag{1.6}$$

($\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index, and $|\alpha| := \alpha_1+\ldots+\alpha_n$; analogous definitions for β, and $D_x := -i\hbar\partial_x$). By (1.6) the differential equation (1.3) can be recursively solved in the space of the formal power series in $\hbar$ (for details see [Ro], Chapt.IV.10). The result, known as the semiclassical Egorov theorem, is the formal semiclassical expansion of the symbol b_t:

$$b_t(x,\xi;\hbar) \sim (b\circ\phi_t^a)(x,\xi) + \sum_{j=2}^{\infty} b_j(x,\xi;t)\hbar^j \tag{1.7}$$

Here the term of order zero in $\hbar$, by definition the *principal symbol* of B_t, is just the evolution of the observable b along the Hamiltonian flow generated by $\mathcal{H}$, i.e. the solution of the Liouville equation $\dot{b}_t = \{\mathcal{H}, b_t\}$, and

$$b_j(x,\xi;t) = -i\int_0^t \sum_{\substack{|\alpha+\beta|+l=j+1\\ 0\le l\le j-1}} (1-(-1)^{|\alpha+\beta|})\Gamma(\alpha,\beta)\left(\partial_\xi^\alpha \mathcal{H} D_x^\beta b_l\right)\circ \phi_{t-\tau}^{\mathcal{H}}(x,\xi)\,d\tau \tag{1.8}$$

The higher order terms $b_j(x,\xi;t)$ are thus completely determined by the classical evolution but have a polynomial dependence on the derivatives of the flow $\phi_t^{\mathcal{H}}(x,\xi)$ with respect to the inital conditions (x,ξ) up to order $j-1$. If, as it happens in general, there are initial conditions (x,ξ) generating a flow with positive Lyapunov exponents, the difference between the symbol $b_t(x,\xi;\hbar)$ of B_t and any prescribed approximation $(b\circ\phi_t^{\mathcal{H}})(x,\xi)+\sum_{j=2}^N b_j(x,\xi;t)\hbar^j$ is expected to increase exponentially in time: hence it can vanish as $\hbar\to 0$ only for a time interval not exceeding $-\log\hbar$. Put in a different way: the non-local nature of quantum mechanics, embodied in the symbol expansion (1.7), (1.8), can be dominated by its local approximation, the principal symbol $b\circ\phi_t^{\mathcal{H}}(x,\xi)$, only if the the remainder is small. This can be obtained only within the above time span.

In this paper we work out, in the analytic case, the estimates implying the validity of the above "Ehrenfest time" for a class of flows somewhat restricted but in a sense natural as discussed below. More precisely, for any fixed $\sigma>0$ set $|z| := \sup|z_k|$ and $\mathcal{G}_\sigma := \left\{z\in\mathbf{C}^{2n} \;:\; |\mathrm{Im}\, z|<\sigma\right\}$. The Hamiltonian $\mathcal{H}(x,\xi)\equiv\mathcal{H}(z)$, $z:=(x,\xi)$ is required to fulfill the following properties:

(A1) There exists $\nu>0$ such that $\mathcal{H}$ is real-holomorphic on $\mathcal{G}_\nu$.

(A2) Let $Jd\mathcal{H}$ be the symplectic gradient of $\mathcal{H}$. Then there are $A_1>0, A_2>0, \alpha>0$ such that $|Jd\mathcal{H}(z+iy)|\le A_1+A_2|z|\ \forall z,y\in\mathbb{R}^{2n}$, $|y|\le\sigma$. Moreover $|Jd^2\mathcal{H}(z)|\le\alpha$ on $\mathcal{G}_\sigma$.

(A3) Denote $(\hat{\mathcal{H}})(k)$ the Fourier transform of $\mathcal{H}(z)$. Then there are $\rho>0,\sigma>0$ such that $\hat{\mathcal{H}}(k_1+ik_2)$ is holomorphic on $\mathcal{G}_\rho\setminus(0,0)$; moreover $k^3\hat{\mathcal{H}}(k)$ is holomorphic on $\mathcal{G}_\rho$ and

$$|k_1|^3|\hat{\mathcal{H}}(k_1+ik_2)|\le Ce^{-\sigma|k_1|}\quad \text{for } |k_2|\le\rho$$

Remarks.

1 Under the above assumptions $H = Op^W(\mathcal{H})$ defined by (1.1) is essentially self-adjoint in $L^2(\mathbb{R}^n)$. By a standard abuse of notation we denote H also its self-adjoint closure.

2 Within the analyticity and decay assumptions (A1)-(A3), (A2) is the quadratic growth condition ensuring the existence of the Fourier integral operator representing the propagator $\exp\dfrac{iHt}{\hbar}$ [Cha] and thus the existence of the symbol of B_t [Ro].

3 In the phase variables $z=(x,\xi)$ Assumption (A3) means that there are $\sigma,\rho>0$ such that

$$\sup_{z+iy\in\mathcal{G}_\sigma}\left|\partial_z^{|\alpha|+|\beta|=3}\mathcal{H}(z+iy)\right|e^{\rho|z|}<+\infty. \tag{1.9}$$

To state the main result of the paper we need some further notation. For b as above set:

$$\Delta_{\mathcal{H}}b:=\frac{1}{\hbar^2}\left[\{b,\mathcal{H}\}-\{b,\mathcal{H}\}_M\right]\ . \tag{1.10}$$

and define recursively the two sequences $r_k^t, b_k^t : k\geq 1$ in the following way:

$$r_1^{t-\tau_1}:=\Delta_{\mathcal{H}}(b\circ\phi^{t-\tau_1}),\quad r_{k+1}^{t-\tau_1,\tau_1,\tau_2,\dots,\tau_k}:=\Delta_{\mathcal{H}}\left[r_k^{t-\tau_1,\tau_1,\tau_2,\dots,\tau_{k-1}}\circ\phi^{\tau_k}\right] \tag{1.11}$$

$$b_k^t:=\int_0^t d\tau_1\int_0^{t-\tau_1}d\tau_2\int_0^{t-\tau_2}d\tau_3...\int_0^{t-\tau_{k-1}}d\tau_k r_k^{t-\tau_1,\tau_1,\tau_2,\dots,\tau_{k-1}}\circ\phi^{\tau_k};b_0:=b\circ\phi^t$$

Moreover, let $b:\mathcal{G}_\sigma\to\mathbf{C}$ be holomorphic. Set:

$$|b|_{\sigma,\rho}:=\sup_{x+iy\in\mathcal{G}_\sigma}|b(x+iy)|\,e^{\rho|x|}\ . \tag{1.12}$$

Denote $\mathcal{A}_{\sigma,\rho}$ the set of all functions f holomorphic on $\mathcal{G}_{\sigma,\rho}$ such that $|f|_{\sigma,\rho}<+\infty$. Then:

Theorem 1.1. *Let there exist $\sigma>0,\rho>0$ and $0<\overline{B}<+\infty$ such that $|b|_{\sigma,\rho}<\overline{B}$. Then:*

(1) The operators $B_t^j:=Op^W(b_j^t)$ are continuous in L^2 and the Heisenberg operator $B_t=U(t)BU(-t)$ admits the expansion

$$B_t=\sum_{j=0}^N B_j^t\hbar^{2j}+\hbar^{2(N+1)}S_N^t\ ,$$

where

$$S_N^t:=\int_0^t d\tau_1\int_0^{t-\tau_1}d\tau_2\int_0^{t-\tau_2}d\tau_3...\int_0^{t-\tau_{k-1}}d\tau_kU(\tau_k)Op^w\left(r_k^{t-\tau_1,\tau_1,\tau_2,\dots,\tau_{k-1}}\right)U(-\tau_k)$$

(2) There are positive constants E,F independent of j,N and $\hbar$ such that for all $j\geq 1$, $N\geq 2$, $t\geq 0$ the following estimates hold:

$$\left\|B_j^t\right\|_{L^2\to L^2}\leq\left[eEe^{7\alpha t}j^{6n+3}\right]^j\frac{\overline{B}F}{j!}e^{(4n+2)\alpha t}\left[\exp\left(\alpha\frac{j(j-1)}{2}t\right)\right]^{6n+3} \tag{1.13}$$

$$\left\|S_N^t\right\|_{L^2\to L^2}\leq\left[eEe^{7\alpha t}N^{6n+3}t\right]^N\frac{\overline{B}F}{N!}e^{(4n+2)\alpha t}\left[\exp\left(\alpha\frac{N(N-1)}{2}t\right)\right]^{6n+3} \tag{1.14}$$

Remark. The holomorphy assumptions are needed to control the remainder to order $\hbar^N$ for all N. If we limit ourselves to $N = 1$ more general classes of Hamiltonians and of observables can be considered. More precisely, let for instance $\mathcal{H}(x,\xi)$ be a polynomial of order $2p$ such that the subgraph $\Sigma_E := \{(x,\xi) \in \mathbb{R}^{2n} | \mathcal{H}(x,\xi) \leq E\}$ is compact for some E, and let $b(x,\xi) \in C_0^\infty(\Sigma_E)$. Then (proof in the next section) there are $\Gamma > 0$ and $\Delta > 0$ such that:

$$\left\| B^t - B_0^t \right\|_{L^2 \to L^2} = \left\| B^t - Op^W(b \circ \phi^t) \right\|_{L^2 \to L^2} \leq \Gamma \hbar^2 t e^{\Delta t} . \tag{1.15}$$

The symbols b_j^t and hence the operators B_j^t are completely determined by the classical flow ϕ^t via (1.11). The quantum evolution will then stay close to the (semi) classical one as long as the error S_N^t stays small. The estimate (1.14) yields indeed, through a straightforward computation:

Corollary 1.2. *Let* $T_N(\hbar) := -\dfrac{2\log\hbar}{\alpha(N-1)}$, $B_t^N := \sum_{j=O}^{N-1} B_j^t \hbar^j$. *Then, for* $0 \leq t \leq T_N(\hbar)$*:*

$$\left\| B_t - B_t^N \right\| \leq \left(\frac{2e^2 E}{\alpha} \right)^N N^{(6n+1)N} BF \hbar^{2-15/\alpha-(8n+4)/\alpha N} (-\hbar \log\hbar)^N \tag{1.16}$$

Remarks.

1 If the Lyapunov numbers are zero for any initial datum (x,ξ), then we can take $\alpha = 0$ in formula (1.14), and by Assertion 2 of Theorem 1.2 one has

$$\left\| B_t - B_t^{N(\hbar)} \right\| = O(\hbar^N) \qquad 0 \leq t \leq \tilde{T}_N(\hbar)$$

where $\tilde{T}_N(\hbar) := e^{-1} N^{-6n-1} h^{-1}$.

2 Estimates valid for a time interval of duration $-C_N \log\hbar$ for Hamiltonians admitting polynomial growth of any order (but without control of the constant C_N), have been obtained by Combescure and Robert [Co-Ro1] in a weaker sense, i.e. comparing classical and quantum evolutions along coherent states (according to ideas introduced in [He], [BZ] and developed in [Ha], [BIZ], [Co-Ro2]).

3 The symbol expansion generated by Assertion (1) of Theorem 1.1, namely

$$b_t(x,\xi;\hbar) = \sum_{j=0}^{N} b_j^t \hbar^{2j} + O(\hbar^{2N+1})$$

differs from (1.7) in all terms with $j>0$. This difference makes the present expansion a non formal one, so that its remainder can be estimated.

4 Finally, let $T(\hbar)\in C([0,1];\mathbb{R}_+)$ be an increasing function such that $\lim_{\hbar\to 0}\frac{T(\hbar)}{-\log\hbar}=0$, and let $N(\hbar):=\left[-\frac{\log\hbar}{T(\hbar)}\right]$. Then clearly $\left\|B_t-B_t^{N(\hbar)}\right\|=O(\hbar^\infty)$, $0\le t\le T(\hbar)$.

To put this result into a more quantitative version, define the function sequence $\{\log^{[k]}(x)\}:k\in\mathbb{N}$ by $\log^{[1]}(x):=\log(x)$, $\log^{[k]}(x):=\log(\log^{[k-1]}(x))$.

Corollary 1.3. *For any integer $k\ge 1$ define $N_k(\hbar):=[\log^{[k]}(|\log(\hbar)|)]$. Then there exist positive constants $C,\overline{\hbar}$ such that, for $0<\hbar\le\overline{\hbar}$ one has*

$$\left\|B_t-B_t^{N_k(\hbar)}\right\|\le C\,\hbar^{\log^{[k]}(|\log(\hbar)|)}$$

for

$$0\le t\le\frac{|\log(\hbar)|}{\log^{[k]}(|\log(\hbar)|)}$$

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2. Proofs

Let b be a Weyl symbol of class Σ_0^1, and $\mathcal{H}$ an admissible semiclassical symbol (For these notions, see [Ro], Chapter 2; particular examples are all bounded observables $b(x,\xi)\in C^\infty(\mathbb{R}^{2n})$ and the Hamiltonians $\mathcal{H}\in C^\infty(\mathbb{R}^{2n})$ of polynomial growth at infinity). Denote ϕ^t the flow generated by $Jd\mathcal{H}$, J the unit $2n\times 2n$ symplectic matrix; let $H=Op^w(\mathcal{H})$ be essentially self-adjoint in $L^2(\mathbb{R}^n)$ and denote also $U(t):=\exp(itH/\hbar)$, $B:=Op^w(b)$, $B_t:=U(t)BU(-t)$, and

$$\Delta_{\mathcal{H}}b:=\frac{1}{\hbar^2}\left[\{b,\mathcal{H}\}-\{b,\mathcal{H}\}_M\right]\ . \tag{2.1}$$

Our semiclassical expansion is generated by the following simple remark:

Lemma 2.1. *The following formula holds*

$$B_t := Op^w\left(b\circ\phi^t\right) + \hbar^2\int_0^t d\tau U(\tau) Op^w\left(r_1^{t-\tau}\right) U(-\tau)\ ,$$

where $r_1^s := \Delta_{\mathcal{H}}\left(b\circ\phi^s\right)$

Proof. Denote $\beta_t := b\circ\phi^t$. Then:

$$\frac{d}{dt}\left[Op^w\left(\beta_t\right)\right] = Op^w\left(\frac{d}{dt}\beta_t\right) = Op^w\left(\{\beta_t,\mathcal{H}\}\right)$$
$$= Op^w\left(\{\beta_t,\mathcal{H}\} - \{\beta_t,\mathcal{H}\}_M\right) + \frac{i}{\hbar}\left[Op^w\left(\beta_t\right), Op^w\left(\mathcal{H}\right)\right]$$
$$= \frac{i}{\hbar}\left[Op^w\left(\beta_t\right), Op^w\left(\mathcal{H}\right)\right] + \hbar^2 Op^w\left(r_1^t\right)\ .$$

It follows

$$\frac{d}{dt}\left[Op^w\left(\beta_t\right) - B_t\right] = \frac{i}{\hbar}\left[Op^w\left(\beta_t\right) - B_t, Op^w\left(\mathcal{H}\right)\right] + \hbar^2 Op^w\left(r_1^t\right)\ ,$$

and by the variation of parameters formula

$$Op^w\left(\beta_t\right) - B_t = \hbar^2\int_0^t U(t-s) Op^w\left(r_1^s\right) U(-(t-s)) ds\ . \tag{2.2}$$

The assertion is now proved performing the change of variable $\tau = t-s$ in the integral. □

Proof of formula (1.15). Since $\mathcal{H}$ is a polynomial of degree $2p$

$$r_1^t = \Delta_{\mathcal{H}}(b\circ\phi^t) = \frac{1}{\hbar^2}\left[\left\{b\circ\phi^t,\mathcal{H}\right\} - \left\{b\circ\phi^t,\mathcal{H}\right\}_M\right] = \sum_{\substack{|k|=1\\ k=(k_1,\dots,k_{2n})}}^{2p} c_k\hbar^k\frac{\partial^{|k|} b\circ\phi^t}{\partial z^k}$$

where $c_k(x,\xi)$ is a polynomial of degree $2p-|k|$. Now the smooth functions $\theta_k(x,\xi) := c_k(x,\xi)\dfrac{\partial^{|k|} b\circ\phi^t}{\partial z^k}$ have compact support in $\mathbb{R}^{2n}$ and hence define bounded operators in L^2 upon Weyl quantization. Denote $\lambda(x,\xi)$ the Lyapunov number of the trajectory ϕ^t with any initial datum $(x,\xi)\in\Sigma_E$. Since $\phi^t(x,\xi)$ is bounded $\forall\, t\in\mathbb{R}$ we have (see e.g.[Ce], 3.12) $\delta := \sup_{\Sigma_E}\lambda(x,\xi) < +\infty$. Hence there are $\gamma_k > 0$ such that $\sup_{\Sigma_E}\left|\dfrac{\partial^{|k|}\phi^t}{\partial z^k}\right| \le \gamma_k e^{\delta t}$. Since $\dfrac{\partial^{|k|} b\circ\phi^t}{\partial z^k}$ is a polynomial of degree $|k|$ in the variables $\dfrac{\partial^{|s|}\phi^t}{\partial z^s}$, $s = 1,\dots,|k|$ with coefficients depending on $\dfrac{\partial^{|s|} b}{\partial z^s}$, $s = 1,\dots,|k|$, for any fixed $q\in\mathbf{N}$ depending only on n there are $\Gamma_k(n) > 0$ such $\sup_{\mathbb{R}^n}\left|\dfrac{\partial^{|l|}\theta_k}{\partial z^l}\right| \le \Gamma_k e^{|k|q\delta t}$, $|l|\le q$. Hence by the Calderon-Vaillancourt theorem there exists $q > 0$ such that $\|Op^W(\theta_k)\|_{L^2 \to L^2} \le \Gamma_k e^{|k| q \delta t}$. Inserting this estimate in (2.2) we get (1.15) with $\Delta = 2pq, \Gamma = \mathrm{Max}_k \Gamma_k$. ☐

Recall now the definition of the sequences $r_k^{t-\tau_1, \tau_1, \tau_2, \dots, \tau_{k-1}}$ $(k \ge 2)$ and $\{b_k^t\}_{k \ge 0}$, $b_0 := b$:

$$r_{k+1}^{t-\tau_1, \tau_1, \tau_2, \dots, \tau_k} := \Delta_{\mathcal{H}} \left[r_k^{t-\tau_1, \tau_1, \tau_2, \dots, \tau_{k-1}} \circ \phi^{\tau_i} \right]$$

$$b_k^t := \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \int_0^{t-\tau_2} d\tau_3 \dots \int_0^{t-\tau_{k-1}} d\tau_k r_k^{t-\tau_1, \tau_1, \tau_2, \dots, \tau_{k-1}} \circ \phi^{\tau_k} \quad k \ge 1$$

Lemma 2.2. *Let $B_j^t = Op^W(b_j^t)$. Then:*

$$B_t = \sum_{j=0}^{N} B_j^t \hbar^{2j} + \hbar^{2(N+1)} S_N \ ,$$

where

$$S_N := \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \int_0^{t-\tau_2} d\tau_3 \dots \int_0^{t-\tau_{k-1}} d\tau_k U(\tau_k) Op^w \left(r_k^{t-\tau_1, \tau_1, \tau_2, \dots, \tau_{k-1}} \right) U(-\tau_k)$$

Proof. Just iterate the proof of lemma 2.1 ☐

Let $b : \mathcal{G}_\sigma \to \mathbf{C}$ be an analytic function; recall the definitions

$$|b|_{\sigma,\rho} := \sup_{x+iy \in \mathcal{G}_\sigma} |b(x+iy)| \, e^{\rho |x|} \ . \tag{2.3}$$

and $\mathcal{A}_{\sigma,\rho} := \{ b \text{ holomorphic in } \mathcal{G}_\sigma : |b|_{\sigma,\rho} < +\infty \}$. We will estimate the sequence r_k in the above norm. Clearly we have to estimate the norm of $b \circ \phi^t$ and of $\Delta_{\mathcal{H}} b$. We first prove the following

Lemma 2.3. *There exists a positive σ such that ϕ^t extends to a complex analytic function*

$$\phi^t : \mathcal{G}_{\sigma e^{-\alpha t}} \to \mathcal{G}_\sigma$$

Proof. Denote $f := J d\mathcal{H}$, and consider, on $\mathcal{G}_\sigma$, the system of equations

$$\dot{z} = f(z) \ . \tag{2.4}$$

Writing $z = x + iy$ and $f = f_1 + if_2$, one has $\dot{y} = f_2(x + iy)$. Since $f_2 = 0$ on the real axis, by assumption A2 one has $|f_2(x + iy)| \leq \alpha|y|$. It follows that the inequalities

$$|y|^{\cdot} \leq \alpha|y| \quad \Longrightarrow |y(t)| \leq |y_0|e^{\alpha|t|} \tag{2.5}$$

hold. So one has $\phi^t(\mathcal{G}_{\sigma e^{-\alpha|t|}}) \subset \mathcal{G}_\sigma$.

Fix $\tilde{t}$. Given $\bar{z} \in \mathcal{G}_{\sigma e^{-\alpha|\tilde{t}|}}$ we prove that $\phi^{\tilde{t}}$ is analytic at $\bar{z}$. By the Cauchy- Kowaleskaya theorem (see e.g.[Pe]) there exists a neighbourhood $\mathcal{U}$ of $\bar{z}$ and a time $\bar{t}$ such that, for any $|\tau| < \bar{t}$, ϕ^τ is analytic on $\mathcal{U}$. Assume that $\bar{t}$ is the supremum of such times (so that $\phi^{\bar{t}}$ is not analytic in $\mathcal{U}$). Assume by contradiction $\bar{t} < \tilde{t}$. By (2.5) the limit $\lim_{\tau\to\bar{t}}\phi^\tau(z)$ exists on $\mathcal{U}$. Denote $w := \lim_{\tau\to\bar{t}}\phi^\tau(\bar{z})$. Again by the Cauchy-Kowakeskaya thoerem there exists a neighbourhood $\mathcal{V}$ of w and a $t_1 > 0$ such that ϕ^τ is analytic on $\mathcal{V}$ for $|\tau| < t_1$. Assume that $\mathcal{U}$ is so small that for fixed ϵ small enough one has $\phi^{\bar{t}-\epsilon}(\mathcal{U}) \subset \mathcal{V}$, then one has

$$\phi^{\bar{t}+\epsilon}(\mathcal{U}) = \phi^{2\epsilon}\left(\phi^{\bar{t}-\epsilon}(\mathcal{U})\right) ,$$

which is analytic since it is the composition of two analytic functions, against the assumption that $\bar{t}$ is the last time of analyticity. $\square$

Lemma 2.4. *Let $b \in \mathcal{A}_{\sigma,\rho}$, then, for any t, and for σ small enough, one has $b \circ \phi^t \in \mathcal{A}_{\sigma e^{-\alpha|t|},\rho e^{-\alpha|t|}}$, and*

$$\left|b \circ \phi^t\right|_{\sigma e^{-\alpha|t|},\rho e^{-\alpha|t|}} \leq |b|_{\sigma,\rho} .$$

Proof. By the above lemma $b \circ \phi^t$ has the required analyticity properties. Denote $\rho_t := \rho e^{-\alpha|t|}$, $\sigma_t := \sigma e^{-\alpha|t|}$, $\varphi_1 + i\varphi_2 = \phi^t(x + iy)$, then one has

$$\left|b \circ \phi^t\right|_{\sigma_t,\rho_t} = \sup_{x+iy\in\mathcal{G}_{\sigma_t}} \left|b\left(\phi^t(x + iy)\right) e^{\rho_t|x|}\right|$$

$$\leq \sup_{\varphi_1+i\varphi_2\in\mathcal{G}_{\sigma_t e^{\alpha t}}} \left|b(\varphi_1 + i\varphi_2)e^{\rho_t|Re(\phi^{-t}(\varphi_1+i\varphi_2))|}\right| ;$$

using the equation of motion and A2, one has

$$|Re(\phi^{-t}(\varphi_1 + i\varphi_2))| < |\varphi_1|e^{\alpha|t|} ,$$

which implies the assertion. $\square$

We will estimate the norm of $\Delta_{\mathcal{H}}$ using the Fourier transform. For this reason the following lemma is useful

Lemma 2.5. *One has*

$$\left|\hat{b}\right|_{\rho-\delta,\sigma} \leq \left(\frac{2}{\pi}\right)^n \frac{1}{\delta^{2n}} |b|_{\sigma,\rho} \ . \tag{2.6}$$

Proof. Fix $k_1 = \kappa e_1$ where e_1 is the unit vector of the first axis and κ a positive number; fix also k_2 with $|k_2| < \rho - \delta$. One has

$$(2\pi)^n \left|\hat{b}(k_1 + ik_2)\right| = \left|\int_{\mathbb{R}^{2n}} b(x) e^{i(k_1+ik_2)x} dx\right|$$
$$= \left|\int_{\mathbb{R}^{2n}} b(x + ie_1\sigma) e^{i(k_1+ik_2)(x+ie_1\sigma)} dx\right| \leq \int_{\mathbb{R}^{2n}} |b|_{\sigma,\rho} e^{-\rho|x|} e^{-\kappa\sigma} e^{|k_2||x|}$$
$$\leq |b|_{\sigma,\rho} e^{-\kappa\sigma} \int_{\mathbb{R}^{2n}} e^{-\delta|x|} dx = \left(\frac{2}{\delta}\right)^{2n} e^{-\kappa\sigma} |b|_{\sigma,\rho} \ ,$$

which by definition of $|b|_{\sigma,\rho}$ is the thesis in the particular case just considered. The general case can be dealt with in a similar way. □

Lemma 2.6. *Let $b \in \mathcal{A}_{\sigma,\rho}$ with $\sigma \leq \nu$ small enough. Then there exists a positive constant A such that, $\forall d < \sigma$, $\delta < \rho$:*

$$|\Delta_{\mathcal{H}} b|_{\sigma-d,\rho-\delta} \leq \frac{A}{\delta^{2n} d^{4n+3}} |b|_{\sigma,\rho} \ .$$

Proof. To obtain the estimate via the Fourier transform we first recall that

$$\{b, \mathcal{H}\}_M^{\wedge}(k) = \frac{2}{\hbar} \int_{\mathbb{R}^{2n}} \hat{b}(k-s) \hat{\mathcal{H}}(s) \sin \frac{(k-s)\wedge s}{\hbar/2} ds$$

where $(k_p, k_q) \wedge (s_p, s_q) := k_p \cdot s_q - k_q \cdot s_p$, whence

$$\widehat{\Delta_{\mathcal{H}} b}(k) = \frac{2}{\hbar} \int_{\mathbb{R}^{2n}} \hat{b}(k-s) \hat{\mathcal{H}}(s) \left(\sin \frac{(k-s)\wedge s}{\hbar/2} - \frac{(k-s)\wedge s}{\hbar/2}\right) ds$$

Since $|\sin z - z| \leq C_1 |z^3|$ for all $z \in \mathcal{G}_\sigma$, one has, for $|\mathrm{Im}\, k| < \rho - \delta$,

$$\begin{aligned} \left|\widehat{\Delta_{\mathcal{H}} b}(k)\right| &\leq \int_{\mathbb{R}^{2n}} \left|\hat{b}(k-s)\right| \left|\hat{\mathcal{H}}(s)\right| C_1 |k-s|^3 |s|^3 ds \\ &\leq C_2 \frac{|b|_{\sigma,\rho}}{\delta^{2n}} \int_{\mathbb{R}^{2n}} |k_1 - s|^3 e^{-\sigma|k_1-s|} e^{-\sigma|s|} ds \ , \end{aligned} \tag{2.7}$$

where use has been made of (2.6) and Assumption A2. Now $||k_1 - s| + |s|| \geq |k_1|$ and $||k_1 - s| + |s|| \geq |k_1 - s|$. Hence ( ) does not exceed

$$\begin{aligned} &\frac{|b|_{\sigma,\rho}}{\delta^{2n}} C_3 e^{-(\sigma-d)|k_1|} \int_{\mathbb{R}^{2n}} |k_1 - s|^3 e^{-d|k_1-s|} ds \\ &= \frac{|b|_{\sigma,\rho}}{\delta^{2n} d^{2n+3}} C_3 e^{-(\sigma-d)|k_1|} \int_{\mathbb{R}^{2n}} |s|^3 e^{-|s|} ds \ , \end{aligned}$$

which gives

$$\left|\widehat{\Delta_{\mathcal{H}} b}\right|_{\rho-\delta,\sigma-d} \leq \frac{C_4}{d^{2n+3}\delta^{2n}} |b|_{\sigma,\rho} \ .$$

Using again (2.6) to antitransform $\widehat{\Delta_{\mathcal{H}} b}$ the assertion is proved. □

Lemma 2.7. *Assume $|b|_{\sigma,\rho} \leq \overline{B}$ for some positive $\overline{B}, \sigma, \rho$. Then, for $k \geq 1$ and $0 < \tau_k < t$, one has*

$$\left| r_k^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{k-1}} \right|_{(\sigma-k\delta)e_k,(\rho-kd)e_k} \leq \Gamma_k$$

Here the sequence e_k is defined by

$$e_1 := e^{-\alpha t} \ , \quad e_k := e_1 \exp\left(-\alpha t(k-1)\right) \ , \quad k \geq 2 \tag{2.8}$$

and the sequence Γ_k by

$$\begin{aligned} \Gamma_1 &:= \frac{A\overline{B}}{\delta^{2n} d^{4n+3}} \frac{1}{e_1^{6n+3}} \\ \Gamma_k &:= \Gamma_1 \left(\frac{Ae_1}{d^{4n+3}\delta^{2n}}\right)^{k-1} \left[\exp\left(\alpha \frac{k(k-1)}{2}\right)\right]^{6n+3} . \end{aligned} \tag{2.9}$$

Proof. The expressions of e_1 and Γ_1 are a direct consequence of lemmas 2.4 and 2.6. By induction assume that the estimates of the lemma are true for k we prove them for $k+1$. By lemmas 2.4 and 2.6 we have

$$\left| r_k^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{k-1}} \circ \phi^{\tau_k} \right|_{(\sigma-k\delta)e_k e^{-\sigma\tau_k},(\rho-kd)e_k e^{-\sigma\tau_k}} \leq \Gamma_k$$

and therefore

$$\begin{aligned} &\left| r_{k+1}^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_k} \right|_{(\sigma-(k+1)\delta)e_k e^{-\sigma\tau_k},(\rho-(k+1)d)e_k e^{-\sigma\tau_k}} \\ &= \left| \Delta_{\mathcal{H}}(r_k^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{k-1}} \circ \phi^{\tau_k}) \right|_{(\sigma-(k+1)\delta)e_k e^{-\sigma\tau_k},(\rho-(k+1)d)e_k e^{-\sigma\tau_k}} \\ &\leq \Gamma_k \frac{A}{(e_k e^{-\alpha t})^{6n+3} d^{4n+3}\delta^{2n}} \end{aligned}$$

This yields $e_{k+1} = e_k e^{-\alpha t}$, and therefore (2.8); moreover

$$\Gamma_{k+1} = \Gamma_k \frac{A}{(e_k e^{-\alpha t})^{6n+3} d^{4n+3} \delta^{2n}} \ ,$$

whence

$$\Gamma_k = \left(\frac{A}{d^{4n+3}\delta^{2n}} \right)^{k-1} \Gamma_1 \left(\prod_{i=2}^{k} \frac{1}{e_i} \right)^{6n+3} \ ,$$

This proves (2.9) upon insertion of (2.8). This proves the lemma. □

Lemma 2.8. *For any $N \geq 2$ one has*

$$\begin{aligned} \left\| Op^w \left(r_N^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{N-1}} \right) \right\|_{L^2 \to L_2} &\leq \\ \left[E e^{7\alpha t} N^{6n+3} \right]^N BF e^{(4n+2)\alpha t} \left[\exp \left(\alpha \frac{N(N-1)}{2} t \right) \right]^{6n+3} & \ , \end{aligned} \tag{2.10}$$

where E, F are positive constants independent of N.

Proof. We estimate the l.h.s. of (2.10) by the L^1 norm of the Fourier transform of r_N. By lemma2.5 we have, for $k \in \mathbb{R}^{2n}$,

$$|\hat{r}_N^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{N-1}}(k)| \leq \frac{C_5 \Gamma_N}{(\rho - Nd)^{2n} e_N^{2n}} \exp\left[-(\sigma - N\delta) e_N |k| \right] \ ,$$

and therefore

$$\begin{aligned} \left\| \hat{r}_N^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{N-1}} \right\|_{L^1} &\leq \\ \frac{C_5 \Gamma_N}{(\rho - Nd)^{2n} e_N^{2n}} \int_{\mathbb{R}^{2n}} \exp\left[-(\sigma - N\delta) e_N |k| \right] dk &= \frac{C_5 \Gamma_N}{(\rho - Nd)^{2n} e_N^{2n}} \frac{C_6}{(\sigma - N\delta)^{2n} e_N^{2n}} \ . \end{aligned}$$

Choosing $\delta = \sigma/2N$ and $d = \rho/2N$, and inserting the expressions of e_N and Γ_N the assertion is proved because, (see e.g.[Ro], Corollary II.19)

$$\left\| Op^w \left(r_N^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{N-1}} \right) \right\|_{L^2 \to L_2} \leq \left\| \hat{r}_N^{t-\tau_1,\tau_1,\tau_2,\ldots,\tau_{N-1}} \right\|_{L^1}$$

□

Lemma 2.9. *For all $N \geq 2$ and $t \geq 0$ the following estimate holds*

$$\|S_N\|_{L^2 \to L_2} \leq \left[E e^{7\alpha t} N^{6n+3} t \right]^N \frac{BF}{N!} e^{(4n+2)\alpha t} \left[\exp \left(\alpha \frac{N(N-1)}{2} t \right) \right]^{6n+3} \ ,$$

Proof. One has

$$\|S_N\| \leq \sup_{0<\tau_k<t} \left\| Op^w \left(r_N^{t-\tau_1,\tau_1,\tau_2,\dots,\tau_{N-1}} \right) \right\| \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \int_0^{t-\tau_2} d\tau_3 ... \int_0^{t-\tau_{k-1}} d\tau_k \ ;$$

the norm of the Weyl quantization of r_N is estimated by the above lemma. To compute the integral it is convenient to make a change of variables introducing the new variables $s_1, ..., s_N$ defined by

$$s_1 = t - \tau_1 \ , \quad s_1 + s_2 = t - \tau_2 \ ... \quad s_1 + s_2 + ... + s_N = t - \tau_N \ ,$$

This transforms the integral into

$$\int_0^t ds_1 \int_0^{t-s_1} ds_2 \int_0^{t-(s_1+s_2)} ds_3 ... \int_0^{t-(s_1+s_2+...s_{N-1})} ds_N \ . \tag{2.11}$$

To see this fact it is enough to remark that

$$\int_0^{t-\tau_{N-1}} d\tau_N = \int_0^{t-\tau_{N-1}} ds_N = \int_0^{t-(s_1+s_2+...s_{N-1})} ds_N \ ,$$

and to iterate the argument. Denote I_N the integral in (2.11). We claim that $I_N(t) = t^N/N!$. To this end remark that one has

$$I_{N+1}(t) = \int_0^t I_N(t-s) ds.$$

Hence the assertion is proved by induction and the result is thus obtained. □

Proof of Theorem 1.1. It is enough to apply Lemma 2.9.

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