Non-Linear Stability Analysis of Higher Order Dissipative Partial Differential Equations

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Abstract. We extend the invariant manifold method for analyzing the asymptotics of dissipative partial differential equations on unbounded spatial domains to treat equations in which the linear part has order greater than two. One important example of this type of equation which we analyze in some detail is the Cahn-Hilliard equation. We analyze the marginally stable solutions of this equation in some detail. A second context in which such equations arise is in the Ginzburg-Landau equation, or other pattern forming equations, near a codimension-two bifurcation.

1. Introduction and statement of results

In this paper, we extend the methods developed in [W1], [W2], [EWW], to study the asymptotic behavior of marginally stable non-linear PDE's. These are PDE's such as

$$\partial_t u \;=\; P(-i\nabla_x)u + W'(u)\;,$$

where $u = u(x,t)$, with $x \in \mathbf{R}^d$, and where P is a polynomial. In the papers cited above, we have treated essentially parabolic problems, *i.e.*, the case where $P(\xi) = -\xi^2$. In this paper, we extend the problem to non-parabolic cases such as $P(\xi) = -\xi^4$, where $P(-i\nabla_x)$ has continuous spectrum all the way up to 0. We deal in particular with the stability analysis of the Cahn-Hilliard equation [CH] in an infinite domain. Where appropriate, we indicate how to formulate the assumptions for more general differential operators and non-linearities.

The Cahn-Hilliard equation models the dynamics of a material with the following 3 properties:

i) The material prefers one of two concentrations that can coexist at a given temperature.
ii) The material prefers to be spatially uniform.
iii) The total mass is conserved.

The first point above means that we should consider a potential with 2 minima with equal critical values, and for concreteness, we will choose $W(u) = (1-u^2)^2$.* The Cahn-Hilliard equation is then

$$\partial_t u \;=\; \Delta\big(-\Delta u + W'(u)\big)\;, \tag{1.1}$$

or, expanding,

$$\partial_t u \;=\; -\Delta^2 u - 4\Delta u + 4\Delta u^3\;. \tag{1.2}$$

We will be interested specifically in the *non-linear stability* of the spatially uniform states, $u(x,t) \equiv u_0$.

It is obvious that constants are solutions of Eq.(1.2), for any u_0. Furthermore, it is easy to check that these solutions are (locally) linearly stable for $|u_0| > 3^{-1/2}$, and linearly unstable for $|u_0| < 3^{-1/2}$. We concentrate our analysis on the remaining case, namely $u_0 = \pm 3^{-1/2}$. In this case, linearizing about $u_0 = 3^{-1/2}$ leads to the linear equation

$$\partial_t v \;=\; -\Delta^2 v\;, \tag{1.3}$$

which has spectrum in $(-\infty, 0]$ and corresponds to the case $P(\xi) = -\xi^4$. It is obvious that bounded initial data lead to solutions which tend to 0 as $t \to \infty$ and the purpose of this paper is to study under which conditions the addition of the nonlinear terms does *not* change the stability of the solutions. This is more difficult, for two reasons: First, as we have said, the spectrum of the linearized problem extends all the way to 0, and second, the nonlinearity does not have a sign.

* In our example, the curvatures of the two minima are equal. This does not seem to be necessary for our proofs.

Another, more complicated, example of a similar nature is provided by time-independent solutions of the Ginzburg-Landau equations (on $\mathbf{R}$)

$$\partial_t u \;=\; \partial_x^2 u + u - u|u|^2 \;, \tag{1.4}$$

which are exactly on the borderline between being Eckhaus stable and Eckhaus unstable. These solutions are

$$u_q(x) \;=\; e^{iqx}\sqrt{1-q^2}\;,$$

with $q = 1/\sqrt{3}$, *cf.* [EG]. We will not prove that this problem scales like the Cahn-Hilliard equations, but only describe a program which we believe would lead to a proof. The first part of the analysis of this problem would follow rather closely that given in [EEW] for the Swift-Hohenberg equation. Letting $u^* = u_q$ for the critical value $q = 1/\sqrt{3}$, and writing $u = u^* + v$, the equation for v is

$$\partial_t v \;=\; \partial_x^2 v + v - 2v|u^*|^2 - \bar{v}(u^*)^2 + \mathcal{O}(v^2)\;. \tag{1.5}$$

It has a linear part which is like a Schrödinger operator in a periodic potential (the inhomogeneity u^*). This can be handled by going to Floquet variables, namely setting

$$v(x,t) \;=\; \int_{-q}^{q} \mathrm{d}k\, e^{ikx} v_k(x,t)\;,$$

where v_k is π/q-periodic in x:

$$v_k(x,t) \;=\; \sum_{m\in\mathbf{Z}} e^{2imqx} v_{k,m}(t)\;.$$

The linear part of Eq.(1.5) leaves the subspace spanned by the v_k invariant, and has discrete spectrum in each such subspace. The spectrum is in $\sigma \leq 0$ and the largest eigenvalue is $-\mathcal{O}(k^4)$ when q equals its critical value $q = 1/\sqrt{3}$ (which is the case we discuss here). In this sense, the problem of the marginal Eckhaus instability resembles the problem of the Cahn-Hilliard equation. At this point, the discussion of the problem follows the techniques we developed in [EWW]. We would like to rescale as we will do below for the Cahn-Hilliard equation and its generalizations, but the problem will be more complicated because the Brillouin zone is restricted to $k \in [-q, q]$. We then have to check that the non-linearity is "irrelevant" in the terminology developed below. Again, as in [EWW], we believe that this will not be quite the case, but the saving grace will be that the projection of the potentially non-irrelevant modes onto the eigenstates corresponding to the $-\mathcal{O}(k^4)$ term vanish to some higher degrees because of translation invariance of the original problem, *cf.* [EWW, Section 4], and [S].

We place our examples in the following more general setting. Consider equations of the form

$$\frac{\partial u}{\partial t} \;=\; (-1)^{n+1}\Delta^n u + F(u, \{\partial_x^\alpha u\})\;, \tag{1.6}$$

where the multi-indices α satisfy $|\alpha| \leq 2n-1$, and $x \in \mathbf{R}^d$, $t \geq 1$. Furthermore, F is a polynomial in u and its derivatives. We wish to study the asymptotics of the solution u of (1.6) as $t \to \infty$. First, one introduces scaling variables by defining

$$u(x,t) \;=\; \frac{1}{t^{d/(2n)}}\, v\big(\frac{x}{t^{1/(2n)}}, \log t\big)\;. \tag{1.7}$$

Introducing new variables $\xi = x/t^{1/(2n)}$ and $\tau = \log(t+t_0)$, with t_0 an arbitrary positive constant, the Eq.(1.6) is transformed to the non-autonomous problem

$$\frac{\partial v}{\partial \tau} \;=\; (-1)^{n+1}\Delta_\xi^n v + \frac{1}{2n}\xi\cdot\nabla_\xi v + \frac{d}{2n}v + e^{(\frac{2n+d}{2n})\tau}F(e^{-\frac{d\tau}{2n}}v, \{e^{-(\frac{|\alpha|+d}{2n})\tau}\partial_\xi^\alpha v\})\;. \tag{1.8}$$

The analysis of this equation involves two steps:
i) An analysis of the linear operator
ii) A determination of which non-linear terms are relevant.

As we will see, the term $1/(2n)\xi\cdot\nabla_\xi$ plays an important rôle in the analysis of this linear operator as it allows us to push the continuous spectrum of the operator more and more into the stable region by working in Sobolev spaces with higher and higher polynomial weights. These weights force the functions to decrease more and more rapidly near $|x| = \infty$. Taking Fourier transforms on both sides of Eq.(1.8) we obtain:

$$\frac{\partial \tilde v}{\partial \tau} \;=\; -(p\cdot p)^n\tilde v - \frac{1}{2n}p\cdot\nabla_p\tilde v + e^{(\frac{2n+d}{2n})\tau}F^*(e^{-\frac{d\tau}{2n}}\tilde v, \{e^{-(\frac{|\alpha|+d}{2n})\tau}(-ip)^\alpha\tilde v\})\;, \tag{1.9}$$

where F^* is the polynomial F, written in terms of convolution products, (see the discussion of the non-linearities below).

We will discuss the form of the non-linear terms below, and consider first the linear operator

$$\mathcal{L} \;=\; -(p\cdot p)^n - \frac{1}{2n}p\cdot\nabla_p\;. \tag{1.10}$$

A straightforward calculation shows that $\mathcal{L}$ has the countable set of eigenvalues

$$\lambda_j \;=\; -\frac{j}{2n}\;,\quad j = 0,1,2,\dots\;, \tag{1.11}$$

with eigenfunctions (written in multi-index notation),

$$\varphi_\alpha(p) \;=\; p^\alpha e^{-(p\cdot p)^n}\;, \tag{1.12}$$

and $|\alpha| = j$.

If we consider $\mathcal{L}$ as acting on the Sobolev spaces

$$\tilde H_{\ell,m} \;=\; \Big\{\tilde v \;:\; \|p^\alpha\partial_p^\beta\tilde v\|_{\mathrm{L}^2} < \infty\;,\ \text{for all } |\alpha|\leq\ell, |\beta|\leq m\Big\}\;,$$

then $\mathcal{L}$ will have continuous spectrum in the half-plane $\mathrm{Re}\,\lambda < -\sigma_m$ in addition to the eigenvalues above. By choosing m appropriately, we can force this continuous spectrum arbitrarily far into the left half-plane, and the dominant behavior of the linear operator will be dictated by the eigenvalues with the largest real part.

Remark. In order to switch back and forth from the Fourier transform representation of $\mathcal{L}$ to the un-Fourier transformed representation of this operator with ease, we also consider the Sobolev spaces

$$H_{\ell,m} \;=\; \Big\{ v \;:\; \|\partial_x^\alpha x^\beta v\|_{\mathrm{L}^2} < \infty \;,\; \text{for all } |\alpha| \le \ell, |\beta| \le m \Big\} \;.$$

Note that Fourier transformation is an isomorphism from $\tilde{H}_{\ell,m}$ to $H_{\ell,m}$.

Note that $\mathcal{L}$ is *not* sectorial, and therefore we know of no way to bound the semi-group generated by $\mathcal{L}$ by spectral information alone. However, in Appendix A, we develop an integral representation of the semi-group and we then show that it satisfies the estimates needed for the invariant manifold theorem.

We next discuss which terms in the non-linearity are "relevant." Consider a monomial

$$A \;=\; \prod_{j=0}^{s} \Big(\partial_x^{\alpha^{(j)}} u \Big)^{k_j} \;, \tag{1.13}$$

where the $\alpha^{(j)}$ are distinct multi-indices. After rescaling and taking Fourier transforms this becomes

$$\begin{aligned} \tilde{A} \;=\; & \exp\Big((\frac{2n+d}{2n})\tau \Big) \;\exp\left(-\sum_{j=0}^{s} (\frac{|\alpha^{(j)}|+d}{2n}) k_j \tau \right) \\ & \times \Big((-ip)^{\alpha^{(0)}} \tilde{v} \Big)^{*k_0} * \cdots * \Big((-ip)^{\alpha^{(s)}} \tilde{v} \Big)^{*k_s} \;. \end{aligned} \tag{1.14}$$

Here, $*$ denotes the convolution product. If we combine the powers of τ in the exponential, we see that if

$$2n + d \;<\; \sum_{j=0}^{s} \big(|\alpha^{(j)}| + d \big) k_j \;, \tag{1.15}$$

then the coefficient of this term will go to zero exponentially fast in τ, and hence it will be irrelevant from the point of view of the long time behavior of the solutions.

Definitions. A monomial like (1.14) is called *irrelevant* if it satisfies the inequality (1.15). It is called *critical* if the l.h.s. of Eq.(1.15) is equal to the r.h.s, and *relevant* in the remaining case.

These definitions are suggested by the following which is our first main result:

Theorem 1.1. *Assume all terms in the non-linearity in Eq.(1.6) are irrelevant. For any solution $u(x,t)$ of Eq.(1.6) with sufficiently small initial conditions in $H_{m,\ell}$ (with $\ell > (2n-1) + d/2$ and $m > (n+2)/(2n)$), there is a constant B^*, depending on the initial conditions, such that for every $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{(\frac{d+1}{2n} - \varepsilon)} \left\| u(x,t) - \frac{B^*}{t^{d/(2n)}} f^*(\frac{x}{t^{1/(2n)}}) \right\|_{\mathrm{L}^\infty} \;=\; 0 \;.$$

Here,

$$f^*(\xi) \;=\; \frac{1}{(2\pi)^{d/2}} \int \mathrm{d}^d p\, e^{ip\cdot\xi} e^{-(p\cdot p)^n} \;. \tag{1.16}$$

Remark. This theorem is a special case of a more detailed analysis which will be given below. That analysis will allow us to compute, in principle, the *form* of the solutions of Eq.(1.6) up to $\mathcal{O}(t^{-k})$, for any $k > 0$. We note that if one only wanted the first order asymptotics of the solution, one could also use the renormalization group analysis of [BKL].

We now apply the Theorem 1.1 to the Cahn-Hilliard equation. Writing $u = 3^{-1/2} + w$, the function w is seen to satisfy

$$\frac{\partial w}{\partial t} \;=\; -\Delta^2 w + \sqrt{3}\Delta(w^2) + \Delta(w^3) \;. \tag{1.17}$$

Upon expanding $\Delta(w^2)$ we obtain two types of terms—those of the form $w(\partial_{x_i}^2 w)$ and those of the form $(\partial_{x_i} w)^2$. In both cases,

$$\sum (|\alpha^{(j)}| + d)k_j \;=\; 2d + 2 \;.$$

Since $n = 2$ in this example, these terms will be irrelevant if $4+d < 2d+2$, that is in dimensions $d > 2$. Also, the term $\Delta(w^3)$ is irrelevant for $d > 1$. Thus, as a corollary to Theorem 1.1 we get immediately

Corollary 1.2. *Solutions of the Cahn-Hilliard equation in dimension $d \geq 3$, with initial conditions sufficiently close (in $H_{1,1}$) to the constant solution $u \equiv 3^{-1/2}$ behave asymptotically as*

$$u(x,t) \;=\; \frac{1}{3^{1/2}} + \frac{B^*}{t^{d/4}} f^*\big(\frac{x}{t^{1/4}}\big) + \mathcal{O}\big(\frac{1}{t^{(d+1)/4-\varepsilon}}\big) \;. \tag{1.18}$$

Remark. We will examine below what happens in the cases $d = 1, 2$. The case $d = 2$ is of particular interest because its non-linearity is critical in the renormalization group terminology.

2. Invariant manifolds

Note that spectral subspaces corresponding to eigenvalues of $\mathcal{L}$ are automatically invariant manifolds for the semi-flow defined by the linear part of Eq.(1.9). The aim of this section is to demonstrate that the full non-linear problem has similar invariant manifolds in a neighborhood of the origin. This then shows that the conceptual understanding of what is happening can be gained purely from a knowledge of $\mathcal{L}$, (and the scaling behavior of the non-linearity).

We begin with a proposition concerning the linear semi-group generated by $\mathcal{L}$.

Proposition 2.1. *Let P_k denote the projection onto the spectral subspace associated with the eigenvalues $\{\frac{-j}{2n}\}_{j=0}^{k}$, and let $Q_k = P_k^{\perp}$ (in $H_{\ell,m}$). If $m > (n+k+1)/(2n)$, then there exists $C_k > 0$ such that the semi-group generated by $\mathcal{L}$ satisfies*

$$\|Q_k e^{\tau\mathcal{L}} Q_k v\|_{\ell,m} \;\leq\; \frac{C_k}{t^{q/2n}} \exp\big(\frac{k+1}{2n}\tau\big) \|v\|_{\ell-q,m} \;, q = 0, 1, \ldots, 2n-1 \;. \tag{2.1}$$

Proof. The proof, which is presented in Appendix A, is modeled on the proof in [EWW] which treats the case $n = 1$.

Given such estimates on the linear evolution, the construction of invariant manifolds is straightforward. Denote by y the coordinates on the (finite-dimensional) range of P_k, and let $z = Q_k \tilde{v}$. Finally let $\eta = e^{-\tau/(2n)} = (t + t_0)^{-1/(2n)}$. Then, applying the projection operators P_k and Q_k to Eq.(1.9), it can be written as the system of equations

$$\begin{aligned} \dot{y} &= \Lambda_k y + f(y, z, \eta) \,, \\ \dot{z} &= Q_k \mathcal{L} z + g(y, z, \eta) \,, \\ \dot{\eta} &= -\tfrac{1}{2n}\eta \,, \end{aligned} \tag{2.2}$$

where $\dot{}$ denotes differentiation w.r.t. τ. We next need a bound on the non-linearity:

Lemma 2.2. *Assume $u \in H_{m,\ell}$ with $\ell > 2n - 1 + d/2$, and assume*

$$2n + d \;\leq\; \sum_{j=0}^{s} |\alpha^{(j)} + d| k_j \;.$$

Then the non-linear term Eq.(1.14) has $H_{m,\ell-2n+1}$ norm bounded by

$$\eta^p \prod_{j=0}^{s} \|u\|_{m,\ell-2n+1} \;=\; \eta^p \|u\|^K_{m,\ell-2n+1} \;,$$

with $p = \sum_{j=0}^{s} |\alpha^{(j)} + d| k_j - (2n + d)$ and $K = \sum_{j=0}^{s} k_j$.

Proof. Taking the inverse Fourier transform of Eq.(1.14), and substituting $\eta = e^{-\tau/(2n)}$, Eq.(1.14) becomes

$$\eta^p \prod_{j=0}^{s} \left(\partial_\xi^{\alpha^{(j)}} v \right)^{k_j} \;.$$

The result then follows from the Sobolev embedding theorem. Note that the lemma has the immediate corollary (because F is a polynomial):

Corollary 2.3. *Under the hypotheses of Lemma 2.2, for every $r \geq 1$, the non-linear term in Eq.(1.9) is a $\mathcal{C}^r$ function from $\mathbf{R} \times H_{m,\ell}$ to $H_{m,\ell-2n+1}$.*

This corollary in turn implies that the terms in Eq.(1.14) and Eq.(2.2) are all $\mathcal{C}^r$ functions. This, in conjunction with the estimates on the linear semi-group is sufficient to establish the following

Theorem 2.4. *Suppose that $\ell > 2n - 1 + d/2$ and $m > (n + k + 1)/(2n)$. Suppose further that all terms in the nonlinearity satisfy*

$$2n + d \;\leq\; \sum_{j=0}^{s} |\alpha^{(j)} + d| k_j \;. \tag{2.3}$$

Then there exists a $\mathcal{C}^{1+\alpha}$ function $h(\eta, y)$, with $\alpha > 0$, defined in some neighborhood of the origin in $\mathbf{R} \times \mathbf{R}^{\dim \text{ range}(P_k)}$, such that the manifold $z = h(\eta, y)$ is left invariant by the semi-flow of Eq.(2.2). Furthermore, any solution of Eq.(2.2) which remains near the origin for all times approaches a solution of (2.2)—restricted to the invariant manifold—at a rate $\mathcal{O}\big(e^{\frac{k+1-\varepsilon}{2n}\tau}\big)$.

Proof. The existence of the invariant manifold, given the assumptions on the linear semi-group and the non-linearity, seems, to our knowledge, not to be explicitly spelled out in the literature. The formulation which comes closest to our needs is the one given in [H], where the assumptions on the non-linearity are those we have in our case, but the semi-group is supposed to be analytic. However, Henry's construction of the invariant manifold only uses certain bounds on the decay of the semi-group, and not the stronger assumption of analyticity. Those bounds *are* true in our case, by Lemma 2.2. Thus, existence of the invariant manifold follows in fact from Henry's proof.

Once one knows that the manifold exists, it is also easy to show that any solution near the origin must approach a solution on the invariant manifold (see, *e.g.* [C]). Note that even though our non-linearity is quite smooth, we cannot hope, in general, to obtain an invariant manifold whose smoothness is greater than $\mathcal{C}^{1+\alpha}$, since this smoothness is related to the gap between the spectrum of Λ_k, and that of $Q_k \mathcal{L} Q_k$, (see, *e.g.* [LW]).

3. Applications

Here, we show how the existence of the invariant manifold implies Theorem 1.1 and related results. To prove Theorem 1.1, we assume that all terms in the non-linearity are irrelevant. This means that Eq.(2.3) holds. Suppose further that $k = 1$ and that $\ell > 2n - 1 + d/2$ and $m > (n+2)/(2n)$. These hypotheses guarantee that Theorem 2.4 applies and hence any solution near the origin must approach a solution on the invariant manifold, at a rate $\mathcal{O}\big(e^{\frac{2-\varepsilon}{2n}\tau}\big)$ in $H_{m,\ell}$.

The equations on the invariant manifold can be written as a system of ordinary differential equations:

$$\begin{aligned} \dot{y}_0 &= \big\langle \varphi_0^* | f\big(y, h(\eta, y), \eta\big)\big\rangle \,, \\ \dot{y}_{1,j} &= -\tfrac{1}{2n} y_{1,j} + \big\langle \varphi_{1,j}^* | f\big(y, h(\eta, y), \eta\big)\big\rangle \,, \quad j = 1, \ldots, d \,, \\ \dot{\eta} &= -\tfrac{1}{2n}\eta \,, \end{aligned} \tag{3.1}$$

where φ_0^* and $\varphi_{1,j}^*$ are the projections onto the spectral subspace of λ_0 and $\lambda_1 = -1/(2n)$, respectively. Note that λ_1 has a d-dimensional spectral subspace.

The important observation to make at this point is that since the non-linearity is assumed to be irrelevant, there exist constants C_0 and C_1 such that

$$\big|\big\langle \varphi_0^* | f\big(y, h(\eta, y), \eta\big)\big\rangle\big| \le C_0 \eta^p \,, \quad \big|\big\langle \varphi_1^* | f\big(y, h(\eta, y), \eta\big)\big\rangle\big| \le C_1 \eta^p \,,$$

for some $p \ge 1$. Since $\eta(\tau) = e^{-\tau/(2n)}\eta(0)$, this implies immediately that solutions of Eq.(3.1) behave as

$$\begin{aligned} y_0(\tau) &= B^* + \mathcal{O}(e^{-\tau/(2n)}) \,, \\ y_{1,j}(\tau) &= \mathcal{O}(e^{-\tau/(2n)}) \,. \end{aligned}$$

The eigenfunction with eigenvalue 0 of $\mathfrak{L}$ is $e^{-p\cdot p}$, or taking inverse Fourier transform, f^*, *cf.* Eq.(1.16). Thus, in $H_{m,\ell}$ solutions of Eq.(1.9) behave as

$$\tilde{v}(p,\tau) \;=\; B^* e^{-p\cdot p} + \mathcal{O}(e^{-\tau/(2n)}) \;.$$

Reverting from scaling variables to the unscaled variables $u(x,t)$ and using the Sobolev lemma to estimate the L^∞ norm in terms of the $H_{m,\ell}$ norm, we obtain Theorem 1.1. Since we observed above that the non-linearity in the Cahn-Hilliard equation is irrelevant when $d \geq 3$, we immediately see in this case that Eq.(1.18) holds for initial conditions which are close to $u \equiv 3^{-1/2}$, which yields Corollary 1.2.

4. The critical case

We now consider the Cahn-Hilliard equation in dimension $d = 2$, which is the critical case in terms of the renormalization group terminology [BKL]. This means that in some non-linear terms the inequality Eq.(1.15) becomes an equality.

In the Cahn-Hilliard equation, when $d = 2$ (and $n = 2$), we see that the quadratic term is critical, and the cubic term is irrelevant. Note that Theorem 2.4 still implies the existence of an invariant manifold tangent at the origin to the eigenspace of λ_0. This means that when written in the form of Eq.(2.2), the non-linearity can be written as the sum of 2 pieces—one quadratic in y and z which is independent of η (and hence critical) and a cubic piece in y and z which is linear in η (and hence irrelevant). This implies that the Eqs.(3.1), when reduced to the invariant manifold, take the form

$$\begin{aligned}
\dot{y}_0 \;&=\; \big\langle \varphi_0^* | f^{(2)}\big(y, h(\eta,y),\eta\big)\big\rangle + \big\langle \varphi_0^* | f^{(3)}\big(y, h(\eta,y),\eta\big)\big\rangle \;, \\
\dot{y}_{1,j} \;&=\; -\tfrac{1}{4} y_{1,j} + \big\langle \varphi_{1,j}^* | f^{(2)}\big(y, h(\eta,y),\eta\big)\big\rangle + \big\langle \varphi_{1,j}^* | f^{(3)}\big(y, h(\eta,y),\eta\big)\big\rangle \;, \quad j = 1,2 \;, \\
\dot{\eta} \;&=\; -\tfrac{1}{4}\eta \;.
\end{aligned} \tag{4.1}$$

We now exploit the form of the non-linear term in Eq.(1.17), namely $3^{1/2}\Delta(w^2) + \Delta(w^3)$, plus the fact that the eigenfunction $\varphi_0^* \equiv 1$. Thus if we integrate by parts, we find that

$$\big\langle \varphi_0^* | f^{(2)}\big(y, h(\eta,y),\eta\big)\big\rangle + \big\langle \varphi_0^* | f^{(3)}\big(y, h(\eta,y),\eta\big)\big\rangle \;=\; 0 \;,$$

so that in Eq.(4.1), $\dot{y}_0 \equiv 0$ and thus $y_0(t) = y_0(0)$. This reflects the fact that the Cahn-Hilliard equation conserves mass.

Since from Eq.(4.1) we also see that $y_{1,j} = \mathcal{O}(e^{-\tau/4})$, we find upon reverting to the unscaled variables the second main result:

Theorem 4.1. *For $d = 2$, if the initial conditions of the Cahn-Hilliard equation are sufficiently close in $H_{1,4}$ to the stationary state $u \equiv 3^{-1/2}$, then the solution behaves asymptotically as*

$$u(x,t) \;=\; \frac{1}{3^{1/2}} + \frac{B^*}{t^{1/2}} f^*\big(\frac{x}{t^{1/4}}\big) + \mathcal{O}\big(\frac{1}{t^{3/4-\varepsilon}}\big) \;.$$

5. The relevant case

Here, we consider the case of $d = 1$ where one term of the non-linearity is relevant. This necessitates a change of strategy, because the quadratic term is proportional to η^{-1} and hence the non-linear terms in Eq.(2.2) are not smooth enough to apply the invariant manifold theorem. In order to circumvent this difficulty, we choose a scaling different from Eq.(1.7). Consider again the Cahn-Hilliard equation, Eq.(1.17), with $u = 3^{-1/2} + w$. In $d = 1$, we get

$$\frac{\partial w}{\partial t} \;=\; -\frac{\partial^4}{\partial x^4} w + 3^{1/2} \frac{\partial^2}{\partial x^2}\big(w^2\big) + \frac{\partial^2}{\partial x^2}\big(w^3\big)\;. \tag{5.1}$$

Now let $w(x,t) = t^{-1/2} W(x/t^{1/4}, \log t)$. Then W satisfies

$$\frac{\partial W}{\partial \tau} \;=\; -\partial_\xi^4 W + \frac{1}{4}\xi\cdot\partial_\xi W + \frac{1}{2} W + 3^{1/2}\partial_\xi^2\big(W^2\big) + e^{-\tau/2}\partial_\xi^2\big(W^3\big)\;. \tag{5.2}$$

Proceeding as in the other cases, we define the linear operator $L_1 = -\partial_\xi^4 + \frac{1}{4}\xi\partial_\xi + \frac{1}{2}$, which in Fourier variables becomes

$$\mathcal{L}_1 \;=\; -p^4 - \tfrac{1}{4} p\partial_p + \tfrac{1}{4}\;,$$

so that it has eigenvalues $\mu_m = \frac{1-j}{4}$, $j = 0, 1, \dots$. Thus, unlike the operator $\mathcal{L}$, we have one eigenvalue lying in the right half-plane. Let $\tilde\eta = e^{-\tau/8}$, and let y_0 and y_1 denote the amplitudes of the eigenvectors with eigenvalues μ_0 and μ_1. Then Eq.(5.2) takes the form

$$\begin{aligned}
\dot y_0 \;&=\; \tfrac{1}{4} y_0 + f_0(y_0, y_1, \eta, y^\perp)\;,\\
\dot y_1 \;&=\; f_1(y_0, y_1, \eta, y^\perp)\;,\\
\dot\eta \;&=\; -\tfrac{1}{8}\eta\;,\\
\dot y^\perp \;&=\; QL_1 y^\perp + f^\perp(y_0, y_1, \eta, y^\perp)\;.
\end{aligned} \tag{5.3}$$

Here, Q is the projection onto the complement of the eigenspaces corresponding to μ_0 and μ_1, $y^\perp = QW$, and f_0, f_1, and $f^\perp$ are the projections of the non-linearity onto the various subspaces.

Since the spectrum of QL_1Q lies in the half-plane $\operatorname{Re}\mu \le \frac{1}{4}$, we can construct an invariant manifold for Eq.(5.3) which is the graph of a function $h^\perp(y_0, y_1, \eta)$, and every solution of Eq.(5.3) which remains in a neighborhood of the origin will approach this manifold at a rate $\mathcal{O}(e^{-\tau/4})$. What is more, the equations on the invariant manifold are extremely simple in this case, since the projections onto the "0" and "1" components correspond to integrating with respect to the functions 1 and x, respectively. Applying these projections to the non-linearity and integrating once, resp. twice by parts, we see that these projections of the non-linear terms vanish. Thus, the equations on the invariant manifold of Eq.(5.3) are simply

$$\dot y_0 \;=\; \tfrac{1}{4} y_0\;, \quad \dot y_1 \;=\; 0\;, \quad \dot{\tilde\eta} = -\tfrac{1}{8}\tilde\eta\;.$$

Thus, *as long as the solution of Eq.(5.3) remains in a neighborhood of the origin*, it will be of the form

$$\begin{aligned} y_0(\tau) &= e^{\tau/4} y_0(0) \,, \\ y_1(\tau) &= y_1(0) \,, \\ \tilde{\eta}(\tau) &= e^{-\tau/8} \tilde{\eta}(0) \,, \\ y^{\perp}(\tau) &= \mathcal{O}(e^{-\tau/4}) \,. \end{aligned} \tag{5.4}$$

Thus, we see that the solution either leaves the neighborhood of the origin, or its asymptotic behavior can be read off from Eq.(5.4). Note that the solutions that remain near the origin must have $y_0 = 0$. Thus:

Theorem 5.1. *Suppose that the initial condition of the Cahn-Hilliard equation is of the form $u_0 = 3^{-1/2} + w_0$ with w_0 small in the $H_{m,\ell}$ norm for some $\ell \geq 4$ and $m \geq 2$. Assume furthermore that $\int_{-\infty}^{\infty} \mathrm{d}x\, w_0(x) = 0$. Then the solution is of the form*

$$u(x,t) = \frac{1}{3^{1/2}} + \frac{B^{**}}{t^{1/2}} f^{**}\big(\frac{x}{t^{1/4}}\big) + \mathcal{O}\big(\frac{1}{t^{3/4-\varepsilon}}\big) \,,$$

where

$$B^{**} = \int_{-\infty}^{\infty} \mathrm{d}x\, x w_0(x) \,, \quad f^{**}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ip\xi} e^{-p^4} \,.$$

Remark. The constant B^* in Theorem 1.1 is not as easy to describe because there, the non-linearity in the equation for y_0 did *not* necessarily disappear.

Proof. The proof is an obvious modification of the one of Theorem 1.1, taking into account the special form of the eigenfunctions corresponding to the eigenvalues μ_0 and μ_1.

Appendix. Bounds on the linear semi-group

In this appendix, we sketch the proof of Proposition 2.1. The proof is quite similar to the estimates on the linear semi-group in Appendix B of [EWW], (which was given for the case of a one-dimensional Laplacian, or in the present notation $n = d = 1$) so we concentrate only on the points where the present argument differs from the one in [EWW].

We begin with the representation

$$\big(e^{\tau L}\big)(x) = \frac{e^{\frac{\tau d}{2n}}}{2\pi^d} \int \mathrm{d}^d z\, g(z,\tau) v\big(e^{\frac{\tau}{2n}}(x+z)\big) \,, \tag{A.1}$$

where

$$g(z,\tau) = \int \mathrm{d}^d k\, e^{ik\cdot z} \exp\big(-(k\cdot k)^n (1-e^{-\tau})\big) \,. \tag{A.2}$$

As in [EWW], the action of the semi-group is analyzed by considering separately the behavior of the part far from the origin and that close to the origin. The new difficulty here is that we do not

have an explicit representation of g as in the case $n = 1$. However, the technique of estimating the long-time behavior will remain essentially the same. Let χ_R be a smooth characteristic function which vanishes for $|x| < R$ and is equal to 1 for $|x| > 4R/3$. We start by studying the region far from the origin. The analog of Proposition B.2 of [EWW] is

Proposition A.1. *For every $\ell \geq 1$ and every $m \geq 0$, there exist a $\gamma > 0$ and a $C(\ell, m) < \infty$ such that for all $v \in H_{\ell,m}$ one has*

$$\left\|\chi_R e^{\tau L} v\right\|_{\ell,m} \leq \frac{C(\ell,m)}{\big(a(\tau)\big)^{\frac{q}{2n}}} e^{(\frac{\tau}{2n})(d+\ell)} \left(e^{-\tau m/2} + e^{-\gamma R^{2n/(2n-1)}}\right) \|v\|_{\ell-q,m} \,, \tag{A.3}$$

for $q = 0, 1, \ldots, 2n-1$. Here, $a(\tau) = 1 - e^{-\tau}$.

The crucial step in proving this estimate is to derive the asymptotics of $g(z, \tau)$ for large z. This will replace the explicit (Gaussian) estimates for the $d = 1$, $n = 1$ case analyzed in [EWW]. This estimate is provided by the following

Proposition A.2. *The kernel $g(z, \tau)$ decays faster than any inverse power of z for $|z|$ large. In fact, one has the estimate*

$$|g(z,\tau)| \leq C a(\tau)^{-\frac{d}{2n}} \exp\big(-\gamma(|z|^{2n}/a(\tau))^{\frac{1}{2n-1}}\big) \,, \tag{A.4}$$

for some $\gamma = \gamma(n, d) > 0$.

Remark. If $n = 1$, we recover the explicit bound on the Green's function:

$$\frac{C}{\sqrt{a(\tau)}} e^{-\gamma(2,d) z^2/a(\tau)} \,.$$

Proof. We need to estimate the quantity

$$I_{n,d} = \int \mathrm{d}^d k \, e^{-a(\tau)\left(\sum_{j=1}^d k_j^2\right)^n} e^{i\sum_{j=1}^d k_j x_j} \,. \tag{A.5}$$

By rotational symmetry, it suffices to bound the preceding expression for x lying on the positive real axis. Setting $x = 2na(\tau)z^{2n-1}$, and $k = (p, q)$, with $p \in \mathbf{R}$, and $q \in \mathbf{R}^{d-1}$, this means that we must bound

$$X = \int \mathrm{d}p \, \mathrm{d}^{d-1} q \exp\big(-a(\tau)(p^2 + q \cdot q)^n + 2inpa(\tau)z^{2n-1}\big) \,.$$

If we rescale the variables as $p = zt$, and $q = zs$, then we have

$$X = z^d \int \mathrm{d}t \, \mathrm{d}^{d-1} s \exp\big(-a(\tau) z^{2n} \big((t^2 + s \cdot s)^n + 2int\big)\big) \,.$$

Remark. Note that the polynomial $(t^2 + s \cdot s)^n + 2int$ is independent of z.

We will bound X by taking advantage of the fact that the integrand is an entire function and translate the contour of integration so that it passes through at least one critical point of the exponent. These critical points occur at $s = 0$ and the roots of $t^{2n-1} = i$ – *i.e.*, at the points $t_k = \exp(i\frac{\pi(4k+1)}{2(2n-1)})$, $k = 0, 1, 2, \ldots, 2n-2$.

Inserting this expression into the exponent of the integrand of X, we see that the value of the polynomial at the critical points is

$$\begin{aligned} -a(\tau)z^{2n}\left(\exp(i\frac{2n\pi(4k+1)}{2(2n-1)}) - 2ni\exp(i\frac{\pi(4k+1)}{2(2n-1)})\right) \\ = (2n-1)a(\tau)z^{2n}\exp(i\frac{\pi(4k+2n)}{2(2n-1)}) \,. \end{aligned}$$

In particular, if we take $k = 0$, then the real part of the critical value is

$$(2n-1)a(\tau)z^{2n}\cos\big(\pi/2 + \pi/(4n-2)\big) \;\approx\; -a(\tau)z^{2n}\cdot\frac{\pi}{2} \,,$$

when n is large (and is negative for all $n > 0$). Integrating over the region $\mathbf{R} + t_0$ and observing that there is only one critical point on this line, we get, using the techniques of Hörmander:

$$X \;\approx\; a(\tau)^{d/(2n)}e^{-C_n a(\tau)z^{2n}} \,,$$

with $C > 0$ and $C_n \to \pi/2$ as $n \to \infty$, when $z \to \infty$. Reverting to the original variables, this leads to

$$I_{n,d} \;\approx\; a(\tau)^{d/(2n)}e^{-D_n x^{2n/(2n-1)}/a(\tau)^{1/(2n-1)}} \,,$$

where $D_n = C_n/(2n)^{1/(2n-1)}$.

We now consider the action of the semi-group on functions localized inside a ball of radius R. A key observation here is the following lemma. Let $\varphi_0(x)$ denote the eigenfunction of L with eigenvalue 0 and let $T(x) = \tilde{\varphi}_0(x)^{1/2}$ (note that $\tilde{\varphi}_0(x) > 0$ for all x).

Lemma A.3. *The operator $H = T^{-1}LT$ is self-adjoint on (a dense domain in) $\mathrm{L}^2(\mathbf{R}^d)$ and has the same eigenvalues as L.*

Proof. The proof is a straightforward calculation. We note further that if φ_α are the eigenfunctions of L then the eigenfunctions of H are $\psi_\alpha(x) = \big(T^{-1}\varphi_\alpha\big)(x)$.

If we take the inverse Fourier transform of the eigenfunctions $\varphi_\alpha(p)$ of Eq.(1.12), we see that

$$|\tilde{\varphi}_\alpha(x)| \;\approx\; C|x|^{|\alpha|}\exp(-\gamma|x|^{\frac{2n}{2n-1}}) \,,$$

for some $\gamma > 0$, using the same sort of estimates as those used to bound the kernel g of the semi-group. Thus, for $|x|$ sufficiently large, we get

$$|\tilde{\psi}_\alpha(x)| \;\approx\; C|x|^{|\alpha|}\exp(-\tfrac{1}{2}\gamma|x|^{\frac{2n}{2n-1}}) \,.$$

The usefulness of introducing the operator H is that it is sectorial, since it is self-adjoint and bounded below. Therefore, the associated semi-group can be estimated from spectral information alone. In particular, if P_k denotes the projection onto the spectral subspace spanned by the eigenfunctions with eigenvalues $0, -\frac{1}{2n}, \frac{-2}{2n}, \ldots, \frac{-k}{2n}$, and Q_k is defined by $Q_k = 1 - P_k$, then we have a bound on the operator norm of $Q_k e^{\tau H} Q_k$

$$\|Q_k e^{\tau H} Q_k\| \leq C_k e^{-\tau k/(2n)} . \tag{A.6}$$

We can use this information to bound the semi-group associated with L. Note that if we denote by $P_k^{(0)}$ and $Q_k^{(0)}$ the projection associated with the spectral subspaces of L (as we did for H), the we have the identity:

$$e^{\tau L} Q_k^{(0)} v = e^{\tau L} Q_k^{(0)} (1 - \chi_R) v + e^{\tau L} Q_k^{(0)} \chi_R v .$$

Since $\chi_R v$ is localized away from the origin, it can be studied with the help of Proposition A.1, so we only focus on the other term. And there we get

$$\begin{aligned} \|e^{\tau L} Q_k^{(0)} (1 - \chi_R) v\|_{q,r} &= \|T T^{-1} e^{\tau L} T T^{-1} Q_k^{(0)} T T^{-1} (1 - \chi_R) v\|_{q,r} \\ &= \big\|T\big(e^{\tau H} Q_k\big)\big(T^{-1}(1 - \chi_R) v\big)\big\|_{q,r} \\ &\leq C \exp\big(-\tau \frac{k+1}{2n}\big) \big\|T^{-1}(1 - \chi_R) v\big\|_{q,r} . \end{aligned}$$

Using now the information that $|T^{-1}(x)| \leq C \exp(\gamma |x|^{\frac{2n}{2n-1}})$ and that $(1 - \chi_R v)(x) = 0$ when $|x| > 4R/3$, we get

$$\big\|T^{-1}(1 - \chi_R) v\big\|_{q,r} \leq C \exp\big(\gamma (4R/3)^{\frac{2n}{2n-1}}\big) \|v\|_{q,r} ,$$

so that finally

$$\Big\|e^{\tau L} Q_k^{(0)} (1 - \chi_R) v\Big\|_{\ell,m} \leq C \exp\big(\gamma (4R/3)^{\frac{2n}{2n-1}}\big) \exp\big(-\tau \frac{k+1}{2n}\big) \|v\|_{\ell,m} .$$

Thus we have proven:

Proposition A.4. *Under the hypotheses of Proposition A.1, there exist constants $C(\ell, m) > 0$ and $\gamma > 0$, such that for all $v \in H_{\ell,m}$, one has*

$$\Big\|e^{\tau L} Q_k^{(0)} (1 - \chi_R) v\Big\|_{\ell,m} \leq C \exp\big(\gamma (4R/3)^{\frac{2n}{2n-1}}\big) \exp\big(-\tau \frac{k+1}{2n}\big) \|v\|_{\ell,m} .$$

We now return to the:

Proof of Proposition 2.1. As in [EWW] it is only necessary to consider the term with highest derivative in $\|\chi_R e^{\tau \mathcal{L}} v\|_{\ell,m}$. All other terms are easier to estimate. Also, as in that paper, we use the fact that

$$\mathrm{D}^\ell e^{\tau \mathcal{L}} = e^{\tau \ell/(2n)} e^{\tau \mathcal{L}} \mathrm{D}^\ell , \tag{A.7}$$

where D^ℓ is a shorthand notation for a product of derivatives w.r.t. the x_j of total degree ℓ. Thus,

$$\big(e^{\tau\mathcal{L}}\mathrm{D}^\ell v\big)(x) \;=\; \frac{e^{\tau d/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\, g(z,\tau)\big(\mathrm{D}^\ell v\big)(e^{\tau/(2n)}(x+z))\;. \tag{A.8}$$

First consider the $q=0$ case of (A.3). Then

$$\|\chi_R e^{\tau\mathcal{L}} v\|_{\ell,m} \;\le\; \frac{e^{\tau d/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\,|g(z,\tau)|\,\big\|w^m\chi_R\big(\mathrm{D}^\ell v\big)(e^{\tau/(2n)}(.+z))\big\|_2\;, \tag{A.9}$$

where w is the operator of multiplication by $(1+x\cdot x)^{1/2}$. Note that the conclusions of Lemma B.4 of [EWW] do not depend on the exact form of g and so it also holds in the present situation and we have

Lemma A.5. *One has the bounds*

$$\big\|w^r\chi_R v(e^{\tau/(2n)}(.+z))\big\|_2^2 \;\le\; \begin{cases} Ce^{-\tau m}\|v\|_{0,m}^2\;, & \text{if } |z|\le 7R/8\;,\\ C(1+|z|^2)^r\|v\|_{0,m}^2, & \text{if } |z|>7R/8\;. \end{cases}$$

Remark. Note that the proof in [EWW] is also unaffected by the dimension d in which we work.

Now use Lemma A.5 to bound the integral in (A.9) by writing it as an integral over $|z|\le 7R/8$ and an integral over $|z|>7R/8$. The integral over $|z|\le 7R/8$ is bounded with the aid of Lemma A.5 as

$$C\frac{e^{\tau d/(2n)}}{(2\pi)^d}\int_{|z|\le 7R/8}|g(z,\tau)|e^{-\tau m/2}\|v\|_{\ell,m} \;\le\; C(n)e^{\tau d/2n}e^{-\tau m}\|v\|_{\ell,m}\;, \tag{A.10}$$

where the last step used the estimates of Theorem 4.1 to show that $\int \mathrm{d}z|g(z,\tau)|\le C$, with C independent of τ.

To estimate the integral in the outer region, we use the second part of Lemma A.5 and then bound it by

$$C\frac{e^{\tau d/(2n)}}{(2\pi)^d}\int_{|z|>7R/8}|g(z,\tau)|e^{-\tau r/2}\|v\|_{\ell,m} \;\le\; C(n,m)e^{\tau d/(2n)}\exp\big(-\gamma R^{\frac{2n}{2n-1}}\big)\|v\|_{\ell,m}\;, \tag{A.11}$$

for some $\gamma>0$, where, again, we have used the estimates of decay in Theorem 4.1 both to extract the factor of $\exp\big(-\gamma R^{\frac{2n}{2n-1}}\big)$ as well as to bound the integral over z. Combining Eqs.(A.1), (A.10), and (A.11), we get the $q=0$ case of Eq.(A.3).

We next indicate how to treat the $q>0$ cases of Eq.(A.3). Consider the case $q=1$. We can rewrite Eq.(A.8) by integrating by parts once w.r.t. one component of z, for example z_1. Then,

$$\big(e^{\tau\mathcal{L}}\mathrm{D}^\ell v\big)(x) \;=\; \frac{e^{\tau d/(2n)}e^{-\tau/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\big(\mathrm{D}_{z_1}g\big)(z,\tau)\big(\mathrm{D}^{\ell-1}v\big)(e^{\tau/(2n)}(x+z))\;. \tag{A.12}$$

Differentiating (A.2) w.r.t. z_1 gives

$$\big(\mathrm{D}_{z_1} g\big)(z,\tau) \;=\; \int \mathrm{d}^d k\, ik_1 \,\exp(iq\cdot z)\exp\big((k\cdot k)^n(1-e^{-\tau})\big)\;. \tag{A.13}$$

To estimate (A.13), first replace k by $p_j = a(\tau)^{1/(2n)}k_j$, where, as before $a(\tau) = 1-e^{-\tau}$. Then,

$$\big(\mathrm{D}_{z_1} g\big)(z,\tau) \;=\; \frac{i}{a(\tau)^{d/(2n)}}\frac{1}{a(\tau)^{1/(2n)}}\int \mathrm{d}^d p\, p_1 \exp\big((p\cdot p)^n\big)\exp\big(ip\cdot z/a(\tau)^{1/(2n)}\big)\;. \tag{A.14}$$

The estimate of the integral in Eq.(A.14) now follows as before, since the extra factor of p_1 does not cause any trouble as it is easily offset by the exponentially decaying terms.

One now uses the Schwarz inequality to rewrite

$$\|\chi_R e^{\tau\mathcal{L}}\mathrm{D}^\ell v\|_{\ell,m} \;\le\; \frac{e^{\tau d/(2n)}}{(2\pi)^d}\int \mathrm{d}^d z\;\big|\mathrm{D}_{z_1} g(z,\tau)\big|\;\Big\|w^m\chi_R\big(\mathrm{D}^{\ell-1}v\big)(e^{\tau/2}(.+z))\Big\|_2\;, \tag{A.15}$$

and then proceeds as in the case when $q=0$, breaking the integral over z into the same two pieces as before. These two pieces are then estimated with the aid of Lemma A.5. Note that while the factor $a(\tau)^{-d/(2n)}$ of Eq.(A.14) will be absorbed when one integrates w.r.t z, the remaining factor of $a(\tau)^{-1/(2n)}$ will remain in the final bound of Eq.(A.3). The bounds for $q = 2,3,\ldots 2n-1$ follow in a similar fashion.

To complete the proof of Proposition 2.1, first rewrite

$$e^{\tau\mathcal{L}}Q_k \;=\; e^{\tau\mathcal{L}/2}Q_k e^{\tau\mathcal{L}/2} \;=\; e^{\tau\mathcal{L}/2}Q_k\chi_R e^{\tau\mathcal{L}/2} + e^{\tau\mathcal{L}/2}Q_k(1-\chi_R)e^{\tau\mathcal{L}/2}\,. \tag{A.16}$$

The second of these terms involves an estimate of the action of $e^{\tau\mathcal{L}/2}Q_k$ on a function localized near the origin, so by Proposition A.4, we get a bound

$$\|e^{\tau\mathcal{L}/2}Q_k(1-\chi_R)e^{\tau\mathcal{L}/2}v\|_{\ell,m} \;\le\; C_q\exp\left(\gamma\big(\frac{4R}{3}\big)^{2n/(2n-1)} - \tfrac{1}{2}\frac{k+1}{2n}\right)\|v\|_{\ell-q,m}\;. \tag{A.17}$$

We use Proposition A.1 to bound the first term of (A.16):

$$\frac{C(\ell,m)}{a(\tau)^{\frac{q}{(2n)}}}e^{\tau\ell/2}\left(e^{-\frac{1}{4}\tau m} + e^{-\gamma R^{2n/(2n-1)}}\right)\|v\|_{\ell-q,m}\;. \tag{A.18}$$

As a preliminary step, we note that if we first choose r and R such that

$$e^{\tau q/2}\left(e^{-\frac{1}{4}\tau r} + e^{-\gamma R^{2n/(2n-1)}}\right) \;\le\; e^{-\mu((k+1)/(2n))}\;,$$

then for sufficiently small μ (roughly speaking $\mu \sim \frac{1}{2}\big(1+(4/3)^{2n/(2n-1)}\big)^{-1}$), the Eqs.(A.17) and (A.18) imply

$$\|e^{\tau\mathcal{L}}Q_k\| \;\le\; \frac{C}{a(\tau)^{q/(2n)}}e^{-\mu\frac{k+1}{2n}}\|v\|_{\ell-q,k} \;=\; \frac{C}{a(\tau)^{q/(2n)}}e^{-\mu|\lambda_{k+1}|}\|v\|_{\ell-q,k}\;.$$

This shows that the projection of the semi-group onto the complement of the eigenspace spanned by the first k eigenvalues decays with a rate proportional to the eigenvalue λ_{k+1}. We can sharpen the decay rate so that we obtain a rate like $\exp\big(-(1-\varepsilon)|\lambda_{k+1}|\big)$ by the techniques of [EWW], (see Eq. B.14 and following) and this completes the proof of Proposition 2.1

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