Absence of Singular Spectrum for a Perturbation of a Two-dimensional Laplace-Beltrami Operator with Periodic Electro-Magnetic Potential

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Abstract

Let Γ be a lattice on $\mathbf{R}^2$. We consider a metric g, a one-form A and a real function V on $\mathbf{R}^2$, all Γ periodic. We prove that the spectrum of the Schrödinger operator on $L^2(\mathbf{R}^2)$, $u \mapsto P_g(D-A)u + Vu = (id + A)^\star(idu + uA) + Vu$, is absolutely continuous.

The same result holds with the related generalized Dirac operator.

1 Introduction

Let Γ be a lattice on $\mathbf{R}^2$, and $g = (g_{jk})$ be a C^∞, Γ-periodic metric on $\mathbf{R}^2$,

$$g_{x-a} = g_x, \ \forall a \in \Gamma, \ g_{jj} > 0 \text{ and } \mid g \mid = det(g_{jk}) > 0. \tag{1.1}$$

We consider a real C^∞, Γ-periodic one-form A, (a magnetic potential),

$$A = A_1(x)dx_1 + A_2(x)dx_2, \quad A_j(x-a) = A_j(x), \quad \forall a \in \Gamma, \tag{1.2}$$

and a Γ-periodic electrical potential

$$V: \ \mathbf{R}^2 \ \mapsto \ \mathbf{R}, \ \ V \ \in \ L^\infty(\mathbf{R}^2), \ \ V(x-a) = V(x) \ \ \forall \ a \in \Gamma. \tag{1.3}$$

Hence the Schrödinger operator $P_g(D-A) + V = (id + A)^\star(id + A) + V$,

$$P_g(D-A) = \sum_{1 \le j,k \le 2} \mid g_x \mid^{-1/2} (D_{x_j} - A_j(x)) g^{jk}(x) \mid g_x \mid^{1/2} (D_{x_k} - A_k(x)), \tag{1.4}$$

is self-adjoint on $L^2_g(\mathbf{R}^2) = L^2(\mathbf{R}^2; \mid g \mid^{1/2} dx)$, $(dx = dx_1 \wedge dx_2)$,
with domain the Sobolev space of order two, $\mathcal{D}(P_g(D-A) + V) = H^2(\mathbf{R}^2)$.
$(D_x := -i\frac{\partial}{\partial x} = (D_{x_1}, D_{x_2}), \ \ D_{x_j} = -i\partial_{x_j} = -i\frac{\partial}{\partial x_j})$.

We will identify the magnetic field $\widetilde{\mathrm{B}} = \mathrm{dA}$ with the real function

$$B(x) = \frac{\partial}{\partial x_1} A_2(x) - \frac{\partial}{\partial x_2} A_1(x), \quad (\widetilde{\mathrm{B}} = \mathrm{B(x)dx_1 \wedge dx_2}). \tag{1.5}$$

E.L. Green points out in [G] some spectral differences between $P_g(D) + V$ and $P_{g_0}(D) + V$, with $g_0 = (\delta_{jk})$ the flat metric. Among other things, one can find in [G] examples of conformal metric $g = c_N g_0$ with more than N gaps in the spectrum of $P_{c_N g_0}(D) + V$.

Other such examples can be constructed easily for general metric and with particular large magnetic field, using the localisation of the spectrum of $P_g(D - \lambda A) + V$, as an operator on $L^2_g(\mathbf{T}^2) = L^2(\mathbf{T}^2, \mid g \mid^{1/2} dx)$, for large constant λ, established in [H-M-1] .

We let $\mathbf{T}^2$ be the $\Gamma-$related torus, $\mathbf{T}^2 := \mathbf{R}^2/\Gamma$, equipped with the Lebesgue measure associated to the metric g, $\mid g \mid^{1/2} dx$.

Let us remark that $P_{g_0}(D-A)+V$ has a finite number of gaps in its spectrum, see [Mh] (see also [H-M-2] for the case without magnetic field). A spectral question is important. Are there any eigenvalues for $P_g(D-A)+V$? For the flat metric the answer is no. It is well known when $A=0$, see [R-S], and recently proved for $P_{g_0}(D-A)+V$ in [B-S], (see also [S] for higher dimensions).

Theorem 1.1 : *Under the above assumptions, the spectrum of* $P_g(D-A)+V$ *is absolutely continuous.*

It comes from this theorem that $P_g(D-A)+V$ has no eigenvalues and its spectrum is formed by intervals, (with positive lengths).

The strategy of our proof is the Thomas one [T]. It differs from the usual one by requiring gauge invariance for the analytic extension and by using Schrödinger operator as a perturbation of square of Dirac operator. The difficulty does not come from the magnetic field, because of the dimension, but from the metric. We hope that our method can be applied for higher dimensions.

Now then we will establish, in chapter 4, the absolute continuity of the spectrum of the related generalized Dirac operator and in the three-dimensional case for some particular metric and magnetic field.

Let us remark that later on, we may take $\Gamma = \mathbf{Z^2}$ if we modify g consequently, using the invariance of the spectrum by the action of the linear group $GL(2;\mathbf{R})$.

2 Some basic facts from Floquet theory

Let $\Gamma^\star$ be the dual lattice, $\Gamma^\star = \{\gamma \in \mathbf{R}^2;\ \gamma a \in 2\pi\mathbf{Z},\ \forall a \in \Gamma\}$.

$\mathbf{K}^\star$ will denote a basic period cell of the dual lattice, i.e. when $\Gamma = \mathbf{Z^2}$, $\mathbf{K}^\star = ([\mathbf{0}, \mathbf{2}\pi[)^\mathbf{2}$.

$\mathbf{K}^\star$ will be equipped with the normalized Lebesgue measure $d\widetilde{\theta} = \frac{d\theta}{|\mathbf{K}^\star|}$.

¿From now on, we will identify every $\Theta = (\Theta_1, \Theta_2) \in \mathbf{C}^2$ with the closed one-form $\Theta_1 dx_1 + \Theta_2 dx_2$.

We recall that the Floquet theory, see [R-S] or [K], is valid for $P_g(D-A)+V$, so there exists a unitary operator U, $U : L^2_g(\mathbf{K}^\star \times \mathbf{T}^2) \mapsto L^2_g(\mathbf{R}^2)$, such that

$$U^{-1}(P_g(D-A)+V)U = P(D-A-\theta)+V = \int_{\mathbf{K}^\star}^{\oplus} P^\theta_{g,\mathbf{T}^2} d\widetilde{\theta}, \tag{2.1}$$

$$P(D-A-\theta) = \sum_{1\le j,k\le 2} \mid g(x) \mid^{-1/2} (D_{x_j} - \theta_j - A_j(x)) g^{jk}(x) \mid g(x) \mid^{1/2} (D_{x_k} - \theta_k - A_k(x))$$

is to be considered as a self-adjoint operator on $L^2(\mathbf{K}^\star; L^2_g(\mathbf{T}^2))$, with domain $L^2(\mathbf{K}^\star; H^2(\mathbf{T}^2))$.

For any fixed $\theta \in \mathbf{R^2}$, $\mathbf{P}^\theta_{\mathbf{g},\mathbf{T^2}}$ is the self-adjoint operator on $L^2_g(\mathbf{T}^2)$ defined by $P(D-A-\theta)+V$, (with domain $H^2(\mathbf{T}^2)$).

For any fixed $\Theta \in \mathbf{C}^2$, we will define in the same way the operator $P^\Theta_{g,\mathbf{T}^2}$, with the same domain. As $P^\Theta_{g,\mathbf{T}^2}$ is a perturbation of the Laplace-Beltrami operator, its spectrum is discrete, (formed by eigenvalues with finite multiplicity and with no accumulation point).

Later on we will adopt the notation

$$P_{g,\mathbf{T}^2}(D-A-\Theta) = P^{\Theta}_{g,\mathbf{T}^2} - V. \tag{2.2}$$

$P_{g,\mathbf{T}^2}(D-A-\Theta)u(x) = P(D-A-\Theta)u(x), \quad \forall u \in H^2(\mathbf{T}^2)$.

As for the case of the flat metric considered in [R-S], see Theorem 4.1.5 of [K] for general elliptic and periodic self-adjoint differential operator, it comes from Floquet theory the following theorem.

Theorem 2.1 *For any real open interval* $(a\ ,\ b) \subset \mathbf{R}, \quad (a < b < +\infty)$, *we have the equivalence*

$$(a\ ,\ b) \cap sp(P_g(D-A)+V) = (a\ ,\ b) \cap sp_{ac}((P_g(D-A)+V) \tag{2.3}$$
$$\Longleftrightarrow \ (a\ ,\ b) \cap sp_p((P_g(D-A)+V) = \emptyset.$$

Moreover

$$\mu \in sp_p((P_g(D-A)+V) \Longleftrightarrow \mu \in sp_d(P_{g,\mathbf{T}^2}(D-A-\theta)+V), \quad \forall \theta \in \mathbf{R}^2 \tag{2.4}$$
$$\Longleftrightarrow \mu \in sp_d(P_{g,\mathbf{T}^2}(D-A-\Theta)+V), \quad \forall \Theta \in \mathbf{C}^2.$$

For an operator T, $sp(T)$, $sp_d(T)$, $sp_p(T)$ and $sp_{ac}(T)$ denote the spectrum, the discrete spectrum, the point spectrum (eigenvalues), and the absolutely continuous spectrum of T.

We recall that the spectrum of $P_g(D-A)+V$ is gauge invariant, the same is true for $P_{g,\mathbf{T}^2}(D-A)+V$,

$$sp(P_g(D-A)+V) = sp(P_g(D-A-d\varphi)+V), \quad \forall\ \varphi(x) \in C^2(\mathbf{R^2};\mathbf{R}) \tag{2.5}$$

and $sp(P_{g,\mathbf{T}^2}(D-A)+V) = sp(P_{g,\mathbf{T}^2}(D-A-d\varphi)+V), \quad \forall\ \varphi(x) \in C^2(\mathbf{T}^2;\mathbf{R})$.

We will use the Corollary below.

Corollary 2.2 *Let* $\varphi(x) \in C^2(\mathbf{T}^2;\mathbf{R})$ *and* $\omega \in \mathbf{S}^1$ *be given.*

Then $\mu \in sp_p((P_g(D-A)+V)$ *iff*

$$\mu \in sp_d(P_{g,\mathbf{T}^2}(D-A-\theta-z(d\varphi+\omega))+V), \quad \forall (z,\theta) \in \mathbf{C} \times \mathbf{R^2}. \tag{2.6}$$

This comes from (2.4) when $\varphi = 0$ and the fact that $P_{g,\mathbf{T}^2}(D-A-\Theta)$ and $P_{g,\mathbf{T}^2}(D-A-\Theta-d\psi)$ have the same eigenvalues, for any $\Theta \in \mathbf{C}$ and any $\psi \in C^2(\mathbf{T}^2;\mathbf{C})$.

3 Proof of Theorem 1.1

The proof of Theorem 1.1 comes from the assumption $V \in L^\infty(\mathbf{T}^2)$, (2.6) of Corollary (2.2) and the Theorem below.

Theorem 3.1 *For any* $\omega \in \mathbf{S}^1$, *the unit sphere of* $\mathbf{R^2}$, *there exist* $\theta_\omega \in \mathbf{R^2}$, $\mathbf{c}_\omega > \mathbf{0}$, *an integer* $m = m(\omega) \in \mathbf{N}$ *and a sequence of non negative real numbers* (λ_k), *with* $\lim\limits_{k \mapsto \infty} \lambda_k = +\infty$, *such that for any* k,

$$\| P_{g,\mathbf{T}^2}(D-A-\theta_\omega - i\lambda_k G(\omega))u \| \geq \lambda_k^{\frac{1}{m+1}} c_\omega \| u \|, \quad \forall u \in C^\infty(\mathbf{T}^2). \tag{3.1}$$

$$G(\omega) = d(\omega x + \psi_\omega), \ \text{with}\ \psi_\omega \in C^\infty(\mathbf{T}^2;\mathbf{R}), \ \int_{\mathbf{T^2}} \psi_\omega \mathbf{dx} = \mathbf{0}, \ \text{s.t.}\ \mathbf{\Delta_g}(\omega\mathbf{x}+\psi_\omega) = \mathbf{0}. \tag{3.2}$$

($\| \ . \ \|$ denotes the L^2_g-norm on $\mathbf{T}^2$, and $\Delta_g = -P_g(D)$ is the g-Laplace-Beltrami operator and as $\Delta_g(\omega x)$ is always a periodic function orthogonal to the constants in $L^2_g(\mathbf{T}^2)$, the function ψ_ω exists and is unique).

Remark 3.2 *We have just to prove (3.1) for the special conformal metric* $\mid g \mid^{-1/2} g$.

If $\widetilde{g} =\mid g \mid^{-1/2} g$, then $g^{jk} \mid g \mid^{1/2} = \widetilde{g}^{jk}$ and $\mid \widetilde{g} \mid = 1$.
The estimate (3.1) becomes

$$\int_{\mathbf{T}^2} \mid g \mid^{-1/2} \mid P_{\widetilde{g},\mathbf{T}^2}(D - A - \theta_\omega - i\lambda_k G(\omega))u \mid^2 dx \geq \lambda_k^{\frac{2}{m+1}} c_\omega^2 \int_{\mathbf{T}^2} \mid g \mid^{1/2} \mid u \mid^2 dx.$$

As $\mid g \mid$ and $\mid g \mid^{-1}$ are bounded, the estimate is equivalent to

$$\int_{\mathbf{T}^2} \mid P_{\widetilde{g},\mathbf{T}^2}(D - A - \theta_\omega - i\lambda_k G(\omega))u \mid^2 dx \geq \lambda_k^{\frac{2}{m+1}} \widetilde{c}_\omega^2 \int_{\mathbf{T}^2} \mid u \mid^2 dx.$$

So from now on we will assume $\mid g \mid = 1$ and, due to (2.5) we will work in the particular gauge $div(A^g) = 0$, if A^g is the vector field associated to the one-form A by the metric g, $(g(A^g, \ .) = A)$,

$$\mid g \mid = 1, \quad A = J^\star(d\phi) = -\sum_k g^{2k} \partial_{x_k}\phi(x)dx_1 \ + \ \sum_k g^{1k}\partial_{x_k}\phi(x)dx_2, \tag{3.3}$$

where $J^\star$ is the natural involution on $T^\star(\mathbf{T}^2)$ and $\phi(x)$ is the unique periodic function satisfying, (we can choose a gauge A such that $\int_{\mathbf{T}^2} A^g \wedge \theta dx = 0, \ \forall \theta \in \mathbf{R}^2$),

$$\Delta_g \phi(x) = B(x) \ \text{ and } \ \int_{\mathbf{T}^2} \phi(x)dx = 0. \tag{3.4}$$

For the proof of Theorem 3.1, let us introduce the Pauli operators on $(L^2(\mathbf{T}^2))^2$.
For any complex one-form $N = \sum_j N_j(x)dx_j, \ N_j \in C^\infty(\mathbf{T}^2; \mathbf{C})$, we let

$$M^\mp(D - N) = \sum_k h^{1k}(D_{x_k} - N_k(x)) \ \mp \ i\sum_k h^{2k}(D_{x_k} - N_k(x)), \tag{3.5}$$

and

$$\widetilde{\mathrm{M}}^\mp(\mathrm{D} - \mathrm{N}) = \sum_{\mathrm{k}}(\mathrm{D}_{\mathrm{x_k}} - \mathrm{N_k}(\mathrm{x}))\mathrm{h}^{1\mathrm{k}} \ \mp \ \mathrm{i}\sum_{\mathrm{k}}(\mathrm{D}_{\mathrm{x_k}} - \mathrm{N_k}(\mathrm{x}))\mathrm{h}^{2\mathrm{k}}. \tag{3.6}$$

The matrix (h^{jk}) is the square root of (g^{jk}) :

$$h^{jj} > 0, \quad h^{jk} = h^{kj}, \ \ \sum_q h^{jq}h^{qk} = g^{jk}.$$

$((M^\mp(D - N))^\star = \widetilde{\mathrm{M}}^\pm(\mathrm{D} - \bar{\mathrm{N}}))$.
With the choice of A in (3.3) we get, $\forall \Theta \in \mathbf{C^2}$,

$$e^{\mp\phi}M^\mp(D - A - \Theta)e^{\pm\phi} = M^\mp(D - \Theta), \quad e^{\mp\phi}\widetilde{\mathrm{M}}^\mp(\mathrm{D} - \mathrm{A} - \Theta)\mathrm{e}^{\pm\phi} = \widetilde{\mathrm{M}}^\mp(\mathrm{D} - \Theta). \tag{3.7}$$

(We used that, if h is the matrix $h = (h_{jk})$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ the Pauli matrix, then $h^{-1}\sigma_2 = \sigma_2 h$).

Let us define the two Pauli operators

$$\mathcal{D}(D-N)=\begin{pmatrix}0 & M^-(D-N)\\ M^+(D-N) & 0\end{pmatrix}, \tag{3.8}$$

$$\widetilde{\mathcal{D}}(D-N)=\begin{pmatrix}0 & \widetilde{\mathrm{M}}^-(\mathrm{D}-\mathrm{N})\\ \widetilde{\mathrm{M}}^+(\mathrm{D}-\mathrm{N}) & 0\end{pmatrix}. \tag{3.9}$$

If $dN=B_N dx_1\wedge dx_2$, then

$$\widetilde{\mathcal{D}}(D-N)\mathcal{D}(D-N)=\begin{pmatrix}P(D-N)-B_N & 0\\ 0 & P(D-N)+B_N\end{pmatrix} \tag{3.10}$$

and $\widetilde{\mathcal{D}}(D-\bar{N})=(\mathcal{D}(D-N)^\star$.

Lemma 3.3 *If $\mu_1(\lambda,\omega,\theta)$ is the first eigenvalue of $P_{g,\mathbf{T}^2}(D-A-\theta)+\lambda^2\mid G(\omega)\mid^2_{g_x}$, then*

$$\parallel \mathcal{D}(D-A-\theta-i\lambda G(\omega))\mathcal{I}(u)\parallel\geq(2\mu_1(\lambda,\omega,\theta))^{1/2}\parallel u\parallel,\ \forall u\in C^1(\mathbf{T}^2) \tag{3.11}$$

and $\forall(\theta,\omega,\lambda)\in\mathbf{R^2}\times\mathbf{S^1}\times\mathbf{R}$, if $\mathcal{I}(u)=\begin{pmatrix}u\\u\end{pmatrix}$, for any $u\in L^2(\mathbf{T}^2)$.

<u>Proof of the Lemma</u> Taking into account that $\mid g\mid=1$ by (3.3), we check the identities

$$M^\mp(D-A-\theta-i\lambda G(\omega))=M^\mp(D-A-\theta\pm\lambda R(\omega)), \tag{3.12}$$

$$\widetilde{\mathrm{M}}^\mp(\mathrm{D}-\mathrm{A}-\theta-\mathrm{i}\lambda\mathrm{G}(\omega))=\widetilde{\mathrm{M}}^\mp(\mathrm{D}-\mathrm{A}-\theta\pm\lambda\mathrm{R}(\omega));$$

with $R(\omega)$ the real closed one-form, (thanks to (3.2)), linear in ω, defined by

$$R(\omega)=J^\star(G(\omega));\qquad(dR(\omega)=\Delta_g(\omega x+\psi_\omega)=0). \tag{3.13}$$

So, as for (3.10) we get

$$(\mathcal{D}(D-A-\theta-i\lambda G(\omega)))^\star\mathcal{D}(D-A-\theta-i\lambda G(\omega))= \tag{3.14}$$

$$\begin{pmatrix}P_{g,\mathbf{T}^2}(D-A-\theta-\lambda R(\omega))-B & 0\\ 0 & P_{g,\mathbf{T}^2}(D-A-\theta+\lambda R(\omega))+B\end{pmatrix},$$

and then, for any function $u\in C^2(\mathbf{T}^2)$,

$$\parallel\mathcal{D}(D-A-\theta-i\lambda G(\omega))\mathcal{I}(u)\parallel^2=2<P_{g,\mathbf{T}^2}(D-A-\theta)u\mid u>_{L^2(\mathbf{T}^2)}\ + \tag{3.15}$$

$$2\lambda^2\int_{\mathbf{T}^2}\mid G(\omega)\mid^2_g\times\mid u\mid^2 dx.$$

We used that $J^\star$ is isometric, $\mid R(\omega)\mid_g=\mid J^\star(G(\omega))\mid_g=\mid G(\omega)\mid_g$ •

Lemma 3.4 *Let $\omega\in\mathbf{S}^1$. There exists $\theta^0=\theta^0(\omega)\in\mathbf{R^2}$ satisfying: for any $\eta>0$ there exists $c_2(\eta)=c_2(\eta,\omega)>0$, such that, $\forall\theta\in\mathbf{R^2}$ such that $d_0(\theta\pm(\theta^0+\lambda\theta^1(\omega));\Gamma^\star)\geq\eta$, then*

$$\parallel\widetilde{\mathcal{D}}(D-A-\theta-i\lambda G(\omega))U\parallel\geq c_2(\eta)\parallel U\parallel,\quad\forall U\in(C^1(\mathbf{T}^2))^2,\ \forall\lambda\in\mathbf{R}, \tag{3.16}$$

$d_0(\ .\ ;\ .\ )$ denotes the standard euclidian distance and

$$\theta^1(\omega)=(\theta_1(\omega),\theta_2(\omega)=(-\sum_j g_0^{2j}w_j,\sum_j g_0^{1j}w_j);$$

$$g_0^{kj}=\frac{1}{\mid\mathbf{T}^2\mid}\int_{\mathbf{T}^2}[g_x^{kj}+(\sum_m\partial_{x_m}g_x^{km})(\Delta_g)^{-1}(\sum_m\partial_{x_m}g_x^{jm})]dx.$$

Proof of the Lemma Let $R(\omega)$ be the one-form defined by (3.13) and let N_g be be the one

$$N_g = \sum_{j,k} \partial_{x_j} h^{kj}(-h^{2k} dx_1 + h^{1k} dx_2). \tag{3.17}$$

Then, if $U = \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in (C^1(\mathbf{T}^2))^2$,

$$\| \widetilde{\mathcal{D}}(D - A - \theta - i\lambda G(\omega))U \|^2 = \| M^+(D - A - N_g - \lambda R(\omega) - \theta)u^+ \|^2 + \tag{3.18}$$

$$\| M^-(D - A + N_g + \lambda R(\omega) - \theta)u^- \|^2 .$$

But, as for the definition of $\phi(x)$ in (3.3) and (3.4), we can find $\theta^0 \in \mathbf{R^2}$, two real, periodic functions $\psi(x)$ and $\varphi(x)$ such that

$$N_g = d(\theta^0 x) + d\varphi(x) + J^\star(d\psi). \tag{3.19}$$

(ψ , φ $\in C^\infty(\mathbf{T}^2; \mathbf{R})$). By (3.13),

$$R(\omega) = d(\theta^1(\omega)x + \varphi_\omega), \quad \text{with } \varphi_\omega(x) \in C^\infty(\mathbf{T}^2; \mathbf{R}). \tag{3.20}$$

Taking into account (3.7) and (3.18), we get

$$\| \widetilde{\mathcal{D}}(D - A - \theta - i\lambda G(\omega))U \|^2 = \| e^{-(\phi+\psi-i\varphi-i\lambda\varphi_\omega)} M^+(D - \theta^0 - \lambda\theta^1(\omega) - \theta) e^{(\phi+\psi-i\varphi-i\lambda\varphi_\omega)} u^+ \|^2$$

$$+ \| e^{-(-\phi+\psi+i\varphi+i\lambda\varphi_\omega)} M^-(D + \theta^0 + \lambda\theta^1(\omega) - \theta) e^{(-\phi+\psi+i\varphi+i\lambda\varphi_\omega)} u^- \|^2 . \tag{3.21}$$

So there exists a constant $C_{A,g} > 0$ such that

$$\| \widetilde{\mathcal{D}}(D - A - \theta - i\lambda G(\omega))U \|^2 \geq C_{A,g}\{\| M^+(D - \theta^0 - \lambda\theta^1(\omega) - \theta) e^{(\phi+\psi-i\varphi-i\lambda\varphi_\omega)} u^+ \|^2$$

$$+ \ \| M^-(D + \theta^0 + \lambda\theta^1(\omega) - \theta) e^{(-\phi+\psi+i\varphi+i\lambda\varphi_\omega)} u^- \|^2\} \tag{3.22}$$

$$= C_{A,g}\{< P_{g,\mathbf{T}^2}(D - \theta^0 - \lambda\theta^1(\omega) - \theta) e^{(\phi+\psi-i\varphi-i\lambda\varphi_\omega)} u^+ \mid e^{(\phi+\psi-i\varphi-i\lambda\varphi_\omega)} u^+ >_{L^2(\mathbf{T}^2)}$$

$$+ < P_{g,\mathbf{T}^2}(D + \theta^0 + \lambda\theta^1(\omega) - \theta) e^{(-\phi+\psi+i\varphi+i\lambda\varphi_\omega)} u^- \mid e^{(-\phi+\psi+i\varphi+i\lambda\varphi_\omega)} u^- >_{L^2(\mathbf{T}^2)}\}.$$

Changing eventually $C_{A,g}$, it comes from (3.22) the estimate

$$\| \widetilde{\mathcal{D}}(D - A - \theta - i\lambda G(\omega))U \|^2 \geq C_{A,g}\{\| (D - \theta^0 - \lambda\theta^1(theta^1(\omega) - \theta) e^{(\phi+\psi-i\varphi-i\lambda\varphi_\omega)} u^+ \|^2 + \tag{3.23}$$

$$\| (D + \theta^0 + \lambda\theta^1(\omega) - \theta) e^{(-\phi+\psi+i\varphi+i\lambda\varphi_\omega)} u^- \|^2\},$$

which proves (3.16)•

Lemma 3.5 *For any $\omega \in \mathbf{S}^1$, there exists an integer $m \in \mathbf{N}$ and a constant C_ω such that*

$$\mu_1(\lambda, \omega, \theta) \geq \lambda^{\frac{2}{m+1}}/C_\omega, \quad \forall\lambda > 1/C_\omega, \ \theta \in \mathbf{K}. \tag{3.24}$$

$\mu_1(\lambda, \omega, \theta)$ denotes the first eigenvalue of $P_{g,\mathbf{T}^2}(D - A - \theta) + \lambda^2 \mid G(\omega) \mid^2_{gx}$.

Proof of the Lemma The min-max principle, (see [R-S]), gives the formula of the ground state energy

$$\mu_1(\lambda,\omega,\theta)=\inf_{\|u\|=1}\{\| M^+(D-A-\theta)u\|^2+\int_{\mathbf{T}^2}(\lambda^2\mid G(\omega)\mid_{g_x}^2+B(x))\mid u\mid^2 dx\}. \tag{3.25}$$

It comes from (3.25), (3.18) and (3.21) that it is sufficient to prove (3.24) when $A=0$, (B is bounded). If $\mid G(\omega)\mid_{g_x}>0,\ \ \forall x\in\mathbf{T}^2$, then (3.24) is obvious with $m=0$.

But the zeros of the function $x\mapsto\mid G(\omega)\mid_{g_x}$, if there exist, are isolated:

$$\mid G(\omega)\mid_{g_x}=0 \iff x\in\mathcal{Z}=\{z_1,\ldots,z_N\}. \tag{3.26}$$

More precisely, for each zero z_j there exists an integer $m_j\in\mathbf{N}$ and a constant C_j such that

$$C_j^{-1}\mid x-z_j\mid^{m_j}\leq\mid G(\omega)\mid_{g_x}\leq C_j\mid x-z_j\mid^{m_j},\ \text{ if }\ \mid x-z_j\mid^{m_j}\leq C_j^{-1}. \tag{3.27}$$

To be convinced, recall that $G(\omega)=df_\omega$ with $f_\omega(x)=\omega x+\psi_\omega(x)$ which is a g-harmonic function. But it is well known that for any $x_0\in\mathbf{T}^2$, there exist local coordinates in a neighbourhood of x_0 such that the metric g becomes conformal to the flat one. (Take for example $y=(y_1(x),y_2(x)$ with $y_j(x)$ g-harmonic functions, $dy_{1,x_0}\neq 0$ and $dy_2=J^\star(dy_1)$).

So the function f_ω is (locally) the real part of some holomorphic function, for some locally complex structure. And then (3.26) and (3.27) are valid.

We get from (3.26) and (3.27) that, if $A=0$, there exists C_ω such that

$$\inf_j\{\mu_{1,j}(\lambda,\omega)\}\leq\inf_j\{\mu^0_{1,j}(\lambda,\omega)\}\leq C^0_\omega\mu_1(\lambda,\omega,\theta),\ \forall\lambda>C_\omega,\ \theta\in\mathbf{K}, \tag{3.28}$$

for some constant C^0_ω, if $\mu_{1,j}(\lambda,\omega)$ is the first eigenvalue of the Schrödinger operator $-\Delta+\lambda^2C_\omega^{-1}\mid x-z_j\mid^{2m_j}$ on $L^2(\mathbf{R^2})$, and if $\mu^0_{1,j}(\lambda,\omega)$ is the one for the Dirichlet problem on the ball $B(z_j;C_\omega^{-1})$ of the same operator.

So by scaling we get from (3.28)

$$\inf_j\{\lambda^{\frac{2}{m_j+1}}\mu_{1,j}\}\leq C_\omega\mu_1(\lambda,\omega,\theta),\ \ \forall\lambda>C_\omega,\ \theta\in\mathbf{K}; \tag{3.29}$$

if $\mu_{1,j}$ is the first eigenvalue of the Schrödinger operator $-\Delta+\mid x\mid^{2m_j}$ on $L^2(\mathbf{R^2})$.

The second estimate of (3.28), (the right one), can be obtained in the following way.

We take a smooth partition of unity $\{\varphi_0,\varphi_1,...,\varphi_N\}$ on $\mathbf{T}^2$ such that

$$\sum_{j=0}^N\varphi_j^2(x)=1,\ \ Supp(\varphi_j)\subset B(z_j;C_\omega^{-1}),\ Supp(\varphi_0)\cap B(z_j;\frac{1}{2}C_\omega^{-1})=\emptyset,\ \text{for } j=1,...,N.$$

For any $u\in H^1(\mathbf{T}^2)$, using that $M^+(D)$ is a one-form, so

$$\sum_{j=0}^N\varphi_jM^+(D)(\varphi_j)=0 \text{ and } M^+(D)(\varphi_ju)=\varphi_jM^+(D)(u)+uM^+(D)(\varphi_j),$$

it comes that

$$\| M^+(D)u\|^2=\sum_{j=0}^N\|\varphi_jM^+(D)u\|^2=\sum_{j=0}^N[\|M^+(D)(\varphi_ju)-uM^+(D)(\varphi_j)\|^2$$

$$= \sum_{j=0}^{N} [\| M^+(D)(\varphi_j u) \|^2 + \| u M^+(D)(\varphi_j) \|^2] -$$

$$\sum_{j=0}^{N} [< M^+(D)(\varphi_j u) \mid u M^+(D)(\varphi_j) > + < u M^+(D)(\varphi_j) \mid M^+(D)(\varphi_j u) >].$$

Writing $< M^+(D)(\varphi_j u) \mid u M^+(D)(\varphi_j) > + < u M^+(D)(\varphi_j) \mid M^+(D)(\varphi_j u) >$
$= 2 \| u M^+(D)(\varphi_j) \|^2 + < M^+(D)(u) \mid u\varphi_j M^+(D)(\varphi_j) > +$

$$< u\varphi_j M^+(D)(\varphi_j) \mid M^+(D)(u) >,$$

then summing, we get the equality

$$\| M^+(D)u \|^2 = \sum_{j=0}^{N} \| M^+(D)(\varphi_j u) \|^2 - \int_{\mathbf{T}^2} | u |^2 \sum_{j=0}^{N} | M^+(D)(\varphi_j) |^2 \, dx.$$

We recall that $\mu^0_{1,j}(\lambda,\omega) \leq \lambda^2 C_\omega^{-2m_j-1} + \mu^0_1(\omega) \leq \lambda^2 C'_\omega$,
for some constant $C'_\omega > 1$, if $\lambda > C_\omega$, ($\mu^0_1(\omega)$ is the first eigenvalue of the Dirichlet problem associated to $-\Delta$ on $B(z_j; C_\omega^{-1})$).

We recall also that

$$\| M^+(D)(\varphi_0 u) \|^2 + \lambda^2 \| | G(\omega) |_g \varphi_0 u \|^2$$

$$\geq \lambda^2 (C''_\omega)^{-1} \| \varphi_0 u \|^2 \geq (C^1_\omega)^{-1} \inf_k \mu^0_{1,k}(\lambda,\omega) \| \varphi_0 u \|^2$$

for some constant $C''_\omega > 1$, if $C^1_\omega = C'_\omega C''_\omega$.

So, for any $j = 0, ..., N$,

$$\| M^+(D)(\varphi_j u) \|^2 + \lambda^2 \| | G(\omega) |_g \varphi_j u \|^2 \geq (C^1_\omega)^{-1} \inf_k \mu^0_{1,k}(\lambda,\omega) \| \varphi_j u \|^2,$$

and then

$$\| M^+(D)(u) \|^2 + \lambda^2 \| | G(\omega) |_g u \|^2 \geq [(C^1_\omega)^{-1} \inf_k \mu^0_{1,k}(\lambda,\omega) - C_{\omega,2}] \| \varphi_j u \|^2$$

with $C_{\omega,2}$ the maximum on $\mathbf{T}^2$ of $\sum_j | M^+(D)(\varphi_j) |^2$.

The second estimate of (3.28) follows for large λ, ($\mu^0_{1,j}(\lambda,\omega) \mapsto +\infty$ when $\lambda \mapsto +\infty$) •

End of the proof of Theorem (3.1)

We have seen that it is sufficient to take $\Gamma = \mathbf{Z^2}$. We take θ_ω such that $\theta_{\omega,1} \pm \theta^0_1 \notin 2\pi\mathbf{Z}$.
Let $\mu > 0$ such that $\mu\theta^1_1(\omega) \in 2\pi\mathbf{Z}$.

Then we can take $(\lambda_k) = (n(k)\mu)$ for any increasing sequence of non negative integer $(n(k))$, there exists $\eta > 0$ such that $d_0(\theta_\omega \mp (\theta^0 + \lambda_k\theta^1(\omega)); \Gamma^\star) > \eta$.

With $\lambda = \lambda_k$, applying Lemma 3.4 to $U = \mathcal{D}(D - A - \theta_\omega - i\lambda_k G(\omega))\mathcal{I}(u)$ and then using Lemma 3.3 to estimate U above, we get (3.1) from (3.10), (3.16), (3.11) , Lemma 3.5 and from $B \in L^\infty(\mathbf{T}^2)$ •

Remark 3.6 *There exists* $\omega \in \mathbf{S}^1$ *such that* $m = 0$ *in (3.1), iff*

$F^\star(g) = c(y)g_0$, *for some non-negative and* $u(\Gamma)$*- periodic function* $c(y) \in C^\infty(\mathbf{R^2})$, *with* $u \in GL(2;\mathbf{R})$ *and* F *is a* $\mathbf{R^2}$*-diffeomorphisme of the form* $F(x) = u(x) + (\psi_1(x), \psi_2(x))$, *with* $\psi_j(x) \in C^\infty(\mathbf{T}^2)$.

If $\omega \in \mathbf{S}^1$ is such that $\mid G(\omega) \mid_{g_x} > 0$, then $\psi_1(x) = \psi_\omega(x) - \psi_\omega(0)$ and $F = (f_1, f_2)$, $f_1(x) = \omega x + \psi_1(x)$, and $f_2(x) = \tilde{\omega} x + \psi_2(x)$ is the g-harmonic function such that $df_2 = J^\star(df_1)$ and $f_2(0) = 0$.

The f_j are g-harmonic and $F(\Gamma) = u(\Gamma)$, $u(x) = (\omega x, \tilde{\omega} x)$.

Otherwise, using semi-classical analysis method as in [H-M-1] , one can get easily the equivalence in (3.24): $\mu_1(\lambda, \omega, \theta)/\lambda^{\frac{2}{m+1}}$ is also bounded.

4 Some applications of the method

The two-dimensional generalized Dirac operator

Let $\mathcal{P}_g(D-A)$ be the generalized Dirac operator defined on $(L^2_g(\mathbf{R}^2))^2$,

$$\mathcal{P}_g(D-A) = \frac{1}{2} \sum_{1 \le j,k \le 2} \sigma_k \{ h^{jk}(D_{x_j} - A_j) + \mid g_x \mid^{-1/2} (D_{x_j} - A_j) \mid g_x \mid^{1/2} h^{jk} \}, \tag{4.1}$$

with domain the Sobolev space $(H^1(\mathbf{R}^2))^2$.

The σ_k are the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and (h_{jk}) is the square root of the metric $g = (g_{jk})$.

Let $V_0(x)$ and $V_1(x)$ be real functions such that

$$V_j, \; D_{x_k} V_j \; \in L^\infty(\mathbf{R}^2); \quad V_j(x-a) = V_j(x), \; \forall a \in \Gamma, \text{ and } \forall \; j,k. \tag{4.2}$$

Then $\mathcal{P}_g(D-A) + V_0(x)\sigma_0 + V_1(x)\sigma_3$ is a self-adjoint operator and Theorem 2.1 is still valid for $\mathcal{P}_g(D-A) + V_0(x)\sigma_0 + V_1(x)\sigma_3$.

Theorem 4.1 *Under the assumptions of Theorem 1.1 and (4.2), the spectrum of* $\mathcal{P}_g(D-A) + V_0(x)\sigma_0 + V_1(x)\sigma_3$ *is absolutely continuous.*

Proof We remark that

$$(\mathcal{P}_g(D-A))^2 = \begin{pmatrix} P_g(D-A-A^g) & 0 \\ 0 & P_g(D-A+A^g) \end{pmatrix} - \tag{4.3}$$

$$\mid g_x \mid^{-1/2} B(x)\sigma_3 - \frac{1}{2} \mid g_x \mid^{-1/2} \sum_{1 \le j,k,m \le 2} \partial_{x_j}(h^{jk}\partial_{x_m}(\mid g_x \mid^{1/2} h^{mk}))\sigma_0;$$

with $A^g = \sum A^g_j dx_j$, $A^g_j(x) = \frac{1}{2} \mid g_x \mid^{-1/2} \sum_{1 \le k,m \le 2} \{(-1)^k h_{jk}(x) \partial_{x_m}(\mid g_x \mid^{1/2} h^{mJ(k)}(x))\}$,

and J is difined by $J(1) = 2$ and $J(2) = 1$.

For any $\Theta \in \mathbf{C}^2$, we consider the operator $\mathcal{P}^\Theta_g(D-A) = P_g(D-A-\Theta)$ on $(L^2_g(\mathbf{T}^2))^2$, (with domain $(H^1(\mathbf{T}^2))^2$).

So, if μ is an eigenvalue of $\mathcal{P}_g(D-A) + V_0(x)\sigma_0 + V_1(x)\sigma_3$, μ is also an eigenvalue of $\mathcal{P}^\Theta_g(D-A) + V_0(x)\sigma_0 + V_1(x)\sigma_3$, $\forall \; \Theta \in \mathbf{C}^2$.

Let $u^\Theta(x)$ be an associated eigenfunction, $u^\Theta = (u^\Theta_+, u^\Theta_-)$ is of $(L^2_g(\mathbf{T}^2))^2$-norm 1.
So, as $\sigma_3 \mathcal{P}^\Theta_g(D-A)\sigma_3 = -\mathcal{P}^\Theta_g(D-A)$, we get

$$(\mathcal{P}^\Theta_g(D-A))^2 u^\Theta = ((\mu - V_0(x))\sigma_0 - V_1(x)\sigma_3)^2 u^\Theta -$$

$$[\mathcal{P}_g^\Theta(D-A);V_0(x)\sigma_0]u^\Theta+\sigma_3[\mathcal{P}_g^\Theta(D-A);V_1(x)\sigma_0]u^\Theta.$$

But the $[\mathcal{P}_g^\Theta(D-A);V_j(x)\sigma_0]=[\mathcal{P}_g(D-A);V_j(x)\sigma_0]$ are bounded matrices, so $P_g(D-A-A^g-\Theta)u_+^\Theta$ and $P_g(D-A+A^g-\Theta)u_-^\Theta$ will be $L^2_g(\mathbf{T}^2)$-bounded by a constant independent of Θ. This will be in contradiction with Theorem 3.1 •

The three dimensional case

We add a third variable x_3. If $c(x_3)$ is a function on $\mathbf{R}$,

$$c(x_3)>0,\quad c(x_3-1)=c(x_3)\ \text{ and } c(x_3)\ \in C^\infty(\mathbf{R}), \tag{4.4}$$

Let $\widetilde{g}_x=(\widetilde{g}_{jk}(x))_{1\le j,k\le 3}$ be the riemmann metric on $\mathbf{R}^3$,

$$\widetilde{g}_{jk}(x)=g_{jk}(x_1,x_2),\quad \text{for}\ \ 1\le j,k\le 2,\quad g_{3,3}(x_3)=c(x_3)\ \text{ and } g_{3,1}=g_{3,2}=0. \tag{4.5}$$

Theorem 4.2 *Under the assumptions (1.1), (1.2) on g and A, (4.4) and (4.5), if V_0 and V_1 are real functions*

$$V_j(x)\in L^\infty(\mathbf{R}^3),\quad V_j(x-a)=V_j(x),\ \forall\ a\in\Gamma\times\mathbf{Z},\ \text{and } j=0,\ 1, \tag{4.6}$$

then the Schrödinger operator

$$P_{\widetilde{g}}(D_{x'}-A(x'),D_{x_3})+V_0(x)=P_g(D_{x'}-A(x'))+c^{-1/2}(x_3)D_{x_3}(c^{-1/2}(x_3)D_{x_3})+V_0(x)$$

is essentially self-adjoint on $L^2_{\widetilde{g}}(\mathbf{R}^3)$ and its spectrum is absolutely continuous.

The same result holds for the generalized Dirac operators,

$$\mathcal{P}_{\widetilde{g}}(D_{x'}-A(x'),D_{x_3})+V_0(x)\mathbf{1}_2\quad\text{on}\quad (L^2_{\widetilde{g}}(\mathbf{R}^3))^2$$

and

$$\widetilde{\mathcal{P}}_{\widetilde{g}}(D_{x'}-A(x'),D_{x_3})+V_0(x)\mathbf{1}_4+V_1(x)\alpha_4\quad\text{on}\quad (L^2_{\widetilde{g}}(\mathbf{R}^3))^4,$$

with

$$\mathcal{P}_{\widetilde{g}}(D_{x'}-A(x'),D_{x_3})=\mathcal{P}_g(D_{x'}-A(x'))+c^{-1/2}(x_3)D_{x_3}\sigma_3,$$

and

$$\widetilde{\mathcal{P}}_{\widetilde{g}}(D_{x'}-A(x'),D_{x_3})=\begin{pmatrix}0 & \mathcal{P}_{\widetilde{g}}\\ \mathcal{P}_{\widetilde{g}} & 0\end{pmatrix}\quad\text{with } \mathcal{P}_{\widetilde{g}}=\mathcal{P}_{\widetilde{g}}(D_{x'}-A(x'),D_{x_3})$$

(For any $x=(x_1,x_2,x_3)\ \in\ \mathbf{R}^3$, $x'=(x_1,x_2)$ and $\alpha_4=\begin{pmatrix}\mathbf{1}_2 & 0\\ 0 & -\mathbf{1}_2\end{pmatrix}$).

For the proof, let $\Theta=(0,0,\rho+i\lambda)\in\mathbf{C}^3$, $(\rho,\lambda)\in\ \mathbf{R}^2$. Consider on $(L^2_{\widetilde{g}}(\mathbf{T}^3))^2$, $(\mathbf{T}^3=\mathbf{R}^3/(\Gamma\times\mathbf{Z}))$, the operator

$$\mathcal{P}_{\widetilde{g}}^\Theta(D-A+A^g)=\mathcal{P}_{\widetilde{g}}^\Theta(D_{x'}-A(x')+A^g(x'),D_{x_3})= \tag{4.7}$$

$$\mathcal{P}_{\widetilde{g}}(D_{x'}-A(x')+A^g(x'),D_{x_3}-\rho-i\lambda).$$

Then as in (4.3) we get

$$(\mathcal{P}_{\widetilde{g}}^\Theta(D-A+A^g))^2=$$

$$\begin{pmatrix} P_g(D_{x'}-A(x'),D_{x_3}-\rho-i\lambda) & 0 \\ 0 & P_g(D_{x'}-A(x')+2A^g(x'),D_{x_3}-\rho-i\lambda) \end{pmatrix} - \tag{4.8}$$

$$\mid g_{x'} \mid^{-1/2} B(x')\sigma_3 - \frac{1}{2} \mid g_{x'} \mid^{-1/2} \sum_{1\leq j,k,m\leq 2} \partial_{x_j}(h^{jk}(x'\partial_{x_m}(\mid g_{x'} \mid^{1/2} h^{mk}(x'))\sigma_0.$$

Let us write

$$\mathcal{P}_g(D_{x'}-A(x')+A^g(x')) = \begin{pmatrix} 0 & M^\star \\ M & 0 \end{pmatrix}.$$

Then for any $u=(u_+,u_-)\in (H^2(\mathbf{T}^3))^2$,

$$\| \mathcal{P}_{\tilde g}^{\Theta}(D_{x'}-A(x')+A^g(x'),D_{x_3})u \|^2 = \| c^{-1/2}(x_3)(D_{x_3}-\rho)\sigma_0 u \|^2 + \lambda^2 \| u \|^2 + \tag{4.9}$$

$$\| Mu_+ \|^2 + \| M^\star u_- \|^2 + 2i\lambda\{< M^\star u_- \mid u_+ > - < Mu_+ \mid u_- >\}.$$

($< \, . \mid . \, >$ is the hermitian form on $L^2_{\tilde g}(\mathbf{T}^3)$ and $\| \, . \, \|$ its associated norm).

As $M^\star$ is the adjoint of M, we get that

$$2\lambda\{< M^\star u_- \mid u_+ > - < Mu_+ \mid u_- >\} = 2\{< \lambda u_- \mid Mu_+ > - < \lambda u_+ \mid M^\star u_- >\}$$

so

$$\| \mathcal{P}_{\tilde g}^{\Theta}(D_{x'}-A(x')+A^g(x'),D_{x_3})u \|^2 \geq \| c^{-1/2}(x_3)(D_{x_3}-\rho)\sigma_3 u \|^2 . \tag{4.10}$$

Then we get from (4.9) and (4.10) that there exists a constant $C>1$ such that

$$\| (\mathcal{P}_{\tilde g}^{\Theta}(D_{x'}-A(x')+A^g(x'),D_{x_3}))^2(u_+,0) \| \geq C^{-1} \mid \lambda \mid d_0(\rho;\mathbf{Z}) \| u_+ \| .$$

(d_0 is the standard euclidian distance).

If C is large enough, by (4.8) we get

$$\| P_{\tilde g}^{\Theta}(D_{x'}-A(x'),D_{x_3})u_+ \| = \| P_{\tilde g}(D_{x'}-A(x'),D_{x_3}-\rho-i\lambda)u_+ \| \geq \tag{4.11}$$

$$[C^{-1} \mid \lambda \mid d_0(\rho;\mathbf{Z}) - C] \| u_+ \| .$$

This estimate proves that $P_{\tilde g}^{\Theta}(D_{x'}-A(x'),D_{x_3})+V_0(x)$ has no eigenvalue independent of Θ in $\mathbf{C}^3$. As Theorem 2.1 is still valid for $P_{\tilde g}^{\Theta}(D_{x'}-A(x'),D_{x_3})+V_0(x)$, we get the absolute continuity of its spectrum.

The proof for $\mathcal{P}_{\tilde g}^{\Theta}(D_{x'}-A(x'),D_{x_3})+V_0(x)\mathbf{1}_2$
and for $\widetilde{\mathcal{P}}_{\tilde g}^{\Theta}(D_{x'}-A(x'),D_{x_3})+V_0(x)\mathbf{1}_4+V_1(x)\alpha_4$
are the same as in Theorem 4.1, taking into account (4.11) •

For the flat metric the absence of singular spectrum for the Dirac operator was proved in [D].

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