Arnold diffusion in perturbations of analytic integrable Hamiltonian systems

Ernest Fontich
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via, 585, 08007 Barcelona, Spain

Pau Martín
Departament de Matemàtica Aplicada i Telemàtica
Universitat Politècnica de Catalunya
Ed-C3, Jordi Girona, 1-3, 08034 Barcelona Spain

Abstract

Given an analytic integrable Hamiltonian with three or more degrees of freedom, we construct, arbitrarily close to it, an analytic perturbation with transition chains whose lengths only depend on the unperturbed Hamiltonian. Then we deduce that the perturbed system has Arnold diffusion. We provide the technical details of the tools we use.

AMS Subject Clasification (1991): 58F05, 58F30, 34C37, 70H05.

Keywords: Hamiltonian systems, invariant tori, heteroclinic solutions, Arnold diffusion.

1 Introduction

Under general assumptions, Kolmogorov's Theorem asserts that the phase space of Hamiltonian systems close to integrable contains a set of large measure filled by invariant tori of the system. The actions of orbits lying in the complementary of this set may have a drift of order 1 independently of the size of the perturbation. However, if this happens, according to Nekhoroshev's Theorem, the drift must be very slow [18, 6, 15, 20]. Arnold's example [1] shows that the drift can actually occur, through a mechanism that could be quite general: chains of partially hyperbolic invariant tori with unstable and stable invariant manifolds intersecting each other transversally. These chains are shadowed by actual orbits of the Hamiltonian system. However, the extension of this example to a wider class of Hamiltonians has been troublesome (see [14]). If one considers analytic systems, one of the main difficulties is the fact that, in general, the splitting of separatrices of partially hyperbolic tori is exponentially small with respect to the perturbative parameter and the standard Poincaré-Melnikov techniques do not apply. A way to overcome this difficulty consists in considering perturbations with two parameters, one of them exponentially small with respect to he other. However, very interesting recent works [22, 21, 7] deal with the splitting of separatrices with one parameter. Bessi [2] studies Arnold's example using variational techniques. The case of C^∞ systems has been considered by Douady [8], see also [9].

In this work we shall consider an integrable analytic Hamiltonian in $\mathbb{T}^n \times \mathbb{R}^n$

$$H_0(\theta, r) = H_0(r)$$

satisfying general standard non degeneracy conditions. From H_0 we choose suitable paths in the space of the action variables. Then we construct arbitrary small analytic perturbations such that the perturbed systems have orbits whose perturbation over the action variables are close to the prescribed path. For the sake of completeness we provide the details of the results we need on invariant manifolds and the inclination lemma.

Section 2 contains the main result and the construction of the perturbations. Section 3 is devoted to the study of the existence of invariant manifolds related to invariant partially hyperbolic tori, and their dependence on parameters. Analogous results can be found in [10] and [23], where flows instead of maps are considered. Section 4 presents a version of the Inclination Lemma for partially hyperbolic tori.

2 Perturbations of Hamiltonian Systems

2.1 The main result

Let $B^n \subset \mathbb{R}^n$ be an open set, and $H_0 : \mathbb{T}^n \times B^n \to \mathbb{R}$ be an analytic integrable Hamiltonian, $H_0(\theta, r) = H_0(r)$. Let $\omega = \partial H_0/\partial r : B^n \to \mathbb{R}^n$ be the frequency vector. Since the equations of motion generated by H_0 are

$$\dot{\theta} = \omega(r), \qquad \dot{r} = 0,$$

the sets $\{r = \text{const.}\}$ are invariant n tori, and the motion on each torus is a rotation with frequency vector $\omega(r)$. If ω is non-resonant, that is, $\langle k, \omega\rangle \neq 0$ for all $k \in \mathbb{Z}^n \setminus \{0\}$, the sets are minimal (they cannot be decomposed into other invariant sets by the flow except the empty set and themselves). If there exists $k \in \mathbb{Z}^n \setminus \{0\}$ such that $\langle k, \omega\rangle = 0$, then the torus corresponding to the frequency ω is foliated with $n-1$ invariant tori, which are minimal, if k is the only resonance, or decompose into lower dimensional tori if ω has other resonances.

Theorem 2.1 *Let $B^n \subset \mathbb{R}^n$ open and let $H_0 : \mathbb{T}^n \times B^n \to \mathbb{R}$ be an analytic integrable Hamiltonian, $H_0(\theta, r) = H_0(r)$, $n \geq 3$, such that the frequency vector $\omega = \partial H_0/\partial r : B^n \to \mathbb{R}^n$ is a local diffeomorphism. Then, for all $r \in B^n$ there exist an open neighborhood B' of r, an analytic path $\gamma : [0,1] \subset \mathbb{R} \to B'$, $\gamma(0)$ close to r, $\gamma(0) \neq \gamma(1)$, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a Hamiltonian H_ε, ε-close to H_0 in the topology of analytic functions over $\mathbb{T}^n \times B'$ given by the supremum norm, such that for any two neighborhoods, U_0 and U_1, of $\gamma(0)$ and $\gamma(1)$ respectively there exists an orbit $(\theta(t), r(t))$ of H_ε with $r(0) \in U_0$ and $r(1) \in U_1$ and $r(t)$ ε-close to $\gamma(t)$.*

Remark 2.2 *Notice that the path γ in the theorem is independent of ε, a necessary condition to have actual diffusion to a distance that does not decrease with ε. Its length may be small, but it is fixed.*

Next subsections are devoted to prove this theorem.

2.2 Single resonances

We are interested in the design of perturbations which produce $n-1$ invariant tori with some hyperbolicity. For this reason we must find the proper resonances associated to the frequency vector ω. Next lemma states the existence of points where the path stated in the theorem may start.

Lemma 2.3 *Let $n \geq 3$. Let $\omega = \frac{\partial H_0}{\partial r} : B^n \to \mathbb{R}^n$ be a local diffeomorphism. Then there exist a dense set in B^n such that for any r^* belonging to it there exists $k^0 \in \mathbb{Z}^n \setminus \{0\}$ such that*

1. $\langle k^0, \omega(r^*) \rangle = 0$.
2. $\langle k^0, \dfrac{\partial \omega}{\partial r}(r^*) k^0 \rangle \neq 0$.
3. $\langle k', \omega(r^*) \rangle \neq 0$, $\forall k' \in \mathbb{Z}^n \setminus \{k^0 \mathbb{Z}\}$.

Proof. Let $\Omega = \omega(B^n)$. Ω is an open set in $\mathbb{R}^n$. We choose $\omega^0 \in \Omega$, $\omega^0 \neq 0$, such that all its components, ω_i^0, are rational numbers. In this way $\langle x, \omega^0 \rangle = 0$ defines a $n-1$ dimensional subspace V in $\mathbb{R}^n$ which can be spanned by $n-1$ vectors $k^1, \ldots, k^{n-1}$, with integer coordinates. Any linear combination over $\mathbb{Z}$ of these vectors gives a $k \in \mathbb{Z}^n$ with $\langle k, \omega^0 \rangle = 0$. Let $r^0 \in B^n$ be such that $\omega(r^0) = \omega^0$ and consider $\frac{\partial \omega}{\partial r}(r^0)$ as a quadratic form, which is non degenerate. Suppose that $\langle k, \frac{\partial \omega}{\partial r}(r^0) k \rangle = 0$, for any $k \in V$ with integer coordinates. The same will happen for any $k \in V$ with rational coordinates. Then, using the density of the points with rational coordinates in V, $\langle x, \frac{\partial \omega}{\partial r}(r^0) x \rangle = 0$, $\forall x \in V$. Therefore, since $\partial \omega / \partial r$ is symmetric we have that $\langle x, \frac{\partial \omega}{\partial r}(r^0) y \rangle = 0$, $\forall x, y \in V$ and, in particular, for the basis $k^1, \ldots, k^{n-1}$, it will hold

$$\langle k^i, \frac{\partial \omega}{\partial r}(r^0) k^j \rangle = 0, \qquad i, j = 1, \ldots, n-1.$$

Taking a suitable $k^n \in \mathbb{Z}^n$ to complete the basis of V to a basis of $\mathbb{R}^n$ and writing $\frac{\partial \omega}{\partial r}(r^0)$ in this basis we get

$$\frac{\partial \omega}{\partial r}(r^0) = \begin{pmatrix} 0 & \cdots & 0 & a_{n1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn-1} \\ a_{n1} & \cdots & a_{nn-1} & a_{nn} \end{pmatrix}.$$

Hence, as $n \geq 3$, $\det(\frac{\partial \omega}{\partial r}(r^0)) = 0$, which contradicts the fact that ω is a local diffeomorphism. Then, there exists $k^0 \in \mathbb{Z}^n \setminus \{0\}$ such that $\langle k^0, \omega(r^0)\rangle = 0$ and $\langle k^0, \frac{\partial \omega}{\partial r}(r^0)k^0\rangle \neq 0$, and this happens for any r^0 such that all the components of $\omega(r^0)$ are rational numbers, which is a dense subset of B^n.

Now consider the sets $\Omega_{k^0} = \{\omega \in \Omega : \ \langle k^0, \omega\rangle = 0\}$ and $\tilde{\Omega}_{k^0} = \{\omega \in \Omega : \langle k^0, \omega\rangle = 0, \ \langle k', \omega\rangle \neq 0, \ \forall k' \in \mathbb{Z}^n \setminus k^0\mathbb{Z}\}$. Since

$$\tilde{\Omega}_{k^0} = \bigcap_{k' \in \mathbb{Z}^n \setminus k^0 \mathbb{Z}} \{\omega \in \Omega : \ \langle k^0, \omega\rangle = 0, \ \langle k', \omega\rangle \neq 0\}$$

we have that $\tilde{\Omega}_{k^0}$ is a residual set in Ω_{k^0}, and, hence, is dense. We take a sequence ω^j in $\tilde{\Omega}_{k^0}$ with limit ω^0. Let r^j be the images of ω^j by a local inverse of the map ω. Clearly, $r^j \to r^0$. Since $\langle k^0, \frac{\partial \omega}{\partial r}(r^0)k^0\rangle \neq 0$, we have that $\langle k^0, \frac{\partial \omega}{\partial r}(r^j)k^0\rangle \neq 0$ for all $j > j_0$. We choose $r^* = r^j$, for j large enough. $\square$

Remark 2.4 *Lemma 2.3 is not true if* $n = 2$. *The function* $H(r_1, r_2) = \ln r_1 - \ln r_2$ *is a counterexample: if* $k = (k_1, k_2)$ *is such that* $\langle k, \omega(r)\rangle = 0$, *then* $\langle k, \frac{\partial \omega}{\partial r}(r)k\rangle = 0$.

2.3 The diffusion path

Let $r^* \in B^n$ be as in Lemma 2.3, i.e., $\omega^* = \omega(r^*)$ has a single non-degenerate resonance $k^0 \in \mathbb{Z}^n \setminus \{0\}$. After a translation, we can suppose that $r^* = 0$ and B^n is a neighborhood of $0 \in \mathbb{R}^n$. Adding a constant to the Hamiltonian, it can also be assumed that $H_0(0) = 0$.

Now we consider the change of variables defined as follows (see [8]): if $k^0 = (k_1^0, \ldots, k_n^0)$, let $q = \gcd(k_1^0, \ldots, k_n^0)$, and $k_i' = k_i^0/q$, $i = 1, \ldots, n$. There exists a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ with $k' = (k_1', \ldots, k_n')$ as its first row (see [3]). We take the change given by the linear map $\phi_A(\theta, r) = (A^{-1}\theta, {}^tAr)$, where ${}^tA$ stands for the transpose of the matrix A. It is symplectic. Notice that ϕ_A leaves $\{r = 0\}$ invariant. The Hamiltonian in the new variables will be $\tilde{H}_0 = H_0 \circ \phi_A$. The new variables will be denoted again by (θ, r). The frequency vector is now

$$\tilde{\omega}(r) = \frac{\partial \tilde{H}_0}{\partial r}(r) = A\omega({}^tAr),$$

and, at $r = 0$, we have that

$$\tilde{\omega}_1(0) = \frac{\partial \tilde{H}_0}{\partial r_1}(0) = \sum_{j=1}^{n} k_i' \omega_j(0) = \frac{1}{q}\langle k^0, \omega^* \rangle = 0.$$

Moreover,

$$\frac{\partial \tilde{\omega}_1}{\partial r_1}(0) = \frac{\partial^2 \tilde{H}_0}{\partial r_1^2}(0) = \frac{1}{q^2} \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial r_j}(0) k_i k_j \neq 0.$$

Notice that, since $\omega^* = \omega(0)$ has a simple resonance, the $n-1$ vector

$$\Big(\frac{\partial \tilde{H}_0}{\partial r_2}(0), \dots, \frac{\partial \tilde{H}_0}{\partial r_n}(0)\Big)$$

is non-resonant. Indeed, if $k \in \mathbb{Z}^{n-1}$, $k \neq 0$, and $\tilde{k} = (0, k)$ we have that

$$\sum_{j=2}^{n} k_j \frac{\partial \tilde{H}_0}{\partial r_j}(r) = \langle \tilde{k}, \tilde{\omega}(r) \rangle = \langle \tilde{k}, A\omega({}^t A r) \rangle = \langle {}^t A \tilde{k}, \omega({}^t A r) \rangle \neq 0,$$

since ${}^t A \tilde{k}$ is a linear combination of integer vectors linearly independent of k^0.

We shall denote again by H_0 and ω the new Hamiltonian and the new frequency vector. Also, we shall write $r = (r_1, \hat{r})$, where $\hat{r} = (r_2, \dots, r_n)$ and $\omega = (\omega_1, \hat{\omega})$, with $\hat{\omega} = \frac{\partial H_0}{\partial \hat{r}}$. The fact that $\tilde{\omega}(0)$ is non resonant implies, in particular, that $\omega_j(0) = \frac{\partial H_0}{\partial r_j}(0) \neq 0$, $j = 2, \dots, n$.

Therefore, we can suppose that the resonance is $k = (1, 0, \dots, 0)$ with the non-degeneracy condition transformed into $\frac{\partial^2 H_0}{\partial r_1^2}(0) \neq 0$, i.e., the Hamiltonian is written in the new variables, near $r = 0$, as

$$H_0(r) = c_0(\hat{r}) + c_1(\hat{r}) r_1 + c_2(\hat{r}) r_1^2 + \cdots, \quad r = (r_1, \hat{r}) \in \mathbb{R} \times \mathbb{R}^{n-1},$$

with $c_0(0) = c_1(0) = 0$ and $c_2(0) \neq 0$. Next we shall use the Weierstrass Preparation Theorem:

Theorem 2.5 *Suppose $f : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ is an analytic function in a neighborhood of $(0, 0)$ satisfying*

1) $f(w, 0) = w^l g(w)$, g analytic in a neighborhood of $0 \in \mathbb{C}$, *2)* $g(0) \neq 0$.

Then there exists a function q analytic in a neighborhood V of zero of $\mathbb{C} \times \mathbb{C}^n$ and functions $a_0(z), \dots, a_{l-1}(z)$ analytic in a neighborhood of zero in $\mathbb{C}^n$ such that $a_0(0) = \cdots = a_{l-1}(0) = 0$, $q(0,0) \neq 0$ and

$$q(w,z) f(w,z) = w^l + \sum_{j=0}^{l-1} a_j(z) w^j, \qquad (w,z) \in V.$$

In our case, it holds that $H_0(r_1, 0) = r_1^2 g(r_1)$, with $g(0) \neq 0$. Hence there exist analytic functions, $a_0(\hat{r})$, $a_1(\hat{r})$, $q(r_1, \hat{r})$, defined around $r = 0$, with $a_0(0) = a_1(0) = 0$, $q(0,0) \neq 0$ and $H_0 = F_0(r)/q(r)$ where

$$F_0(r) = \frac{1}{2} r_1^2 + a_0(\hat{r}) + a_1(\hat{r}) r_1. \tag{2.1}$$

The factor $1/2$ is introduced to follow the traditional notation. We define the functions

$$\hat{\alpha}^i = (\alpha_2^i, \dots, \alpha_n^i) = \frac{\partial a_i}{\partial \hat{r}}, \qquad i = 0, 1. \tag{2.2}$$

Let $B' \subset B^n$ be a neighborhood of $0 \in B^n$ where $q(r_1, \hat{r})$ stays away from 0. Consider the $n-1$ manifolds $\Sigma_0 = \{r \in B' : H_0(r) = 0\} = \{r \in B' : F_0(r) = 0\}$, the 0 energy level of the Hamiltonian, and $\Gamma_0 = \{r \in B' : \omega_1(r) = 0\}$, the manifold where the frequency vector is resonant with resonance k. From the formula (2.1) we deduce that

$$\Sigma_0 \cap \Gamma_0 = \{r : r_1 = -a_1(\hat{r}),\ a_0(\hat{r}) - \frac{1}{2} a_1^2(\hat{r}) = 0\}. \tag{2.3}$$

Lemma 2.6 *$\Sigma_0 \cap \Gamma_0$ is an $n-2$ dimensional analytic manifold (in a neighborhood of $0 \in B'$).*

Proof. The points of $\Sigma_0 \cap \Gamma_0$ are defined by the set of equations

$$\varphi_1(r_1, \hat{r}) = r_1 + a_1(\hat{r}) = 0, \qquad \varphi_2(r_1, \hat{r}) = a_0(\hat{r}) - \frac{1}{2} a_1^2(\hat{r}) = 0.$$

We have that $\varphi_i(0,0) = 0$, $i = 1, 2$. Also, by (2.1),

$$\frac{\partial a_0}{\partial r_j}(0) = q(0,0) \frac{\partial H_0}{\partial r_j}(0,0) = q(0,0)\, \omega_j(0,0) \neq 0, \quad j = 2, \dots, n,$$

and then

$$\frac{\partial}{\partial r_j}(a_0(\hat{r}) - \frac{1}{2}a_1^2(\hat{r}))_{|\hat{r}=0} = \frac{\partial a_0}{\partial r_j}(0) \neq 0, \quad j = 2, \ldots, n.$$

Then, by the Implicit Function Theorem, we can parameterize $\Sigma_0 \cap \Gamma_0$ by any $n-2$ variables the set $\{r_2, \ldots, r_n\}$, and the lemma is proved. $\square$

Lemma 2.7 *Any analytic curve* $\gamma : [0,1] \to \Sigma_0 \cap \Gamma_0$ *such that* $\gamma(0) = r^* = 0$ *verifies that* $\hat{\omega}(\gamma(s))$ *is non-resonant for a dense subset of* s *in* $[0,1]$.

Proof. Let $\gamma : [0,1] \to \Sigma_0 \cap \Gamma_0$ be an analytic curve such that $\gamma(0) = r^* = 0$. We recall that $\omega_1(\gamma(s)) = 0$, by definition of Γ_0. For any $k \in \mathbb{Z}^{n-1} \setminus \{0\}$ we define the function $\varphi_k(s) = \langle \hat{\omega}(\gamma(s)), k \rangle$, which is analytic in $[0,1]$. Since $\varphi_k(0) = \langle \hat{\omega}(0), k \rangle \neq 0$, we have that φ_k has, at most, a finite number of zeros in $[0,1]$. Let $S_k = \{s \in [0,1] : \varphi_k(s) \neq 0\}$. S_k is open and dense in $[0,1]$. Therefore $S_\infty = \bigcap_{k \in \mathbb{Z}^n \setminus \{0\}} S_k$ is a residual set in $[0,1]$ and, hence, dense. $\square$

Then, we just choose any $\gamma : [0,1] \to \Sigma_0 \cap \Gamma_0$ such that $\gamma(0) = r^* = 0$.

In [8] there is a "dual" result in the C^∞ case. Instead of finding a curve satisfying the conditions of Lemma 2.3 for a dense subset of the domain of the parameter describing the curve, there it is proven that the Hamiltonians for which a given curve satisfy those conditions form a residual set in the Fréchet space of C^∞ functions from $\mathbb{R}^n$ to $\mathbb{R}$. In both cases the length of the curve only depends on H_0.

By technical reasons we will add some conditions to the curve that may restrict its length in the following way: we shall suppose that, for a fixed j, $j = 2, \ldots, n$, $2\alpha_j^1(\hat{\gamma}(s)) \notin \mathbb{Z}$, for all $s \in [0,1]$ (which can be achieved if $\alpha_j^1(\hat{r})$) is non constant) or $2\alpha_j^1(\hat{\gamma}(s)) = k$, $k \in \mathbb{Z}$, for all $s \in [0,1]$. That is, $\alpha_j^1(\hat{\gamma}(s))$ has to be integer for all s or it has to stay away from integers. Also, we shall suppose that $\omega_j(\gamma(s)) \neq 0$, $j = 2, \ldots, n$, for all $s \in [0,1]$. This condition can be fulfilled since $\omega_j(\gamma(0)) \neq 0$, $j = 2, \ldots, n$. Notice that both conditions only depend on H_0.

2.4 Construction of the Perturbations

Let us consider first a perturbation, introducing a pendulum in the (θ_1, r_1) variables, H_ε of H_0 which destroys the tori associated to the resonance

$(1, 0, \dots, 0)$ but it is still integrable:

$$H_\varepsilon(\theta, r) = \frac{1}{q(r)} F_\varepsilon(\theta, r), \qquad F_\varepsilon(\theta, r) = F_0(r) + \varepsilon \cos(2\pi\theta_1) - \varepsilon.$$

The equations of motion are

$$\begin{aligned}
\dot{\theta}_1 &= \frac{\partial}{\partial r_1}\Big(\frac{1}{q(r)}\Big) F_\varepsilon(\theta, r) + \frac{1}{q(r)}(r_1 + a_1(\hat{r})) \\
\dot{\hat{\theta}} &= \frac{\partial}{\partial \hat{r}}\Big(\frac{1}{q(r)}\Big) F_\varepsilon(\theta, r) + \frac{1}{q(r)}(\hat{\alpha}^0(\hat{r}) + \hat{\alpha}^1(\hat{r}) r_1) \\
\dot{r}_1 &= \frac{2\pi\varepsilon}{q(r)} \sin(2\pi\theta_1) \\
\dot{\hat{r}} &= 0,
\end{aligned}$$

where $\theta = (\theta_1, \hat{\theta})$. Since $\dot{\hat{r}} = 0$, for each $\hat{r}$ fixed we can consider the motion in the (θ_1, r_1) variables

$$\begin{aligned}
\dot{\theta}_1 &= \frac{\partial}{\partial r_1}\Big(\frac{1}{q(r)}\Big) F_\varepsilon(\theta, r) + \frac{1}{q(r)}(r_1 + a_1(\hat{r})) \\
\dot{r}_1 &= \frac{2\pi\varepsilon}{q(r)} \sin(2\pi\theta_1).
\end{aligned} \tag{2.4}$$

If fact, this is a family of one degree of freedom Hamiltonian systems depending on $\hat{r}$. Each one is integrable. Now we consider points on the curve γ. For sake the of simplicity, we define $r_s = (r_{1s}, \hat{r}_s) = \gamma(s) = (\gamma_1(s), \hat{\gamma}(s))$. The point $\theta_1 = 0$, $r = r_s$ is a fixed point of the system (2.4). Therefore, the sets

$$\mathbb{T}_s = \{(\theta, r) : \ \theta_1 = 0, \ r = \gamma(s)\} \tag{2.5}$$

are invariant $n-1$ tori of H_ε. Moreover, since the derivatives of H_ε and H_0 coincide on $\mathbb{T}_s$, the dynamics on it is a rotation with the same frequency vector as the corresponding unperturbed one

$$\hat{\omega}_s = \frac{\partial H_\varepsilon}{\partial \hat{r}}_{|(\theta_1 = 0, r_s)} = \frac{\partial H_0}{\partial \hat{r}}_{|(\theta_1 = 0, r_s)},$$

which is non-resonant for a dense set of s.

Let $\Sigma_\varepsilon = \{(\theta, r) : H_\varepsilon(\theta, r) = 0\} = \{(\theta, r) : F_\varepsilon(\theta, r) = 0\}$ be the 0 energy level of H_ε. By construction, $\mathbb{T}_s \subset \Sigma_\varepsilon$, $\forall s \in [0, 1]$. Being Σ_ε an energy level of the Hamiltonian, it is invariant under the flow and the equations of motion on Σ_ε reduce to

$$\begin{aligned} \dot\theta_1 &= \frac{1}{q(r)}(r_1 + a_1(\hat r)), & \dot r_1 &= \frac{2\pi\varepsilon}{q(r)}\sin(2\pi\theta_1), \\ \dot{\hat\theta} &= \frac{1}{q(r)}(\hat\alpha^0(\hat r) + \hat\alpha^1(\hat r) r_1), & \dot{\hat r} &= 0. \end{aligned} \tag{2.6}$$

The tori $\mathbb{T}_s$ are partially hyperbolic with invariant manifolds $W^u\mathbb{T}_s$ and $W^s\mathbb{T}_s$ contained on $H_\varepsilon(\theta, r) = 0$. Considering the projection of the motion on Σ_ε to the (θ_1, r_1) variables, we have

$$\dot\theta_1 = \frac{1}{q(r)}(r_1 + a_1(\hat r)), \qquad \dot r_1 = \frac{2\pi\varepsilon}{q(r)}\sin(2\pi\theta_1). \tag{2.7}$$

Having $\hat r = \hat\gamma(s)$ fixed, $(0, \gamma_1(s))$ is a hyperbolic fixed point of (2.7), with two homoclinic connections ($(0, \gamma_1(s))$ and $(1, \gamma_1(s))$ are identified) given by the curves determined by the equation

$$f_\varepsilon(\theta_1, r_1) = a_0(\hat r) + a_1(\hat r) r_1 + \frac{1}{2} r_1^2 + \varepsilon(\cos(2\pi\theta_1) - 1) = 0, \tag{2.8}$$

which we shall denote by Γ_s^+ and Γ_s^-. Taking into account that $\gamma(s) \in \Sigma_0 \cap \Gamma_0$, we have that

$$a_0(\hat\gamma(s)) = \frac{1}{2} a_1^2(\hat\gamma(s)),$$

and then we can represent the invariant curves $\Gamma^\pm$ by

$$r_1 = -a_1(\hat\gamma(s)) \pm 2\sqrt{\varepsilon}\sin(\pi\theta_1), \qquad \theta_1 \in (0,1),$$

if ε is small enough to make them contained in the domain where $q(r)$ stays away from 0. Moreover, they intersect transversally the plane $\theta_1 = 1/2$.

In the original system, the invariant manifolds of the tori $\mathbb{T}_s$ are given by

$$\begin{aligned} W^u\mathbb{T}_s = W^s\mathbb{T}_s \;\; = \;\; & \{(\theta, r) :\, \hat r = \hat\gamma(s),\, (\theta_1, r_1) \in \Gamma_s^+, \hat\theta \in \mathbb{T}^{n-1}\} \\ & \cup \{(\theta, r) :\, \hat r = \hat\gamma(s),\, (\theta_1, r_1) \in \Gamma_s^-, \hat\theta \in \mathbb{T}^{n-1}\}. \end{aligned}$$

Notice that $W^{u,s}\mathbb{T}_s$, which coincide, intersect transversely the plane $\{\theta_1 = 1/2, \hat\theta = 0\}$. Any orbit on the invariant manifolds $W^{u,s}\mathbb{T}_s$ can be obtained in the following way: in the (θ_1, r_1) plane, let $(\theta_1(t), r_1(t))$ be the solution of (2.7) that passes through $(1/2, -a_1(\hat{\gamma}(s)) \pm 2\sqrt{\varepsilon})$ at $t = 0$. Then, for any $t_0 \in \mathbb{R}$, the orbit that passes through $(\theta_1(-t_0), r_1(-t_0))$ at $t = 0$ is $(\theta_1(t - t_0), r_1(t - t_0))$. Now, for the whole system, the orbit on $W^{u,s}\mathbb{T}_s$ that passes through $(\theta_1(-t_0), r_1(-t_0)) \times (\hat{\theta}^0, \hat{\gamma}(s))$ at $t = 0$ is given by $(\theta_1(t - t_0), \hat{\theta}(t), r_1(t - t_0), \hat{\gamma}(s))$, where

$$\hat{\theta}(t) = \hat{\theta}^0 + \int_0^t \frac{1}{q(r_1(z - t_0), \hat{\gamma}(s))}(\hat{\alpha}^0(\hat{\gamma}(s)) + \hat{\alpha}^1(\hat{\gamma}(s)) r_1(z - t_0))\, dz.$$

In this way, we can use $(t_0, \hat{\theta}^0) \in \mathbb{R} \times \mathbb{T}^{n-1}$ to parameterize the points of $W^{u,s}\mathbb{T}_s$.

Now, let us consider a second perturbation of the Hamiltonian

$$H_{\varepsilon\mu}(\theta, r) = \frac{1}{q(r)} F_{\varepsilon\mu}(\theta, r), \qquad F_{\varepsilon\mu}(\theta, r) = F_\varepsilon(\theta, r) + \mu P(\theta, r), \tag{2.9}$$

where

$$P(\theta, r) = (r_1 + a_1(\hat{r}))^2 \sum_{j=2}^{n} \cos(2\pi\theta_j).$$

The factor $(r_1 + a_1(\hat{r}))^2$ makes the perturbation and its derivatives to vanish on $\mathbb{T}_s$. In this way, for all μ, these tori persist as partially hyperbolic tori, and, as a consequence of Theorem 3.8 applied to the time τ map of the system, for any $\tau > 0$, their invariant manifolds vary smoothly in μ. Therefore, if μ is small enough, they cross transversely the section $\{\theta_1 = 1/2, \hat{\theta} = 0\}$.

Lemma 2.8 *Let $W^{u,s}\mathbb{T}_s$ be the invariant manifolds of the tori $\mathbb{T}_s$ of the Hamiltonian (2.9). Then $W^u\mathbb{T}_s$ intersects $W^s\mathbb{T}_s$ at two points of the form $p_s = (\theta_1 = 1/2, \hat{\theta} = 0, r = \rho_s)$. Moreover, if $(\theta(t), r(t))$ is a homoclinic orbit such that $(\theta(0), r(0)) = p_s$, then $\hat{\theta}$ is an odd function and r is an even one.*

Proof. It is straightforward using that $H_{\varepsilon\mu}(\theta_1, \hat{\theta}, r) = H_{\varepsilon\mu}(1 - \theta_1, -\hat{\theta}, r)$. □

2.5 Transversal splitting and Poincaré-Melnikov functions

We have already proved that, for each $s \in [0,1]$, the Hamiltonian $H_{\varepsilon\mu}$ has $\mathbb{T}_s$ as partially hyperbolic $n-1$ tori with invariant manifolds $W^{u,s}\mathbb{T}_s$ which intersect at some point of the form $(1/2, 0, \rho_s)$. In fact, they intersect at least along a curve, the solution of the Hamilton equations through this point. In order to prove that their intersection is only that curve we shall use the Poincaré-Melnikov functions considering $H_{\varepsilon\mu}$ as a perturbation of H_ε.

Therefore, for a fixed $s \in [0,1]$, we define the scalar functions

$$M_j(t_0, \hat{\theta}^0) = -\int_{-\infty}^{\infty} D_{\theta_j}\Big(\frac{1}{q}P\Big)_{|(\theta(t), r(t))}\, dt, \qquad j = 2, \ldots, n, \tag{2.10}$$

where D_{θ_j} denotes the derivatives with respect to θ_j and $(\theta(t), r(t))$ is the orbit of H_ε that passes through the point parameterized by $(t_0, \hat{\theta}^0)$ of $W^{u,s}\mathbb{T}_s$ at $t = 0$, and the vector function

$$M(t_0, \hat{\theta}^0) = (M_2(t_0, \hat{\theta}^0), \ldots, M_n(t_0, \hat{\theta}^0)). \tag{2.11}$$

An application of the Implicit Function Theorem gives

Theorem 2.9 *Let $(t_0^*, \hat{\theta}^{0*})$ be such that*

1) $M(t_0^*, \hat{\theta}^{0*}) = 0$.

2) $\mathrm{rang} D_{\hat{\theta}^0} M(t_0^*, \hat{\theta}^{0*}) = n - 1$.

Then, the invariant manifolds $W^{u,s}\mathbb{T}_s$ intersect transversely (in the energy level) in a point in a neighborhood of $(t_0^*, \hat{\theta}^{0*})$.

We already know, from Lemma 2.8, that there exists a homoclinic orbit of $H_{\varepsilon\mu}$. But evaluating M at $(0,0) \in \mathbb{R} \times \mathbb{T}^{n-1}$ we have that (2.10) is

$$M_j(0,0) = 2\pi \int_{-\infty}^{\infty} \frac{(r_1(t) + a_1(\hat{\gamma}(s)))^2}{q(r_1(t), \hat{\gamma}(s))} \sin(2\pi\theta_j(t))\, dt, \qquad j = 2, \ldots, n,$$

where $(\theta(t), r(t))$ is the homoclinic orbit of H_ε such that $(\theta(0), r(0)) = (1/2, 0, -a_1(\hat{\gamma}(s)) + 2\sqrt{\varepsilon}, \hat{\gamma}(s))$. By Lemma 2.8 we have that $\hat{\theta}(t) = -\hat{\theta}(-t)$

and $r(t) = r(-t)$, for all $t \in \mathbb{R}$. Hence $M_j(0,0) = 0$, $j = 2, \ldots, n$, since it is the integral over $\mathbb{R}$ of an odd function.

To check that the intersection of the manifolds is transversal, we compute

$$D_{\theta^0} M(0,0) = \begin{pmatrix} g_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_n \end{pmatrix},$$

where

$$g_j = 4\pi^2 \int_{-\infty}^{\infty} \frac{(r_1(t) + a_1(\hat\gamma(s)))^2}{q(r_1(t), \hat\gamma(s))} \cos(2\pi\theta_j(t))\, dt, \qquad j = 2, \ldots, n. \tag{2.12}$$

Therefore, $\det D_{\hat\theta^0} M(0,0) = g_2 \cdots g_n \neq 0$ if and only if $g_j \neq 0$, $j = 2, \ldots, n$.

In order to calculate the value of (2.12) we need a suitable expression of the orbit $(\theta(t), r(t))$.

Lemma 2.10 *Let ρ_1 be the function*

$$\begin{aligned} \rho_1(z) &= -a_1(\hat r_s) + \frac{2\sqrt{\varepsilon}}{\cosh(2\pi\sqrt{\varepsilon}\, z)}, \\ \phi_j(z) &= (\alpha_j^0(\hat r_s) - \alpha_j^1(\hat r_s) a_1(\hat r_s)) z + \frac{2}{\pi} \alpha_j^1(\hat r_s) \arctan(\tanh(\pi\sqrt{\varepsilon}\, z)), \end{aligned}$$

for $2 \leq j \leq n$. Then, there exists $u : \mathbb{R} \to \mathbb{R}$, an analytic, strictly monotone function, with $u(0) = 0$ and $\lim_{|t|\to\infty} |u(t)| = \infty$ such that

$$\begin{aligned} r_1(t) &= \rho_1(u(t)) \\ \theta_j(t) &= \phi_j(u(t)), \qquad 2 \leq j \leq n. \end{aligned}$$

Proof. Denoting $x = (\theta, r)$, the system (2.6) can be written in the form

$$\dot x = \frac{1}{q(x)} f(x). \tag{2.13}$$

Now, if $y = (\phi, \rho) \in \mathbb{T}^n \times \mathbb{R}^n$, we consider the auxiliary system $\dot y = f(y)$, that is

$$\begin{aligned} \dot\phi_1 &= \rho_1 + a_1(\hat\rho), & \dot\rho_1 &= 2\pi\varepsilon \sin(2\pi\phi_1), \\ \dot{\hat\phi} &= \hat\alpha^0(\hat\rho) + \hat\alpha^1(\hat\rho)\rho_1, & \dot{\hat\rho} &= 0. \end{aligned} \tag{2.14}$$

The geometry of the phase space of (2.6) is the same as that of (2.14), the latter being the unperturbed pendulum cross a rotation on $\mathbb{T}^{n-1}$. In the (ϕ_1, ρ_1) variables, the solution of (2.14) along the pendulum separatrix that passes though the point $(1/2, -a_1(\hat\rho) + 2\sqrt{\varepsilon})$ at $z = 0$ is

$$\begin{aligned}\phi_1(z) &= \frac{2}{\pi}\arctan(e^{2\pi\sqrt{\varepsilon}\,z}) = \frac{1}{2} + \frac{2}{\pi}\arctan(\tanh(\pi\sqrt{\varepsilon}z))\\ \rho_1(z) &= -a_1(\hat r_s) + \frac{2\sqrt{\varepsilon}}{\cosh(2\pi\sqrt{\varepsilon}\,z)}.\end{aligned}$$

Therefore, if $\phi_j(0) = 0$, for $j = 2, \ldots, n$,

$$\begin{aligned}\phi_j(z) &= \int_0^z (\alpha_j^0(\hat\rho) + \alpha_j^1(\hat\rho)\rho_1(\xi))\,d\xi\\ &= \Big(\alpha_j^0(\hat\rho) - \alpha_j^1(\hat\rho)a_1(\hat r_s)\Big)z + \frac{2}{\pi}\alpha_j^1(\hat\rho)\arctan(\tanh(\pi\sqrt{\varepsilon}\,z)).\end{aligned}$$

Now we are going to compare the solutions $x(t)$ of $\dot x = f(x)/q(x)$ and $y(t)$ of $\dot y = f(y)$ such that $x(0) = y(0)$. They must be related by a reparametrization of time: $x(t) = y(u(t))$, where u is the solution of

$$\dot u = \frac{1}{q(y(u))} = \frac{1}{q(\rho(u))}, \tag{2.15}$$

such that $u(0) = 0$. Equation (2.15) is well defined if ε is small enough because ρ stays in B'. Moreover, since $q \neq 0$, $1/q$ has upper and lower bounds in B', both non zero, and, hence, u is monotone, is defined for all $t \in \mathbb{R}$, and tends to $\pm\infty$ when $|t| \to \infty$. □

The integral (2.12) can be rewritten as

$$g_j = 4\pi^2 \int_{-\infty}^{\infty} \frac{1}{q(\rho_1(u(z)), \hat r_s)} \left(\frac{2\sqrt{\varepsilon}}{\cosh(2\pi\sqrt{\varepsilon}\,u(z))}\right)^2 \cos(2\pi\phi_j(u(z)))\,dz.$$

Since $t = u(z)$ is a well defined change of variables we obtain the following expression for g_j (except for the sign, which depends on the sign of the function q)

$$g_j = 4\pi^2 \int_{-\infty}^{\infty} \left(\frac{2\sqrt{\varepsilon}}{\cosh(2\pi\sqrt{\varepsilon}\,t)}\right)^2 \cos(2\pi\phi_j(t))\,dt. \tag{2.16}$$

The asymptotic expressions of these integrals are explicitly calculated in Appendix A, and the expression obtained there depends on α_j^1. If $2\alpha_j^1 \notin \mathbb{Z}$ we have that

$$g_j = -e^{-\frac{\pi}{2}\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}}}\sin(2\alpha_j^1\pi)\left(\frac{\Gamma(2\alpha_j^1-1)}{(2\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}})^{2\alpha_j^1-1}}+\mathcal{O}\left(\frac{1}{(\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}})^{2\alpha_j^1}}\right)\right),$$

if $2\alpha_j^1 = 2, 3, \dots$ then

$$g_j = 2\pi e^{-\frac{3\pi}{2}\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}}}\left(\frac{(-2)^{2\alpha_j^1}}{(2\alpha_j^1+1)!}\left(\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}}\right)^{2\alpha_j^1+1}+\mathcal{O}\left((\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}})^{2\alpha_j^1}\right)\right),$$

if $2\alpha_j^1 = 1, 0, -1, \dots$ then

$$g_j = 2\pi e^{-\frac{\pi}{2}\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}}}\left(\frac{2^{-2\alpha_j^1}}{(1-2\alpha_j^1)!}\left(\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}}\right)^{1-2\alpha_j^1}+\mathcal{O}\left((\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}})^{-2\alpha_j^1}\right)\right).$$

We recall that

$$q\omega_j = \alpha_j^0 - \alpha_j^1 a_1,$$

which, by the hypotheses over the curve γ stated in the end of Subsection 2.3, is nonzero along the curve. Therefore, the exponent

$$\frac{\alpha_j^0-\alpha_j^1 a_1}{\sqrt{\varepsilon}}$$

stays away from zero. Also, since α_j^1 is constant integer or, otherwise, stays away from integers, we can conclude that the coefficients g_j are bounded from below along the curve γ.

Remark 2.11 *The value of g_j is exponentially small in the first perturbative parameter ε. It is necessary, then, to assume the second perturbative parameter μ to be also exponentially small in ε in order to assure that the Melnikov function gives the right asymptotic value of the splitting. Moreover, g_j are a measure of the angles between both invariant manifolds, which is also exponentially small in ε in every direction.*

2.6 The transition chain

Now, for a fixed value of ε, there is a value $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$ and for all $s \in [0,1]$ the invariant tori $\mathbb{T}_s$ of $H_{\varepsilon\mu}$ have invariant manifolds $W^{u,s}\mathbb{T}_s$ which intersect transversally in the energy level.

Consider $s \in [0,1]$. Since the unstable manifold, $W^u\mathbb{T}_s$ of $\mathbb{T}_s$ intersects transversally its stable manifold, $W^u\mathbb{T}_s$ must intersect transversely $W^s\mathbb{T}_{s'}$, for all s' such that $|s - s'|$ is small enough (and depending on s). In this way, and using that the subset of $[0,1]$ such that the corresponding $\mathbb{T}_s$ has an irrational dynamics on it is dense in $[0,1]$, it is possible to choose a finite sequence $s_0, \ldots, s_N$ in $[0,1]$, N depending on ε, such that

1. $s_0 = 0$, $s_N = 1$,
2. the dynamics on $\mathbb{T}_j$ is an irrational rotation, $j = 0, \ldots, N-1$,
3. the manifold $W^u\mathbb{T}_j$ intersects transversally in the energy level $W^s\mathbb{T}^{j+1}$, $j = 0, \ldots, N-1$.

The corresponding sequence of tori is called transition chain.

Since the intersection of the invariant manifolds is transversal, we can apply the Inclination Lemma (Theorem 4.5) to the time τ map, where τ plays an auxiliary role and it is chosen in such a way that the rotation on the tori remains ergodic for the map. Indeed, the rotation for the map is ergodic for a G_δ dense set in $\mathbb{R}$. Hence, $W^u\mathbb{T}_{j+1} \subset \overline{W^u\mathbb{T}_j}$, $j = 0, \ldots, N-1$. Therefore, $W^u\mathbb{T}_N \subset \overline{W^u\mathbb{T}_0}$, which implies Theorem 2.1.

3 Invariant manifolds of partially hyperbolic tori

3.1 Existence of invariant manifolds. Lipschitz case

In this section we study the existence of invariant manifolds of partially hyperbolic tori of families of maps which are of the form a linear map f in $\mathbb{R}^m \times \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^n$ plus a Lipschitz map with certain conditions.

Let $\delta > 0$, $\mathcal{M}_\delta = \mathbf{B}^m_\delta(0) \times \mathbb{T}^n \times \mathbf{B}^m_\delta(0) \times \mathbf{B}^n_\delta(0)$ and $\mathcal{M} = \mathbb{R}^m \times \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^n$, where $\mathbf{B}^k_\delta(0)$ stands for the ball of radius δ centered at 0 in $\mathbb{R}^k$. We shall denote by $\mathbb{T}_0$ the n dimensional torus $\{(0,\theta,0,0),\ \theta \in \mathbb{T}^n\}$. We want to consider maps of the form $f_\nu : \mathcal{M}_\delta \to \mathcal{M}$, $\nu \in [-\mu,\mu]$, defined by $(x^*,\theta^*,y^*,r^*) = f_\nu(x,\theta,y,r)$, where f_ν is a small perturbation of

$$\begin{aligned} x^* &= \Lambda_+ x, & y^* &= \Lambda_- y,\\ \theta^* &= \theta + \omega(r), & r^* &= r, \end{aligned}$$

where Λ_+, $\Lambda_- \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)$, or even the more general case where ω depends on r, θ and ν. In fact we shall assume the slightly more general case where f_ν is defined by

$$\begin{aligned} x^* &= \Lambda_+ x + \mathbf{X}(x,\theta,y,r,\nu), & y^* &= \Lambda_- y + \mathbf{Y}(x,\theta,y,r,\nu),\\ \theta^* &= \theta + \mathbf{\Theta}(x,\theta,y,r,\nu), & r^* &= r + \mathbf{R}(x,\theta,y,r,\nu), \end{aligned} \tag{3.1}$$

where $\mathbf{\Theta}$ is not necessarily small and $\nu \in [-\mu,\mu]$ is a parameter.

We shall assume the following hypotheses on f_ν:

H1 f_ν is continuous and depends continuously with respect to ν.

H2 If $\lambda_- = \|\Lambda_-\|$ and $\lambda_+^{-1} = \|\Lambda_+{}^{-1}\|$, then $0 < \lambda_-,\ \lambda_+^{-1} < 1$.

H3 For $F = \mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}$ we have $F(0,\theta,0,0,\nu) = 0$ for all $\theta \in \mathbb{T}^n$.

H4 For $F = \mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}$ there exists $K > 0$ such that

$$\begin{aligned} &\|F(x,\theta,y,r,\nu) - F(x',\theta',y',r',\nu)\| \le \\ &\qquad K(\delta+\mu)(\|x-x'\| + \|\theta-\theta'\| + \|y-y'\| + \|r-r'\|) \end{aligned}$$

and

$$\|F(x,\theta,y,r,\nu) - F(x,\theta',y,r,\nu)\| \le K(\delta+\mu)\,(\|x\| + \|y\| + \|r\|)\,\|\theta-\theta'\|$$

in $\mathcal{M}_\delta$.

H5 The function $\mathbf{\Theta}$ verifies

$$\begin{aligned} &\|\mathbf{\Theta}(x,\theta,y,r,\nu) - \mathbf{\Theta}(x',\theta',y',r',\nu)\| \le \\ &\qquad K(\delta+\mu)(\|x-x'\| + \|\theta-\theta'\| + \|y-y'\|) + K\|r-r'\| \end{aligned}$$

in $\mathcal{M}_\delta$.

Remark 3.1 *1) In general, μ and δ will be small. We let the bounds on hypotheses H4 and H5 depend on μ and δ instead of ν, x and y. Therefore they are larger and independent of the parameter and the variables x, y.*

2) As a consequence of the invariance, we have that the points belonging to the stable manifold do converge to the torus by the successive images of f_ν, and the same happens to the points in the unstable manifold under iterations of f_ν^{-1}.

Theorem 3.2 *Let $f_\nu : \mathcal{M}_\delta \to \mathcal{M}$ satisfying hypotheses H1 to H5. Then there exist δ_0 and μ_0 such that, if $0 < \delta < \delta_0$ and $|\mu| < \mu_0$, the invariant torus $\mathbb{T}_0$ of the map f_ν defined by (3.1) has invariant manifolds related to the x and y directions which can be represented as graphs of Lipschitz functions $\gamma_\nu^u(x,\theta)$, the unstable one, and $\gamma_\nu^s(y,\theta)$, the stable one. Their Lipschitz constants are of order $\delta + \mu$.*

Proof. We shall prove the existence of $W^u\mathbb{T}_0$. The existence of $W^s\mathbb{T}_0$ is then easily obtained by considering f_ν^{-1}. We follow the ideas in [11, 12] to prove the theorem by means of the *graph transform*. Also, the norm we shall use is suggested in [1]. Although f_ν depends on ν, for the sake of simplicity we shall not write it explicitely.

First we introduce some notation. Throughout the proof we shall consider f defined on $\mathcal{M}_\delta = \mathcal{M}_\delta^1 \times \mathcal{M}_\delta^2$, where $\mathcal{M}_\delta^1 = \mathbf{B}_\delta^m(0) \times \mathbb{R}^n$, $\mathcal{M}_\delta^2 = \mathbf{B}_\delta^m(0) \times \mathbb{R}^n$. That is, we shall consider $\theta \in \mathbb{R}^n$ and all the functions that will appear will be 2π periodic with respect each component of θ.

In $\mathcal{M}_\delta^1$ we take the norm $\|(x,\theta)\| = \max(\|x\|, \|\theta\|)$ and in $\mathcal{M}_\delta^2$, $\|(y,r)\| = \max(\|y\|, \|r\|)$, where $\|\cdot\|$ represents any norm on $\mathbb{R}^k$. We introduce the Banach space

$$\begin{aligned}\Gamma \;=\; & \{\gamma : \mathcal{M}_\delta^1 \to \mathcal{M}_\delta^2;\ \gamma \text{ is Lipschitz, } 2\pi \text{ periodic in } \theta, \\ & \sup_{\substack{x,\theta \\ x\neq 0}} \|x\|^{-1}\|\gamma(x,\theta)\| < \infty\},\end{aligned}$$

with the norm $\|\gamma\| = \sup_{\substack{x,\theta \\ x\neq 0}} \|x\|^{-1}\|\gamma(x,\theta)\|$. Notice that if $\gamma \in \Gamma$, $\|\gamma(x,\theta)\| \leq \|x\|\,\|\gamma\|$ and therefore

$$\gamma(0,\theta) = 0, \quad \forall \theta \in \mathbb{R}^n. \tag{3.2}$$

We also consider

$$\begin{aligned}\Gamma_c = \quad & \{\gamma \in \Gamma;\ \text{lip}\ \gamma \le c, \|\gamma(x,\theta) - \gamma(x,\theta')\| \le c\|x\|\,\|\theta - \theta'\|,\\ & \forall x \in \mathbf{B}^m_\delta(0), \forall \theta, \theta' \in \mathbb{R}^n\},\end{aligned}$$

which is a closed subset of Γ. We shall write $\gamma = (\gamma_1, \gamma_2)$, with $\gamma_1 : \mathcal{M}^1_\delta \longrightarrow \mathbb{R}^m$, $\gamma_2 : \mathcal{M}^1_\delta \longrightarrow \mathbb{R}^n$, and $f = (f_1, f_2)$, with $f_1 : \mathcal{M}_\delta \to \mathbb{R}^m \times \mathbb{R}^n$, $f_2 : \mathcal{M}_\delta \to \mathbb{R}^m \times \mathbb{R}^n$.

If the graph of $\gamma(x,\theta)$ is invariant, it must verify $f(\text{graph}(\gamma)) \cap \mathcal{M}_\delta \subset \text{graph}(\gamma)$. With the previous notation the condition of invariance has the form

$$f_2 \circ (I, \gamma) = \gamma \circ f_1 \circ (I, \gamma),$$

where I stands for the identity map in $\mathbb{R}^m \times \mathbb{R}^n$. We shall look for γ as a fixed point of L on Γ_c, where L is defined by

$$L(\gamma) = f_2 \circ (I, \gamma) \circ (f_1 \circ (I, \gamma))^{-1}.$$

To apply the Fixed Point Theorem we shall prove that if c, δ and μ are small enough, L is actually well defined and it is a contraction on Γ_c.

We write $L(\gamma) = \phi_2(\gamma) \circ \phi_1(\gamma)^{-1}$, with

$$\phi_1(\gamma)\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} \Lambda_+ x \\ \theta \end{pmatrix} + \begin{pmatrix} \mathbf{X}(x,\theta,\gamma(x,\theta)) \\ \boldsymbol{\Theta}(x,\theta,\gamma(x,\theta)) \end{pmatrix} = A\begin{pmatrix} x \\ \theta \end{pmatrix} + Q\begin{pmatrix} x \\ \theta \end{pmatrix} \tag{3.3}$$

and

$$\begin{aligned}\phi_2(\gamma)\begin{pmatrix} x \\ \theta \end{pmatrix} &= \begin{pmatrix} \Lambda_- \gamma_1(x,\theta) \\ \gamma_2(x,\theta) \end{pmatrix} + \begin{pmatrix} \mathbf{Y}(x,\theta,\gamma(x,\theta)) \\ \mathbf{R}(x,\theta,\gamma(x,\theta)) \end{pmatrix} \\ &= B\begin{pmatrix} x \\ \theta \end{pmatrix} + P\begin{pmatrix} x \\ \theta \end{pmatrix}. \end{aligned} \tag{3.4}$$

The previous formulas define the maps A, B, Q, P. Notice that B is not linear. We introduce the parameter $k = 2K(\delta + \mu + c)$, and we suppose from now on that $\delta + \mu \le 1$.

Lemma 3.3 *Let $\gamma \in \Gamma_c$. Then*

1. $\text{lip}\ Q, \text{lip}\ P < k$.
2. $\text{lip}\ \phi_2(\gamma) \le c + k$.

Proof. If $F = \mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}$, from H4, we have

$$\|F(x,\theta,\gamma(x,\theta)) - F(x',\theta',\gamma(x',\theta'))\| \leq 2K(\delta+\mu)(1+c)\|(x,\theta)-(x',\theta')\|,$$

and, from H5,

$$\|\mathbf{\Theta}(x,\theta,\gamma(x,\theta)) - \mathbf{\Theta}(x',\theta',\gamma(x',\theta'))\| \leq (K(\delta+\mu)(2+c)+Kc)\|(x,\theta)-(x',\theta')\|.$$

Therefore 1 is proved. Statement 2 follows immediately from 1 and the fact that lip $A \leq \lambda_+$ and lip $B \leq c$. □

Lemma 3.4 *Let δ, μ and c be positive numbers such that $\delta + \mu \leq 1$ and $k \leq \frac{\lambda_+ - 1}{2\lambda_+ - 1} < 1$. Then for any $\gamma \in \Gamma_c$, $\phi_1(\gamma)$ is invertible, $\phi_1(\gamma)^{-1}$ is also defined on $\mathcal{M}^1_\delta$ and has Lipschitz constant less or equal than $(1-k)^{-1}$.*

Proof. Throughout this proof we shall denote $\phi_1(\gamma)$ by ϕ, Λ_+ by Λ and λ_+ by λ. Note that we need a global inverse of ϕ.

It is clear that the function Q defined in (3.3) verifies $Q_1(0,\theta) = 0$, $\forall\theta \in \mathbb{R}^n$. To check that the image of ϕ covers $\mathcal{M}^1_\delta$ we seek for a right inverse of it. We look for such an inverse $\psi(y,\varphi)$ defined on $\mathbf{B}^m_\delta(0) \times \mathbb{R}^n$, 2π periodic with respect to any component of $\varphi \in \mathbb{R}^n$ and having the form $\psi(y,\varphi) = A^{-1}(y,\varphi) + R(y,\varphi)$. The condition $\phi \circ \psi = I$ is equivalent to

$$(\Lambda R_1(y,\varphi), R_2(y,\varphi)) + Q(\Lambda^{-1}y + R_1(y,\varphi), \varphi + R_2(y,\varphi)) = 0.$$

We introduce the space

$$\begin{aligned}\Omega &= \{R : \mathbf{B}^m_\delta(0) \times \mathbb{R}^n \longrightarrow \mathbb{R}^m \times \mathbb{R}^n;\ R \text{ is Lipschitz },\\ &\quad 2\pi \text{ periodic in } \varphi,\ \text{lip } R \leq m_0,\ R_1(0,\varphi) = 0, \forall\varphi \in \mathbb{R}^n\},\end{aligned}$$

with $m_0 = k(1-k)^{-1}$ and the norm $\|R\| = \max(\|R_1\|, \|R_2\|)$, and we define $H : \Omega \longrightarrow \Omega$ by $H(R)(y,\varphi) = (H_1(R)(y,\varphi), H_2(R)(y,\varphi))$, where

$$\begin{aligned}H_1(R)(y,\varphi) &= -\Lambda^{-1}Q_1(\Lambda^{-1}y + R_1(y,\varphi), \varphi + R_2(y,\varphi)),\\ H_2(R)(y,\varphi) &= -Q_2(\Lambda^{-1}y + R_1(y,\varphi), \varphi + R_2(y,\varphi)).\end{aligned}$$

We check that H is well defined. First, if $\|y\| \leq \delta$ we have that

$$\|\Lambda^{-1}y + R_1(y,\varphi)\| \leq (\lambda^{-1} + m_0)\delta.$$

Also, for all $\varphi \in \mathbb{R}^n$, $H_1(R)(0,\varphi) = -\Lambda^{-1}Q_1(R_1(0,\varphi),\varphi + R_2(0,\varphi)) = 0$. On the other hand

$$\begin{aligned}
&\|H_1(R)(y,\varphi) - H_1(R)(y',\varphi')\| \\
&\leq \frac{k}{\lambda}\max\{\|\Lambda^{-1}(y-y') + R_1(y,\varphi) - R_1(y',\varphi')\|, \\
&\qquad\qquad \|\varphi - \varphi' + R_2(y,\varphi) - R_2(y',\varphi')\|\} \\
&\leq \frac{k}{\lambda}(1+m_0)\|(y,\varphi) - (y',\varphi')\|
\end{aligned}$$

and, analogously,

$$\|H_2(R)(y,\varphi) - H_2(R)(y',\varphi')\| \leq k(1+m_0)\|(y,\varphi) - (y',\varphi')\|.$$

Notice that $k(1+m_0) = m_0$. To see that H is a contraction we check that

$$\|H_1(R) - H_1(R')\| \leq \frac{k}{\lambda}\max(\|R_1 - R_1'\|, \|R_2 - R_2'\|)$$

and analogously

$$\|H_2(R) - H_2(R')\| \leq k\max(\|R_1 - R_1'\|, \|R_2 - R_2'\|).$$

Then the Fixed Point Theorem gives us a unique fixed point of H in Ω which is a right inverse of ϕ. On the other hand, ϕ is injective because

$$\|\phi(x,\theta) - \phi(x',\theta')\| \geq (1-k)\|(x,\theta) - (x',\theta')\|.$$

Hence ϕ has a global inverse in $\mathbf{B}^k_\delta(0)\times\mathbb{R}^n$ so that the restriction to $\mathbf{B}^k_\delta(0)\times\mathbb{T}^n$ is injective. Its Lipschitz constant is bounded by $1+m_0 = (1-k)^{-1}$. □

We shall write $(\tilde{x},\tilde{\theta}) = \phi_1(\gamma)^{-1}(x,\theta)$ so that it must be understood that $\tilde{x} = \tilde{x}(x,\theta)$ and $\tilde{\theta} = \tilde{\theta}(x,\theta)$.

Lemma 3.5 *If $\gamma \in \Gamma_c$ and $(x,\theta) \in \mathcal{M}^1_\delta$ we have*

$$\|\tilde{x}\| \leq (\lambda_+ - k)^{-1}\|x\|. \tag{3.5}$$

Furthermore

$$\sup_{(x,\theta)} \|x\|^{-1}\,\|\gamma(\tilde{x},\tilde{\theta})\| \leq \frac{c}{\lambda_+ - k} \tag{3.6}$$

and, for $F = \mathbf{Y}$, $\mathbf{R}$,

$$\sup_{(x,\theta)} \|x\|^{-1}\,\|F(\tilde{x},\tilde{\theta},\gamma(\tilde{x},\tilde{\theta}))\| \leq \frac{k}{\lambda_+ - k}. \tag{3.7}$$

Proof. The inequalities in (3.5) follow from the relation

$$\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} \Lambda_+\tilde{x} + \mathbf{X}(\tilde{x}, \tilde{\theta}, \gamma(\tilde{x}, \tilde{\theta})) \\ \tilde{\theta} + \mathbf{\Theta}(\tilde{x}, \tilde{\theta}, \gamma(\tilde{x}, \tilde{\theta})) \end{pmatrix},$$

the hypotheses H2, H3, H4 and (3.2). Indeed

$$\|x\| \geq \lambda_+\|\tilde{x}\| - \|\mathbf{X}(\tilde{x}, \tilde{\theta}, \gamma(\tilde{x}, \tilde{\theta}))\| \geq (\lambda_+ - k)\|\tilde{x}\|.$$

Inequality (3.6) follows from the definition of Γ_c,

$$\|x\|^{-1}\|\gamma(\tilde{x}, \tilde{\theta})\| = \|x\|^{-1}\|\gamma(\tilde{x}, \tilde{\theta}) - \gamma(0, \tilde{\theta})\| \leq c\|x\|^{-1}\|\tilde{x}\| \leq c(\lambda_+ - k)^{-1}.$$

Finally for statement (3.7) we use that $F(0, \tilde{\theta}, 0, 0) = 0$ and hypothesis H4,

$$\begin{aligned} \|x\|^{-1}\|F(\tilde{x}, \tilde{\theta}, \gamma(\tilde{x}, \tilde{\theta}))\| &\leq \|x\|^{-1}\|F(\tilde{x}, \tilde{\theta}, \gamma(\tilde{x}, \tilde{\theta})) - F(0, \tilde{\theta}, \gamma(0, \tilde{\theta}))\| \\ &\leq K(\delta + \mu)(1 + 2c)(\lambda_+ - k)^{-1}. \end{aligned}$$

□

In order to check that $\tilde{\gamma} = L(\gamma)$ satisfies $\|\tilde{\gamma}(x, \theta) - \tilde{\gamma}(x, \theta')\| \leq c\|x\|\,\|\theta - \theta'\|$, we need the following lemma:

Lemma 3.6 *Let $\gamma \in \Gamma_c$ and $\psi(\gamma) = \phi_1(\gamma)^{-1}$. Let $(x, \theta), (x, \theta') \in \mathcal{M}_\delta^1$, $(\tilde{x}, \tilde{\theta}) = \psi(\gamma)(x, \theta)$ and $(\tilde{x}', \tilde{\theta}') = \psi(\gamma)(x, \theta')$. We have*

1. $\|\gamma \circ \psi(x, \theta) - \gamma \circ \psi(x, \theta')\| \leq c_1(\delta, \mu, c)\|x\|\,\|\theta - \theta'\|$,
2. *for* $F = \mathbf{X}, \mathbf{Y}, \mathbf{R}$,

$$\|F(\tilde{x}, \tilde{\theta}, \gamma(\tilde{x}, \tilde{\theta})) - F(\tilde{x}', \tilde{\theta}', \gamma(\tilde{x}', \tilde{\theta}'))\| \leq c_2(\delta, \mu, c)\|x\|\,\|\theta - \theta'\|,$$

3. *if* δ, μ *are small enough,* $\|\tilde{\gamma}(x, \theta) - \tilde{\gamma}(x, \theta')\| \leq c\|x\|\,\|\theta - \theta'\|$,

where

$$c_1(\delta, \mu, c) = \frac{c\lambda_+}{(\lambda_+ - k)(1 - k)(\lambda_+ - (c + 1)k)}$$

and

$$c_2(\delta, \mu, c) = K(\delta + \mu)\left(\frac{\lambda_+ + 2c(\lambda_+ - k)}{(1 - k)(\lambda_+ - k)^2} + \frac{(2\lambda_+ - k)c_1(\delta, \mu, c)}{\lambda_+ - k}\right).$$

Proof. 1. We begin by establishing the following bound for $F = \mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}$,

$$\begin{aligned}
&\|F(\tilde{x},\tilde{\theta},\gamma(\tilde{x},\tilde{\theta})) - F(\tilde{x}',\tilde{\theta}',\gamma(\tilde{x}',\tilde{\theta}'))\| \\
&\quad\leq \|F(\tilde{x},\tilde{\theta},\gamma(\tilde{x},\tilde{\theta})) - F(\tilde{x},\tilde{\theta}',\gamma(\tilde{x},\tilde{\theta}))\| \\
&\qquad + \|F(\tilde{x},\tilde{\theta}',\gamma(\tilde{x},\tilde{\theta})) - F(\tilde{x}',\tilde{\theta}',\gamma(\tilde{x}',\tilde{\theta}'))\| \\
&\quad\leq K(\delta+\mu)\frac{1+2c}{(1-k)(\lambda_+ - k)}\|x\|\,\|\theta-\theta'\| \\
&\qquad + K(\delta+\mu)(\|\tilde{x}-\tilde{x}'\| + 2\|\gamma(\tilde{x},\tilde{\theta}) - \gamma(\tilde{x}',\tilde{\theta}')\|)
\end{aligned} \tag{3.8}$$

where we have used H4, (3.5), (3.6) and the fact that $\tilde{\theta}$, as a component of $\psi(\gamma)$, is Lipschitz with Lipschitz constant bounded by $(1-k)^{-1}$.

On the other hand, we have that

$$\begin{aligned}
\|\gamma\circ\psi(x,\theta) - \gamma\circ\psi(x,\theta')\| &\leq \|\gamma(\tilde{x},\tilde{\theta}) - \gamma(\tilde{x},\tilde{\theta}')\| + \|\gamma(\tilde{x},\tilde{\theta}') - \gamma(\tilde{x}',\tilde{\theta}')\| \\
&\leq c\|\tilde{x}\|\,\|\tilde{\theta}-\tilde{\theta}'\| + c\|\tilde{x}-\tilde{x}'\|.
\end{aligned} \tag{3.9}$$

The first term of (3.9) is bounded by

$$c\|\tilde{x}\|\,\|\tilde{\theta}-\tilde{\theta}'\| \leq \frac{c(1-k)^{-1}}{\lambda_+ - k}\|x\|\,\|\theta-\theta'\|. \tag{3.10}$$

To get a proper bound for the second term, we take into account that, as a consequence of the definition of $\tilde{x}$, $\tilde{\theta}$, $\tilde{x}'$ and $\tilde{\theta}'$,

$$x = \Lambda_+\tilde{x} + \mathbf{X}(\tilde{x},\tilde{\theta},\gamma(\tilde{x},\tilde{\theta})), \qquad x = \Lambda_+\tilde{x}' + \mathbf{X}(\tilde{x}',\tilde{\theta}',\gamma(\tilde{x}',\tilde{\theta}')).$$

Subtracting and using (3.8)

$$\begin{aligned}
\|\Lambda_+(\tilde{x}-\tilde{x}')\| &= \|\mathbf{X}(\tilde{x},\tilde{\theta},\gamma(\tilde{x},\tilde{\theta})) - \mathbf{X}(\tilde{x}',\tilde{\theta}',\gamma(\tilde{x}',\tilde{\theta}'))\| \\
&\leq K(\delta+\mu)\frac{1+2c}{(1-k)(\lambda_+ - k)}\|x\|\,\|\theta-\theta'\| \\
&\quad + K(\delta+\mu)\|\tilde{x}-\tilde{x}'\| + 2K(\delta+\mu)\|\gamma(\tilde{x},\tilde{\theta}) - \gamma(\tilde{x}',\tilde{\theta}')\|
\end{aligned} \tag{3.11}$$

and then

$$\|\tilde{x}-\tilde{x}'\| \leq \frac{k}{(1-k)(\lambda_+ - k)^2}\|x\|\,\|\theta-\theta'\| + \frac{k}{\lambda_+ - k}\|\gamma(\tilde{x},\tilde{\theta}) - \gamma(\tilde{x}',\tilde{\theta}')\|. \tag{3.12}$$

Putting (3.12) and (3.10) into (3.9) we get 1.

2 follows from (3.8), (3.12) and 1.

3. We have that

$$\tilde{\gamma}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} \Lambda_{-}\gamma_1 \circ \psi(x,\theta) + \mathbf{Y} \circ (I,\gamma) \circ \psi(x,\theta) \\ \gamma_2 \circ \psi(x,\theta) + \mathbf{R} \circ (I,\gamma) \circ \psi(x,\theta) \end{pmatrix}.$$

Then, from 1 and 2 we have

$$\|\tilde{\gamma}_1(x,\theta) - \tilde{\gamma}_1(x,\theta')\| \leq (\lambda_{-}c_1(\delta,\mu,c) + c_2(\delta,\mu,c))\|x\|\,\|\theta-\theta'\|,$$

and analogously

$$\|\tilde{\gamma}_2(x,\theta) - \tilde{\gamma}_2(x,\theta')\| \leq (c_1(\delta,\mu,c) + c_2(\delta,\mu,c))\|x\|\,\|\theta-\theta'\|.$$

Hence since $c_1(\delta,\mu,c) = c\lambda_{+}^{-1} + \mathcal{O}(\delta+\mu)$ and $c_2(\delta,\mu,c) = \mathcal{O}(\delta+\mu)$, we obtain the desired inequality. □

The last step in order to prove that L is a well defined map is to check that $\tilde{\gamma} = L(\gamma)$ is also c-Lipschitz.

Lemma 3.7 *Let $\gamma \in \Gamma_c$ and $\tilde{\gamma} = L(\gamma)$. Let (x,θ), $(x',\theta) \in \mathcal{M}_\delta^1$, $(\overline{x},\overline{\theta}) = \psi(\gamma)(x,\theta)$ and $(\overline{x}',\overline{\theta}') = \psi(\gamma)(x',\theta)$. Then*

1. for $F = \mathbf{X}$, $\mathbf{Y}$ and $\mathbf{R}$ we have

$$\|F(\overline{x},\overline{\theta},\gamma(\overline{x},\overline{\theta})) - F(\overline{x}',\overline{\theta}',\gamma(\overline{x}',\overline{\theta}'))\| \leq \frac{2K(\delta+\mu)(1+c)}{1-k}\|x-x'\|,$$

2. if δ, μ are small enough we have

$$\|\tilde{\gamma}(x,\theta) - \tilde{\gamma}(x',\theta')\| \leq c\|(x,\theta)-(x',\theta')\|.$$

The constant c may be taken $\mathcal{O}(\delta+\mu)$

Proof. 1. We have

$$\|F(\overline{x},\overline{\theta},\gamma(\overline{x},\overline{\theta})) - F(\overline{x}',\overline{\theta}',\gamma(\overline{x}',\overline{\theta}'))\| \leq \frac{2K(\delta+\mu)(1+c)}{1-k}\|x-x'\|,$$

where we have used H4 and that $\operatorname{lip}\psi(\gamma) \leq (1-k)^{-1}$.

We have bound $\|\gamma(\overline{x},\overline{\theta}) - \gamma(\overline{x}',\overline{\theta}')\|$ in a rough manner. Next we shall bound it more carefully in order to gain, essentially, a factor λ_+^{-1} in the bound.

2. We have

$$\begin{aligned} \|\gamma(\overline{x},\overline{\theta}) - \gamma(\overline{x}',\overline{\theta}')\| &\leq \|\gamma(\overline{x},\overline{\theta}) - \gamma(\overline{x},\overline{\theta}')\| + \|\gamma(\overline{x},\overline{\theta}') - \gamma(\overline{x}',\overline{\theta}')\| \\ &\leq \frac{c\delta}{(1-k)(\lambda_+ - k)}\|x - x'\| + c\|\overline{x} - \overline{x}'\|. \end{aligned}$$

Although we can use $\operatorname{lip}\psi(\gamma)$ to bound $\|\overline{x} - \overline{x}'\|$ here we need a better estimate. From

$$x = \Lambda_+\overline{x} + \mathbf{X}(\overline{x},\overline{\theta},\gamma(\overline{x},\overline{\theta})), \qquad x' = \Lambda_+\overline{x}' + \mathbf{X}(\overline{x}',\overline{\theta}',\gamma(\overline{x}',\overline{\theta}')),$$

we have, using 1,

$$\|\overline{x} - \overline{x}'\| \leq \lambda_+^{-1}\|x - x'\| + \frac{\lambda_+^{-1}k}{1-k}\|x - x'\| = \frac{\lambda_+^{-1}}{1-k}\|x - x'\|, \tag{3.13}$$

and therefore

$$\|\gamma(\overline{x},\overline{\theta}) - \gamma(\overline{x}',\overline{\theta}')\| \leq \frac{c}{1-k}\left(\frac{\delta}{\lambda_+ - k} + \frac{1}{\lambda_+}\right)\|x - x'\|. \tag{3.14}$$

On the other hand

$$\begin{aligned} \|\tilde{\gamma}(x,\theta) - \tilde{\gamma}(x',\theta')\| &\leq \|\tilde{\gamma}(x,\theta) - \tilde{\gamma}(x,\theta')\| + \|\tilde{\gamma}(x,\theta') - \tilde{\gamma}(x',\theta')\| \\ &\leq c\delta\|\theta - \theta'\| + \|\tilde{\gamma}(x,\theta') - \tilde{\gamma}(x',\theta')\|, \end{aligned} \tag{3.15}$$

where we have used 3 of Lemma 3.6. We bound the second difference, writing $(\hat{x},\hat{\theta}) = \psi(\gamma)(x,\theta')$, $(\hat{x}',\hat{\theta}') = \psi(\gamma)(x',\theta')$. Since these points are images by $\psi(\gamma)$ of points which only differ on the first variable we can use part 1 of the lemma and (3.14)

$$\begin{aligned} &\|\tilde{\gamma}_1(x,\theta') - \tilde{\gamma}_1(x',\theta')\| \\ &\quad\leq \lambda_-\|\gamma_1(\hat{x},\hat{\theta}) - \gamma_1(\hat{x}',\hat{\theta}')\| + \|\mathbf{Y}(\hat{x},\hat{\theta},\gamma(\hat{x},\hat{\theta})) - \mathbf{Y}(\hat{x}',\hat{\theta}',\gamma(\hat{x},\hat{\theta}))\| \\ &\quad\leq \frac{1}{1-k}\left(\frac{\lambda_-\delta c}{\lambda_+ - k} + \frac{\lambda_- c}{\lambda_+} + 2K(\delta+\mu)(1+c)\right)\|x - x'\| \end{aligned} \tag{3.16}$$

and analogously

$$\|\tilde{\gamma}_2(x,\theta') - \tilde{\gamma}_2(x',\theta')\| \le \frac{1}{1-k}\left(\frac{\delta c}{\lambda_+ - k} + \frac{c}{\lambda_+} + 2K(\delta+\mu)(1+c)\right)\|x - x'\|. \tag{3.17}$$

Putting (3.16) and (3.17) into (3.15) we finally get

$$\|\tilde{\gamma}(x,\theta) - \tilde{\gamma}(x',\theta')\| \le c_3(\delta,\mu,c)\|(x,\theta) - (x',\theta')\|$$

with $c_3(\delta,\mu,c) = \max\{c\delta, \frac{1}{1-k}\left(\frac{\delta c}{\lambda_+ - k} + \frac{c}{\lambda_+} + 2K(\delta+\mu)(1+c)\right)\}$ which is less than c if δ and μ are small enough. From this condition we see that c can be taken of order $\delta+\mu$ but $c > 2K(\delta+\mu)$. $\square$

Finally we prove that L is a contraction. Bearing in mind that $L(\gamma) = \phi_2(\gamma)\circ\psi(\gamma)$, consider γ, $\gamma' \in \Gamma_c$. Note that by H4 we have that if $\gamma,\gamma' \in \Gamma_c$ then

$$\|\phi_1(\gamma)(x,\theta) - \phi_1(\gamma')(x,\theta)\| \le 2K\|\gamma(x,\theta) - \gamma'(x,\theta)\|.$$

We write

$$\begin{aligned}\|L(\gamma)(x,\theta) - L(\gamma')(x,\theta)\| \;&\le\; \|\phi_2(\gamma)\circ\psi(\gamma)(x,\theta) - \phi_2(\gamma)\circ\psi(\gamma')(x,\theta)\| \\ &+\; \|\phi_2(\gamma)\circ\psi(\gamma')(x,\theta) - \phi_2(\gamma')\circ\psi(\gamma')(x,\theta)\|.\end{aligned}$$

The first term of the last inequality is easily bounded using lemmas 3.3, 3.4 and the fact that if f and g are Lipschitz invertible maps between Banach spaces such that f^{-1} is Lipschitz then $\|f^{-1} - g^{-1}\| \le \text{lip}\,(f^{-1})\|f - g\|$. We have

$$\|\phi_2(\gamma)\circ\psi(\gamma)(x,\theta) - \phi_2(\gamma)\circ\psi(\gamma')(x,\theta)\| \le \frac{c+k}{1-k}2K\|\gamma(x,\theta) - \gamma'(x,\theta)\|.$$

To bound the second term we have to be more careful. We write $(\tilde{x},\tilde{\theta}) = \psi(\gamma')(x,\theta)$, $(\tilde{y},\tilde{r}) = \phi_2(\gamma)(\tilde{x},\tilde{\theta})$ and $(\tilde{y}',\tilde{r}') = \phi_2(\gamma')(\tilde{x},\tilde{\theta})$. For the y coordinate we have

$$\begin{aligned}\|x\|^{-1}\|\tilde{y} - \tilde{y}'\| \;&\le\; \|x\|^{-1}\|\tilde{x}\|\Big(\lambda_-\|\tilde{x}\|^{-1}\|\gamma_1(\tilde{x},\tilde{\theta}) - \gamma_1(\tilde{x},\tilde{\theta})\| \\ &\quad +2K(\delta+\mu)\|\tilde{x}\|^{-1}\|\gamma(\tilde{x},\tilde{\theta}) - \gamma'(\tilde{x},\tilde{\theta})\|\Big) \\ &\le\; \frac{\lambda_- + 2K(\delta+\mu)}{\lambda_+ - k}\|\gamma - \gamma'\|.\end{aligned}$$

For the r coordinate we proceed in a similar way

$$\begin{aligned}\|x\|^{-1}\|\tilde{r}-\tilde{r}'\| &\leq \|x\|^{-1}\|\gamma_2(\tilde{x},\tilde{\theta})-\gamma_2(\tilde{x},\tilde{\theta})\| \\ &\quad + \|x\|^{-1}\|\mathbf{R}(\tilde{x},\tilde{\theta},\gamma(\tilde{x},\tilde{\theta}))-\mathbf{R}(\tilde{x},\tilde{\theta},\gamma'(\tilde{x},\tilde{\theta}))\| \\ &\leq \frac{1+2K(\delta+\mu)}{\lambda_+-k}\|\gamma-\gamma'\|.\end{aligned}$$

Summarizing,

$$\begin{aligned}\|L(\gamma)-L(\gamma')\| &= \sup_{(x,\theta)}\|x\|^{-1}\|L(\gamma)(x,\theta)-L(\gamma')(x,\theta)\| \\ &\leq \left(\frac{c+k}{1-k}2K+\frac{1+2K(\delta+\mu)}{\lambda_+-k}\right)\|\gamma-\gamma'\|.\end{aligned}$$

Hence, for a fixed c small enough, which can be taken of order $\delta+\mu$ as we have seen in the proof of Lemma 3.7 there exist δ_0 and μ_0 such that if $0<\delta<\delta_0$ and $0<\mu<\mu_0$, L is a contraction in Γ_c. This ensures the existence of an invariant Lipschitz manifold. Also, since the contraction is uniform in ν, by the Uniform Contraction Principle (see [13] for instance), the manifold has a Lipschitz dependence in ν. It is interesting to remark that the contraction factor in the y direction is approximately $\lambda_-\lambda_+^{-1}$, but in the r direction is only λ_+^{-1}. □

3.2 Existence of invariant manifolds. Analytic case

In this section we study the analyticity of the invariant manifolds for analytic maps. Several steps are an identical repetition of what was done in the previous section. However, in dealing with analytic functions, it is necessary to consider complex domains. Also we can work with derivatives instead of Lipschitz constants. We shall only present the definitions and the changes one has to do in the previous proof.

Let $U_\delta=\{(x,\theta,y,r)\in\mathbb{C}^m\times\mathbb{C}^n\times\mathbb{C}^m\times\mathbb{C}^n;\|\Im\theta\|+\|x\|<\delta,\|y\|<\delta,\|r\|<\delta\}$, and let $f_\nu:U_\delta\to\mathbb{C}^m\times\mathbb{C}^n\times\mathbb{C}^m\times\mathbb{C}^n$, $|\nu|<\mu$, be a family of maps of the form (3.1).

Given $\mu>0$, we shall assume the following hypotheses on f_ν:

HA1 f_ν is real analytic and depends analytically with respect to ν.

HA2 If $\lambda_- = \|\Lambda_-\|$ and $\lambda_+^{-1} = \|\Lambda_+{}^{-1}\|$, then $0 < \lambda_-,\ \lambda_+^{-1} < 1$.

HA3 For $F = \mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}$ we have $F(0, \theta, 0, 0) = 0$ for all θ with $\|\Im\theta\| < \delta$. Also, $\mathbf{\Theta}(0, \theta, 0, 0) = \omega$, constant, for all θ with $\|\Im\theta\| < \delta$. All these functions are 2π-periodic in each component of θ.

HA4 If $F = \mathbf{X}$, $\mathbf{Y}$, $\mathbf{R}$ there exists $K > 0$ such that the norm of the derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial r$ is bounded by $K(\delta + \mu)$ and the norm of $\partial F/\partial\theta$ is bounded by $K(\delta + \mu)(\|x\| + \|y\| + \|r\|)$ in U_δ, $|\nu| < \mu$.

HA5 The norm of the derivatives $\partial\mathbf{\Theta}/\partial x$, $\partial\mathbf{\Theta}/\partial y$, $\partial\mathbf{\Theta}/\partial\theta$ is bounded by $K(\delta + \mu)$ and the norm of $\partial\mathbf{\Theta}/\partial r$ is bounded by K in U_δ, $|\nu| < \mu$.

Theorem 3.8 *Let $f_\nu : U_\delta \to \mathbb{C}^{2(n+m)}$ be a family of maps satisfying hypotheses HA1 to HA5. Then there exist δ_0 and μ_0 such that, if $0 < \delta < \delta_0$ and $0 < \mu < \mu_0$, the invariant torus $\mathbb{T}_0$ of the map f_ν has invariant manifolds related to the x and y directions which can be parameterized with analytic functions $\gamma^u(x, \theta, \nu)$, $\|x\| < \delta$, $\theta \in \mathbb{T}^n$, the unstable one, and $\gamma^s(y, \theta, \nu)$, $\|y\| < \delta$, $\theta \in \mathbb{T}^n$, the stable one. Their derivatives are of order $\delta + \mu$.*

Sketch of the proof. Again, for the sake of simplicity, we drop ν. We consider f defined on $U_\delta = D_\delta \times U'_\delta$, where $D_\delta = \{(x, \theta) \in \mathbb{C}^m \times \mathbb{C}^n; \|\Im\theta\| + \|x\| < \delta\}$ and $U'_\delta = \{(y, r) \in \mathbb{C}^m \times \mathbb{C}^n; \|y\| < \delta, \|r\| < \delta\}$. In D_δ we take the norm $\|(x, \theta)\| = \max(\|x\|, \|\theta\|)$ and in U'_δ, $\|(y, r)\| = \max(\|y\|, \|r\|)$. Also, let G_c be the set $G_c = \{\gamma : D_\delta \to U'_\delta;\ \gamma$ real analytic, 2π-periodic in θ, c-Lipschitz, $\gamma(0, \theta) = 0$, $\|\gamma(x, \theta) - \gamma(x, \theta')\| \le c\|x\|\,\|\theta - \theta'\|\}$. Notice that all the elements of G_c are uniformly bounded in D_δ.

Then, using the notations introduced in the preceding subsection, given $\gamma \in G_c$, we define $L(\gamma) = f_2 \circ (I, \gamma) \circ (f_1 \circ (I, \gamma))^{-1}$, which can also be written as $L(\gamma) = \phi_2(\gamma) \circ \phi_1(\gamma)^{-1}$.

Lemma 3.9 *Let $k = 2K(\delta + \mu + c)$. Let δ, μ and c be such that $\delta + \mu \le 1$ and $k \le \frac{\lambda_+ - 1}{3\lambda_+ - 1} < 1$. Then for any $\gamma \in G_c$, $\phi_1(\gamma)$ is invertible and $\phi_1(\gamma)^{-1}$ is also defined on D_δ and has Lipschitz constant less or equal than $(1 - k)^{-1}$.*

The proof of this lemma follows the same steps of Lemma 3.4. The only difference is that now we must check that the inverse is defined in D_δ.

Lemmas 3.5 to 3.7 also hold in the analytic case. Then we conclude that given $\gamma \in G_c$, $L(\gamma) \in G_c$.

Given $\gamma_0 \in G_c$, the sequence γ_k defined inductively by $\gamma_k = L(\gamma_{k-1})$ is in G_c. Since the functions in G_c are uniformly bounded, (γ_k) itself is uniformly bounded. Furthermore, by the preceding subsection, the sequence is convergent over the reals. Hence, applying Vitali's Theorem (see, for instance, [17]) to each component of γ_k, we have that (γ_k) is convergent over compacts sets in the complex domain. Clearly, the limit function is a fixed point of L and therefore its graph gives the unstable manifold. Moreover, considering ν as a complex parameter the uniform estimates with respect to ν give that the limit function of the sequence γ_k depends analytically on ν.
□

4 Inclination Lemma

4.1 The Settings and the Inclination Lemma

In this section we consider version of the Inclination Lemma (see [19]) for partially hyperbolic tori bearing a non-resonant rotation. We need such a statement to prove the existence of an actual orbit following the transitions chain.

The proof is quite similar to the one of the classical Inclination Lemma and it is divided in the same steps. The only difference stays in the control of the $n-1$ variables conjugated to those of the torus. There is a very similar result in [16] where also $n-1$ tori are considered. A recent work [5] considers the case of $n-k$ partially hyperbolic invariant tori.

We shall consider two different cases. In the first one, we shall suppose that there exists transversal intersection between n dimensional manifolds in a $2n$ dimensional space in the usual sense, that is, that the tangent spaces of the manifolds at the point of intersection span the tangent space of the whole space. However, when we study the intersections of invariant manifolds of $n-1$ tori of Hamiltonian systems, since their are contained in the $2n-1$ dimensional energy level, their intersection can not be transversal in that sense. We shall perform the corresponding modifications to cover this last case.

Let $U_\delta = \{(x,\theta,y,r) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n; |x| < \delta, |y| < \delta, \|r\| < \delta\}$. We recall that in the previous section we have considered families of maps $f_\nu : U_\delta \to \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$ of the form (3.1). We shall assume the hypotheses HA1 to HA5 in subsection 3.2. Under these hypotheses we have found that $\mathbb{T}_0 = \{(x,\theta,y,r) : \ x = y = 0, r = 0\}$ is a partially hyperbolic invariant n torus with invariant $(n+1)$-dimensional stable and unstable manifolds $W^s\mathbb{T}_0$ and $W^u\mathbb{T}_0$ parameterized by analytic functions $(y,r) = \gamma^u(x,\theta)$ and $(x,r) = \gamma^s(y,\theta)$, depending analytically on ν.

The main result of this section is

Theorem 4.1 (Inclination Lemma) *Let Γ be an $(n+1)$-dimensional analytic manifold intersecting transversally $W^s\mathbb{T}_0$. If $f_{|\mathbb{T}_0}$ is a non-resonant rotation, then $W^u\mathbb{T}_0 \subset \overline{\cup_{n\geq 0} f^n(\Gamma)}$.*

Remark 4.2 *In fact, we only need Γ to be C^1 in order to prove the theorem. However, since all the functions involved are analytic, we add this hypothesis. As a matter of fact, we shall apply the result when Γ is the stable invariant manifold of an invariant torus, which is analytic.*

4.2 Normal form around the invariant torus

Although many quantities actually depend on ε and μ, in order to simplify the notation, we shall not write this dependence explicitly, unless necessary.

First we look for suitable new coordinates so that the local study of f near $\mathbb{T}_0$ becomes easier.

Lemma 4.3 *Let $f_\nu : U_\delta \to \mathbb{R}^{2+2n}$ be a map of the form (3.1) satisfying hypotheses HA1 to HA5. If δ and ν are small enough, there exists a close to the identity analytic change of variables g defined in a neighborhood of $\mathbb{T}_0$ such that $g^{-1}\circ f\circ g$ takes the same form (3.1)*

$$g^{-1}\circ f\circ g\begin{pmatrix} x\\ \theta\\ y\\ r\end{pmatrix} = \begin{pmatrix} \lambda_+ x + \tilde{\mathbf{X}}(x,\theta,y,r,\nu)\\ \theta + \tilde{\mathbf{\Theta}}(x,\theta,y,r,\nu)\\ \lambda_- y + \tilde{\mathbf{Y}}(x,\theta,y,r,\nu)\\ r + \tilde{\mathbf{R}}(x,\theta,y,r,\nu)\end{pmatrix}$$

and

1. $\mathbb{T}_0$ *is represented by* $\{(0,\theta,0,0)\}$ *and* g *restricted to* $\mathbb{T}_0$ *is the identity.*
2. *The stable and unstable invariant manifolds of* $\mathbb{T}_0$*,* $W^u\mathbb{T}_0$ *and* $W^s\mathbb{T}_0$*, are given by* $\{(0,\theta,y,0)\}$ *and* $\{(x,\theta,0,0)\}$ *respectively.*
3. $\tilde{\mathbf{\Theta}}_{|W^u\mathbb{T}_0\cup W^s\mathbb{T}_0} = \tilde{\mathbf{\Theta}}_{|\mathbb{T}_0} = \omega$.
4. *The derivative of* $g^{-1}\circ f\circ g$ *on* $\mathbb{T}_0$ *has the form*

$$D(g^{-1}\circ f\circ g)(0,\theta,0,0) = \begin{pmatrix} \lambda_+(\theta) & 0 & 0 & \tilde{\mathbf{X}}_r(\theta) \\ 0 & I & 0 & \tilde{\mathbf{\Theta}}_r(\theta) \\ 0 & 0 & \lambda_-(\theta) & \tilde{\mathbf{Y}}_r(\theta) \\ 0 & 0 & 0 & I+\tilde{\mathbf{R}}_r(\theta) \end{pmatrix}, \tag{4.1}$$

with $\lambda_\pm(\theta) = \lambda_\pm + \mathcal{O}(\delta+\mu)$ *and* $\tilde{\mathbf{X}}_r$*,* $\tilde{\mathbf{Y}}_r$ *and* $\tilde{\mathbf{R}}_r$ *have order* $\delta+\mu$*.*

Proof. Let f be as in the statement. The torus $\mathbb{T}_0$ is given by $(0,\theta,0,0)$. By Theorem 3.8, its local unstable and stable invariant manifolds, $W^{u,s}\mathbb{T}_0$, can be represented in the form

$$(x,\theta,\gamma_1^u(x,\theta),\gamma_2^u(x,\theta)), \qquad (x,\theta)\in(-\delta,\delta)\times\mathbb{T}^n \tag{4.2}$$

$$(\gamma_1^s(y,\theta),\theta,y,\gamma_2^s(y,\theta)), \qquad (y,\theta)\in(-\delta,\delta)\times\mathbb{T}^n \tag{4.3}$$

respectively. Also, the functions $\gamma^{u,s}$ verify that

$$\gamma_1^u(0,\theta) = \gamma_1^s(0,\theta) = 0, \qquad \gamma_2^u(0,\theta) = \gamma_2^s(0,\theta) = 0 \tag{4.4}$$

and the derivatives of $\gamma^{u,s}$ are bounded by some constant c which is $\mathcal{O}(\delta+\mu)$ (see Lemma 3.7).

First we shall straighten the invariant manifolds. Consider the change of variables $(x,\theta,y,r) = g_1(\tilde{x},\tilde{\theta},\tilde{y},\tilde{r})$ defined by the relations

$$\begin{aligned} x &= \tilde{x} + \gamma_1^s(\tilde{y},\tilde{\theta}), & y &= \tilde{y} + \gamma_1^u(\tilde{x},\tilde{\theta}), \\ \theta &= \tilde{\theta}, & r &= \tilde{r} + \gamma_2^s(\tilde{y},\tilde{\theta}) + \gamma_2^u(\tilde{x},\tilde{\theta}). \end{aligned} \tag{4.5}$$

which is well defined in some neighborhood of $\mathbb{T}_0$, since the derivatives of $\gamma_{1,2}^{u,s}$ are small. Indeed, injectivity is easily checked as follows: suppose that

$g_1(\tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{r}) = g_1(\tilde{x}', \tilde{\theta}', \tilde{y}', \tilde{r}')$. Then, the second equation in (4.5) implies $\tilde{\theta} = \tilde{\theta}'$. From the first component of (4.5), we obtain that

$$|\tilde{x} - \tilde{x}'| = |\gamma_1^s(\tilde{y}, \tilde{\theta}) - \gamma_1^s(\tilde{y}', \tilde{\theta})| \leq c|\tilde{y} - \tilde{y}'|.$$

In the same way, from the third component $|\tilde{y} - \tilde{y}'| \leq c|\tilde{x} - \tilde{x}'|$. Since $0 < c < 1$, we obtain that $\tilde{x} = \tilde{x}'$ and $\tilde{y} = \tilde{y}'$. Using the last equation in (4.5), we finally get $\tilde{r} = \tilde{r}'$. Exhaustivity is checked along the lines of the proof of Lemma 3.4.

In the $(\tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{r})$ variables, $W^s\mathbb{T}_0$ is given by $\tilde{x} = 0$, $\tilde{r} = 0$ and $W^u\mathbb{T}_0$ is given by $\tilde{y} = 0$, $\tilde{r} = 0$. Indeed, we recall that a point (x, θ, y, r) belongs to $W^s\mathbb{T}_0$, with $|x|, |y| < \delta$, $\|r\| < \delta$, if and only if $x = \gamma_1^s(y, \theta)$ and $r = \gamma_2^s(y, \theta)$. Now, consider a point of the form $(0, \tilde{\theta}, \tilde{y}, 0)$ and $(x, \theta, y, r) = g_1(0, \tilde{\theta}, \tilde{y}, 0)$. From (4.4) and (4.5) we have $y = \tilde{y}$ and $\theta = \tilde{\theta}$, which implies, using again (4.4), $x = \gamma_1^s(\tilde{y}, \tilde{\theta}) = \gamma_1^s(y, \theta)$ and $r = \gamma_2^s(\tilde{y}, \tilde{\theta}) + \gamma_1^u(0, \tilde{\theta}) = \gamma_2^s(y, \theta)$. The proof for $W^u\mathbb{T}_0$ is analogous.

We shall denote again the new variables by (x, θ, y, r) and $g_1^{-1} \circ f \circ g_1$ by f.

Since g_1 is the identity over $\mathbb{T}_0$ and $Dg_1 = I + \mathcal{O}(\delta + \mu)$, f can be written again in the form (3.1) with functions $\tilde{\mathbf{X}}$, $\tilde{\mathbf{Y}}$, $\tilde{\mathbf{\Theta}}$, $\tilde{\mathbf{R}}$ instead of $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{\Theta}$, $\mathbf{R}$. Furthermore, the bounds of hypotheses HA5 also hold for f with some constant K' instead of K. The invariance conditions imply

$$f_1(0, \theta, y, 0) = f_3(x, \theta, 0, 0) = 0, \qquad f_4(0, \theta, y, 0) = f_4(x, \theta, 0, 0) = 0.$$

Moreover, the condition $\mathbf{\Theta}(0, \theta, 0, 0) = \omega$ gives $\tilde{\mathbf{\Theta}}(0, \theta, 0, 0) = \omega$, which implies

$$Df(0, \theta, 0, 0) = \begin{pmatrix} \lambda_+(\theta) & 0 & 0 & \tilde{\mathbf{X}}_r(\theta) \\ a(\theta) & I & b(\theta) & \tilde{\mathbf{\Theta}}_r(\theta) \\ 0 & 0 & \lambda_-(\theta) & \tilde{\mathbf{Y}}_r(\theta) \\ 0 & 0 & 0 & I + \tilde{\mathbf{R}}_r(\theta) \end{pmatrix},$$

where a and b have order $\delta + \mu$,

$$\begin{aligned} \lambda_+(\theta) &= \lambda_+ + \frac{\partial \tilde{\mathbf{X}}}{\partial x}(0, \theta, 0, 0) = \lambda_+ + \mathcal{O}(\mu + \delta), \\ \lambda_-(\theta) &= \lambda_- + \frac{\partial \tilde{\mathbf{Y}}}{\partial y}(0, \theta, 0, 0) = \lambda_- + \mathcal{O}(\mu + \delta), \end{aligned}$$

and the subindex r means derivative with respect to r.

Now we seek for a change of variables such that, in the new variables, $\tilde{\mathbf{\Theta}} \equiv \omega$ on $W^s\mathbb{T}_0$. Since $W^s\mathbb{T}_0$ is invariant by f we look for g_2 of the form $g_2(x,\theta,y,r) = (x, \theta + c(\theta,y)y, y, r)$ (which leaves $W^s\mathbb{T}_0$ invariant) verifying the condition

$$\tilde{f} \circ g_{2|W^s} = g_2 \circ f_{|W^s} \tag{4.6}$$

where $\tilde{f}_{|W^s}(\theta,y) = (\theta+\omega, \tilde{f}_1(\theta,y))$. Since

$$f_{|W^s}(\theta,y) = (\theta + \tilde{\mathbf{\Theta}}(0,\theta,y,0), \lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0))$$

the first equation of condition (4.6) can be written as

$$\begin{aligned} \theta+\omega+yc(\theta,y) &= \theta + \tilde{\mathbf{\Theta}}(0,\theta,y,0) + \\ & \quad c(\theta + \tilde{\mathbf{\Theta}}(0,\theta,y,0), \lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0)) \\ & \quad \times(\lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0)), \end{aligned} \tag{4.7}$$

from which we obtain

$$\begin{aligned} c(\theta,y) &= \frac{1}{y}(\tilde{\mathbf{\Theta}}(0,\theta,y,0) - \omega) + \frac{1}{y}(\lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0)) \\ & \quad \times c(\theta + \tilde{\mathbf{\Theta}}(0,\theta,y,0), \lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0)). \end{aligned} \tag{4.8}$$

To find an analytic solution of (4.8) we define the set $\mathcal{D}_\delta = \{(y,\theta) \in \mathbb{C}\times\mathbb{C}^n : \|\Im\theta\| + |y| < \delta\}$ and the space $\Sigma^\delta = \{c : \mathcal{D}_\delta \to \mathbb{C}^n;\ \text{bounded and analytic }\}$, with the supremum norm. Let $\lambda = \sup_{\theta\in D_\delta} \lambda_-(\theta)$. We introduce $\eta = K'(\delta + \mu)$. Making δ and μ smaller, if necessary, we can suppose that $\lambda + 2\eta < 1$. We also define $\alpha = \frac{\eta}{1-(\lambda+\eta)} > 0$. Let Σ^δ_α be the closed ball of radius α of Σ^δ and $G : \Sigma^\delta_\alpha \longrightarrow \Sigma^\delta_\alpha$ the map defined by

$$\begin{aligned} G(c)(\theta,y) &= \frac{1}{y}(\tilde{\mathbf{\Theta}}(0,\theta,y,0) - \omega) + \frac{1}{y}(\lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0)) \\ & \quad \times c(\theta + \tilde{\mathbf{\Theta}}(0,\theta,y,0), \lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0)). \end{aligned}$$

$G(c)$ is a well defined function on D_δ since

$$\begin{aligned} & \|\Im(\theta + \tilde{\mathbf{\Theta}}(0,\theta,y,0))\| + |\lambda_-(\theta)y + \tilde{\mathbf{Y}}(0,\theta,y,0)| \\ \leq & \ \|\Im\theta\| + \eta|y| + (\lambda+\eta)|y| \leq \delta. \end{aligned}$$

Moreover

$$\begin{aligned}\|G(c)(\theta,y)\| &\leq \|\frac{1}{y}(\tilde{\mathbf{\Theta}}(0,\theta,y,0)-\omega)\| + \|\frac{1}{y}(\lambda_-(\theta)y+\tilde{\mathbf{Y}}(0,\theta,y,0))\\ &\quad \times c(\theta+\tilde{\mathbf{\Theta}}(0,\theta,y,0),\lambda_-(\theta)y+\tilde{\mathbf{Y}}(0,\theta,y,0))\|\\ &\leq \eta+(\lambda+\eta)\|c\| \leq \eta+(\lambda+\eta)\alpha=\alpha\end{aligned}$$

implies that $G(c)\in\Sigma_\alpha^\delta$. Therefore, G is a well defined map from Σ_α^δ to itself.

Furthermore, G is a contraction, since $\lambda+m<1$ and

$$\|G(c)-G(c')\| \leq (\lambda+m)\|c-c'\|.$$

Therefore, there exists a unique fixed point of G in Σ_α^δ, which gives the desired change of variables.

To get that $\tilde{\mathbf{\Theta}}\equiv\omega$ over $W^u\mathbb{T}_0$ we follow the analogous procedure. We look for a change g_3 of the form $g_3(x,\theta,y,r)=(x,\theta+c(\theta,x)x,y,r)$. Now it is more convenient to write the invariance condition in terms of f^{-1}, that is,

$$\tilde{f}^{-1}\circ g_{3|W^u}=g_3\circ f^{-1}{}_{|W^u}$$

where $\tilde{f}^{-1}_{|W^u}(x,\theta)=(h(x,\theta),\theta-\omega)$. Since g_3 is the identity over $W^s\mathbb{T}_0$, in the new variables, the fact that $\tilde{\mathbf{\Theta}}\equiv\omega$ over $W^s\mathbb{T}_0$ remains true.

Finally, (4.1) follows from the fact that $\tilde{\mathbf{\Theta}}$ is constant over $W^s\mathbb{T}_0\cup W^u\mathbb{T}_0$.

□

4.3 Inclination Lemma

To prove the Inclination Lemma we recall that Γ is an $(n+1)$-dimensional manifold, intersecting transversally $W^s\mathbb{T}_0$. Let $p_0\in W^u(\mathbb{T}_0)$. We want to prove that $\cup_{n\geq 0}f^n(\Gamma)$ intersects any neighborhood of p_0. Theorem 4.1 follows from the next lemma

Lemma 4.4 *Given $p_0\in W^u(\mathbb{T}_0)$ there exists $q(t)=(t,\theta(t),y(t),r(t))$, an analytic curve, $t\in[-t_0,t_0]$, with $q(0)=q_0\in\Gamma\cap W^s(\mathbb{T}_0)$ and $q(t)\in\Gamma$, $\forall t\in[-t_0,t_0]$, such that for all $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $B(p_0,\varepsilon)\cap\{f^n(q(t));\,t\in[-t_0,t_0]\}\neq\emptyset$.*

Proof. The proof has four steps. In the first one some constants that later will play a role are defined. In the second step, the curve $q(t)$ is chosen and it is proven that the tangent vectors of $q(t)$ at the points where it intersects $W^s(\mathbb{T}_0)$ tend to be tangent to $W^u(\mathbb{T}_0)$. In the third step it is proven that actually the curve tend to accumulate on $W^u(\mathbb{T}_0)$. Finally, using the fact that $f_{|\mathbb{T}_0}$ is an irrational rotation, the lemma is proven. We follow [19] in steps 2 and 3.

In this proof we shall denote the map by f, which have the form (3.1). Iterating by f^{-1} , if necessary, we can assume that p_0 is in the neighborhood U of $\mathbb{T}_0$ where the coordinates given by Lemma 4.3 are defined. On the other hand, the point q_0, where Γ intersects transversally $W^s(\mathbb{T}_0)$, can also be supposed to be in U, iterating by f, if necessary. Since we shall work with the coordinates given by Lemma 4.3 we can assume that p_0 and q_0 have coordinates of the form $(x_0, \varphi_0, 0, 0)$ and $(0, \theta_0, y_0, 0)$ respectively, with $|x_0|$ and $|y_0|$ arbitrarily small (but fixed *a priori*).

Now we fix $\varepsilon > 0$. We can assume that $U = \{(x, \theta, y, r); |x|, |y|, \|r\| < \rho\}$ for some positive ρ. From (4.1) we have that Df has the following form in U

$$Df(x,\theta,y,r) = \begin{pmatrix} \lambda_+ & \alpha_\theta & \alpha_y & \alpha_r \\ \beta_x & I+\beta_\theta & \beta_y & \beta_r \\ \gamma_x & \gamma_\theta & \lambda_- & \gamma_r \\ \delta_x & \delta_\theta & \delta_y & I+\delta_r \end{pmatrix}, \tag{4.9}$$

where all matrix elements depend analytically on (x, θ, y, r), $\lambda_+(x, \theta, y, r) > 1$, $\lambda_-(x, \theta, y, r) < 1$, and, denoting by z any of the variables x, y, θ, the functions α_z, β_z, γ_z and δ_z tend to 0 when $(x, y, r) \to 0$. Let

$$\lambda_+^* = \inf\{\lambda_+(x,\theta,y,r);\ (x,\theta,y,r) \in U\}$$

and let λ be such that $1 < \lambda < \lambda_+^*$. Then, given $\eta_0 > \varepsilon > 0$, let $\kappa > 0$ be such that

$$\lambda_+^* - 3\kappa\eta_0 > \lambda > 1 + 3\kappa, \tag{4.10}$$

$$0 < \frac{\kappa}{\lambda - (1+\kappa)} < \eta_0, \tag{4.11}$$

$$\frac{1+3\kappa}{\lambda} < 1, \tag{4.12}$$

$$\lambda_- + \kappa < 1. \tag{4.13}$$

Next we give some estimates on the matrix elements of $Df(x, \theta, y, r)$ on U.

By hypothesis, we have that $\|\beta_r\| < K$. Since the local unstable manifold is represented by $\{y = 0,\ r = 0\}$ and $\mathbf{\Theta} \equiv \omega$ over it, we have

$$\mathbf{Y}(x, \theta, 0, 0) = 0, \qquad \mathbf{R}(x, \theta, 0, 0) = 0, \qquad \mathbf{\Theta}(x, \theta, 0, 0) = \omega.$$

Hence the derivatives of $\mathbf{Y}$, $\mathbf{R}$ and $\mathbf{\Theta}$ with respect to x, θ are zero on $\{y = 0,\ r = 0\}$. Then, there exists $K' > 0$ such that $|\gamma_x|$, $\|\beta_x\|$, $\|\delta_x\| < K'\|(y, r)\|$. Next, given $C > 0$, we consider another change given by the directional scaling $g(x, \theta, y, r) = (Cx, \theta, Cy, Cr)$. We write $\tilde{f} = g^{-1} \circ f \circ g$. It has also the form (3.1). Let $\tilde{\mathbf{X}}$, $\tilde{\mathbf{\Theta}}$, $\tilde{\mathbf{Y}}$, $\tilde{\mathbf{R}}$, α_z, β_z, γ_z, δ_z, $z = x,\ \theta,\ y,\ r$, be its corresponding functions when one expresses $\tilde{f}$ in the mentioned form. It holds that for $\tilde{F} = \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}, \tilde{\mathbf{R}}$, $\tilde{F}(x, \theta, y, r) = C^{-1}F(Cx, \theta, Cy, Cr)$ and $\tilde{\mathbf{\Theta}}(x, \theta, y, r) = \mathbf{\Theta}(Cx, \theta, Cy, Cr)$. Then we have that $\|\tilde{\beta}_r\| \leq CK$ and $|\tilde{\gamma}_x|$, $\|\tilde{\beta}_x\|$, $\|\tilde{\delta}_x\| < CK'\|(y, r)\|$. We choose C such that $CK, CK' < 1$.

Taking U and ν smaller, if necessary, we can assume, by 4 of Lemma 4.3 that, except for $\|\tilde{\beta}_r\|$, we have $\|\tilde{\alpha}_z\|, \|\tilde{\beta}_z\|, \|\tilde{\gamma}_z\|, \|\tilde{\delta}_z\| < \kappa$, for $z = x, \theta, y, r$, in U.

We further introduce $\tau > 1$ such that

$$\tau > \frac{2}{\lambda - (1 + 3\kappa)} > 0, \qquad \frac{1}{\tau} < \rho. \tag{4.14}$$

Since, as we have said, we can take q_0 as near to the torus as we want, we take $p_0 = (x_0, \theta_0, 0, 0)$ with $|x_0| < 1/\tau$. Notice that all these conditions are independent of ε.

On the other hand, as $q_0 \in W^s(\mathbb{T}_0) \cap U$, using the invariance condition $\mathbf{X}(0, \theta, y, 0) = 0$, $\mathbf{R}(0, \theta, y, 0) = 0$, and the condition $\mathbf{\Theta}(0, \theta, y, 0) = \omega$, we have that

$$Df(q_0) = Df(0, \theta_0, y_0, 0) = \begin{pmatrix} \lambda_+ & 0 & 0 & \alpha_r^0 \\ \beta_x^0 & I & 0 & \beta_r^0 \\ \gamma_x^0 & \gamma_\theta^0 & \lambda_- & \gamma_r^0 \\ \delta_x^0 & 0 & 0 & I + \delta_r^0 \end{pmatrix}. \tag{4.15}$$

Let $q_k = f^k(q_0)$. Since $q_k \in W^s\mathbb{T}_0 \cap U$, $\forall k \geq 0$, $Df(q_k)$ has the form (4.15). Let α_z^k, β_z^k, γ_z^k and δ_z^k be their corresponding elements.

We define $\kappa_k = \max\{\|\beta_x^k\|, \|\gamma_x^k\|, \|\delta_x^k\|, \|\gamma_\theta^k\|\}$ which is less than κ because the iterates q_k remain in U. Furthermore notice that, since $q_k \to \mathbb{T}_0$, $\kappa_k \to 0$ as $k \to \infty$.

Since the intersection of the two manifolds is transversal, $T_{q_0}\Gamma \oplus T_{q_0}W^s\mathbb{T}_0$ span $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. But since

$$T_{q_0}W^s\mathbb{T}_0 = \{v = (v_x, v_\theta, v_y, v_r) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n :\ v_x = 0, v_r = 0\}$$

there exists $\tilde{v} \in T_{q_0}\Gamma$ such that $\|\tilde{v}_r\| \leq \eta_0$ and $\tilde{v}_x = 1$. Therefore, there exists an analytic curve in Γ of the form $q_0(t) = (t, \theta_0(t), y_0(t), r_0(t))$, $t \in [-t_0, t_0]$, with $q_0(0) = q_0$, $q_0'(0) = (1, \theta_0', y_0', r_0')^t$ and $\|r_0'\| \leq \eta_0$. We define the number $m_0 = \max\{\|\theta_0'\|, |y_0'|\}$. We are going to consider iterates of the curve q_0 by f in two steps: first we iterate a large enough number of times so that the tangent vector at $t = 0$ (after reparametrization) is close enough to the vector $(1, 0, 0, 0)$. Next we continue iterating until we get the condition of the statement.

Let $\bar{q}_1 = f \circ q_0$. Then

$$\bar{q}_1'(0) = Df(q_0(0))q_0'(0) = \begin{pmatrix} \lambda_+ + \alpha_r^0 r_0' \\ \beta_x^0 + \theta_0' + \beta_r^0 r_0' \\ \gamma_x^0 + \gamma_\theta^0\theta_0' + \lambda_- y_0' + \gamma_r^0 r_0' \\ \delta_x^0 + (I + \delta_r^0) r_0' \end{pmatrix}.$$

We have that $\lambda_+ + \alpha_r^0 r_0' > \lambda_+ - \kappa\eta_0 > \lambda$, and therefore, there exists a reparametrization q_1 of the curve $\bar{q}_1$ of the form $q_1 = (x_1(t), \theta_1(t), y_1(t), r_1(t))$, with $q_1'(0) = (1, \theta_1', y_1', r_1')^{\mathrm{T}}$ and

$$\begin{aligned} \theta_1' &= \frac{\beta_x^0 + \theta_0' + \beta_r^0 r_0'}{\lambda_+ + \alpha_r^0 r_0'} \\ y_1' &= \frac{\gamma_x^0 + \lambda_- y_0' + \gamma_\theta^0\theta_0' + \gamma_r^0 r_0'}{\lambda_+ + \alpha_r^0 r_0'} \\ r_1' &= \frac{\delta_x^0 + (I + \delta_r^0) r_0'}{\lambda_+ + \alpha_r^0 r_0'}. \end{aligned}$$

Then

$$|y_1'|, \|\theta_1'\| \leq \frac{\kappa + m_0 + \kappa\eta_0}{\lambda}, \qquad \|r_1'\| \leq \frac{\kappa + (1+\kappa)\eta_0}{\lambda} < \eta_0,$$

where the last inequality follows from (4.11). Now, iterating this procedure we consider the family of curves $\bar{q}_k = f \circ q_{k-1}$, and a reparametrization q_k of each one, such that

$$q_k'(0) = (1, \theta_k', y_k', r_k')^t = \frac{1}{\lambda_+(q_{k-1}) + \alpha_r^{k-1} r_{k-1}'} \bar{q}_k'(0).$$

Let $m_k = \max\{\|\theta_k'\|, |y_k'|\}$ and $\eta_k = \|r_k'\|$. The following relationships hold:

$$m_k \leq \frac{\kappa_{k-1}}{\lambda} + \frac{m_{k-1}}{\lambda} + \frac{\kappa_{k-1}}{\lambda}\eta_{k-1} \quad (4.16)$$

$$\eta_k \leq \frac{\kappa_{k-1}}{\lambda} + \frac{1+\kappa_{k-1}}{\lambda}\eta_{k-1}, \quad (4.17)$$

where, by induction, using again (4.11), $\eta_k < \eta_0$.

We are going to use the following fact about sequences of real numbers: let (a_k), (b_k), (η_k) be sequences of positive real numbers such that $a_k \to 0$, $b_k \leq b < 1$ and $\eta_{k+1} \leq a_k + b_k\eta_k$, $k \geq 1$. Then $\eta_k \to 0$.

Applying this fact to (η_k), since $\kappa_k \to 0$ when $k \to \infty$ and

$$0 < \frac{1+\kappa_k}{\lambda} < \frac{1+\kappa}{\lambda} < 1,$$

we have that $\eta_k \to 0$. Also since

$$\frac{\kappa_{k-1}}{\lambda} + \kappa_{k-1}\frac{\eta_{k-1}}{\lambda}$$

also tends to 0 and $1/\lambda < 1$ we get that m_k tends to 0.

Therefore, there exists k_0 such that if $k \geq k_0$, then $m_k, \eta_k < \varepsilon/4$ and $\|(y_k(0), r_k(0))\| < \varepsilon/(2\tau)$. We consider the curve $q_{k_0}(t)$, which we denote again by $q_0(t)$. Now we shall perform the second step where we shall iterate this $q_0(t)$. We can suppose that this curve has a parameterization of the form $(t, \theta(t), y(t), r(t))$, for t in some neighborhood of 0. We consider the neighborhood of $W^u(\mathbb{T}_0)$ defined by $V_{\varepsilon/\tau} = \{(x, \theta, y, r) \in U; \|(y, r)\| < \varepsilon/\tau\}$.

By the continuity of q_0 and its derivatives, we can assume that there exists t_0 such that

$$m_0 = \sup_{t \in [-t_0, t_0]} \{\|\theta_{k_0}'(t)\|, |y_{k_0}'(t)|, \|r_{k_0}'(t)\|\} < \frac{\varepsilon}{2},$$

and that $q_0(t) \in V_{\varepsilon/\tau}$, for $t \in [-t_0, t_0]$. Notice that $x'_{k_0}(t) = 1$. Also in $V_{\varepsilon/\tau}$ we have $\|\beta_x\|, \|\gamma_x\|, \|\delta_x\| < \varepsilon/\tau$.

Consider $\bar{q}_1(t) = f(q_0(t))$. Then, if $\bar{q}'_1(t) = (\bar{x}'_1(t), \bar{\theta}'_1(t), \bar{y}'_1(t), \bar{r}'_1(t))$, we have that

$$|\bar{x}'_1(t)| > \lambda^*_+ - \frac{3}{2}\kappa\varepsilon > \lambda,$$

where we have used (4.10). Hence there exists a reparametrization $q_1(t) = (x_1(t), \theta_1(t), y_1(t), r_1(t))$ of $\bar{q}_1(t)$ such that $x_1(t) = t$ and $q_1(t)$ is defined in $[-\lambda t_0, \lambda t_0]$.

Let $m_1 = \sup_{t\in[-\lambda t_0, \lambda t_0]}\{\|\theta'_1(t)\|, |y'_1(t)|, \|r'_1(t)\|\}$. We have

$$m_1 < \frac{1}{\lambda}\left(\kappa + (1+3\kappa)m_0\right) < \frac{\varepsilon}{2}.$$

We have $q_1(t) \in V_{\varepsilon/\tau}$. Indeed, using the bounds of the derivatives:

$$|y_1(t) - y_1(0)|, \|r_1(t) - r_1(0)\| < \frac{\varepsilon}{2}|t|,$$

and that $|y_1(0)| < \varepsilon/(2\tau)$, $r_1(0) = 0$ we have that $q_1(t) \in V_{\varepsilon/\tau}$ if $t \in [-t_1, t_1]$, where $t_1 = \min\{\lambda t_0, 1/\tau\}$.

We repeat the procedure, and we obtain a family of curves in $V_{\varepsilon/\tau}$, $q_k(t) = (t, \theta_k(t), y_k(t), r_k(t))$, $t \in [-t_k, t_k]$, with $t_k = \min\{\lambda t_{k-1}, 1/\tau\}$, and $m_k < \varepsilon/2$. Therefore, if k is large enough, we can assume that the curves are defined in $[-1/\tau, 1/\tau]$.

We recall that $p_0 = (x_0, \varphi_0, 0, 0)$. As $q_k(0)$ tends to the torus and the rotation on the torus is irrational, there exists some large enough k_1 such that $\|\theta_{k_1}(0) - \varphi_0\| < \varepsilon/2$. Then taking $t = x_0 < 1/\tau$

$$\begin{aligned}
\|q_{k_1}(x_0) - p_0\| &= \|(0, \theta_{k_1}(x_0) - \varphi_0, y_{k_1}(x_0), r_{k_1}(x_0))\| \\
&\le \max\{\frac{\varepsilon}{\tau}, \|\theta_{k_1}(x_0) - \theta_{k_1}(0)\| + \|\theta_{k_1}(0) - \varphi_0\|\} \\
&\le \max\{\frac{\varepsilon}{\tau}, \frac{\varepsilon}{2}\frac{1}{\tau} + \frac{\varepsilon}{2}\} < \varepsilon.
\end{aligned}$$

□

4.4 Inclination Lemma. Hamiltonian case

Let f be of the form (3.1) verifying the same hypotheses. Additionally we will suppose that $W^s\mathbb{T}_0 \cup W^u\mathbb{T}_0 \subset M$, where M is a $2n-1$ dimensional analytic manifold. Furthermore, we shall assume that M is defined in a neighborhood of $\mathbb{T}_0$ by the equation $G(\theta, r) = 0$, G being an analytic scalar function such that $\nabla G \neq 0$.

Theorem 4.5 (Inclination Lemma. Hamiltonian case) *Let $\Gamma \subset M$ be an n-dimensional (analytic) manifold intersecting transversally $W^s\mathbb{T}_0$ in M, that is, the tangent spaces of $W^s\mathbb{T}_0$ and Γ at their intersection span the tangent space of M. If $f_{|\mathbb{T}_0}$ is a non-resonant rotation, then $W^u\mathbb{T}_0 \subset \overline{\cup_{n\geq 0} f^n(\Gamma)}$.*

To prove this lemma it is only necessary to repeat the steps made in the proof of Lemma 4.1. The unique difficulty in this version of the lemma lies in the fact that the intersection between Γ and $W^s\mathbb{T}_0$ is only transversal in M. Since the intersection is no longer transversal in the whole space, we cannot use the same arguments in order to ensure the existence of a tangent vector to Γ in its intersection with $W^s\mathbb{T}_0$ with the suitable properties. Next we give the arguments which show how to obtain such a vector.

More concretely, using the same notation of the proof of Lemma 4.4, let $q_0 \in \Gamma \cap W^s\mathbb{T}_0 \cap U$, where U is the domain where the bounds of the derivatives hold. We want to prove that there exists $v \in T_{q_0}\Gamma$ such that $v = (v_x, v_\theta, v_y, v_r)$ with $v_x = 1$ and $\|v_r\| < \eta_0$.

We recall that the invariant manifolds are expressed in U in the form

$$W^u\mathbb{T}_0 = \{(x, \theta, y, r) \in U : y = 0,\ r = 0\} \tag{4.18}$$
$$W^s\mathbb{T}_0 = \{(x, \theta, y, r) \in U : x = 0,\ r = 0\}. \tag{4.19}$$

Then, since $W^u\mathbb{T}_0 \subset M$, for any $p \in W^u\mathbb{T}_0$ we have that $T_pW^u\mathbb{T}_0 \subset \langle \nabla G(p)\rangle^\perp$, which implies that $\nabla G(p) = (0, 0, g_y(p), g_r(p))$. Therefore, for any $\delta > 0$, there exists a neighborhood U_δ of $W^u\mathbb{T}_0$ such that for any $p \in U_\delta$ we have that

$$\nabla G(p) = (g_x(p), g_\theta(p), g_y(p), g_r(p)), \tag{4.20}$$

with $|g_x(p)| < \delta$ and, since $\nabla G \neq 0$ in U, $\|g_r(p)\| > d$, for some $d > 0$. Notice that making δ smaller does not imply to take a smaller d.

Iterating by f, if necessary, we can assume that $q_0 \in U_\delta$. Since the intersection between Γ and $W^s\mathbb{T}_0$ is transversal in M we have that

$$\langle \nabla G(q_0)\rangle^\perp = T_{q_0}\Gamma + T_{q_0}W^s\mathbb{T}_0. \tag{4.21}$$

From (4.19) and (4.20) we have that $\nabla G(q_0) = (g_x(q_0), 0, 0, g_r(q_0))$, verifying $|g_x(q_0)| < \delta$. Taking into account that $T_{q_0}W^s\mathbb{T}_0 = \{(v_x, v_\theta, v_y, v_r) : v_x = 0,\ v_r = 0\}$, and formula (4.21) we have that, given any pair $(v_x, v_r) \in \mathbb{R}\times\mathbb{R}^{n-1}$ satisfying

$$v_x g_x(q_0) + \langle v_r, g_r(q_0)\rangle = 0, \tag{4.22}$$

there exists $\tilde v = (\tilde v_x, \tilde v_\theta, \tilde v_y, \tilde v_r,) \in T_{q_0}\Gamma$ such that $\tilde v_x = v_x$ and $\tilde v_r = v_r$. Therefore, taking $v_x = 1$ and

$$v_r = -\frac{g_x(q_0)}{\|g_r(q_0)\|} g_r(q_0)$$

formula (4.22) holds and, if $\delta < d\eta_0$, we have that

$$\|v_r\| \le \frac{\delta}{d} < \eta_0.$$

$\square$

5 Appendix A. Calculation of the integrals

This appendix is devoted to calculate the functions g_j defined in (2.16). Substituting the expression of $\phi_j(t)$ into the integral we get

$$\begin{aligned} g_j &= (2\pi)^2 \int_{-\infty}^{\infty} \left(\frac{2\sqrt{\varepsilon}}{\cosh(2\pi\sqrt{\varepsilon}\, t)}\right)^2 \\ &\quad \times \cos\left(4\alpha_j^1(\hat r_s)\arctan(\tanh(\pi\sqrt{\varepsilon}\, t)) + 2\pi\left(\alpha_j^0(\hat r_s) - \alpha_j^1(\hat r_s)a_1(\hat r_s)\right)t\right) dt. \end{aligned}$$

To simplify the notation we introduce

$$\sigma = 2\alpha_j^1(\hat r_s), \qquad \beta = \alpha_j^0(\hat r_s) - \alpha_j^1(\hat r_s)a_1(\hat r_s).$$

Using that the integral of and odd function in $\mathbb{R}$ is zero and making the change $s = 2\pi\sqrt{\varepsilon}\, t$, we obtain that $g_j = 8\pi\sqrt{\varepsilon} I(\sigma, \beta, \sqrt{\varepsilon})$ where

$$I(\sigma, \beta, \sqrt{\varepsilon}) = \int_{-\infty}^{\infty} \frac{\exp(2\sigma i \arctan(\tanh(s/2)))}{\cosh^2 s} e^{i\frac{\beta}{\sqrt{\varepsilon}}s}\, ds.$$

We further introduce $\gamma = \beta/\sqrt{\varepsilon}$ and we denote $I(\sigma, \beta, \sqrt{\varepsilon})$ by $I(\sigma, \gamma)$.

Lemma 5.1 *The asymptotic expressions of $I(\sigma, \gamma)$ when $\gamma \to \infty$ are: if $\sigma \in \mathbb{R} \setminus \mathbb{Z}$,*

$$I(\sigma, \gamma) = -\frac{e^{-\gamma\frac{\pi}{2}}}{1 - e^{-2\pi\gamma}} \sin(\sigma\pi) \left(\frac{\Gamma(\sigma - 1)}{(2\gamma)^{\sigma-1}} + \mathcal{O}\left(\frac{1}{\gamma^\sigma}\right)\right),$$

where Γ is the analytic extension of the Gamma function to $\mathbb{C}\setminus\{0, -1, -2, \ldots\}$, if $\sigma \in \mathbb{Z}$,

$$I(\sigma, \gamma) = \begin{cases} 2\pi \dfrac{e^{-\gamma\frac{3\pi}{2}}}{1 - e^{-2\pi\gamma}} \left((-1)^\sigma 2^\sigma \dfrac{\gamma^{\sigma+1}}{(\sigma + 1)!} + \mathcal{O}\left(\gamma^\sigma\right)\right), & \text{if } \sigma \geq 2, \\ 2\pi \dfrac{e^{-\gamma\frac{\pi}{2}}}{1 - e^{-2\pi\gamma}} \left(\dfrac{1}{2^\sigma} \dfrac{\gamma^{1-\sigma}}{(1 - \sigma)!} + \mathcal{O}\left(\gamma^{-\sigma}\right)\right), & \text{if } \sigma \leq 1. \end{cases}$$

The asymptotic expressions when $\gamma \to -\infty$ can be obtained by using $I(\sigma, \gamma) = I(-\sigma, -\gamma)$.

Proof. Consider the function $g(z) = \arctan(\tanh(z/2))$. Notice first that

$$g(z) = \frac{\ln(\tan(iz/2 + \pi/4))}{2i} = \frac{1}{2i} \ln \frac{1 + i \sinh z}{\cosh z},$$

where we are considering the principal branch of the logarithm in $\mathbb{C} \setminus \{\Im z = 0, \Re z \leq 0\}$.

Therefore, we have to calculate

$$I(\sigma, \gamma) = \int_{-\infty}^{\infty} \frac{e^{2i\sigma g(s)}}{\cosh^2 s} e^{i\gamma s}\, ds.$$

If $\sigma \in \mathbb{Z}$, the function $e^{2i\sigma g(z)}$ is meromorphic in $\mathbb{C}$. Otherwise, it has branching points. Hence, we shall distinguish two cases.

Case $\sigma \in \mathbb{Z}$. Let $\Omega_R = \{z \in \mathbb{C}; 0 \leq \Im z \leq 2\pi,\ |\Re z| \leq R\}$. The integral to calculate now is

$$I(\sigma, \gamma) = \int_{-\infty}^{\infty} \frac{\tan^\sigma(is/2 + \pi/4)}{\cosh^2 s} e^{i\gamma s}\, ds.$$

The number of poles in Ω_R of the function to integrate depends on σ: if $\sigma \geq 2$ there is a unique pole at $i3\pi/2$, since $i\pi/2$ is a removable singularity. If $\sigma \leq -2$ there is a unique pole at $i\pi/2$, since in this case $i3\pi/2$ is a removable singularity. Finally, if $\sigma = -1, 0, 1$, both poles are present. The integrals are calculated via residue theory in the usual way.

Case $\sigma \in \mathbb{R} \setminus \mathbb{Z}$. Let $\tilde{\Omega}_R = \Omega_R \setminus \{z \in \mathbb{C}; \Re z = 0, \pi/2 \leq \Im z \leq 3\pi/2\}$. The function

$$f(z) = \frac{e^{\sigma \ln(\tan(iz/2 + \pi/4))}}{\cosh^2 z} e^{i\gamma z}$$

is analytic in $\tilde{\Omega}_R$ and, hence, the integrals along two paths around the segment $\{z \in \mathbb{C}; \Re z = 0, \pi/2 \leq \Im z \leq 3\pi/2\}$ (having the same index with respect to a point of the segment) must coincide. We shall consider two paths. The first one will be the border of Ω_R. Taking the limit when $R \to \infty$, the value of the corresponding integral will essentially be, if γ is large, the integral over $\mathbb{R}$ since the integral over $2\pi i + \mathbb{R}$ will have a factor $e^{-2\pi\gamma}$. The second path will be $\gamma_\rho = \gamma_{\rho 1} + \cdots + \gamma_{\rho 8}$ with $0 < \rho < \pi/4$ and (see Fig. 1)

$$\begin{array}{rcl}
\gamma_{\rho 1}(t) = & i\frac{\pi}{2} + \rho e^{i(t - 3\pi/2)}, & t \in [0, 2\pi] \\
\gamma_{\rho 2}(t) = & i(\pi/2 + t), & t \in [\rho, \pi/2 - \rho] \\
\gamma_{\rho 3}(t) = & i\pi + \rho e^{i(t - \pi/2)}, & t \in [0, \pi] \\
\gamma_{\rho 4}(t) = & i(\pi + t), & t \in [\rho, \pi/2 - \rho] \\
\gamma_{\rho 5}(t) = & i\frac{3\pi}{2} + \rho e^{i(t - \pi/2)}, & t \in [0, 2\pi] \\
\gamma_{\rho 6}(t) = & i(3\pi/2 - t), & t \in [\rho, \pi/2 - \rho] \\
\gamma_{\rho 7}(t) = & i\pi + \rho e^{i(t + \pi/2)}, & t \in [0, \pi] \\
\gamma_{\rho 8}(t) = & i(\pi - t), & t \in [\rho, \pi/2 - \rho].
\end{array}$$

To simplify the notation we have taken $\gamma_{\rho j}$, $j = 2, 4, 6, 8$ on the border of $\tilde{\Omega}_R$. The values of $f(z)$ on these parts of the path will be the obvious limit values.

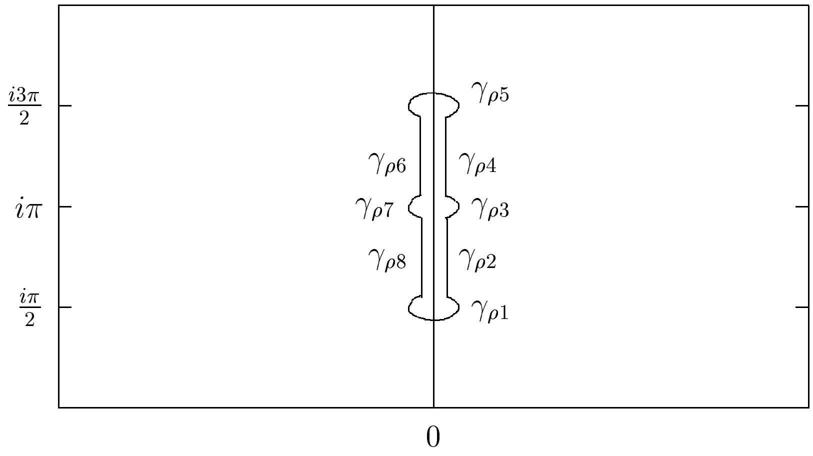

Figure 1: Integration paths in $\tilde{\Omega}_R$.

Hence

$$(1 - e^{-2\pi\gamma}) \int_{\infty}^{\infty} f(t)\, dt = \int_{\gamma_\rho} f(z)\, dz.$$

In calculating the integrals $\int_{\gamma_{\rho k}} f(z)\, dz$, $k = 1, \ldots, 8$, we shall see that they depend on ρ in the following way: for $\rho > 0$

$$\int_{\gamma_{\rho k}} f(z)\, dz = \sum_{l=l_0}^{j_0} a_l^k \rho^{\sigma+l} + \sum_{l=l_1}^{0} b_l^k \rho^l + r(\rho), \tag{5.1}$$

where $j_0 = -[\sigma]$ if $\sigma < 0$, $j_0 = 0$ if $\sigma > 0$ and $r(\rho) = \mathcal{O}(\rho)$ ($[\cdot]$ denotes the integer part). Since $\int_{\mathbb{R}} f(t)\, dt$ is convergent we must have that $\lim_{\rho\to 0} \int_{\gamma_\rho} f(z)\, dz$ is a finite value. Then, the coefficients corresponding to negative powers of ρ have to compensate and, since $\sigma \in \mathbb{R} \setminus \mathbb{Z}$,

$$(1 - e^{-2\pi\gamma}) \int_{\infty}^{\infty} f(t)\, dt = \lim_{\rho\to 0} \int_{\gamma_\rho} f(z)\, dz = b_0^1 + \cdots + b_0^8.$$

Integral over $\gamma_{\rho 1}$. We can write

$$\tan(iz/2 + \pi/4) = \frac{1 + i\sinh z}{\cosh z} = i(z - i\frac{\pi}{2})\, r(z),$$

where $r(z)$ is an analytic function around $i\pi/2$ with $r(i\pi/2) = 1/2$. In particular, there is a neighborhood of $i\pi/2$ where $r(z)$ stays away from 0. Therefore $\ln(r(z))$ is analytic around $i\pi/2$ and we can write f in the form $f(z) = \exp(\sigma \ln(i(z - i\frac{\pi}{2})))u(z)$, with $u(z) = e^{\sigma \ln(r(z)) + i\gamma z}/\cosh^2 z$. The function u has a pole at $i\pi/2$ which is of order two. Let $u(z) = \sum_{j=-2}^{\infty} c_j (z - i\frac{\pi}{2})^j$ be its Laurent series. Substituting into the integral

$$\begin{aligned}\int_{\gamma_{\rho 1}} f(z)\,dz &= \int_0^{2\pi} e^{\sigma \ln(i\rho \exp(i(t-3\pi/2)))} \Big(\sum_{j=-2}^{\infty} c_j \rho^j e^{ij(t-3\pi/2)}\Big) i\rho e^{i(t-3\pi/2)}\,dt \\ &= i \sum_{j=-2}^{\infty} c_j \rho^{j+\sigma+1} \int_0^{2\pi} e^{\sigma i(t-\pi)} e^{i(j+1)(t-3\pi/2)}\,dt.\end{aligned}$$

Therefore, the corresponding coefficient b_0^1 is zero and there is no contribution of this term to the limit when $\rho \to 0$.

The same analysis applies to $\gamma_{\rho 5}$, and $b_0^5 = 0$.

Integral over $\gamma_{\rho 2} + \gamma_{\rho 8}$. Taking care of the determination of the logarithm, the integral can be written as

$$\begin{aligned}&\int_{\gamma_{\rho 2}+\gamma_{\rho 8}} \frac{e^{\sigma \ln(\tan(iz/2+\pi/4))}}{\cosh^2 z} e^{i\gamma z}\,dz \\ &= ie^{-\gamma\frac{\pi}{2}} \Big(\int_\rho^{\pi/2-\rho} \frac{e^{\sigma(\ln(\tan(s/2))+i\pi)}}{\sin^2 s} e^{-\gamma s}\,ds - \int_\rho^{\pi/2-\rho} \frac{e^{\sigma(\ln(\tan(s/2))-i\pi)}}{\sin^2 s} e^{-\gamma s}\,ds \Big) \\ &= -2e^{-\gamma\frac{\pi}{2}} \sin(\sigma\pi) \int_\rho^{\pi/2-\rho} \frac{\tan^\sigma(s/2)}{\sin^2 s} e^{-\gamma s}\,ds.\end{aligned}$$

Let j_0 be as in formula (5.1). We have $\tan^\sigma(t/2) = u(t)\, t^\sigma/2^\sigma$, where $u(t)$ is an analytic function around 0 (in the disk of radius π) with $u(0) = 1$, and since $\sin^{-2}(t)$ has a pole of order two at zero we can write

$$\frac{\tan^\sigma(t/2)}{\sin^2 t} = \frac{t^\sigma}{2^\sigma}\Big(\sum_{j=-2}^{j_0} c_j t^j + r(t)\Big), \qquad t \in (0, \pi/2],$$

where $r(t)$ is an analytic function such that $|r(t)| < K t^{j_0+1}$, where K is a constant depending only in j_0 (and, hence, in σ) and $c_{-2} = 1$. Therefore,

$$\int_\rho^{\frac{\pi}{2}-\rho} \frac{\tan^\sigma(t/2)}{\sin^2 t} e^{-\gamma t}\,dt = \int_\rho^{\frac{\pi}{2}-\rho} \frac{t^\sigma}{2^\sigma}\Big(\sum_{j=-2}^{j_0} c_j t^j + r(t)\Big) e^{-\gamma t}\,dt$$

$$
\begin{aligned}
&= \frac{1}{2^\sigma}\Big(\sum_{j=-2}^{j_0} c_j \int_\rho^{\frac{\pi}{2}-\rho} t^{\sigma+j} e^{-\gamma t} dt + \int_\rho^{\frac{\pi}{2}-\rho} t^\sigma r(t) e^{-\gamma t} dt\Big)\\
&= \frac{1}{2^\sigma}\Big(\sum_{j=-2}^{j_0} c_j \int_\rho^{\frac{\pi}{2}-\rho} t^{\sigma+j} e^{-\gamma t} dt\Big) + R_1. \qquad (5.2)
\end{aligned}
$$

We can easily bound R_1 as follows

$$
\begin{aligned}
|R_1| &\leq \int_\rho^{\frac{\pi}{2}-\rho} |t^\sigma r(t) e^{-\gamma t}| dt \leq \int_0^{\frac{\pi}{2}} |t^\sigma r(t) e^{-\gamma t}| dt\\
&\leq K \int_0^{\frac{\pi}{2}} t^{j_0+\sigma+1} e^{-\gamma t} dt \leq K \frac{\Gamma(j_0+\sigma+2)}{\gamma^{j_0+\sigma+2}}. \qquad (5.3)
\end{aligned}
$$

On the other hand, integrating by parts $(j_0 - j)$ times we obtain that

$$
\int_\rho^{\frac{\pi}{2}-\rho} t^{\sigma+j} e^{-\gamma t}\, dt = J_1^j(\rho) + J_2^j(\pi/2-\rho) - J_2^j(\rho), \qquad (5.4)
$$

where

$$
\begin{aligned}
J_1^j(\rho) &= \frac{\gamma^{j_0-j}}{(\sigma+j+1)\cdots(\sigma+j_0)} \int_\rho^{\frac{\pi}{2}-\rho} t^{\sigma+j_0} e^{-\gamma t}\, dt, \qquad (5.5)\\
J_2^j(t) &= e^{-\gamma t}\left(\frac{t^{\sigma+j+1}}{\sigma+j+1} + \cdots + \frac{\gamma^{j_0-j-1} t^{\sigma+j_0}}{(\sigma+j+1)\cdots(\sigma+j_0)}\right). \qquad (5.6)
\end{aligned}
$$

Notice that $J_2^j(\pi/2-\rho) = \mathcal{O}\left(\gamma^{j_0-j-1} e^{-\gamma(\frac{\pi}{2}-\rho)}\right)$. Moreover,

$$
\begin{aligned}
&\int_\rho^{\frac{\pi}{2}-\rho} t^{\sigma+j_0} e^{-\gamma t}\, dt\\
&= \int_0^\infty t^{\sigma+j_0} e^{-\gamma t}\, dt - \int_0^\rho t^{\sigma+j_0} e^{-\gamma t}\, dt - \int_{\frac{\pi}{2}-\rho}^\infty t^{\sigma+j_0} e^{-\gamma t}\, dt\\
&= \frac{\Gamma(\sigma+j_0+1)}{\gamma^{\sigma+j_0+1}} + \mathcal{O}\left(\rho^{\sigma+j_0+1}\right) - \int_{\frac{\pi}{2}-\rho}^\infty t^{\sigma+j_0} e^{-\gamma t}\, dt. \qquad (5.7)
\end{aligned}
$$

We introduce $n_0 = [\sigma + j_0] + 1$ (if $\sigma < 0$ then $n_0 = 1$, otherwise $n_0 = [\sigma] + 1$). We have the following bound

$$
\int_{\frac{\pi}{2}-\rho}^\infty t^{\sigma+j_0} e^{-\gamma t}\, dt < \int_{\frac{\pi}{2}-\rho}^\infty t^{n_0} e^{-\gamma t}\, dt < M \frac{(\pi/2-\rho)^{n_0}}{\gamma} e^{-\gamma(\frac{\pi}{2}-\rho)}
$$

where M is some positive constant which only depends on σ for $\gamma > 1$. Therefore, from (5.4) and the above bounds we obtain

$$\begin{aligned}\int_\rho^{\frac{\pi}{2}-\rho} t^{\sigma+j} e^{-\gamma t}\, dt &= \frac{\gamma^{j_0-j}}{(\sigma+j+1)\cdots(\sigma+j_0)} \frac{\Gamma(\sigma+j_0+1)}{\gamma^{\sigma+j_0+1}} \\ &+ \mathcal{O}\left(\rho^{\sigma+j_0+1}\right) + \mathcal{O}\left(\gamma^{j_0-j-1} e^{-\gamma(\frac{\pi}{2}-\rho)}\right) - J_2^j(\rho). \quad (5.8)\end{aligned}$$

Finally, substituting the expression (5.8) in formula (5.2)

$$\begin{aligned}\int_\rho^{\frac{\pi}{2}-\rho} \frac{\tan^\sigma(t/2)}{\sin^2 t} e^{-\gamma t}\, dt &= \frac{\Gamma(\sigma-1)}{2^\sigma(\sigma-1)\gamma^{\sigma-1}} + \mathcal{O}\left(\frac{1}{\gamma^\sigma}\right) + \mathcal{O}\left(\gamma^{j_0+1} e^{-\gamma(\frac{\pi}{2}-\rho)}\right) \\ &+ \mathcal{O}\left(\rho^{\sigma+j_0+1}\right) - \sum_{j=-2}^{j_0} J_2^j(\rho) + R_1.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\gamma_{\rho 2}+\gamma_{\rho 8}} \frac{e^{\sigma \ln(\tan(iz/2+\pi/4))}}{\cosh^2 z} e^{i\gamma z}\, dz &= -e^{-\gamma\frac{\pi}{2}} \sin(\sigma\pi) \Bigg(\frac{\Gamma(\sigma-1)}{(2\gamma)^{\sigma-1}} + \mathcal{O}\left(\frac{1}{\gamma^\sigma}\right) \\ &+ \mathcal{O}\left(e^{-\gamma(\frac{\pi}{2}-\rho)}\right) + \mathcal{O}\left(\rho^{\sigma+j_0+1}\right) \\ &- \sum_{j=-2}^{j_0} J_2^j(\rho) + R_1 \Bigg)\end{aligned}$$

We recall that $R_1 = \mathcal{O}\left(\frac{1}{\gamma^{\sigma+j_0+2}}\right)$, and that $\sigma + j_0 + 2 \geq \sigma$. Then the contribution to the value of the integral along $\mathbb{R}$ is given by

$$b_0^2 + b_0^8 = -e^{-\gamma\frac{\pi}{2}} \sin(\sigma\pi) \left(\frac{\Gamma(\sigma-1)}{(2\gamma)^{\sigma-1}} + \mathcal{O}\left(\frac{1}{\gamma^\sigma}\right) \right).$$

The same analysis for $\gamma_{\rho 4}$ and $\gamma_{\rho 6}$ gives a similar result for $b_0^4 + b_0^6$ with $e^{-\gamma\pi}$ instead of $e^{-\gamma\frac{\pi}{2}}$.

Integral over $\gamma_{\rho 3} + \gamma_{\rho 7}$. The function $\tan(iz/2 + \pi/4)$ takes the value -1 at $i\pi$, and, therefore, $e^{\sigma \ln(\tan(iz/2+\pi/4))}$ is bounded on $\gamma_{\rho 3}$. The function $e^{i\gamma z}/\cosh^2(z)$ is analytic at $i\pi$. We can conclude that

$$\left|\int_{\gamma_{\rho 3}} \frac{e^{\sigma \ln(\tan(iz/2+\pi/4))}}{\cosh^2 z} e^{i\gamma z}\, dz\right| < M' \operatorname{length}(\gamma_{\rho 3}) < M''\rho,$$

for some positive constants M' and M''. The same analysis is true for $\gamma_{\rho 7}$. Therefore we have that the coefficients $b_0^3 = b_0^7 = 0$.

Finally, from the preceding calculations we have that

$$\int_\infty^\infty f(t)\,dt = \frac{b_0^1 + \ldots + b_0^8}{1 - e^{-2\pi\gamma}} = -\frac{e^{-\gamma\frac{\pi}{2}}}{1 - e^{-2\pi\gamma}} \sin(\sigma\pi) \left(\frac{\Gamma(\sigma - 1)}{(2\gamma)^{\sigma-1}} + \mathcal{O}\left(\frac{1}{\gamma^\sigma}\right)\right).$$

6 Aknowledgements

The second author acknowledges the partial support of the CEE Grant, "Stability and Universality in Classical Mechanics", ERBCHRXCT940460, 1994-1997, the Spanish Grant DGICYT, PB94-0215 and the Catalan Grant CIRIT GRQ93-1135.

References

[1] V.I. Arnold, *Instability of dynamical systems with several degrees of freedom*, Soviet Math. Dokl. **5** (1964), pp. 581-585.

[2] U. Bessi, *An approach to Arnold's diffusion through the Calculus of Variations*, Nonlinear Analisys, T.M.A., **26** (1996), pp. 1115-1135.

[3] P.M. Cohn, *Algebra*, vol 1., second edition, Wiley, Chichester, 1982.

[4] S.N. Chow, J.K. Hale, *Methods of Bifurcation Theory*, Springer Verlag, New York, 1982.

[5] J. Cresson, *A λ-lemma for partially hyperbolic tori and the obstruction property*, preprint, Ins. Mat. de Jusieux (1997).

[6] A. Delshams, P. Gutiérrez, *Effective stability and KAM Theory*, J. Differential Equations, **128** (1996), pp. 415-490.

[7] A. Delshams, A. Jorba, V. Gelfreich, T. Martínez-Seara, *Exponentially small splitting of separatrices under fast quasiperiodic forcing*, Comm. Math. Phys. **189** (1997), pp. 35-71.

[8] R. Douady, *Stabilité ou instabilité des points fixes elliptiques*, Annal. Sci. de l'ENS, **21** (1988), pp. 1-46.

[9] R. Douady, P. Le Calvez, *Exemple de point fixe elliptique non topologiquement stable en dimension 4*, Problèmes Mathématiques de la Mécanique, C.R. Acad. Sc. Paris, **296** (1982), pp. 895-898.

[10] S. Graff, *On the conservation of hyperbolic invariant tori for hamiltonian systems*, J. Differential Equations, **15** (1974), pp. 1-69.

[11] M.W. Hirsch, C.C. Pugh, *Stable Manifolds and Hyperbolic Sets*, Proc. Symp. in Pure Math, vol. 14, AMS, Providence, R.I. (1970), pp. 133-164.

[12] M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1977.

[13] M.C. Irwin, *On the smoothness of the composition map*, Quart. J. Math., Oxford, **23**, (1972), pp. 113-133.

[14] P. Lochak, *Effective speed of Arnold's diffusion and small denominators*, Phys. Lett. A, **143**, 39 (1990).

[15] P. Lochak, A. I. Neishtadt, *Estimates in the theorem of N. N. Nekhoroshev for systems with a quasi-convex Hamiltonian*, Chaos, **2** (1992), pp. 495-499.

[16] J.P. Marco, *Transition le long des chaînes de tores invariants pour les systèmes hamiltoniens analytiques*, Annales de l'I.H.P., Physique Théorique, **64** (1996), pp. 205-252.

[17] R. Narasimhan, *Several complex variables*, Chicago Lectures in Math. Series, University of Chicago Press, 1971.

[18] N.N. Nekhoroshev, *An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems*, Russian Math. Surveys, **32** (1977), pp. 1-65.

[19] N. Palis, W. de Melo, *Geometric Theory of Dynamical Systems*, Springer, New York, 1982.

[20] J. Pöschel, *Nekhoroshev estimates for quasi-convex Hamiltonian systems*, Math. Z., **213** (1993), pp. 187-217.

[21] M. Rudnev, S. Wiggins, *Existence of exponentially small separatrix splittings and homoclinics connections between whiskered tori in weakly hyperbolic near-integrable hamiltonian systems*, to appear in Physica D.

[22] C. Simó, *Averaging under fast quasiperiodic forcing*, in J. Seimenis, editor, Hamiltonian Mechanichs: Integrability and Chaotic Behavior, volume 331 of NATO Adv. Sci. Inst. Ser. Phys. (1994), pp. 13-34.

[23] D.V. Treshev, *The mechanism of destruction of resonance tori of hamiltonian systems*, Math USSR Sbornik, **68** (1991), pp. 181-203.

[24] S. Wiggins, *Global bifurcations and chaos*, Springer, New York, 1988.