Uniqueness for Gibbs States of Quantum Lattices in Small Mass Regime

Sergio Albeverio
Abteilung für Stochastik,
Institut für Angewandte Mathematik, Universität Bonn,
D 53115 Bonn (Germany);
Forschungszentrum BiBoS, Bielefeld (Germany);
SFB 237 (Essen–Bochum–Düsseldorf) (Germany);
CERFIM, Locarno
e-mail albeverio@uni–bonn.de

Yuri Kondratiev
Abteilung für Stochastik,
Institut für Angewandte Mathematik, Universität Bonn,
D 53115 Bonn (Germany);
Forschungszentrum BiBoS, D 33615 Bielefeld (Germany);
Institute of Mathematics, Kiev (Ukraine)
e-mail kondratiev@uni-bonn.de

Yuri Kozitsky
Institute of Mathematics, Maria Curie-Sklodowska University
PL 20-031 Lublin (Poland);
Institute for Condensed Matter Physics, Lviv (Ukraine)
e-mail jkozi@golem.umcs.lublin.pl

Michael Röckner
Fakultet für Mathematik,
Universität Bielefeld, D 33615 Bielefeld (Germany)
e-mail roeckner@mathematik.uni-bielefeld.de

Abstract

The model of interacting quantum particles performing anharmonic one–dimensional oscillations around their unstable equilibrium positions, which form the d– dimensional simple cubic lattice $\mathbb{Z}^d$, is considered. For this model, it is proved that for every fixed value of temperature β^{-1}, there exists a positive $m_*(\beta)$ such that for the values of the physical mass of the particle $m \in (0, m_*(\beta))$, the set of tempered Gibbs states consists of exactly one element.

1 Introduction

In this paper we consider the following quantum lattice model. To each point of the d–dimensional cubic lattice $\mathbb{Z}^d$ there is attached a quantum particle with the physical mass m which has an unstable equilibrium position at this point. Such particles perform one–dimensional oscillations around their equilibrium positions and interact via an attractive potential. Similar objects have been studied for many years as quite realistic models of a crystalline substance undergoing structural phase transitions (see e.g. [10]). In our case the phase transition is connected with the appearance of macroscopic displacements of the particles (a long–range order) if the dimension d, the mass m, temperature β^{-1}, and the parameters of the potential energy satisfy certain conditions. These phenomena were studied mathematically in various papers, see e.g. [8], [13], [20], [23]. The essential problem in this context is to understand the role of quantum effects in the phase transitions in such models. As it was justified on the physical level [25] and observed experimentally (see [28] and Ch.2.5.4.3 of book [10]), the quantum effects may suppress the long–range ordering. This was proved in paper [29]. Later on it was shown in [3], [4] that not only the long–range order but also any critical anomaly of the displacements of particles from the equilibrium positions are suppressed if the model is "strongly quantum", which may occur in particular if the particle's mass is small.

Another important problem of the mathematical theory of models which exhibit such phenomena is the description of their Gibbs states. The uniqueness for such a state implies the absence of all critical anomalies and all the more of the long– range ordering. Therefore, one may expect that the "strong quantumness" of the model should imply the uniqueness of its Gibbs state. So far, the mentioned uniqueness for the model considered in this work was proved to occur under conditions which do not contain the physical mass of the particle and thus are irrelevant to the "quantumness" of the model.

It was done in paper [1] on the base of the logarithmic Sobolev inequalities. An alternative approach to the construction of the Gibbs states for such models lies in the use of cluster expansions. We note here the papers [7], [22] where models of this type with "light" particles were studied by means of different versions of cluster expansions. In particular, in paper [7] the existence of temperature Gibbs states was shown by a proof of the convergence of cluster expansions at fixed temperature and for small masses. In paper [22] for small values of the particle's mass, the convergence of corresponding cluster expansions was proved for all values of temperature including zero. As a result, the existence of temperature and ground states was proved and a number of the properties of these states have been described. It should be pointed out that for systems with unbounded oscillations, as in the case considered in this work, it is impossible to recover the uniqueness of the state from the convergences of the cluster expansion. However, this is possible in the case of quantum particles which move on compact manifolds. Here the uniqueness of the Euclidean Gibbs measure was shown [6] by using the convergence of the cluster expansions.

In this paper we present the proof of the uniqueness for the Gibbs states in the mentioned model for small values of the physical mass of the particle. We follow paper [1] and apply Dobrushin's technique up to the stage where the logarithmic Sobolev inequalities were used to estimate the variances of certain moments, which was crucial in satisfying Dobrushin's uniqueness criterion. Instead, we reduce this problem to the study of spectral properties of the one–particle Hamiltonian which describes the individual oscillations. This study is performed by means of a version of the lattice approximation technique known in the Euclidean quantum field theory [26]. A similar approach has already been used in our works [3], [4]. It enables us to involve the physical mass into the consideration, so that it appears explicitly in the uniqueness conditions. Unfortunately, it is impossible (at least so far) to remove the temperature dependence from these conditions, in contrast to the proof for the suppression of critical fluctuations in papers [3], [4].

2 Temperature loop formalism for Gibbs states

As in [1], [3] we consider a translation invariant system of interacting quantum particles performing one–dimensional oscillations, described by the following formal Hamiltonian

$$H = \sum_k H_k + \frac{J}{4} \sum_{<j,k>} (x_j - x_k)^2, \tag{2.1}$$

where the indices j, k run over the d–dimensional lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ with the Euclidean distance $|\ .\ |$. The second sum in (2.1) is taken over the nearest neighbours, that is restricted to the pairs with $|j-k| = 1$. It describes ferroelectric ($J > 0$) interactions of the particles whereas the first term is a sum of one–particle Hamiltonians

$$H_k = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}x_k^2 + V(x_k), \tag{2.2}$$

where $m > 0$ is the physical mass of the particle and

$$\begin{aligned} V(x) &= P_{2p}(x) - \frac{1}{2}ax^2 \\ P_{2p}(x) &= \sum_{l=2}^{p} b_l x^{2l},\ p \geq 2. \end{aligned} \tag{2.3}$$

Here we suppose: $a > 0$, $b_p > 0$, and $b_l \geq 0$ for all $l \leq p$. For $d \geq 2$, the model has a critical point and long–range order behaviour if its parameters satisfy certain conditions (see e.g. [8], [20]).

Each Hamiltonian H_k is an essentially self–adjoint lower semi–bounded operator on the complex Hilbert space $L^2(\mathbb{R}, dx)$ (see e.g. [9]). We will study temperature Euclidean Gibbs states of the system, i.e. Gibbs states constructed using a probability measure on a loop space, see [2], [19], [16] for this approach. Thus the Gibbs state at temperature β^{-1} is described by the Euclidean (Gibbs) measure on the space of periodic trajectories (loops) taking values in $\mathbb{R}^{\mathbb{Z}^d}$.

Consider the space of continuous periodic functions

$$C(I_\beta) \stackrel{\text{def}}{=} \{\omega \in C([0,\beta] \to \mathbb{R}) \mid \omega(0) = \omega(\beta)\}$$

and let $L^2(I_\beta)$ be the closure of $C(I_\beta)$ in the Hilbert space of real–valued functions on $[0,\beta]$ which are square integrable with respect to Lebesgue measure. Here I_β is the interval $[0,\beta]$ with the ends identified (i.e. a circle of length β). The temperature loop space (TLS) is

$$\begin{aligned} \Omega_\beta &\stackrel{\text{def}}{=} C(I_\beta)^{\mathbb{Z}^d} \\ &= \{\omega = (\omega_k)_{k\in\mathbb{Z}^d} \mid \omega_k \in C(I_\beta)\}. \end{aligned} \tag{2.4}$$

For $\omega \in \Omega_\beta$ and $\Lambda \subset \mathbb{Z}^d$, we set $\omega_\Lambda = (\omega_k)_{k\in\Lambda}$. We shall also consider

$$\Omega_{\beta,\Lambda} \stackrel{\text{def}}{=} \{\omega_\Lambda = (\omega_k)_{k\in\Lambda} \mid \omega_k \in C(I_\beta)\}, \tag{2.5}$$

defined for all subsets $\Lambda \subset \mathbb{Z}^d$. The spaces $\Omega_{\beta,\Lambda}$, Ω_β are equipped with the product topology and with the σ–algebra $\mathcal{B}(\Omega_{\beta,\Lambda})$ generated by the cylinder sets

$$\{\omega \mid (\omega_k)_{k\in\Delta} \in B_\Delta,\ \Delta \in \mathcal{L}\ \},$$

where $\mathcal{L}$ is the set of finite subsets of $\mathbb{Z}^d$, B_Δ is a direct product of Borel subsets $\{B_k \subset C(I_\beta) \mid k \in \Delta\}$. Consider again $L^2(I_\beta) \stackrel{\text{def}}{=} \mathcal{H}_\beta$ equipped with the usual inner product $(.,.)_\beta$ and norm $\|.\|_\beta$. Define on $\mathcal{H}_\beta$ a strictly positive trace class operator $S_\beta = (-m\Delta_\beta + 1)^{-1}$, where Δ_β stands for the Laplace–Beltrami operator on the circle I_β, and m is the physical mass of the particle. In this setting one can define a Gaussian measure γ_β on $\mathcal{H}_\beta$ which has zero mean and S_β as a covariance operator. This measure is uniquely determined by its Fourier transform

$$\int_{\mathcal{H}_\beta} \exp(i(\varphi,\omega)_\beta)\gamma_\beta(d\omega) = \exp(-\frac{1}{2}(\varphi, S_\beta\varphi)_\beta), \quad \varphi \in \mathcal{H}_\beta. \tag{2.6}$$

Actually, the set of continuous loops is a set of full measure (i.e. $\gamma_\beta(C(I_\beta)) = 1$), and the measure γ_β is canonically generated by the oscillator bridge process of length β [27]. By means of this Gaussian measure one can define

$$\sigma_\beta(d\omega) = \frac{1}{X_\beta}\exp\{-\int_{I_\beta} V(\omega(\tau))d\tau\}\gamma_\beta(d\omega) \tag{2.7}$$

as a probability measure on $C(I_\beta)$. Here X_β is a normalizing constant and V is given by eq.(2.3). The measure σ_β has all moments, that is

$$\int_{C(I_\beta)} \|\omega\|^l_{C(I_\beta)}\sigma_\beta(d\omega) < \infty, \quad \forall l \in \mathbb{N}. \tag{2.8}$$

For every $\zeta \in \Omega_\beta$ and each finite subset $\Lambda \subset \mathbb{Z}^d$, we define the Gibbs measure in Λ, subject to a chosen ζ, as the probability measure

$$\nu_{\beta,\Lambda}(d\omega_\Lambda|\zeta) \stackrel{\text{def}}{=} \frac{1}{Z_{\beta,\Lambda}(\zeta)}\exp\{-\Phi_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\prod_{k\in\Lambda}\sigma_\beta(d\omega_k), \tag{2.9}$$

on the loop space $\Omega_{\beta,\Lambda}$. Here

$$\begin{aligned} &\Phi_{\beta,\Lambda}(\omega_\Lambda|\zeta) \qquad (2.10)\\ \stackrel{\text{def}}{=}\ &\frac{J}{4}\int_{I_\beta}\left\{\sum_{j,k\in\Lambda}(\omega_j(\tau)-\omega_k(\tau))^2+2\sum_{j\in\Lambda,k\in\Lambda^c}(\omega_j(\tau)-\zeta_k(\tau))^2\right\}d\tau, \end{aligned}$$

is the pair interaction in Λ, prescribed by the Hamiltonian (2.1), under the external boundary condition

$$\zeta_{\Lambda^c}\in C(I_\beta)^{\Lambda^c},\quad \Lambda^c \stackrel{\text{def}}{=} \mathbb{Z}^d\setminus\Lambda.$$

All sums in (2.10) are taken under the condition $|j-k|=1$, and

$$Z_{\beta,\Lambda}(\zeta)\stackrel{\text{def}}{=}\int_{\Omega_{\beta,\Lambda}}\exp\{-\Phi_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\prod_{k\in\Lambda}\sigma_\beta(d\omega_k), \qquad (2.11)$$

is the finite volume partition function subject to the external boundary condition ζ_{Λ^c}. For $B\in\mathcal{B}(\Omega_\beta)$ and $\omega\in\Omega_\beta$, let $\mathbb{1}_B(\omega)$ take values 1 resp. 0 if ω belongs resp. does not belong to B. Then one can introduce a family of probability kernels $\{\pi_{\beta,\Lambda}\mid\Lambda\in\mathcal{L}\}$ between $(\Omega_\beta,\mathcal{B}(\Omega_\beta))$ and $(\Omega_\beta,\mathcal{B}(\Omega_\beta))$ by

$$\pi_{\beta,\Lambda}(B|\zeta)\stackrel{\text{def}}{=}\int_{\Omega_{\beta,\Lambda}}\mathbb{1}_B(\omega_\Lambda\times\zeta_{\Lambda^c})\nu_{\beta,\Lambda}(d\omega_\Lambda|\zeta). \qquad (2.12)$$

These kernels satisfy the consistency conditions which are set as follows (see e.g. [15]). For arbitrary pair $\Lambda'\subset\Lambda$, each $B\in\mathcal{B}(\Omega_\beta)$, and $\zeta\in\Omega_\beta$,

$$\begin{aligned}\pi_{\beta,\Lambda}\pi_{\beta,\Lambda'}(B|\zeta) &\stackrel{\text{def}}{=} \int_{\Omega_\beta}\pi_{\beta,\Lambda}(d\omega|\zeta)\pi_{\beta,\Lambda'}(B|\omega)\\ &= \pi_{\beta,\Lambda}(B|\zeta). \qquad (2.13)\end{aligned}$$

Definition 2.1 *A probability measure μ_β on $(\Omega_\beta,\mathcal{B}(\Omega_\beta))$ is said to be an Euclidean Gibbs state of the lattice model (2.1), (2.2), (2.3) at temperature β^{-1} if it satisfies the Dobrushin–Lanford–Ruelle (DLR) equilibrium equations:*

$$\mu_\beta\pi_{\beta,\Lambda}=\mu_\beta,$$

that is

$$\int_{\Omega_\beta}\mu_\beta(d\omega)\pi_{\beta,\Lambda}(B|\omega)=\mu_\beta(B), \qquad (2.14)$$

for all $\Lambda\in\mathcal{L}$ and $B\in\mathcal{B}(\Omega_\beta)$.

In order to exclude the states with no physical relevance we impose some *a priori* conditions restricting the growth of the sequences of moments (see [1], [17]). To this end we introduce

$$\mathcal{S}'(\mathbb{Z}^d) \stackrel{\text{def}}{=} \text{ind} \lim_{p \in \mathbb{N}} l^2_{-p}(\mathbb{Z}^d),$$

where

$$l^2_{-p} \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^{\mathbb{Z}^d} \mid \sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-p} x_k^2 < \infty \right\}. \tag{2.15}$$

We then have the following:

Definition 2.2 *The class of tempered Gibbs measures $\mathcal{G}_\beta$ for the model considered consists of the Euclidean Gibbs states defined above which are such that their moment sequences obey the condition*

$$(< \|\omega_k\|_\beta >_{\mu_\beta})_{k \in \mathbb{Z}^d} \in \mathcal{S}'(\mathbb{Z}^d), \tag{2.16}$$

where $\| \, . \, \|_\beta$ is the norm in the Hilbert space $\mathcal{H}_\beta = L^2(I_\beta)$.

Here and further on we write

$$< f >_\mu = \int f d\mu \tag{2.17}$$

if the integral makes sense.

For the existence problem for Euclidean Gibbs measures we refer to [1] and the references therein. Here we only remark that the class $\mathcal{G}_\beta$ is actually nonempty. Our main aim in this paper is to refine the uniqueness condition obtained in [1] to involve the quantum nature of the model.

3 Uniqueness theorem

We start this section with the formulation of the uniqueness theorem.

Theorem 3.1 *For the model considered, let the class of tempered Gibbs measures $\mathcal{G}_\beta$ be as in Definition 2.2. Then for every β, there exists a positive $m_*(\beta)$ such that for all values of the physical mass $m \in (0, m_*(\beta))$, the class $\mathcal{G}_\beta$ consists of exactly one element, that is $|\mathcal{G}_\beta| = 1$.*

Remark 3.1 *The value of m_* depends also on the concrete choice of the function V (2.3) as well as on the value of the interaction intensity J.*

Following the scheme used in [1] we metrize the space of measures. Let $(\mathcal{X}, \rho)$ be a complete separable metric space, $\mathcal{M}$ be the set of all probability measures on $\mathcal{B}(\mathcal{X})$, and

$$\mathcal{M}_1 \stackrel{\mathrm{def}}{=} \left\{ \mu \in \mathcal{M} \mid \int_{\mathcal{X}} \rho(x, x_0)\mu(dx) < \infty \right\}, \tag{3.1}$$

for some $x_0 \in \mathcal{X}$. Further, let $\mathrm{Lip}(\mathcal{X})$ denote the set of all Lipschitz functions $f : \mathcal{X} \to I\!\!R$, and

$$[f]_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} \sup \left\{ \frac{|f(x) - f(y)|}{\rho(x, y)} \mid x, y \in \mathcal{X},\ x \neq y \right\}, \tag{3.2}$$

$$\mathrm{Lip}_1(\mathcal{X}) \stackrel{\mathrm{def}}{=} \{ f : \mathcal{X} \to I\!\!R \mid [f]_{\mathrm{Lip}} \leq 1 \}. \tag{3.3}$$

For given $\mu_1,\ \mu_2 \in \mathcal{M}_1$, we set

$$\begin{aligned} R(\mu_1, \mu_2) & \\ \stackrel{\mathrm{def}}{=} & \sup \left\{ \left| \int_{\mathcal{X}} f(x)\mu_1(dx) - \int_{\mathcal{X}} f(x)\mu_2(dx) \right| \mid f \in \mathrm{Lip}_1(\mathcal{X}) \right\}. \end{aligned} \tag{3.4}$$

A significant role in the proof will be played by Dobrushin's matrix. It is defined in terms of the measures $\nu_{\beta,\Lambda}$ given by eqs. (2.9) – (2.11) with the one–point subset $\Lambda = \{k\}$. To simplify notations we set

$$\xi_{\{j\}^c} = \xi_j^c, \quad \nu_{\beta,\{k\}}(B|\xi) = \nu_k(B|\xi). \tag{3.5}$$

Dobrushin's matrix elements $C_{k,j};\ k, j \in Z\!\!\!Z^d$ are:

$$C_{kj} \stackrel{\mathrm{def}}{=} \sup \left\{ \frac{R(\nu_k(.|\xi), \nu_k(.|\eta))}{\|\xi_j - \eta_j\|_\beta} \mid \xi, \eta \in \Omega_\beta,\ \xi_j^c = \eta_j^c \right\}. \tag{3.6}$$

We will use them to check Dobrushin's uniqueness criterion [11], [12], [15], [21].

Proposition 3.1 (Dobrushin's uniqueness criterion) *For the model considered, let the following condition be satisfied*

$$\sup \left\{ \sum_{j \in Z\!\!\!Z^d \setminus \{k\}} C_{kj} \mid k \in Z\!\!\!Z^d \right\} < 1. \tag{3.7}$$

Then there exists exactly one tempered Gibbs measure, that is $|\mathcal{G}_\beta| = 1$.

Directly from the definitions (2.9) – (2.11), (3.5) one obtains

$$\nu_k(d\omega_k|\xi) = \frac{1}{Z_k(\xi)} \exp\left\{ -Jd\|\omega_k\|_\beta^2 + J \sum_{|i-k|=1} (\omega_k, \xi_i)_\beta \right\} \sigma_\beta(d\omega_k). \quad (3.8)$$

This implies that $\nu_k(d\omega_k|\xi) = \nu_k(d\omega_k|\eta)$ if $\xi_j^c = \eta_j^c$ and $|k-j| \neq 1$. Hence

$$R(\nu_k(.|\xi), \nu_k(.|\eta) = 0, \quad C_{kj} = 0,$$

in this case. For $|k-j| = 1$, all C_{kj} are the same, thus Dobrushin's uniqueness condition is satisfied if:

$$\frac{1}{2d} > C \stackrel{\text{def}}{=} C_{kj}, \quad \text{for } \ |k-j| = 1. \quad (3.9)$$

By the translation invariance of the model one may conclude that each measure $\nu_k(.|\xi)$ is a copy of the following measure defined on the probability space $(\mathcal{H}_\beta, \mathcal{B}(\mathcal{H}_\beta))$:

$$\begin{aligned} \nu^x(d\omega) &= \frac{1}{Z_x} \exp\{-Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta\}\sigma_\beta(d\omega), \quad (3.10) \\ Z_x &= \int_{\mathcal{H}_\beta} \exp\{-Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta\}\sigma_\beta(d\omega), \end{aligned}$$

with

$$\mathcal{H}_\beta \ni x = \sum_{i:\ |i-k|=1} \xi_i.$$

Hence we have an equivalent representation for the Dobrushin coefficients (3.9)

$$C = \sup\left\{ \frac{R(\nu^x, \nu^y)}{\|x-y\|_\beta} \mid x, y \in \mathcal{H}_\beta, \ x \neq y \right\}. \quad (3.11)$$

Now let us estimate the variance of the real–valued function

$$\mathcal{H}_\beta \ni x \mapsto < f >_{\nu^x} = \int_{\mathcal{H}_\beta} f(\omega)\nu^x(d\omega) \in I\!R, \quad (3.12)$$

with certain fixed $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$, in terms of the variance of its argument. To this end we use the mean–value theorem. Due to (2.8), the function (3.12) is Fréchet differentiable and its derivative in the direction $\varphi \in \mathcal{H}_\beta$ is

$$\begin{aligned} (\nabla_x < f >_{\nu^x}, \varphi)_\beta &= J\{< f(., \varphi)_\beta >_{\nu^x} - < f >_{\nu^x} < (., \varphi)_\beta >_{\nu^x}\} \\ &= J\mathrm{Cov}_{\nu^x}(f, (., \varphi)_\beta). \quad (3.13) \end{aligned}$$

Our next aim is to find an estimate of this derivative which does not depend on $x \in \mathcal{H}_\beta$. We start by using the Schwarz inequality

$$|(\nabla_x < f >_{\nu^x}, \varphi)_\beta| \leq J\sqrt{\mathrm{Var}_{\nu^x} f}\sqrt{\mathrm{Var}_{\nu^x}(.,\varphi)_\beta}, \tag{3.14}$$

where

$$\begin{aligned} \mathrm{Var}_{\nu^x} f &= \int_{\mathcal{H}_\beta} f^2(\omega)\nu^x(d\omega) \\ &- \int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} f(\omega)f(\omega')\nu^x(d\omega)\nu^x(d\omega'), \end{aligned} \tag{3.15}$$

$$\begin{aligned} \mathrm{Var}_{\nu^x}(.,\varphi)_\beta &= \int_{\mathcal{H}_\beta} (\omega,\varphi)^2_\beta \nu^x(d\omega) \\ &- \int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} (\omega,\varphi)_\beta(\omega',\varphi)_\beta \nu^x(d\omega)\nu^x(d\omega'), \end{aligned} \tag{3.16}$$

that may also be rewritten as follows

$$\mathrm{Var}_{\nu^x} f = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} (f(\omega) - f(\omega'))^2 \nu^x(d\omega)\nu^x(d\omega'), \tag{3.17}$$

$$\mathrm{Var}_{\nu^x}(.,\varphi)_\beta = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} (\omega - \omega',\varphi)^2_\beta \ \nu^x(d\omega)\nu^x(d\omega'). \tag{3.18}$$

Suppose now that we have estimated the first variance by a constant which continuously depends on β, on the parameters a, b_l of the function (2.3), and on the value of the physical mass m. At the same time let the second variance be bounded by a function of the a, b_l, and m, multiplied by the square of the norm of φ. It is very important that both estimates would hold uniformly for all $x \in \mathcal{H}_\beta$. Then the mean–value theorem together with the representation of the Dobrushin coefficients (3.11) imply that the conditions (3.7), (3.9) may be satisfied provided the product of the mentioned bounds is sufficiently small. Below we shall implement this idea.

It can be observed that the expression (3.18) defines a bounded positive quadratic form on $\mathcal{H}_\beta$. That is

$$\begin{aligned} \mathrm{Var}_{\nu^x}(.,\varphi)_\beta &\stackrel{\mathrm{def}}{=} (\varphi, K_x\varphi)_\beta \\ &\leq \frac{1}{2}\|\varphi\|^2_\beta \int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} \|\omega - \omega'\|^2_\beta \ \nu^x(d\omega)\nu^x(d\omega'). \end{aligned} \tag{3.19}$$

The operator K_x which appears here has the following integral representation

$$(K_x\varphi)(\tau) = \int_{I_\beta} K_x(\tau,\tau')\varphi(\tau')d\tau', \quad \tau \in I_\beta, \tag{3.20}$$

with the kernel $K_x(\tau,\tau')$ given by

$$K_x(\tau,\tau') = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta}[\omega(\tau)-\omega'(\tau)][\omega(\tau')-\omega'(\tau')]\nu^x(d\omega)\nu^x(d\omega'). \tag{3.21}$$

Clearly, for every $x \in \mathcal{H}_\beta$, the operator K_x is symmetric and positive, its kernel (3.21) is a continuous function of τ, $\tau' \in I_\beta$, and

$$\mathrm{trace}(K_x) = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta}\|\omega-\omega'\|_\beta^2\ \nu^x(d\omega)\nu^x(d\omega') < \infty. \tag{3.22}$$

For a bounded linear operator $T : \mathcal{H}_\beta \to \mathcal{H}_\beta$, let $\mathcal{S}_{pp}(T)$ denote the pure point spectrum of T and let $\|T\|$ be its operator norm. For a positive compact operator T, one has

$$\|T\| = \max \mathcal{S}_{pp}(T). \tag{3.23}$$

On the other hand, for the norm of a symmetric positive T, one has (see e.g [24], p.216):

$$\|T\| = \sup\{\ \frac{(\varphi, T\varphi)_\beta}{\|\varphi\|_\beta^2} \mid \varphi \in \mathcal{H}_\beta \setminus \{0\}\ \}. \tag{3.24}$$

Now we modify our notations introducing

$$\tilde{P}_{2p}(x) = \sum_{l=2}^{p} 2^{1-l} b_l x^{2l},\ p \geq 2 \tag{3.25}$$

$$\tilde{V}(x) = \tilde{P}_{2p}(x) - \frac{1}{2}ax^2, \tag{3.26}$$

$$\tilde{H} = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}(2Jd+1)q^2 + \tilde{V}(q). \tag{3.27}$$

The latter operator is defined in $L^2(I\!R, dx)$ and the operator q acts as follows

$$(q\psi)(x) \stackrel{\mathrm{def}}{=} x\psi(x). \tag{3.28}$$

Further, along with the measures defined by (2.7) and (3.10), we introduce

$$\begin{aligned}
\tilde{\sigma}_\beta(d\omega) &= \frac{1}{\tilde{X}_\beta}\exp\left\{-\int_{I_\beta}\tilde{V}(\omega(\tau))d\tau\right\}\gamma_\beta(d\omega), \qquad (3.29)\\
\tilde{X}_\beta &= \int_{\mathcal{H}_\beta}\exp\left\{-\int_{I_\beta}\tilde{V}(\omega(\tau))d\tau\right\}\gamma_\beta(d\omega),\\
\tilde{\nu}^x(d\omega) &= \frac{1}{\tilde{Z}_x}\exp\left\{-Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta\right\}\tilde{\sigma}_\beta(d\omega), \qquad (3.30)
\end{aligned}$$

where the partition function $\tilde{Z}_x$ may be written as in (3.10) but with $\tilde{\sigma}_\beta$ instead of σ_β. Having these new measures we introduce the operator $\tilde{K}_x$ by means of eq.(3.20) with the kernel

$$\tilde{K}_x(\tau,\tau') = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta}[\omega(\tau)-\omega'(\tau)][\omega(\tau')-\omega'(\tau')]\tilde{\nu}^x(d\omega)\tilde{\nu}^x(d\omega'). \quad (3.31)$$

Its trace is finite and can be calculated just as in (3.22) with the measure $\tilde{\sigma}_\beta$ instead of σ_β.

The construction of the bounds mentioned above is based upon the following lemmas, which are proved in Section 4 below.

Lemma 3.1 *For every $x \in \mathcal{H}_\beta$ and all $\varphi \in \mathcal{H}_\beta$*

$$(\varphi, K_x\varphi)_\beta \leq \left\|\tilde{K}_0\right\| \|\varphi\|_\beta^2. \quad (3.32)$$

Lemma 3.2 *For every $x \in \mathcal{H}_\beta$,*

$$\mathrm{trace}(K_x) \leq \mathrm{trace}(\tilde{K}_0). \quad (3.33)$$

It is known (see [9], p.57) that (the closure of) $\tilde{H}$ given by (3.27) is a self–adjoint operator with a discrete spectrum, all its eigenvalues $\tilde{E}_l$, $l \in I\!N$ are simple. We set

$$\tilde{\Delta} \stackrel{\mathrm{def}}{=} \inf\{|\tilde{E}_l - \tilde{E}_{l'}| \mid l, l' \in I\!N, l \neq l\}. \quad (3.34)$$

Lemma 3.3 *For the spectral radius of the operator $\tilde{K}_0$, the following estimate holds*

$$\max\mathcal{S}_{pp}(\tilde{K}_0) \leq \frac{1}{m\tilde{\Delta}^2}. \quad (3.35)$$

Lemma 3.4 *There exists a positive constant χ_0 (independent of m) such that for sufficiently small positive values of the physical mass m, the following estimate holds*

$$\frac{1}{m\tilde{\Delta}^2} \leq \chi_0 m^{\frac{p-1}{p+1}}. \tag{3.36}$$

To estimate the variance of the Lipschitz function f given by eq.(3.17) one may use the logarithmic Sobolev inequality as it was done in [1]. In the case where the function V defined by (2.3) is

$$V(x) = -\frac{1}{2}ax^2 + P_4(x) = -\frac{1}{2}ax^2 + b_2x^4,$$

this estimate looks as follows.

Lemma 3.5 *For all $x \in \mathcal{H}_\beta$ and arbitrary $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$,*

$$\mathrm{Var}_{\nu^x} f \leq \frac{e^{\beta\delta_0}}{2Jd + 1 + a/4}, \tag{3.37}$$

where the exponent δ_0 is defined by the parameters of the function V

$$\delta_0 = \frac{25}{288}\frac{a^2}{b_2}.$$

In the case of V given by eq. (2.3) with $p > 2$ one may use the following estimate of the variance of f, which is linear in β.

Lemma 3.6 *There exists a positive constant ϕ_0 (independent of m and β) such that for all $x \in \mathcal{H}_\beta$, arbitrary $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$, and for sufficiently small m, the following estimate holds*

$$\mathrm{Var}_{\nu^x} f \leq \beta\phi_0 m^{-1/(p+1)}. \tag{3.38}$$

Proof of Theorem 3.1. We start with estimating of the variance $\mathrm{Var}_{\nu^x}(.,\varphi)_\beta$ given by eqs. (3.16), (3.18), (3.19). By means of Lemma 3.1 and (3.23) one obtains

$$\mathrm{Var}_{\nu^x}(.,\varphi)_\beta = (\varphi, K_x\varphi)_\beta \;\leq\; \left\|\tilde{K}_0\right\| \|\varphi\|_\beta^2 = \max\mathcal{S}_{pp}(\tilde{K}_0)\|\varphi\|_\beta^2,$$

and further by means of Lemma 3.3, Lemma 3.4

$$\mathrm{Var}_{\nu^x}(.,\varphi)_\beta \leq \chi_0 m^{\frac{p-1}{p+1}} \|\varphi\|_\beta^2, \tag{3.39}$$

that holds for sufficiently small positive m. In the case $p > 2$ one may use the estimate (3.38) and choose m so small that both latter esimates hold. This yields for the distance (3.4)

$$R(\nu^x, \nu^y) \leq \|x - y\|_\beta J \sqrt{\beta \phi_0 \chi_0} m^{\frac{p-2}{2(p+1)}},$$

which means in view of (3.11), (3.9) that the condition of Dobrushin's criterion (3.7) is satisfied provided

$$\beta m^{\frac{p-2}{p+1}} < \frac{1}{(2Jd)^2 \phi_0 \chi_0}. \tag{3.40}$$

Therefore, in the case $p > 2$ the upper bound for m is

$$m_*(\beta) = \frac{\beta^{-\frac{p+1}{p-2}}}{[(2dJ)^2 \phi_0 \chi_0]^{\frac{p+1}{p-2}}}. \tag{3.41}$$

For $p = 2$, we use the estimate (3.37), choose m so small that (3.39) also holds, and obtain

$$R(\nu^x, \nu^y) \leq \|x - y\|_\beta J e^{\beta \delta_0 / 2} m^{\frac{1}{6}} \sqrt{\frac{\chi_0}{2Jd + 1 + a/4}},$$

which implies in turn that Dobrushin's criterion is satisfied provided

$$m < m_*(\beta) \stackrel{\text{def}}{=} e^{-3\beta\delta_0} \left(\frac{2Jd + 1 + a/4}{(2Jd)^2 \chi_0} \right)^3, \tag{3.42}$$

■

4 Proof of Lemmas

The proof of the first two lemmas is performed by means of a version of the lattice approximation technique known in Euclidean quantum field theory (see [26], Chapter VIII). We start with the following heuristic representation of the Gaussian measure γ_β defined by (2.6) (see e.g. eq. (5.2) in [1]):

$$\gamma_\beta(d\omega) = \frac{1}{C_\beta} \exp\left\{ -\frac{1}{2} m \int_{I_\beta} \dot{\omega}^2(\tau) d\tau - \frac{1}{2} \int_{I_\beta} \omega^2(\tau) d\tau \right\} \prod_{\tau \in I_\beta} d\omega(\tau), \tag{4.1}$$

where $\dot{\omega}(\tau) \stackrel{\rm def}{=} d\omega/d\tau$. Recall that this measure is supported on $C(I_\beta) \subset \mathcal{H}_\beta$. Let us consider the following base $\{e_k(\tau) \mid k \in \mathcal{K}\}$ of $\mathcal{H}_\beta$:

$$e_k(\tau) = \sqrt{\frac{2}{\beta}} \cos k\tau, \quad k > 0, \quad e_k(\tau) = -\sqrt{\frac{2}{\beta}} \sin k\tau, \quad k < 0, \quad e_0(\tau) = \frac{1}{\sqrt{\beta}}, \tag{4.2}$$

where

$$\mathcal{K} \stackrel{\rm def}{=} \{k = \frac{2\pi}{\beta}\kappa \mid \kappa \in \mathbb{Z}\}. \tag{4.3}$$

Then

$$\omega(\tau) = \sum_{k\in\mathcal{K}} \hat{\omega}_k e_k(\tau), \quad \hat{\omega}_k = \int_{I_\beta} \omega(\tau) e_k(\tau) d\tau, \tag{4.4}$$

and the measure (4.1) may be written as the following product

$$\begin{aligned} \gamma_\beta(d\omega) &= \prod_{k\in\mathcal{K}} \gamma_k(d\hat{\omega}_k), \qquad (4.5)\\ \gamma_k(d\hat{\omega}_k) &= \sqrt{\frac{mk^2+1}{2\pi}} \exp\{-\frac{1}{2}(mk^2+1)\hat{\omega}_k^2\} d\hat{\omega}_k. \end{aligned}$$

Sometimes it will be convenient to use the following version of the Fourier transformation (4.4):

$$\omega(\tau) = \sum_{k\in\mathcal{K}} \tilde{\omega}_k \frac{1}{\sqrt{\beta}} e^{ik\tau}, \quad i = \sqrt{-1}, \tag{4.6}$$

with

$$\tilde{\omega}_k = \frac{1}{\sqrt{2}}(\hat{\omega}_k - i\hat{\omega}_{-k}), \ k > 0, \ \ \tilde{\omega}_k = \frac{1}{\sqrt{2}}(\hat{\omega}_{-k} + i\hat{\omega}_k), \ k < 0, \ \ \tilde{\omega}_0 = \hat{\omega}_0. \tag{4.7}$$

Let $C(I_\beta)^*$ be the adjoint space of $C(I_\beta)$. The space of continuous cylinder functions on $C(I_\beta)$ is defined as follows

$$\begin{aligned} \mathcal{FC} \ &= \ \{u : C(I_\beta) \to \mathbb{R} \mid (\exists \theta_1., \ldots, \theta_n \in C(I_\beta)^*), \\ &\quad\ (\exists u_n \in C(\mathbb{R}^n)) \ \ u(\omega) = u_n(\theta_1(\omega), \ldots, \theta_n(\omega))\}. \end{aligned}$$

Given a collection of $\{\tau_1, \ldots, \tau_l\} = \{\tau_j\}$, $\tau_j \in I_\beta$, $l \in \mathbb{N}$, let $\theta_j \in C(I_\beta)^*, \ j = 1, \ldots, l$ be defined as follows $\theta_j(\omega) = \omega(\tau_j)$. Let also

$$\mathcal{FC}(\tau_1, \ldots, \tau_l) \stackrel{\rm def}{=} \{u \in \mathcal{FC} | (\exists u_l \in C(\mathbb{R}^l)) \ u(\omega) = u_l(\omega(\tau_1), \ldots, \omega(\tau_l))\}. \tag{4.8}$$

Furthermore, suppose that for chosen $\{\tau_j\}$, all numbers τ_j/β are rational. Such τ form a dense subset $Q_\beta \subset I_\beta$. Then there exist sequences $\{N^{(k)}, k \in \mathbb{N}\}$, $\{n_j^{(k)}, k \in \mathbb{N}\}$, $j = 1, \ldots, l$ each of which tends to infinity, such that

$$\tau_j = (n_j^{(k)}/N^{(k)})\beta, \quad \forall k \in \mathbb{N}, \text{ and } \forall j = 1, \ldots, l. \tag{4.9}$$

To simplify our notations we drop the symbol (k) assuming that N and n_j tend to infinity in such a way that (4.9) holds. Clearly, we may also suppose that all such N are even. The set of values of N satisfying these conditions depends on the choice of $\{\tau_j\}$, we denote it by $\mathcal{N}(\tau_1, \ldots, \tau_l)$. If the collection $\{\tau_j \in Q_\beta \mid j = 1, \ldots, l\}$ is fixed, we write simply $\mathcal{N}$. For $N = 2L,\ L \in \mathbb{N}$, we introduce

$$\mathcal{K}_N \stackrel{\rm def}{=} \{k = \frac{2\pi}{\beta}\kappa \mid -L+1 \leq \kappa \leq L, \quad \kappa \in \mathbb{Z}\}, \tag{4.10}$$

and

$$\gamma_\beta^{(N)}(d\omega) = \prod_{k \in \mathcal{K}_N} \gamma_k(d\hat{\omega}_k) \prod_{\mathcal{K}\setminus\mathcal{K}_N} \delta(\hat{\omega}_k) d\hat{\omega}_k, \tag{4.11}$$

where δ is the Dirac δ–function.

Proposition 4.1 *The sequence of measures $\{\gamma_\beta^{(N)} \mid N = 2L, L \in \mathbb{N}\}$ weakly converges to the measure (4.5). For every $N = 2L, L \in \mathbb{N}$ and each $x \in \mathcal{H}_\beta$, the functions*

$$F(\omega) \stackrel{\rm def}{=} \exp\left\{-Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta - \int_{I_\beta} V(\omega(\tau))d\tau\right\}, \tag{4.12}$$

$$\tilde{F}(\omega) \stackrel{\rm def}{=} \exp\left\{-Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta - \int_{I_\beta} \tilde{V}(\omega(\tau))d\tau\right\}, \tag{4.13}$$

are $\gamma_\beta^{(N)}$–integrable.

The proof is evident. We denote

$$M_N \stackrel{\rm def}{=} \int_{C(I_\beta)} F d\gamma_\beta^{(N)}, \qquad \tilde{M}_N \stackrel{\rm def}{=} \int_{C(I_\beta)} \tilde{F} d\gamma_\beta^{(N)},$$

and

$$\nu_N^x(d\omega) \stackrel{\rm def}{=} \frac{1}{M_N} F(\omega)\gamma_\beta^{(N)}(d\omega), \tag{4.14}$$

$$\tilde{\nu}_N^x(d\omega) \stackrel{\rm def}{=} \frac{1}{\tilde{M}_N} \tilde{F}(\omega)\gamma_\beta^{(N)}(d\omega), \tag{4.15}$$

The next assertion follows directly from the above one and from the expressions (2.7), (3.10), (3.29), and (3.30).

Proposition 4.2 *Given a choice of* $\tau_1, \dots, \tau_l$, *let* $u \in \mathcal{FC}(\tau_1, \dots, \tau_l)$ *be* γ_β*–integrable. Then for arbitrary* $x \in \mathcal{H}_\beta$ *and every* $N \in \mathcal{N}(\tau_1, \dots, \tau_l)$, *this function is* $\gamma_\beta^{(N)}$*–integrable and the following convergences hold for every* $x \in \mathcal{H}_\beta$

$$\begin{aligned}
\int_{C(I_\beta)} u(\omega)\nu_N^x(d\omega) \quad &\to \quad \int_{C(I_\beta)} u(\omega)\nu^x(d\omega), \\
\int_{C(I_\beta)} u(\omega)\tilde{\nu}_N^x(d\omega) \quad &\to \quad \int_{C(I_\beta)} u(\omega)\tilde{\nu}^x(d\omega).
\end{aligned}$$

Let us consider the integral of the type of

$$\int_{C(I_\beta)} u(\omega)F(\omega)\gamma_\beta^{(N)}(d\omega),$$

with u satisfying the conditions of the latter assertion. Every $\omega(\tau)$ in the arguments of u and F appears as

$$\begin{aligned}
&\text{(i)} \qquad \omega(\tau_j) = \omega(\frac{n_j}{N}\beta), \qquad (4.16)\\
&\text{(ii)} \qquad \int_{I_\beta} x(\tau)\omega(\tau)d\tau, \\
&\text{(iii)} \qquad \int_{I_\beta} \omega^{2r}(\tau)d\tau, \ r \in I\!\!N.
\end{aligned}$$

On the other hand, each such $\omega(\tau)$ has the representation (4.4) in which $\hat{\omega}_k = 0$ for all $k \in \mathcal{K} \setminus \mathcal{K}_N$ because all γ_k with such k are degenerate at zero (see (4.11)). Then

$$\omega(\tau_j) = \sum_{k \in \mathcal{K}_N} \hat{\omega}_k e_k(\frac{n_j}{N}\beta) = \sqrt{\frac{N}{\beta}} \sum_{k \in \mathcal{K}_N} \hat{\omega}_k \epsilon_{k\beta/N}(n_j), \qquad (4.17)$$

where for $q \in \mathcal{Q}_N \stackrel{\rm def}{=} \{q = 2\pi\kappa/N \mid -L+1 \le \kappa \le L, \quad \kappa \in Z\!\!\!Z\}$, and for $n = 0, 1, \dots, N-1$, $\epsilon_q(n)$ are defined as follows

$$\epsilon_q(n) = \sqrt{\frac{2}{N}}\cos qn, \ q > 0, \quad \epsilon_q(n) = -\sqrt{\frac{2}{N}}\sin qn, \ q < 0, \quad \epsilon_0(n) = \frac{1}{\sqrt{N}}. \qquad (4.18)$$

Note that for k varying in $\mathcal{K}_N$, $q = \beta k/N$ runs over the set $\mathcal{Q}_N$.

By means of (4.4), the second expressions in (4.16) may be written in the following form

$$\int_{I_\beta} x(\tau)\omega(\tau)d\tau = \sum_{k\in\mathcal{K}_N} \hat{x}_k\hat{\omega}_k, \;\; \hat{x}_k = \int_{I_\beta} x(\tau)e_k(\tau)d\tau. \tag{4.19}$$

As for the third term in (4.16), it is more convenient to transform it by means of (4.6), (4.7)

$$\begin{aligned}\int_{I_\beta} \omega^{2r}(\tau)d\tau &= \sum_{k_1,\ldots,k_{2r}\in\mathcal{K}_N} \tilde{\omega}_{k_1}\ldots\tilde{\omega}_{k_{2r}}\beta^{-r}\int_{I_\beta}\exp(i(k_1+\ldots+k_{2r})\tau)d\tau \\ &= \sum_{k_1,\ldots,k_{2r}\in\mathcal{K}_N} \tilde{\omega}_{k_1}\ldots\tilde{\omega}_{k_{2r}}\beta^{-r+1}\delta(k_1+\ldots+k_{2r}).\end{aligned} \tag{4.20}$$

Here $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$. Now we introduce

$$\begin{aligned} s_n &= \sqrt{\frac{\beta}{N}}\omega(\frac{n}{N}\beta), \;\; n = 0, 1, \ldots, N-1, \\ \hat{s}_q &= \hat{\omega}_k, \quad \tilde{s}_q = \tilde{\omega}_k \;\; k = \frac{N}{\beta}q, \;\; q \in \mathcal{Q}_N,\end{aligned} \tag{4.21}$$

for which one has

$$s_n = \sum_{q\in\mathcal{Q}_N} \hat{s}_q\epsilon_q(n) = \frac{1}{\sqrt{N}}\sum_{q\in\mathcal{Q}_N}\tilde{s}_q e^{iqn}, \tag{4.22}$$

$$\hat{s}_q = \sum_{n=0}^{N-1} s_n\epsilon_q(n), \;\; \tilde{s}_q = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} s_n e^{-iqn}. \tag{4.23}$$

Then (4.20) may be rewritten as follows

$$\begin{aligned} &\int_{I_\beta} \omega^{2r}(\tau)d\tau \qquad\qquad (4.24) \\ &= \frac{1}{N^r\beta^{r-1}}\sum_{n_1,\ldots,n_{2r}=0}^{N-1} s_{n_1}\ldots s_{n_{2r}} \sum_{k_1,\ldots k_{2r}\in\mathcal{K}_N}\delta(k_1+\ldots+k_{2r}) \\ &\quad \exp(-\frac{i\beta}{N}(k_1n_1+\ldots+k_{2r}n_{2r})) \\ &= \frac{1}{N^r\beta^{r-1}}\sum_{n_1,\ldots,n_{2r}=0}^{N-1} s_{n_1}\ldots s_{n_{2r}} \sum_{\kappa_1,\ldots\kappa_{2r-1}-L+1}^{L}\end{aligned}$$

$$\exp(-\frac{i2\pi}{N}(\kappa_1(n_1-n_{2r}))\dots\exp(-\frac{i2\pi}{N}(\kappa_{2r-1}(n_{2r-1}-n_{2r}))$$

$$= \frac{N^{2r-1}}{N^r\beta^{r-1}}\sum_{n=0}^{N-1}s_n^{2r} = \frac{\beta}{N}\sum_{n=0}^{N-1}\left(\sqrt{\frac{N}{\beta}}s_n\right)^{2r}.$$

Return to (4.19) and rewrite it as follows

$$\int_{I_\beta} x(\tau)\omega(\tau)d\tau = \sum_{k\in\mathcal{K}_N}\hat{x}_k\hat{\omega}_k \tag{4.25}$$

$$= \sum_{n=0}^{N-1}s_nh_n(x), \quad h_n(x) \stackrel{\text{def}}{=} \sum_{k\in\mathcal{K}_N}\hat{x}_k\epsilon_{\beta k/N}(n). \tag{4.26}$$

At last (4.17) now takes the form

$$\omega(\tau_j) = \omega(\frac{n_j}{N}\beta) = \sqrt{\frac{N}{\beta}}s_{n_j}. \tag{4.27}$$

In these new variables $\{s_n \mid n=0,\dots,N-1\}$ the measures $\gamma_\beta^{(N)}$, ν_N^x, and $\tilde{\nu}_N^x$ given by (4.11), (4.14), and (4.15) respectively, may be redefined as the following probability measures on $I\!R^N$:

$$\gamma_\beta^{(N)}(ds) = \frac{1}{C_N}\exp\left\{-\frac{m}{2}(\frac{N}{\beta})^2\sum_{n=0}^{N-1}[s_{n+1}-s_n]^2 - \frac{1}{2}\sum_{n=0}^{N-1}s_n^2\right\}\prod_{n=0}^{N-1}ds_n$$

$$\nu_N^h(ds) = \frac{1}{M_N} \tag{4.28}$$

$$\exp\left\{-\frac{2Jd-a}{2}\sum_{n=0}^{N-1}s_n^2 + J\sum_{n=0}^{N-1}s_nh_n(x) - \sum_{n=0}^{N-1}\frac{\beta}{N}P_{2p}(\sqrt{\frac{N}{\beta}}s_n)\right\}\gamma_\beta^{(N)}(ds),$$

$$\tilde{\nu}_N^h(ds) = \frac{1}{\tilde{M}_N} \tag{4.29}$$

$$\exp\left\{-\frac{2Jd-a}{2}\sum_{n=0}^{N-1}s_n^2 + J\sum_{n=0}^{N-1}s_nh_n(x) - \sum_{n=0}^{N-1}\frac{\beta}{N}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}s_n)\right\}\gamma_\beta^{(N)}(ds).$$

Therewith, Proposition 4.2 may be reformulated as follows.

Proposition 4.3 [Lattice approximation] *Let the collection $\{\tau_1, \dots, \tau_l\}$, $\tau_j \in Q_\beta$, $l \in I\!N$, be fixed. Then for every γ_β–integrable function $u \in \mathcal{FC}(\tau_1, \dots, \tau_l)$, for every $x \in \mathcal{H}_\beta$, and for arbitrary sequence $N \in \mathcal{N}(\tau_1, \dots, \tau_l)$, $N \to \infty$, the following convergences hold*

$$\int_{I\!R^N} u_l(\sqrt{\frac{N}{\beta}} s_{n_1}, \dots, \sqrt{\frac{N}{\beta}} s_{n_l}) \nu_N^h(ds) \quad \longrightarrow \quad \int_{C(I_\beta)} u(\omega) \nu^x(d\omega), \tag{4.30}$$

$$\int_{I\!R^N} u_l(\sqrt{\frac{N}{\beta}} s_{n_1}, \dots, \sqrt{\frac{N}{\beta}} s_{n_l}) \tilde{\nu}_N^h(ds) \quad \longrightarrow \quad \int_{C(I_\beta)} u(\omega) \tilde{\nu}^x(d\omega),$$

where for given $x \in \mathcal{H}_\beta$, $h_n = h_n(x)$ is defined by (4.25).

Our main tools for the proof of the lemmas formulated in the preceding section are given in the following two statements.

Lemma 4.1 [FKG inequality] *For every $x \in \mathcal{H}_\beta$, and for every $\tau, \tau' \in I_\beta$*

$$K_x(\tau, \tau') \geq 0. \tag{4.31}$$

Proof. Let us first choose $\tau, \tau' \in Q_\beta$, so that $\mathcal{FC}(\tau, \tau')$, $\mathcal{N}(\tau, \tau')$, and h_n are defined. Then by Proposition 4.3, the following convergence holds

$$\frac{N}{\beta}(< s_l s_{l'} >_{\nu_N^h} - < s_l >_{\nu_N^h} < s_{l'} >_{\nu_N^h}) \ \longrightarrow \ K_x(\tau, \tau'), \tag{4.32}$$

where $l = (\tau/\beta)N$ and $l' = (\tau'/\beta)N$. But the FKG inequality

$$< s_l s_{l'} >_{\nu_N^h} \ \geq \ < s_l >_{\nu_N^h} < s_{l'} >_{\nu_N^h},$$

holds since ν_N^h , with arbitrary real h_n, is a general ferromagnet (see Theorems VIII.16, VIII.17 p. 280–283, book [26]). Thus the inequality (4.31) holds on the dense subset $Q_\beta \subset I_\beta$. From the continuity of the kernel $K_x(\tau, \tau')$ and (4.32) the proof of (4.31) follows. ■

Lemma 4.2 [GKS inequality] *For every $x \in \mathcal{H}_\beta$, and for every $\tau, \tau' \in I_\beta$*

$$K_x(\tau, \tau') \leq \tilde{K}_0(\tau, \tau'). \tag{4.33}$$

Proof. As in the previous lemma we start with $\tau, \tau' \in Q_\beta$. Then for the kernels K_x, $\tilde{K}_0$, one has the representations (3.21), (3.31). For chosen τ, τ', we construct their lattice approximations

$$
\begin{aligned}
& < s_l s_{l'} >_{\nu_N^h} - < s_l >_{\nu_N^h} < s_{l'} >_{\nu_N^h} \qquad (4.34)\\
= \ & \frac{1}{2}\int\int_{I\!R^{2N}} (s_l - s_{l'})(s'_l - s'_{l'})(\nu_N^h \otimes \nu_N^h)(ds, ds')\\
= \ & \frac{1}{M_N^2}\int\int_{I\!R^{2N}} \frac{s_n - s_{n'}}{\sqrt{2}}\frac{s'_n - s'_{n'}}{\sqrt{2}}\\
& \exp\{-\frac{2Jd-a}{2}\sum_{n=0}^{N-1}(s_n^2 + {s'_n}^2) + J\sum_{n=0}^{N-1}(s_n + s'_n)h_n(x)\\
- \ & \sum_{n=0}^{N-1}\frac{\beta}{N}[P_{2p}(\sqrt{\frac{N}{\beta}}s_n) + P_{2p}(\sqrt{\frac{N}{\beta}}s'_n)]\}(\gamma_\beta^{(N)} \otimes \gamma_\beta^{(N)})(ds, ds'),
\end{aligned}
$$

$$
\begin{aligned}
& < s_l s_{l'} >_{\tilde{\nu}_N^0} - < s_l >_{\tilde{\nu}_N^0} < s_{l'} >_{\tilde{\nu}_N^0} = < s_l s_{l'} >_{\tilde{\nu}_N^0} \qquad (4.35)\\
= \ & \frac{1}{\tilde{M}_N}\int_{I\!R^N} s_l s_{l'} \exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}s_n^2 - \sum_{n=0}^{N-1}\frac{\beta}{N}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}s_n)\}\gamma_\beta^{(N)}(ds),
\end{aligned}
$$

For the polynomials P_{2p} and $\tilde{P}_{2p}$ defined by (2.3), (3.25), one has

$$
P_{2p}(\frac{x+y}{\sqrt{2}}) + P_{2p}(\frac{x-y}{\sqrt{2}}) = \tilde{P}_{2p}(x) + \tilde{P}_{2p}(y) + \Pi_{2p}(x|y), \ x, y \in I\!R, \quad (4.36)
$$

where

$$
\Pi_{2p}(x|y) \ = \ \sum_{k=1}^{p-1}\frac{x^{2k}}{(2k)!}\frac{d^{2k}\tilde{P}_{2p}}{dy^{2k}}(y) \stackrel{\text{def}}{=} \sum_{k=1}^{p-1}\pi_k(y)x^{2k}, \qquad (4.37)
$$

Since $b_k \geq 0$, all coefficients $\pi_k(y)$ are nonnegative for every $y \in I\!R$.

To separate the variables in the integral in (4.34) we use the following orthogonal transformation of $I\!R^N \times I\!R^N$

$$
\begin{aligned}
\xi_n \ &= \ \frac{1}{\sqrt{2}}(s_n - s'_n), \quad \eta_n \ = \ \frac{1}{\sqrt{2}}(s_n + s'_n); \qquad (4.38)\\
s_n \ &= \ \frac{1}{\sqrt{2}}(\xi_n + \eta_n), \quad s'_n \ = \ \frac{1}{\sqrt{2}}(\eta_n - \xi_n).
\end{aligned}
$$

For the Gaussian measure $\gamma_\beta^{(N)} \otimes \gamma_\beta^{(N)}$, the orthogonality implies

$$
\gamma_\beta^{(N)} \otimes \gamma_\beta^{(N)}(ds, ds') = \gamma_\beta^{(N)} \otimes \gamma_\beta^{(N)}(d\xi, d\eta),
$$

which, together with (4.36), yields in (4.34)

$$
\begin{aligned}
& < s_l s_{l'} >_{\nu_N^h} - < s_l >_{\nu_N^h} < s_{l'} >_{\nu_N^h} \qquad (4.39)\\
= \; & \frac{1}{M_N^2} \int\int_{I\!R^{2N}} \xi_l \xi_{l'} \exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}(\xi_n^2+{\eta_n}^2)\\
+ \; & \sqrt{2}J\sum_{n=0}^{N-1}\eta_n h_n(x) - \sum_{n=0}^{N-1} S_{2p}(\xi_n|\eta_n)\\
- \; & \sum_{n=0}^{N-1}\frac{\beta}{N}[\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\xi_n)+\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\eta_n)]\}(\gamma_\beta^{(N)}\otimes\gamma_\beta^{(N)})(d\xi, d\eta),
\end{aligned}
$$

For $\vartheta \in [0,1]$, we set

$$
\begin{aligned}
\Xi_{ll'}(\vartheta) = \; & \frac{1}{Y(\vartheta)}\int_{I\!R^N}\xi_l\xi_{l'}\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}\xi_n^2\\
- \; & \sum_{n=0}^{N-1}\frac{\beta}{N}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\xi_n) - \vartheta\sum_{n=0}^{N-1}\Pi_{2p}(\xi_n|\eta_n)\}\gamma_\beta^{(N)}(d\xi), \qquad (4.40)
\end{aligned}
$$

where

$$
\begin{aligned}
Y(\vartheta) = \; & \int_{I\!R^N}\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}\xi_n^2 - \sum_{n=0}^{N-1}\frac{\beta}{N}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\xi_n)\\
- \; & \vartheta\sum_{n=0}^{N-1}\Pi_{2p}(\xi_n|\eta_n)\}\gamma_\beta^{(N)}(d\xi), \qquad (4.41)
\end{aligned}
$$

and use them in (4.39)

$$
\begin{aligned}
& < s_l s_{l'} >_{\nu_N^h} - < s_l >_{\nu_N^h} < s_{l'} >_{\nu_N^h} \qquad (4.42)\\
= \; & \frac{1}{M_N^2}\int_{I\!R^{2N}}\Xi_{ll'}(1)Y(1)\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}{\eta_n}^2\\
+ \; & \sqrt{2}J\sum_{n=0}^{N-1}\eta_n h_n(x) - \sum_{n=0}^{N-1}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\eta_n)\}\gamma_\beta^{(N)}(d\eta).
\end{aligned}
$$

Both functions $\Xi_{ll'}(\vartheta)$, $Y(\vartheta)$ are continuous on $[0,1]$ and differentiable on $(0,1)$. Furthermore,

$$
\frac{\partial}{\partial\vartheta}\Xi_{ll'}(\vartheta) = -\frac{1}{Y(\vartheta)}[\int_{I\!R^N)}\sum_{n=0}^{N-1}\sum_{k=1}^{p-1}\pi_k(\eta_n)\xi_n^{2k}\xi_l\xi_{l'}
$$

$$\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}\xi_n^2 - \sum_{n=0}^{N-1}\frac{\beta}{N}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\xi_n) - \vartheta\sum_{n=0}^{N-1}\Pi_{2p}(\xi_n|\eta_n)\}\gamma_\beta^{(N)}(d\xi)$$
$$- \quad <\xi_l\xi_{l'}>_\vartheta \int_{I\!R^N}\sum_{n=0}^{N-1}\sum_{k=1}^{p-1}\pi_k(\eta_n)\xi_n^{2k}\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}\xi_n^2$$
$$- \quad \sum_{n=0}^{N-1}\frac{\beta}{N}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\xi_n) - \vartheta\sum_{n=0}^{N-1}\Pi_{2p}(\xi_n|\eta_n)\}\gamma_\beta^{(N)}(d\xi)].$$

In what follows

$$\frac{\partial}{\partial\vartheta}\Xi_{ll'}(\vartheta) = - \frac{1}{Y(\vartheta)}\sum_{n=0}^{N-1}\sum_{k=1}^{p-1}\pi_k(\eta_n)\left(<\xi_n^{2k}\xi_l\xi_{l'}>_\vartheta - <\xi_n^{2k}>_\vartheta<\xi_l\xi_{l'}>_\vartheta\right) \tag{4.43}$$

where the expectations $< . >_\vartheta$ are taken with respect to the measure

$$\rho_\eta^\vartheta(ds) \stackrel{\text{def}}{=} \frac{1}{Y(\vartheta)}\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}\xi_n^2 - \sum_{n=0}^{N-1}\frac{\beta}{N}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\xi_n) - \vartheta\sum_{n=0}^{N-1}\Pi_{2p}(\xi_n|\eta_n)\}\gamma_\beta^{(N)}(d\xi), \tag{4.44}$$

with the Gaussian measure $\gamma_\beta^{(N)}(d\xi)$ given by (4.28). For every $\eta \in I\!R^N$ and $\vartheta \in [0,1]$, the measure $\rho_\eta^\vartheta(ds)$ is an even ferromagnet, thus by Theorem VIII.14 (see pp. 273, 274 of [26]), its moments obey the GKS inequality, in particular

$$<\xi_n^{2k}\xi_l\xi_{l'}>_\vartheta \;\geq\; <\xi_n^{2k}>_\vartheta<\xi_l\xi_{l'}>_\vartheta,$$

which means that the right–hand side of (4.43) is nonpositive, that yields

$$0 \leq \Xi_{ll'}(\vartheta) \leq \Xi_{ll'}(0), \quad \forall\vartheta\in[0,1] \tag{4.45}$$

Now one may employ the latter in (4.42) and obtain

$$\begin{aligned} & <s_ls_{l'}>_{\nu_N^h} - <s_l>_{\nu_N^h}<s_{l'}>_{\nu_N^h} \\ \leq \; & \frac{\Xi_{ll'}(0)}{M_N^2}\int_{I\!R^{2N}}Y(1)\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}{\eta_n}^2 \\ + \; & \sqrt{2}J\sum_{n=0}^{N-1}\eta_nh_n(x) - \sum_{n=0}^{N-1}\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\eta_n)\}\gamma_\beta^{(N)}(d\eta) \end{aligned} \tag{4.46}$$

$$
\begin{aligned}
= & \frac{\Xi_{ll'}(0)}{M_N^2}\int\int_{I\!R^{2N}}\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}(\xi_n^2+{\eta_n}^2)\\
+ & \sqrt{2}J\sum_{n=0}^{N-1}\eta_n h_n(x)-\sum_{n=0}^{N-1}\Pi_{2p}(\xi_n|\eta_n)\\
- & \sum_{n=0}^{N-1}\frac{\beta}{N}[\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\xi_n)+\tilde{P}_{2p}(\sqrt{\frac{N}{\beta}}\eta_n)]\}(\gamma_\beta^{(N)}\otimes\gamma_\beta^{(N)})(d\xi,d\eta).
\end{aligned}
$$

Returning in the latter integral to the initial variables by means of (4.38) one obtains

$$
\begin{aligned}
& < s_l s_{l'} >_{\nu_N^h} - < s_l >_{\nu_N^h} < s_{l'} >_{\nu_N^h} \qquad (4.47)\\
\leq & \frac{\Xi_{ll'}(0)}{M_N^2}\int_{I\!R^{2N}}\exp\{-\frac{1}{2}(2Jd-a)\sum_{n=0}^{N-1}(s_n^2+{s'_n}^2)+J\sum_{n=0}^{N-1}(s_n+s'_n)h_n(x)\\
- & \sum_{n=0}^{N-1}\frac{\beta}{N}[P_{2p}(\sqrt{\frac{N}{\beta}}s_n)+P_{2p}(\sqrt{\frac{N}{\beta}}s'_n)]\}(\gamma_\beta^{(N)}\otimes\gamma_\beta^{(N)})(ds,ds')\\
= & \Xi_{ll'}(0)\int_{I\!R^N}\int_{I\!R^N}\nu_N^h(ds)\nu_N^h(ds')=\Xi_{ll'}(0).
\end{aligned}
$$

Comparing (4.40), (4.44), and (4.29) one gets

$$
\Xi_{ll'}(0) =< \xi_l\xi_{l'} >_0=< \xi_l\xi_{l'} >_{\rho_\eta^0}, \quad \text{and} \quad \rho_\eta^0=\tilde{\nu}_N^0,
$$

that yields in (4.47)

$$
< s_l s_{l'} >_{\nu_N^h} - < s_l >_{\nu_N^h} < s_{l'} >_{\nu_N^h} \ \leq \ < \xi_l\xi_{l'} >_0=< s_l s_{l'} >_{\tilde{\nu}_N^0}, \qquad (4.48)
$$

where (4.35) has also been used. Now passing to the limit $N\to\infty$ one obtains from (4.32) and (4.48) the proof of the stated inequality (4.33) on the dense subset $Q_\beta\subset I_\beta$. To extend the proof to I_β it is enough to employ the continuity of the kernels of K_x and $\tilde{K}_0$. ■

Now we may prove the lemmas stated in the preceding section.

Proof of Lemma 3.1. Applying the FKG inequality (4.31), the GKS inequality (4.33), and the representation of the norm of $\tilde{K}_0$ (3.24) one obtains

$$
\begin{aligned}
& (\varphi,K_x\varphi)_\beta=|(\varphi,K_x\varphi)_\beta|\leq\int_{I_\beta}\int_{I_\beta}K_x(\tau,\tau')|\varphi(\tau)||\varphi(\tau')|d\tau d\tau'\\
\leq & \int_{I_\beta}\int_{I_\beta}\tilde{K}_0(\tau,\tau')|\varphi(\tau)||\varphi(\tau')|d\tau d\tau'\leq\left\|\tilde{K}_0\right\|\||\varphi|\|_\beta^2=\left\|\tilde{K}_0\right\|\|\varphi\|_\beta^2.
\end{aligned}
$$

■

Proof of Lemma 3.2. For the kernels of K_x and $\tilde{K}_0$, one has

$$\mathrm{trace}(K_x) = \int_{I_\beta} K_x(\tau,\tau)d\tau, \quad \mathrm{trace}(\tilde{K}_0) = \int_{I_\beta} \tilde{K}_0(\tau,\tau)d\tau,$$

which gives (3.33) by means of the GKS inequality (4.33). ■

A statement similar to Lemma 3.3 has already been proved earlier (see e.g. Lemma 2.3 of paper [3]), but for the completeness, we repeat it here.

Proof of Lemma 3.3. For the case $x=0$, one has in (3.31)

$$\tilde{K}_0(\tau,\tau') = \int_{\mathcal{H}_\beta} \omega(\tau)\omega(\tau')\tilde{\nu}^0(d\omega). \tag{4.49}$$

The periodicity of the loops $\omega(\tau)$ implies that the kernel of $\tilde{K}_0$ is also periodic

$$\tilde{K}_0(\tau,\tau') = \tilde{K}_0(\tau+\vartheta,\tau'+\vartheta), \quad \forall\vartheta\in[0,\beta] \tag{4.50}$$

where the addition is modulo β. The latter yields in turn

$$\max\mathcal{S}_{pp}(\tilde{K}_0) = \int_{I_\beta} \tilde{K}_0(0,\tau)d\tau = \frac{1}{Z}\int_{I_\beta} \mathrm{trace}\left\{ q e^{-\tau\tilde{H}} q e^{-(\beta-\tau)\tilde{H}} \right\} d\tau, \tag{4.51}$$

where

$$Z = \mathrm{trace}(e^{-\beta\tilde{H}}), \tag{4.52}$$

and the operators $\tilde{H}$, q are defined by (3.27) and (3.28). We recall that the eigenvalues of $\tilde{H}$, $\tilde{E}_l$ are simple. We denote by ψ_l the corresponding eigenfunction and set $q_{ll'} = (\psi_l, q\psi_{l'})_\beta$. For symmetry reasons $q_{ll}=0$, then (4.51) may be written as follows

$$\max\mathcal{S}_{pp}(\tilde{K}_0) = \frac{1}{Z}\sum_{l,l'\in I\!N, l\neq l'} (q_{ll'})^2 \frac{(\tilde{E}_l - \tilde{E}_{l'})(e^{-\beta\tilde{E}_{l'}} - e^{-\beta\tilde{E}_l})}{(\tilde{E}_l - \tilde{E}_{l'})^2},$$

The case of zero denominator is excluded, thus this denominator may be estimated by means of (3.34), which yields

$$\begin{aligned} \max\mathcal{S}_{pp}(\tilde{K}_0) &\leq \frac{1}{Z}\frac{1}{\tilde{\Delta}^2}\sum_{l,l'\in I\!N} (q_{ll'})^2(\tilde{E}_l - \tilde{E}_{l'})(e^{-\beta\tilde{E}_{l'}} - e^{-\beta\tilde{E}_l}) \\ &= \frac{1}{Z}\frac{1}{\tilde{\Delta}^2}\mathrm{trace}[q,[\tilde{H},q]e^{-\beta\tilde{H}} = \frac{1}{m\tilde{\Delta}^2}, \end{aligned}$$

where $[.,.]$ means commutator. ■

Proof of Lemma 3.4. Let us return to the expressions (3.25)–(3.27). For some positive α, consider the unitary operator U_α on $\mathcal{H}_\beta$, which acts as follows

$$(U_\alpha \psi)(x) = \alpha^{1/2}\psi(\alpha x).$$

Then we have on a core for d/dx, resp. q

$$U_\alpha(\frac{d}{dx})U_\alpha^{-1} = \alpha^{-1}(\frac{d}{dx}), \qquad U_\alpha q U_\alpha^{-1} = \alpha q. \tag{4.53}$$

Now we put $\alpha = m^{-1/(2p+2)}$ and conclude that the operators $\tilde{H}$ and $\tilde{H}(m)$

$$\tilde{H}(m) = m^{-\frac{p}{p+1}}T, \quad T \stackrel{\text{def}}{=} T_0 + m^{1/(p+1)}T_1 \tag{4.54}$$

are unitary equivalent. Here

$$T_0 = -\frac{1}{2}\frac{d^2}{dx^2} + 2^{1-p}b_p q^{2p},$$

$$T_1 = \frac{1}{2}(2Jd + 1 - a)m^{\frac{p-2}{p+1}}q^2 + \sum_{l=2}^{p-1} 2^{1-l}m^{\frac{p-l-1}{p+1}}b_l q^{2l}.$$

Let Δ and Δ_0 be defined by the expression (3.34) but with the eigenvalues of the operators T and T_0 respectively. Then

$$\tilde{\Delta} = m^{-\frac{p}{p+1}}\Delta, \tag{4.55}$$

It can be observed (see e.g [18]) that the operator T is a perturbation of T_0, which is analytic at the point $m = 0$. This implies the existence of a certain constant c_0 such that

$$\Delta \geq c_0\Delta_0, \tag{4.56}$$

for sufficiently small positive values of m. These arguments yield the estimate (3.36) with $\chi_0 = 1/(c_0\Delta_0)^2$. ∎

Proof of Lemma 3.5. The estimate of the type of (3.37) has been proved in paper [1] on the base of the logarithmic Sobolev inequality. Its realization in the form of (3.37) is obtained for the choice of $\tilde{V}$ (3.25), (3.26) with $p = 2$. For more details one may see section 6 of [1]. ∎

Proof of Lemma 3.6. Let us return to the definition of the variance of $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$ (3.17). By means of (3.2), (3.3), (3.22), (3.33), (4.49), (4.50) one obtains

$$\begin{aligned}
&\mathrm{Var}_{\nu^x} f && (4.57)\\
&\quad\le \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} \|\omega-\omega'\|_\beta^2 \nu^x(d\omega)\nu^x(d\omega') = \mathrm{trace}(K_x)\\
&\quad\le \mathrm{trace}(\tilde{K}_0) = \int_{\mathcal{H}_\beta} \tilde{K}_0(\tau,\tau)d\tau = \frac{\beta}{Z}\mathrm{trace}(q^2 e^{-\beta\tilde{H}}) \stackrel{\mathrm{def}}{=} \beta < q^2 >,
\end{aligned}$$

where Z is given by (4.52). For $\max\mathcal{S}_{pp}(\tilde{K}_0)$, we have the estimate (3.35). On the other hand, the same quantity may be expressed in terms of the Duhamel two–point function [14] and hence be estimated from below. This estimate comes from Bogolyubov's inequalities, it was obtained in [14] and for our case it reads

$$\beta < q^2 > f\left(\frac{\beta}{4m < q^2 >}\right) \le \max\mathcal{S}_{pp}(\tilde{K}_0), \tag{4.58}$$

where the function f was defined in [14] implicitly by the relation

$$f(x\mathrm{tanh}x) = \frac{1}{x}\mathrm{tanh}x, \quad x \in (0, +\infty),$$

and was estimated there as

$$\frac{1}{x}(1 - e^{-x}) \le f(x). \tag{4.59}$$

Applying this estimate together with (3.35) in (4.58) one gets

$$2m < q^2 > (1 - e^{-\beta/4m<q^2>}) \le \frac{1}{\tilde{\Delta}} \le \frac{m^{\frac{p}{p+1}}}{c_0\Delta_0},$$

where we have used (4.55), (4.56). It follows from the latter that one may find a constant ϕ_0 such that

$$< q^2 >\le \phi_0 m^{-\frac{1}{p+1}},$$

holds for sufficiently small positive values of m. Applying this in (4.57) one obtains (3.38) ■

Acknowledgement Yuri Kozitsky is grateful for kind hospitality extended to him at Bonn and Bochum Universities where his part of this work has mainly been performed. His research was supported in part by the Fundamental Research Fund of Ukraine under the grant 2.4/173 that is also acknowledged.

References

[1] Albeverio, S., Kondratiev, Yu. G., Röckner, M., Tsikalenko, T.V.: *Uniqueness of Gibbs States for Quantum Lattice Systems.* Probab. Theory Relat. Fields **108**, 193–218 (1997)

[2] Albeverio, S., Høegh-Krohn, R.: *Homogeneous Random Fields and Quantum Statistical Mechanics.* J. Funct. Anal. **19**, 242– 272 (1975)

[3] Albeverio, S., Kondratiev, Yu.G., Kozitsky Yu.V.: *Absence of Critical Points for a Class of Quantum Hierarchical Models.* Commun. Math. Phys. **187**, 1–18 (1997)

[4] Albeverio, S., Kondratiev, Yu.G., Kozitsky Yu.V.: *Suppression of Critical Fluctuations by Strong Quantum Effects in Quantum Lattice Systems.* (to appear in Commun. Math. Phys.)

[5] Albeverio, S., Kondratiev, Yu.G., Kozitsky Yu.V.: *Critical Fluctuations and Small Mass Limit for Gibbs States of Quantum Anharmonic Oscillators.* (in preparation)

[6] Albeverio, S., Kondratiev, Yu.G., Minlos, R.A.: *Cluster Expansions for Lattice Models with Spins on Compact Manifolds.* (in preparation)

[7] Albeverio, S., Kondratiev, Yu.G., Minlos, R.A., Rebenko, A.L.: *Small Mass Behaviour of Quantum Gibbs States for Lattice Models with Unbounded Spins.* (to appear in J. Stat. Phys)

[8] Barbulyak, V.S., Kondratiev, Yu.G.: *Functional Integrals and Quantum Lattice Systems: III. Phase Transitions.* Reports Nat. Acad.Sci of Ukraine, No 10, 19–21 (1991)

[9] Berezin, F.A., Shubin, M.A.: *The Schrödinger Equation.* Kluwer Academic Publishers, Dordrecht/Boston/London (1991)

[10] Bruce, A.D., Cowley, R.A.: *Structural Phase Transitions.* Taylor and Francis Ltd (1981)

[11] Dobrushin, R.L.: *Prescribing a System of Random Variables by Conditional Distributions.* Theory Prob. Appl. **15**, 458–486 (1970)

[12] Dobrushin, R.L., Shlosman S.B.: *Constructive Criterion for the Uniqueness of Gibbs field.* In: Statistical Physics and Dynamical Systems. Rigorous Results, pp. 347–370, Birkhaeuser: Basel (1985)

[13] Driesler, W., Landau, L., Perez, J.F.: *Estimates of critical lengths and critical temperatures for classical and quantum lattice systems.* J. Stat. Phys. **20**, 123–162 (1979)

[14] Dyson, F.J., Lieb, E.H., Simon, B.: *Phase transitions in quantum spin systems with isotropic and nonisotropic interactions.* J. Stat. Phys. **18**, 335–383, (1978)

Measures, and

[15] Georgii, H.O.: *Gibbs Measures and Phase Transitions.* vol.9, Walter de Gruyter, Berlin New York: Springer (1988)

[16] Globa, S.A., Kondratiev, Yu.G.: *The construction of Gibbs states of quantum lattice systems.* Selecta Math. Sov. **9**, 297–307 (1990)

[17] Guerra,F., Rosen,L., Simon, B.: *Boundary conditions in the $P(\varphi)_2$ Euclidean field theory.* Ann. Inst. H.Poincaré **15**, 231–334 (1976)

[18] Kato, T.: *Perturbation Theory for Linear Operators.* Berlin–Heidelberg–New York: Springer–Verlag (1966)

[19] Klein, A., Landau, L.: *Stochastic Processes Associated with KMS States.* J. Funct. Anal. **42**, 368–428 (1981)

[20] Kondratiev, Yu.G.: *Phase Transitions in Quantum Models of Ferroelectrics.* In: Stochastic Processes, Physics, and Geometry II, 465–475, World Scientific: Singapore, New Jersey (1994)

[21] Künsch, H.: *Decay of Correlations under Dobrushin's Uniqueness Condition and its Applications.* Commun. Math. Phys. **84**, 207–222 (1982)

[22] Minlos, R.A., Verbeure, A., Zagrebnov, V.A.: *A Quantum Crystal Model in the Light Mass Limit Gibbs State.* Preprint–KUL–TF- -97/16 (1997) (to appear in Reviews in Math. Phys.)

[23] Pastur, L.A., Khoruzhenko, B.A.: *Phase Transitions in Quantum Models of Rotators and Ferroelectrics.* Theoret. Math.Phys. **73**, 111– 124 (1987)

[24] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. I. Functional Analysis.* Academic Press: New York, London (1972)

[25] Schneider, T., Beck, H., Stoll,.E.: *Quantum Effects in an n–component Vector Model for Structural Phase Transition.* Phys.Rev. **B13**, 1123–1130 (1976)

[26] Simon, B.: *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory.* Princeton Univ. Press (1974)

[27] Simon, B.: *Functional Integrals in Quantum Physics.* Academic Press: New York (1986)

[28] Tibbals, J.E., Nelmes, R.J., McIntyre, G.J.:*The Crystal Structure of Tetragonal KH_2PO_4 and KD_2PO_4 as a Function of Temperature and Pressure.* J.Physics **C**: Solid State Phys. **15**, 37–58 (1982)

[29] Verbeure, A., Zagrebnov, V.: *No–Go Theorem for Quantum Structural Phase Transition.* J.Phys.A: Math.Gen. **28**, 5415–5421 (1995)