Multiple measures of maximal entropy and equilibrium states for one-dimensional subshifts

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Abstract

We give simple examples of a one-dimensional mixing subshift with positive topological entropy which have two distinct measures of maximal entropy. We also give examples of subshifts which have two mutually singular equilibrium states for Hölder continuous functions.

1 Introduction

It is a well known fact that a wide class of uniformly hyperbolic transformation that satisfy some irreducibility condition have unique measures of maximal entropy. Irreducible subshifts have unique measures of maximal entropy if they are for instance of finite type or sofic (factors of shifts of finite type). More general hyperbolic systems which satisfy specification have unique measures of maximal entropy [1]. In either case the systems have a 'uniform' mixing property. Hofbauer had shown in [3] that even subshifts of finite type don't necessarily have unique equilibrium states. However the potentials in the provided examples lack (of course) regularity and one of the equilibrium states is supported on a single fixpoint. For a large class of Hölder continuous functions we get here multiple equilibrium states which are supported on embedded subshifts.

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In this note we shall produce a simple example which shows that even in the case of an irreducible subshift with positive entropy one cannot generally expect to have a unique measure of maximal entropy. (It was pointed out that Krieger [4] gave an example with a non-unique measure of maximal entropy—although not as simple and straighforward as this one.) For two-dimensional subshifts of finite type, Burton and Steiff [2] have recently given examples whose measures of maximal entropy form simplices that consist of more than single points. Hence:

Example 1 *There exists a subshift Σ over a finite alphabet with a topologically mixing shift transformation $\sigma : \Sigma \to \Sigma$ such that Σ carries at least two mutually singular measures of maximal entropy.*

An example is provided in definition 2 of the following section. Lemmas 1, 2 and 3 serve to prove the properties claimed above. Extending this example we are then able to produce in section 3 multiple equilibrium states for Hölder continuous functions (see section 3 for Hölder continuity).

Example 2 *There exists a topologically mixing subshift Σ over a finite alphabet and a Hölder continuous function $g : \Sigma \to \mathbf{R}$ such that Σ carries at least two mutually singular equilibirium states for g.*

2 Example

We shall construct a subshift over the symbolset $\mathcal{A} = \{0, 1, 2, 3, 4\}$ and shall refer to the symbols $\{1, 2\}$ as the yellow symbols and denote them by $\mathcal{Y}$. Similarly we call $\{3, 4\}$ the green symbols and write $\mathcal{G}$ for them. Hence $\mathcal{A} = \mathcal{Y} \cup \mathcal{G} \cup \{0\}$. Let us denote by $\Sigma_{\mathcal{Y}} = \mathcal{Y}^{\mathbf{Z}}$ the (yellow) full two-shift over the alphabet $\mathcal{Y}$. Similarly we write $\Sigma_{\mathcal{G}}$ for the (green) full two-shift $\mathcal{G}^{\mathbf{Z}}$.

Let $\tau > 0$. The subshift $\Sigma \subset \mathcal{A}^{\mathbf{Z}}$ will consist of the union of the two monochromatic shift spaces $\Sigma_{\mathcal{Y}}$ and $\Sigma_{\mathcal{G}}$, to which we add bicoloured sequences in the following way. A sequence in Σ in which a word α of one colour (green or yellow) is followed by a word β of the other colour (yellow or green) will have a string of zeroes γ separating the two words α and β. The length $|\gamma|$ of that string of zeros is then at least, $\tau(a + b)$, where $a = |\alpha|, b = |\beta|$ are the lengths of the coloured words to the 'left' and 'right' of γ.

More precisely:

Definition 3 *The shift space $\Sigma \subset \mathcal{A}^{\mathbf{Z}}$ is defined as follows: We have $x \in \Sigma$ if*
(i) either $x \in \Sigma_{\mathcal{Y}} \cup \Sigma_{\mathcal{G}}$, in which case we call x a monochromatic *point.*
(ii) or if bi-coloured blocks are of the form:

$$\ldots 0 y_1 y_2 \ldots y_{a-1} y_a 0^{\lambda} g_1 g_2 \ldots g_{b-s} g_b 0 \ldots$$

or

$$\ldots 0 g_1 g_2 \ldots g_{a-1} g_a 0^{\lambda} y_1 y_2 \ldots y_{b-s} y_b 0 \ldots$$

where $\lambda \geq \tau(a+b)$ and $y_i \in \mathcal{Y}, g_i \in \mathcal{G}$ (0^{λ} is a string of zeros of length λ), $\tau > 0$.

If $\omega = \omega_m \omega_{m+1} \ldots \omega_{m+n-1}$ is an allowed word of some length n in Σ, then we denote by $U(\omega)$ the *cylinder set* $\{x \in \Sigma : x_i = \omega_i, m \leq i < m+n\}$. The shift transformation σ on Σ is defined in the usual way by $\sigma(x)_i = s_{i+1}, i \in \mathbf{Z}$, for $x \in \Sigma$. Notice that the transition rules are shift invariant. Thus Σ is indeed a subshift which we endow with the usual topology generated by the cylinder sets $U(\omega)$, where ω runs over all finite words in Σ.

Notice that a sequence in Σ cannot end in an infinite one-sided string of one colour unless the sequence already belongs to the appropriate monochromatic subshift.

If $\tau = 0$ then Σ is a sofic system in which every string of zeroes connects to coloured symbols of different colours, and whose topological entropy is $\log 4$ (see lemma 2).

Lemma 1 *The shift transformation σ on Σ is topologically mixing for every τ.*

Proof. We have to show that for any two finite words ω and η there exists a number N so that $U(\omega) \cap \sigma^n(U(\eta))$ is non-empty for all $n \geq N$. This is evidently true if ω and η are both monochromatic of the same colour. If the last symbol of η and the first symbol of ω are of different colour then consider the word $\pi = \eta 0^{\lambda} \omega$. Clearly, π is an admissable word for every $\lambda > \tau(|\eta| + |\omega|)$, and the cylinder set $U(\pi) \subset U(\omega) \cap \sigma^n(U(\eta))$ is non-empty for all $n \geq N$, where N can then be chosen to be equal to $(1+\tau)(|\eta| + |\omega|)$.

If the last symbol of η and the first symbol of ω are of the same colour, then let us consider the word $\pi = \eta 0^{\kappa} \varepsilon 0^{\lambda} \omega$, where ε is a symbol (or some other short word) of the other colour. Here we have $\kappa \geq \tau(|\eta| + |\varepsilon|)$ and $\lambda \geq \tau(|\omega| + |\varepsilon|)$. Clearly $U(\pi)$ is non-empty for those choices of λ and κ. Thus $U(\omega) \cap \sigma^n(U(\eta)) \neq \emptyset$ if $n \geq N = (\tau+1)(|\eta| + 2|\varepsilon| + |\omega|)$. $\square$

Lemma 2 *The topological entropy h of the shift $\sigma : \Sigma \to \Sigma$ is* $\max(\log 2, \frac{\log 4}{1+\tau})$.

Proof. Let us first get a lower bound on the topological entropy h of Σ. For that purpose let us observe that since Σ contains two full two shifts, $\Sigma_{\mathcal{Y}}$ and $\Sigma_{\mathcal{G}}$, the topological entropy of Σ must be at least $\log 2$.

Let us now estimate the number of the remaining words of length n from above. We have two cases to consider:
(i) Monochromatic words that might also contain zeroes. According to our rules such words begin or end with strings of zeroes. Thus we obtain purely yellow and green words of lengths $k = 0, \ldots, n$ which on at least one side is framed by strings of zeroes. Their number turns out to be $2\sum_{k=0}^{n} 2^k = 2^{n+1}\sum_{k=0}^{n} 2^{-k}$.
(ii) To estimate the number of words of length n that genuinely contain symbols of both colours, let us observe that since any such word has at least one transition from yellow to green or vice versa, it therefore must also contain at least $n\frac{\tau}{\tau+1}$ zeroes, that is at most $n' = [\frac{n}{\tau+1}]$ ([ ] denotes integer part) coloured symbols (i.e. symbols in $\{1,2,3,4\}$). The coloured symbols come in monochromatic blocks of alternating colour. Denote by $P_{k,\ell}$ the number of possibilities in which one can arrange ℓ symbols in k blocks (separated by the appropriate number of zeroes), where $k = 1, 2, \ldots, \ell$. One finds that

$$P_{k,\ell} = \begin{pmatrix} \ell - 1 \\ k - 1 \end{pmatrix},$$

which is the number of possibilities of picking the first element of every block but the very first one. The number of n-words in Σ which contain ℓ coloured symbols arranged in $k \leq \ell$ monochromatic blocks of alternating colour is $2P_{k,\ell}2^\ell$.

If Q_k denotes the number of n-words in Σ with at most n' coloured symbols in k monochromatic blocks of alternating colour, then

$$Q_k = \sum_{\ell=k}^{n'} 2P_{k,\ell}2^\ell.$$

The number of words of lengths n that contain symbols of both colours is now determined by

$$\sum_{k=2}^{n'} Q_k = 2\sum_{\ell=2}^{n'}\sum_{k=2}^{\ell}\left(\begin{array}{c}\ell-1\\ k-1\end{array}\right)2^\ell = 2\sum_{\ell=2}^{n'}(2^{\ell-1}-1)2^\ell = r_n 2^{2n'}, \qquad (1)$$

where the constants r_n and their reciprocals $1/r_n$ are subexponential as n goes to infinity.

Thus the total number of words of lengths n (large enough) in the subshift Σ has growth rate

$$h = \lim_{n\to\infty}\frac{1}{n}\log(2^n + 2^{2n'}),$$

which implies that Σ has topological entropy $h = \max(\log 2, \frac{\log 4}{\tau+1})$. In particular Σ has topological entropy $\log 2$ if $\tau \geq 1$. □

Lemma 3 *If $\tau \geq 1$, then there are two mutually singular measures of maximal entropy on Σ.*

Proof. It is well known [5] that the measure of maximal entropy on the full two-shift $\{1,2\}^{\mathbf{Z}}$ is the Bernoulli measure with the probability vector $(\frac{1}{2},\frac{1}{2})$. It's metric entropy is $\log 2$. Let us define on Σ a Bernoulli measure $\mu_{\mathcal{Y}}$ which is given by the probability vector $(0,\frac{1}{2},\frac{1}{2},0,0)$ and the measure $\mu_{\mathcal{G}}$ with the probabilities $(0,0,0,\frac{1}{2},\frac{1}{2})$. Evidently both measures are shift invariant and have metric entropies $\log 2$ which by lemma 2 is the topological entropy of Σ. Hence $\mu_{\mathcal{Y}}$ and $\mu_{\mathcal{G}}$ (as well as all their linear combinations) are (distinct) measures of maximal entropy for the subshift Σ. □

In the case when $0 < \tau < 1$, then there is a unique measure of maximal entropy which is given by the probability vector $(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ whose entropy is by lemma 2 equal to $\frac{\log 4}{\tau+1}$.

3 Equilibrium states

Let us construct now subshifts with non unique equilibrium states. We say a function f is Hölder continuous on a shift space if there exists a $\vartheta \in (0,1)$ and a constant C so that for every $n > 0$ one has

$$\sup|f(\vec{x}) - f(\vec{y})| \leq C\vartheta^n$$

where the supremum is over all pairs of sequences $\vec{x} = \cdots x_{-1}x_0x_1x_2\ldots, \vec{y} = \cdots y_{-1}y_0x_1y_2\ldots$ in the shiftspace for which $x_{-n}\cdots x_n = y_{-n}\cdots y_n$ and either $x_{-n} \neq y_{-n}$ or $x_n \neq y_n$.

Let f be a real valued Hölder continuous function on the full two-shift Σ_* (where $* = \mathcal{G}, \mathcal{Y}$) so that $P_*(f) > \sup f$, where $P_*(f)$ is the pressure of f (see [5]).

Let us define a Hölder continuous function g on Σ by putting $g(x) = f(x)$ if x lies in the full two-shifts $\Sigma_{\mathcal{G}}$ or $\Sigma_{\mathcal{Y}}$. We extend g to bi-coloured words in the following way: If $x_0 = 0$ then we put $g(x) = \sup f$. If $x_0 \neq 0$ and $x \in \Sigma$ is a sequence which contains symbols of both colours then let $x_{-n}\cdots x_n$ be the longest monochromatic string of either green or yellow symbols (that is either $x_{-n} = 0$ or $x_n = 0$) and pick a monochromatic point $y \in \Sigma_*$ $(* = \mathcal{G}, \mathcal{Y})$ in the cylinder $U(x_{-n}\cdots x_n)$. Now define $g(x) = f(y)$. In this way g is a Hölder continuous function on Σ. Let us now prove that the pressure of g (on Σ) is in fact equal to $P_*(f)$ (on the two-symbol shiftspace) for a suitable choice of τ.

Lemma 4 *If $\tau \geq \frac{\log 4}{P_*(f) - \sup f} - 1$, then the pressure $P(g)$ of g on Σ is equal to $P_*(f)$.*

Proof. Denote by W_n the collection of Σ-words $\alpha = \alpha_1\cdots\alpha_n$ of lengths n. The pressure of g is then given by the exponential growth rate of the partition function

$$Z_n = \sum_{\alpha \in W_n} e^{g^n(x(\alpha))},$$

where $x(\alpha)$ is an arbitrary point in the cylinder set $U(\alpha) \subset \Sigma$, and $g^n = g + g\sigma + g\sigma^2 + \cdots + g\sigma^{n-1}$ is the n-th ergodic sum of g.

If in the above sum we restrict to monochromatic words α of either colour, then we obtain the green or yellow partition function Z_n^* $(* = \mathcal{G}, \mathcal{Y})$ whose growth rates are exactly $P_*(f)$.

Let us note that the growth rate of $\sum_{k=1}^n Z_k^*$ is also $P_*(f)$. Hence, if we denote by W_n' those words in W_n that contain symbols of exactly one colour and otherwise at least one zero, then it follows that growth rate of $Z_n = \sum_{\alpha \in W_n'} e^{g^n(x(\alpha))}$ is also given by $P_*(f)$.

It thus remains to show that the exponential growth rate of $\sum_{\alpha \in W_n''} e^{g^n(x(\alpha))}$ is $\leq P_*(f)$, where W_n'' denotes the genuinely bi-coloured words in W_n (i.e.

those words that contain green and yellow symbols). In lemma 2 equation (1) we estimated

$$|W''_n| \leq r_n 4^{n/(\tau+1)},$$

where the constants r_n grow at most subexponentially as $n \to \infty$. Since $\sup g = \sup f$, we obtain

$$Z''_n = \sum_{\alpha \in W''_n} e^{g^n(x(\alpha))} \leq |W''_n| e^{n \sup g} \leq r_n 4^{n/(\tau+1)} e^{n \sup f},$$

whose growth rate is bounded by $\frac{\log 4}{\tau+1} + \sup f$, which is $\leq P_*(f)$ exacty if $\tau + 1 \geq \frac{\log 4}{P_*(f) - \sup f}$. □

Lemma 5 *If $\tau \geq \frac{\log 4}{P_*(f) - \sup f} - 1$, then there are two mutually singular equilibrium states for g on Σ.*

Proof. There is an equilibrium state for f on each of the full two-shifts $\Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{Y}}$ both of which, by lemma 4, assume the pressure $P(g) = P_*(f)$ in the variational principle. These two measures are both invariant and mutually singular. □

Remark: This method of constructing examples can be varied and adapted in many ways. Instead of full two-shifts one can of course take any subshift of finite type (or other), and one can have any number of those embedded subshifts yielding a corresponding number of mutually singular measures of maximal entropy or equilibria for suitable potentials. In fact it is possible to have infinitely many embedded subshifts each of which would carry a measure and all of them would be mutually singular. This can be done without sacrificing the mixing property; there still would exist transitive points.

References

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