On the Surface Spectrum in Dimension Two

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Abstract

We study spectral properties of the discrete Laplacian H_ω on the half space $\mathbf{Z}_+^2 = \mathbf{Z} \times \mathbf{Z}_+$ with a random boundary condition $\psi(n, -1) = V_\omega(n)\psi(n, 0)$. Here, $V_\omega(n)$ are independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$. We show that outside the interval $[-4, 4]$ (the spectrum of the Dirichlet Laplacian) the spectrum of H_ω is P-a.s. dense pure point.

1 Introduction

This paper is a part of the program introduced in [JMP]. This program is concerned with spectral and scattering theory of the discrete Laplacian on a half-space with a random boundary condition. We refer the reader to [JMP] for the history of the problem and additional information. In this section we define the model, review some of the known results and state theorems which will be proven in this paper. At the end of the section we will sketch some of the main ideas involved in the proofs of the theorems.

Let $d \geq 1$ be given and let $\mathbf{Z}_+^{d+1} = \mathbf{Z}^d \times \mathbf{Z}_+$, where $\mathbf{Z}_+ = \{0, 1, \ldots\}$. We denote the points in $\mathbf{Z}_+^{d+1}$ by (n, x), $n \in \mathbf{Z}^d$, $x \in \mathbf{Z}_+$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and V_ω, $\omega \in \Omega$, a random process on $\mathbf{Z}^d$ such that $V_\omega(n)$ are independent and identically distributed random variables with density $p(x)$. We denote by $\mathcal{V}$ the support of the probability measure $p(x)dx$. Let H_ω be the discrete Laplacian on $l^2(\mathbf{Z}_+^{d+1})$ with the boundary condition $\psi(n, -1) = V_\omega(n)\psi(n, 0)$. If $V_\omega \equiv 0$, this operator reduces to the Dirichlet Laplacian which we denote by H_0. The operator H_ω acts as

$$(H_\omega \psi)(n, x) = \begin{cases} \sum_{|n-n'|_+ + |x-x'|=1} \psi(n', x') & \text{if } x > 0, \\ \psi(n, 1) + \sum_{|n-n'|_+=1} \psi(n, 0) + V_\omega(n)\psi(n, 0) & \text{if } x = 0, \end{cases}$$

where $|n|_+ = \sum_{j=1}^d |n_j|$. Note that operator H_ω can be viewed as the random Schrödinger operator

$$H_\omega = H_0 + V_\omega, \tag{1.1}$$

where the random potential V_ω acts only along the boundary $\partial \mathbf{Z}_+^{d+1} = \mathbf{Z}^d$. For many purposes, it is convenient to adopt this point of view and we will do so in the sequel. Since H_0 is bounded, the operator H_ω is properly defined as a self-adjoint operator on $l^2(\mathbf{Z}_+^{d+1})$.

It follows from the standard argument (see Section 9.1 of [CFKS] for basic notions concerning random Schrödinger operators) that there are deterministic sets Σ, Σ_{pp}, Σ_{ac} and Σ_{sc} such that P-a.s., $\sigma(H_\omega) = \Sigma$, $\sigma_{pp}(H_\omega) = \Sigma_{pp}$, $\sigma_{ac}(H_\omega) = \Sigma_{ac}$, $\sigma_{sc}(H_\omega) = \Sigma_{sc}$. Obviously, $\Sigma = \Sigma_{pp} \cup \Sigma_{ac} \cup \Sigma_{sc}$. We will use the usual notation $\Sigma_c = \Sigma_{ac} \cup \Sigma_{sc}$, $\Sigma_s = \Sigma_{pp} \cup \Sigma_{sc}$. The set Σ can be explicitely computed (see [JMP], and for detailed proof [JL]). Let

$$\mathcal{S}(\mathcal{V}) \equiv \left\{ E + a + \frac{1}{a} : \ E \in [-2d, 2d], \ a \in \mathcal{V} \text{ and } |a| \geq 1 \right\}. \tag{1.2}$$

Note that $\mathcal{S}(\mathcal{V})$ is a closed set and that $\mathcal{S}(\mathcal{V}) = \emptyset$ if and only if $\mathcal{V} \subset (-1, 1)$. Recall that $\sigma(H_0) = [-2(d+1), 2(d+1)]$. Then

$$\Sigma = \sigma(H_0) \cup \mathcal{S}(\mathcal{V}).$$

Note also that whenever $\mathcal{V} \cap (\mathbf{R} \setminus [-1, 1]) \neq \emptyset$, the set Σ has parts lying outside $\sigma(H_0)$.

The first natural question concerning the spectral theory of H_ω is what is the structure of the sets Σ_{pp}, Σ_{ac}, Σ_{ac}. We briefly summarize the known results.

1) Under very general conditions (e.g. it suffices that $\mathbf{E}(|V_\omega(0)|^\alpha) < \infty$ for some $\alpha > 0$), we have that $\sigma(H_0) \subset \Sigma_{ac}$. The proof of this result is sketched in [JMP]. If the random variables $V_\omega(n)$ are bounded this result is deterministic, i.e. for any bounded real function V on $\mathbf{Z}^d$ we have that $\sigma(H_0) \subset \sigma_{ac}(H_0 + V)$.

2) If $\mathcal{S}(\mathcal{V}) = \emptyset$ then $\Sigma_s = \emptyset$, i.e. if $\mathcal{V} \subset (-1, 1)$ the spectrum of H_ω is P-a.s. purely absolutely continuous. More generaly, if $d = 1$ then $\Sigma_s \cap (\sigma(H_0) \setminus \mathcal{S}(\mathcal{V})) = \emptyset$. The same result holds in any dimension if $\mathcal{V}$ is an interval (this assumption is needed for technical reasons). These results are proven in [JL]. We also remark that these results are deterministic - suitably reformulated they hold for a deterministic boundary potential V.

3) Under some additional technical assumptions on the distribution function $p(x)$ (e.g. it suffices that p is compactly supported and in $L^\infty(\mathbf{R})$), there exists $E_c > 2(d+1)$, which depends on p only, such that

$$\Sigma_c \cap \{E : |E| > E_c\} = \emptyset. \tag{1.3}$$

In other words, P-a.s. the spectrum of H_ω is pure point outside the interval $[-E_c, E_c]$. It is also known that the corresponding eigenfunctions decays exponentially. Similar results hold in the "large disorder regime" - for any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that if $\|p\|_\infty > \delta(\varepsilon)$ then (1.3) holds with $E_c = 2(d+1) + \varepsilon$. The corresponding eigenfunctions also decay exponentially. These results are proven in [AM] and [G]. For some related results see [BS].

In this paper, we are interested in improving the results of 3) in $d = 1$. We will make the following assumptions concerning the random potential V_ω.

(H1) The topological boundary of $\mathcal{V}$ is a discrete set and $p \in L^\infty(\mathbf{R})$.

Our main result is

Theorem 1.1 *Let $d = 1$ and assume that* (H1) *holds. Then*

$$\Sigma_c \cap \{E : |E| > 4\} = \emptyset.$$

In other words, P-a.s. the spectrum of H_ω outside the interval $[-4, 4]$ is pure point.

Remark 1. Our estimates give some control of the decay of the eigenfunctions of H_ω. It follows from our arguments that P-a.s. the eigenfunctions corresponding to the eigenvalues outside $[-4, 4]$ decay as

$$|\psi_{E,\omega}(n, x)| \leq C_{E,\omega,k} \exp(-\gamma_E |x|)(1 + |n|)^{-k}, \tag{1.4}$$

for any $k > 0$. We expect that the estimate (1.4) is not optimal, and that the eigenfunctions decay exponentially in the n-variable. To establish such decay near the edges ± 4 appears to be a difficult technical problem.

Remark 2. We expect that Theorem 1.1 is optimal. More precisely, we expect that if $d = 1$ then $\Sigma_{pp} = \mathbf{R} \setminus (-4, 4)$. This result can be proven in some cases where the boundary potential V is almost periodic [JM1], [KP].

Remark 3. The condition that topological boundary of $\mathcal{V}$ is a discrete set is needed for technical reasons and in some cases it could be relaxed. For example, if the Lebesgue measure of $\mathcal{V}$ is infinite, the result holds under the assumption that $\mathrm{int}(\mathcal{V}) \neq \emptyset$.

Let us summarize the results of 1) and 2) combined with Theorem 1.1. Assume for simplicity that the random variables $V_\omega(n)$ are uniformly distributed on $[\alpha, \alpha + 1]$, where $\alpha > 1$ is a parameter. Let α_c be the solution of the equation $\alpha + 1/\alpha = 6$ such that $\alpha_c > 1$ ($\alpha_c = 5.83...$). We then have the following three regimes. The results are to be understood in the P-a.s. sense.

a) $1 < \alpha < \alpha_c$. In this case $\Sigma = [-4, 2 + (\alpha + 1) + 1/(\alpha + 1)]$. The spectrum on the interval $[-4, -2 + \alpha + 1/\alpha]$ is purely absolutely continuos and on the interval $[4, 2 + (\alpha + 1) + 1/(\alpha + 1)]$ is dense pure point. There exists a component of absolutely continuous spectrum on $[-2 + \alpha + 1/\alpha, 4]$, but the precise nature of the spectrum on this set is unknown.

b) $\alpha = \alpha_c$. In this case, $\sigma(H_0) = [-4, 4]$ and $\mathcal{S}(\mathcal{V}) = [4, 8.68..]$. The spectrum on $[-4, 4]$ is purely absolutely continuous and on $[4, 8.68..]$ dense pure point.

c) $\alpha > \alpha_c$. In this case the spectrum consists of two disjoint components, $\sigma(H_0) = [-4, 4]$ and $\mathcal{S}(\mathcal{V}) = [-2 + \alpha + 1/\alpha, 2 + (\alpha + 1) + 1/(\alpha + 1)]$. The spectrum on $[-4, 4]$ is purely absolutely continuous and on $\mathcal{S}(\mathcal{V})$ dense pure point.

We remark that an interesting open question is what is the structure of the spectrum on the set $\sigma(H_0) \cap \mathcal{S}(\mathcal{V})$. If the existing results (see [JM1]) can serve as a guide, one may expect that the spectrum on this set is also purely absolutely continuous. This problem remains to be investigated in the future.

We finally would like to relate Theorem 1.1 to the discusssion of the surface waves presented in [JMP]. For any boundary potential V we define the surface spectrum of the operator $H_0 + V$ as the closure of the set of energies E for which the equation $(H_0 + V)u(n, x) = Eu(n, x)$ has a non-zero solution which satisfies

$$\sum_{n \in \mathbf{Z}^d} (1 + |n|)^{-k} \sum_{x \geq 0} |u(n, x)|^2 < \infty,$$

for some $k > 0$. Roughly, the surface spectrum consists of the energies whose corresponding generalized eigenfunctions have some decay in the x-variable. We denote the surface spectrum by $\sigma_{surf}(H_0 + V)$. One can show (see [JMP]) that $\sigma(H_0 + V) \setminus \sigma(H_0) \subset \sigma_{surf}(H_0 + V)$. An absolutely continuous surface spectrum exists if V is a constant or a periodic function and $\max_n |V(n)| > 1$. In this case, the generalized eigenfunctions are localized in the x-direction and propagate along the boundary. Theorem 1.1 asserts that if $d = 1$ and the constant boundary condition is replaced with a random boundary condition, then all propagating surface waves with energies outside $[-4,4]$ are localized by the random fluctuations of the boundary. This is physically the most interesting consequence of Theorem 1.1. An interesting open question is whether there are any propagating surface waves with energies inside $\sigma(H_0)$. This problem also remains to be investigated in the future.

In the rest of this introduction we sketch some of the basic ideas involved in the proof of Theorem 1.1.

The first idea concerns dimension reduction ([AM], [G], [JMP]). Roughly speaking, "integrating" the x-variable we will reduce the 2-dimensional spectral problem to an 1-dimensional spectral problem which will depend non-linearly on the spectral parameter E. This reduction could be done in any dimension. For the latter applications, we will describe and prove this result in the general setting.

Let $\mathcal{I}$ be a closed interval on the energy axis such that $\mathcal{I}\cap\sigma(H_0)=\emptyset$. Let $\mathbf{T}=\mathbf{R}/2\pi\mathbf{Z}$ be the circle and $\mathbf{T}^d$ the d-dimensional torus. We denote the points in $\mathbf{T}^d$ by $\phi=(\phi_1,\ldots,\phi_d)$, and by $d\phi$ the usual Lebesgue measure. In the sequel we use a shorthand $\Phi(\phi)=2\sum_{k=1}^d\cos\phi_k$. Let $\lambda(\phi,E)$ be the solution of the quadratic equation

$$\lambda(\phi,E)+\frac{1}{\lambda(\phi,E)}+\Phi(\phi)=E,$$

such that $|\lambda(\phi,E)|<1$. Let

$$\hat{j}(\phi,E)=\lambda(\phi,E)+\Phi(\phi),\qquad j(n,E)=\int_{\mathbf{T}^d}e^{-in\cdot\phi}\hat{j}(\phi,E)d\phi. \tag{1.5}$$

We will prove in Section 2 that there are constants C and γ, which depend only on the distance of $\mathcal{I}$ from $\sigma(H_0)$, such that for $E\in\mathcal{I}$,

$$|j(n,E)|\le C\exp(-\gamma|n|_+).$$

Let $h_0(E)$ be the operator on $l^2(\mathbf{Z}^d)$ defined by

$$(h_0(E)\psi)(n)=\sum_{k\in\mathbf{Z}^d}j(n-k,E)\psi(k). \tag{1.6}$$

We define one parameter family of random operators on $l^2(\mathbf{Z}^d)$ by

$$h_\omega(E)=h_0(E)+V_\omega,\qquad E\in\mathcal{I}. \tag{1.7}$$

Our argument will be based on the following variant of Simon-Wolff theorem [SW]. Let m be the Lebesgue measure on $\mathbf{R}$.

Theorem 1.2 *If for a.e. $(E, \omega) \in \mathcal{I} \times \Omega$ with respect to the product measure $m \otimes P$,*

$$\lim_{\zeta \to 0} \|(h_\omega(E) - E - i\zeta)^{-1}\delta_0\| < \infty, \tag{1.8}$$

then $\Sigma_c \cap \mathcal{I} = \emptyset$.

We will prove this theorem in Section 2.

In comparison with the usual theory of random Schrödinger operators, there are two essential difficulties in studying the quantity $\|(h_\omega(E) - E - i\zeta)^{-1}\delta_0\|$. The first is that $h_0(E)$ is a long-range Laplacian, and the second is that $h_0(E)$ depends on energy. These difficulties are succesfully resolved in the high energy or large coupling regime adopting the techniques of the multiscale analysis and the method of Aizenman-Molchanov [G], [AM]. Of course, in general these results cannot be improved without major new insights into the theory of random Schrödinger operators.

The case $d = 1$ is however special. In this case, the operators $h_\omega(E)$ act on $l^2(\mathbf{Z})$, and there was a hope that the results of 3) could be improved using some of the techniques specific to the theory of one-dimensional Schrödinger operators. As a first part of this program, we have investigated in [JM] the long-range, one dimensional random Schrödinger operators of the form $h_\omega = h_0 + V_\omega(n)$, where $V_\omega(n)$ is as in (1.1), and h_0 is a translationaly invariant self-adjoint operator with some off-diagonal decay. The simplification is that h_0 now does not depend on E,

$$(h_0\psi)(n) = \sum_{k \in \mathbf{Z}} j(n-k)\psi(k).$$

Note that again the spectrum of h_ω and its pp, sc, ac component are P-a.s. deterministic sets. Furthermore, P-a.s. $\sigma(h_\omega) = \sigma(h_0) + \mathcal{V}$.

Before stating a theorem from [JM] which will concern us here, we set some hypothesis on h_0:

(H2) There is $\delta > 0$ such that $\forall n$, $|j(n)| \leq C(1 + |n|)^{-8-\delta}$.

(H3) The function $\hat{j}(\phi) = \sum_n j(n) \exp(in\phi)$ is even, real and strictly monotone on $[0, \pi]$.

The following results was proven in [JM].

Theorem 1.3 *Assume that Hypothesis* (H2)-(H3) *hold and that* $\mathrm{int}(\mathcal{V}) \neq \emptyset$. *Then for a.e. $(E, \omega) \in \mathbf{R} \times \Omega$ with respect to the product measure $m \otimes P$,*

$$\lim_{\zeta \to 0} \|(h_\omega - E - i\zeta)^{-1}\delta_0\| < \infty. \tag{1.9}$$

In particular, P-a.s. the operators h_ω have pure point spectrum.

The techniques used in the proof of this theorem will play a critical role in the proof of Theorem 1.1. For this reason we briefly review some of the basic steps in the argument.

The proof of Theorem 1.3 is based on a geometric approach to localization in $d = 1$ which goes back to [SS], [KMP], [M], [M1], [GJMS]. The principal idea is to show that a particle with given energy E_0 has to tunnel through an infinite sequence of "barriers" to reach infinity. This idea is formalized as follows. Let E_0 be a given point in $\sigma(h_0) + \mathcal{V}$, and $\mathcal{I}$ a small open interval around E_0. Using the structure of the random potential $V_\omega(n)$, one constructs P-a.s. a sequence of intervals (barriers) $I_k(\omega) \subset \mathbf{Z}$, with centers $c_k(\omega)$ and of width $l_k(\omega)$, such that $c_k(\omega) \to \pm\infty$ and $l_k(\omega) \to \infty$ as $k \to \pm\infty$, and such that

$$\mathcal{I} \cap \sigma(h_{I_k(\omega)}) = \emptyset, \tag{1.10}$$

for all k. Here, $h_{I_k(\omega)}$ is the restriction of h_ω to $I_k(\omega)$ with the Dirichlet boundary condition. For barriers to be effective in preventing tunneling, we need that they are sufficently long, namely that $\# I_k(\omega) \geq c_\omega |k|$, and they are not too far apart, namely that $|c_k(\omega)| \leq a_\omega^{|k|}$, for some positive constants c_ω and a_ω. Once such a geometric configuration of the barriers is given, the random parameter ω is fixed, and plays no further role. Thus, we drop subscript ω for the rest of this paragraph. Let Δ_k be the intervals between I_k and I_{k+1}, $M_k = I_k \cup \Delta_k \cup I_{k+1}$ and h_{M_k} the restriction of h to M_k with the Dirichlet boundary condition. One now constructs an iterative expansion of the resolvent $(h - E)^{-1}$ in terms of the resolvents $(h_{M_k} - E)^{-1}$ and $(h_{I_k} - E)^{-1}$. At this point one encounters an analog of the "small divisor problem". The contributions from $(h_{I_k} - E)^{-1}$ are small due to (1.10). Since we do not have any control on the potential within Δ_k's, we need an appriory estimate on $(h_{M_k} - E)^{-1}$ which will make use of the fact that the intervals M_k are not too long. Such an estimate is obtained by randomization of the energy E within interval $\mathcal{I}$. The end result is that for typical $E \in \mathcal{I}$ with respect to the Lebesgue measure, the size of the terms $(h_{M_k} - E)^{-1}$ is compensated by $(h_{I_k} - E)^{-1}$, and this will ultimately yield the estimate (1.8). We remark that although the ideas of the argument are intuitive and transparent, the technical details are involved.

There are two basic mechanism which can yield Relation (1.10). The first is that within intervals $I_k(\omega)$ the absolute value of the potential is sufficently large. This argument is applicable for example in the case where the Lebesgue measure of $\mathcal{V}$ is infinite. In this case, the proof is somewhat simpler and we do not need Hypothesis (H2) for Theorem 1.3 to hold. In more general situations, however, to verify (1.10) we had to construct long periodic approximations $V_{p,\omega}$ of the random potential V_ω such that E_0 is in the spectral gap of the operator $h_0 + V_{p,\omega}$. Since h_0 is long range, this construction is involved and technical. It is precisely in this construction that Hypothesis (H2) enters the game. We refer the reader to Sections 5 and 6 of [JM] for details of the argument.

In this paper we will use the techniques developed in [JM] to show that under the assumptions of Theorem 1.1 Theorem 1.2 holds. Note that for fixed $E \notin [-4, 4]$, $\hat{j}(\phi, E)$ is real, even, analytic and strictly monotone on $[0,\pi]$ so will adopt the strategy of the proof of Theorem 1.3. The main difficulty is that if h_0 depends on E, the randomization of energy used to get an appriory estimate on $(h_{M_k}-E)^{-1}$ is not possible any more. We replace this step in the argument with a construction to which we will refer as a probabilistic reduction. More precisely, we will make suitable partitions of the probability space Ω which will fix the positions of the "barrier" intervals I_k (the intervals for which an analog of (1.10) holds). Within these partitions the random variables $V_\omega(n)$ will be independent but not identically distributed. The appriory estimate on $(h_{M_k}(E)-E)^{-1}$ will be obtained for fixed E within the partitions with the help of a Wegner type result already used in [AM], [M1]. For this reason we need that $p \in L^\infty(\mathbf{R})$, an additional condition which played no role in [JM]. The rest of the argument will follow closely [JM].

We have attempted to give complete proofs, except for the results which are verbatim the same as in [JM].

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2 Dimension reduction

In this section we prove Theorem 1.2.

Let $V : \mathbf{Z}^d \mapsto \mathbf{R}$ be an arbitrary potential. We denote by the same letter the induced multiplication operator on $l^2(\mathbf{Z}_+^{d+1})$ which acts as follows: $(V\psi)(n,x) = 0$ if $x > 0$ and $(V\psi)(n,0) = V(n)\psi(n,0)$. Let $H = H_0 + V$ where H_0 is a Dirichlet Laplacian on $\mathbf{Z}_+^{d+1}$. We recall that the points in $\mathbf{Z}_+^{d+1}$ are denoted by $\mathbf{n} = (n,x)$, $n \in \mathbf{Z}^d, x \in \mathbf{Z}_+$. Let

$$R(\mathbf{m},\mathbf{n};z) = (\delta_{\mathbf{m}}, (H-z)^{-1}\delta_{\mathbf{n}}).$$

If $\mathrm{Im}(z) \neq 0$ and $\mathbf{m}$ is fixed, these matrix elements satisfy the equation

$$R(\mathbf{m},(n,x+1);z)+R(\mathbf{m},(n,x-1);z)+\sum_{|n-n'|_+=1} R(\mathbf{m},(n',x);z) = \delta_{\mathbf{mn}}+zR(\mathbf{m},(n,x);z), \quad (2.11)$$

if $x > 0$, and

$$R(\mathbf{m},(n,1);z) + \sum_{|n-n'|_+=1} R(\mathbf{m},(n',0;z) + (V(n)-z)R(\mathbf{m},(n,0);z) = \delta_{\mathbf{mn}}, \quad (2.12)$$

if $x = 0$. If $\mathbf{m} = (m,0)$ is a point on the boundary, Equation (2.11) can be "integrated". This is most conveniently done in the Fourier representation associated to the variable n. Let $\mathbf{T}^d$ and

$\Phi(\phi)$ be as in (1.5). We define a unitary map $F : l^2(\mathbf{Z}_+^{d+1}) \mapsto L^2(\mathbf{T}^d) \otimes l^2(\mathbf{Z}^+)$ by the formula

$$(F\psi)(\phi, x) \equiv \hat{\psi}(\phi, x) = \frac{1}{(2\pi)^{d/2}} \sum_{n \in \mathbf{Z}^d} \psi(n, x) e^{in\cdot\phi}.$$

In the new representation, Equations (2.11) and (2.12) become (recall that $\mathbf{m} = (m, 0)$),

$$\hat{R}(\mathbf{m}, (\phi, x+1); z) + \hat{R}(\mathbf{m}, (\phi, x-1); z) + (\Phi(\phi) - z)\hat{R}(\mathbf{m}, (\phi, x); z) = 0, \tag{2.13}$$

$$\hat{R}(\mathbf{m}, (\phi, 1); z) + (\Phi(\phi) - z)\hat{R}(\mathbf{m}, (\phi, 0); z) + \widehat{VR}(\mathbf{m}, (\phi, 0); z) - z\hat{R}(\mathbf{m}, (\phi, 0); z) = e_m(\phi), \tag{2.14}$$

where $e_m(\phi) = F(\delta_{mn}) = (2\pi)^{-d/2} \exp(im\phi)$. It follows from Equation (2.13) that if $x > 0$,

$$\hat{R}(\mathbf{m}, (\phi, x); z) = \hat{R}(\mathbf{m}, (\phi, 0); z)\lambda(\phi, z)^x, \tag{2.15}$$

where $\lambda(\phi, z)$ is a solution of the quadratic equation

$$\lambda(\phi, z) + \frac{1}{\lambda(\phi, z)} + \Phi(\phi) = z,$$

which satisfies $|\lambda(\phi, z)| < 1$. Note that for any fixed ϕ, $\lambda(\phi, z)$ is an analytic function in the second variable in the region $\mathbf{C}\backslash\sigma(H_0)$. Substituting (2.15) into (2.14) we get that $\hat{R}(\mathbf{m}, (\phi, 0); z)$ satisfies the equation

$$\hat{R}(\mathbf{m}, (\phi, 0); z)(\lambda(\phi, z) + \Phi(\phi)) + \widehat{VR}(\mathbf{m}, (\phi, 0); z) - z\hat{R}(\mathbf{m}, (\phi, 0); z) = e_m(\phi). \tag{2.16}$$

In the sequel we will use a shorthand

$$\mathcal{R}(m, n; z) \equiv R((m, 0), (n, 0); z) = (\delta_{(m,0)}, (H - z)^{-1}\delta_{(n,0)}). \tag{2.17}$$

Let

$$\hat{j}(\phi, z) = \lambda(\phi, z) + \Phi(\phi), \qquad j(n, z) = \int_{\mathbf{T}^d} e^{-in\cdot\phi}\hat{j}(\phi, z)d\phi. \tag{2.18}$$

Aplying F^{-1} to (2.16), we get that the matrix elements $\mathcal{R}(m, n; z)$ satisfy, for fixed m and z

$$\sum_k j(n-k, z)\mathcal{R}(m, k; z) + (V(n) - z)\mathcal{R}(m, n; z) = \delta_{mn}. \tag{2.19}$$

For any $z \in \mathbf{C} \setminus [-4, 4]$ we define a bounded operator $h_0(z)$ on $l^2(\mathbf{Z})$ by the formula

$$(h_0(z)\psi)(n) = \sum_k j(n-k, z)\psi(k).$$

We furher define $h(z) = h_0(z) + V$. Note that if z is real then $h(z)$ is a self-adjoint operator on $l^2(\mathbf{Z}^d)$. Furthermore, it follows from (2.19) that if $\mathrm{Im}(z) \neq 0$ then $z \notin h(z)$ and

$$(\delta_m, (h(z) - z)^{-1}\delta_n) = \mathcal{R}(m, n; z). \tag{2.20}$$

We will need

Lemma 2.1 *Let $n \in \mathbf{Z}^d$ and $E \notin \sigma(H_0)$ be given. Then*

$$\lim_{\zeta \to 0} \|(h(E+i\zeta) - E - i\zeta)^{-1}\delta_n\| < \infty,$$

if and only if

$$\lim_{\zeta \to 0} \|(h(E) - E - i\zeta)^{-1}\delta_n\| < \infty.$$

Proof: Note first that

$$\|(h(E+i\zeta) - E - i\zeta)^{-1}\| \leq 1/\zeta, \qquad \|(h(E) - E - i\zeta)^{-1}\| \leq 1/\zeta.$$

The second inequality is obvious, and the first follows from (2.17) and (2.20). It now follows from the resolvent identity that

$$\begin{aligned} \|(h(E+i\zeta) - E - i\zeta)^{-1}\delta_n\| &\leq \|(h(E) - E - i\zeta)^{-1}\delta_n\| \,(1 + \|h_0(E+i\zeta) - h_0(E)\|/\zeta) \\ \|(h(E) - E - i\zeta)^{-1}\delta_n\| &\leq \|(h(E+i\zeta) - E - i\zeta)^{-1}\delta_n\| \,(1 + \|h_0(E+i\zeta) - h_0(E)\|/\zeta)\,. \end{aligned}$$

These inequalities combined with the simple estimate

$$\|h_0(E+i\zeta) - h_0(E)\| \leq \sup_{\phi \in \mathbf{T}^d} |\lambda(\phi, E+i\zeta) - \lambda(\phi, E)| = O(\zeta)$$

yield the lemma. □

Remark. Since

$$(\delta_{(m,0)}, (H - E - i\zeta)^{-1}\delta_{(n,0)}) = (\delta_m, (h(E+i\zeta) - E - i\zeta)^{-1}\delta_n),$$

we have that for a.e. E with respect to the Lebesgue measure the limit

$$\lim_{\zeta \to 0} |(\delta_m, (h(E+i\zeta) - E - i\zeta)^{-1}\delta_n)|,$$

exists and is finite. Arguing as in the proof of Lemma 2.1 one can easily show that for a.e. $E \in \mathbf{R}$,

$$\limsup_{\zeta \to 0} |(\delta_m, (h(E) - E - i\zeta)^{-1}\delta_n)| < \infty.$$

This observation will be used latter.

We are now ready for

Proof of Theorem 1.2: Let $\mathcal{H}_n$ be the cyclic subspace of $l^2(\mathbf{Z}_+^{d+1})$ generated by the vector $\delta_{(n,0)}$ and the operator H_ω. It is easy to show that $\mathcal{H}_n$ is the same as the cyclic subspace generated by H_0 and $\delta_{(n,0)}$. Furthermore, the linear span of $\cup_{n\in\mathbf{Z}}\mathcal{H}_n$ is dense in $l^2(\mathbf{Z}_+^{d+1})$. These two simple facts are proven in [JL]. The Simon-Wolff theorem yields that that for a given closed interval $\mathcal{I}$,

$\Sigma_c \cap \mathcal{I} = \emptyset$ if for any $n \in \mathbf{Z}^d$ and for a.e. $(E,\omega) \in \mathcal{I} \otimes \Omega$ with respect to the product measure $m \otimes P$ we have that

$$\lim_{\zeta \to 0} \|(H_\omega - E - i\zeta)^{-1}\delta_{(n,0)}\|^2 = \lim_{\zeta \to 0} \sum_{\mathbf{k} \in \mathbf{Z}_+^{d+1}} |R_\omega((n,0),\mathbf{k}; E+i\zeta)|^2 < \infty. \tag{2.21}$$

Since the family of operators H_ω is ergodic with respect to the usual shift operators on Ω (see e.g. Section 9.1 in [CFKS]), it suffices to establish (2.21) in the case $\mathbf{n} = (0,0)$. The analysis of this section applied to $V = V_\omega$ yields that

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbf{Z}_+^{d+1}} |R_\omega((0,0),\mathbf{k}, E+i\zeta)|^2 &= \frac{1}{(2\pi)^d} \sum_{x \geq 0} \int_{\mathbf{T}^d} \left| \hat{R}_\omega((0,0),(\phi,0); E+i\zeta) \right|^2 |\lambda(\phi,z)|^{2x} d\phi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} \frac{\left| \hat{R}_\omega((0,0),(\phi,0); E+i\zeta) \right|^2}{1 - |\lambda(\phi, E+i\zeta)|^2}. \end{aligned}$$

Since $\mathcal{I} \cap \sigma(H_0) = \emptyset$ there are positive constants c_1 and c_2 such that for $\phi \in \mathbf{T}^d$, $E \in \mathcal{I}$ and $0 \leq \zeta \leq 1$,

$$c_1 \leq (1 - |\lambda(\phi, E+i\zeta)|^2)^{-1} \leq c_2.$$

Thus, (2.21) holds if and only if

$$\lim_{\zeta \to 0} \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} \left| \hat{R}_\omega((0,0),(\phi,0); E+i\zeta) \right|^2 d\phi = \lim_{\zeta \to 0} \sum_{k \in \mathbf{Z}^d} |\mathcal{R}_\omega(0,k; E+i\zeta)|^2 < \infty.$$

Finally, it follows from Lemma 2.1 that the second limit is finite if and only if

$$\lim_{\zeta \to 0} \|(h_\omega(E) - E - i\zeta)^{-1}\delta_0\| < \infty. \tag{2.22}$$

We conclude that if for a.e. $(E,\omega) \in \mathcal{I} \otimes \Omega$ with respect to $m \otimes P$ Relation (2.22) holds then $\Sigma_c \cap \mathcal{I} = \emptyset$. Theorem 1.2 follows. $\square$

We finish this section by collecting a few facts concerning the function $j(n,E)$ which we will use in the sequel. Note that if $n = (n_1, \ldots, n_d)$, and $\tilde{n} = (|n_1|, \ldots, |n_d|)$ then $j(n,E) = j(\tilde{n}, E)$. We also have the following estimate:

Proposition 2.2 *Let $E \notin \sigma(H_0)$ be given. Then there are constants C_E and γ_E such that*

$$|j(n,E)| \leq C_E \exp(-a(E)|n|_+). \tag{2.23}$$

These constants can be chosen as follows. Let $\gamma_E = (|E|-2)/2d$. Then

$$a(E) = \ln \gamma_E, \qquad C_E = (2\pi)^d (1 + 2d\gamma_E).$$

Remark. The estimates (2.23) are crude, but they will suffice for our purposes.

Proof: We denote the points in $\mathbf{C}^d$ by $\mathbf{z} = (z_1, z_2, \ldots, z_d)$. Let $\Phi : \mathbf{C}^d \mapsto \mathbf{C} \cup \{\infty\}$ be defined by

$$\Phi(\mathbf{z}) = \sum_{k=1}^{d} \left(z_k + \frac{1}{z_k} \right),$$

and let $\lambda(\mathbf{z}, E)$ be the solution of the equation

$$\lambda(\mathbf{z}, E) + \frac{1}{\lambda(\mathbf{z}, E)} + \Phi(\mathbf{z}) = E,$$

such that $|\lambda(\mathbf{z}, E)| < 1$. Parametrization $z_k = \exp(i\phi_k)$ and (2.18) yield that

$$j(n, E) = (-i)^d \int_{\mathcal{T}^d} \mathbf{z}^{-n-1} \left[\lambda(\mathbf{z}, E) + \Phi(\mathbf{z}) \right] d\mathbf{z}, \tag{2.24}$$

where $\mathcal{T}^d = \{\mathbf{z} : \forall k, |z_k| = 1\}$, and $\mathbf{z}^{-n-1} = \prod z_k^{-n_k - 1}$. Without loss of generality we can assume that $n_k \geq 0$. Let $\mathcal{S}_\gamma = \{\mathbf{z} : |z_k| \in (1, \gamma)\}$. If

$$\forall \mathbf{z} \in \mathcal{S}_\gamma, \ \ |\Phi(\mathbf{z}) - E)| > 2, \tag{2.25}$$

then the function $\lambda(\mathbf{z}, E)$ is holomorphic in $\mathcal{S}_\gamma$ in each variable separately, and continuous and bounded on $\overline{\mathcal{S}_\gamma}$. It is a simple exercise to show that if $\gamma_E = (|E| - 2)/2d$, then (2.25) holds, and that on the boundary of $\mathcal{S}_{\gamma_E}$ we have an estimate

$$|\lambda(\mathbf{z}, E)| + |\Phi(\mathbf{z})| \leq 1 + 2d\gamma_E.$$

Interchanging the domain of integration in (2.24) we derive the proposition. □

3 Preliminaries

3.1 Path expansions and all that

In this section we collect a few technical results from [JM] concerning the operators $h(E) = h_0(E) + V$ on $l^2(\mathbf{Z})$, where $h_0(E)$ is given by (1.6), E is a fixed point outside $[-4, 4]$, and V is an arbitrary potential. For the proofs we refer the reader to Section 2 of [JM].

A *path* τ connecting n and m is any sequence of sites $\tau = (i_0, i_1, \ldots i_k)$ such that $i_0 = n$, $i_k = m$. The *length* of this path is $|\tau| = k$. To the path τ we associate a sequence of bonds $\tau_b = (b_1, \ldots, b_k)$, where

$$b_1 = (i_0, i_1),\ b_1 = (i_1, i_2), \ldots, b_k = (i_{k-1}, i_k).$$

We write $s \in \tau$ if s is one of the sites of the path τ, and $b = (s, t) \in \tau_b$ if b is one of its bonds. We use the shorthand $j(b) = j(s - t; E)$, and

$$\mathcal{R}(n, m; z) = (\delta_n, (h(E) - z)^{-1} \delta_m).$$

Let

$$j_0(E) = \sum_n |j(n,E)|.$$

Proposition 3.1 *If* $\mathrm{Im}(z) > j_0(E)$ *then*

$$\mathcal{R}(n,m;z) = -\frac{\delta_{nm}}{z-V(n)} - \sum_\tau \left[\prod_{s\in\tau} \frac{1}{z-V(s)}\right] \cdot \left[\prod_{b\in\tau_b} j(b)\right], \tag{3.26}$$

where the sum is over all paths connecting n *and* m*. For each* $\varepsilon > 0$ *the series converges uniformly in the half-plane* $\mathrm{Im}(z) \geq j_0(E) + \varepsilon$*.*

A similar result holds if the system is restricted to a box. Let $I \subset \mathbf{Z}$ be an arbitrary set, and let $h_0^D(E)$ be the operator $h_0(E)$ restricted to I with the Dirichlet boundary condition. This operator is obtained by removing the couplings between the points in I and $\mathbf{Z}\setminus I$, and acts on $l^2(I)$ according to the formula

$$(h_0^D(E)\psi)(n) = \sum_{m\in I} j(n-m,E)\psi(m). \tag{3.27}$$

Remark. For latter applications, we remark that if $E > 4$ then $h_0^D(E) > 0$ and if $E < -4$ then $h_0^D(E) < 0$.

We define the operator $h_I(E)$ on $l^2(I)$ by the formula $h_I(E) = h_0^D(E) + V$. We will refer to $h_I(E)$ as the restriction of $h(E) = h_0(E) + V$ to I with the Dirichlet boundary condition. If $n, m \in I$, let $\mathcal{R}_I(n,m;z) = (\delta_n, (h_I(E)-z)^{-1}\delta_m)$. Then

$$\mathcal{R}_I(n,m;z) = -\frac{\delta_{nm}}{z-V(n)} - \sum_\tau \left[\prod_{s\in\tau} \frac{1}{z-V(s)}\right] \cdot \left[\prod_{b\in\tau_b} j(b)\right], \tag{3.28}$$

where the sum is over all paths which connect n and m and belong to I. If n or $m \notin I$ we set $\mathcal{R}_I(n,m;z) = 0$.

Notation. In the sequel, we will use a shorthand $\langle n\rangle = (1+n^2)^{1/2}$.

Proposition 3.2 *Let* $\mathcal{I}$ *be a closed interval outside* $[-4,4]$ *such that*

$$\sup_{E\in\mathcal{I}} \mathrm{dist}\{E, \sigma(h_I(E))\} = \delta > 0,$$

and let l *be an integer. Then there is a constant* $C_{\delta,l}$*, which depends on* δ *and* l *only, such that*

$$\sup_{E\in\mathcal{I}} |\mathcal{R}_I(n,m;E)| \leq C_{\delta,l}\langle n-m\rangle^{-l}.$$

Remark. The decay of matrix elements $\mathcal{R}_I(n,m;E)$ is probably exponential, but the above weaker result will suffice in our applications.

We will also need

Proposition 3.3 *Let l an integer and $\mathcal{I}$ a closed interval outside $[-4,4]$. Let $I \subset \mathbf{Z}$ be such that for some $\delta > 0$*

$$\inf_{n\in I, E\in\mathcal{I}} |v(n) - E| = j_0(E) + \delta.$$

Then

$$\sup_{E\in\mathcal{I}} |\mathcal{R}_I(n,m;E)| \le C_{\delta,l}\langle n-m\rangle^{-l},$$

where $C_{\delta,l}$ depends on δ and l only. Furthermore, there is a constant C_l which depends on l only, such that for $\delta > 1$, $C_{\delta,l} < C_l/\delta$.

Proposition 3.4 *Let I_ℓ be a sequence of finite intervals such that $I_\ell \uparrow \mathbf{Z}$ as $\ell \to \infty$, and let E be such that $\forall \ell$, $E \notin \sigma(h_\ell(E))$. Then, $\forall n \in \mathbf{Z}$,*

$$\lim_{\zeta\to 0} \|(h(E) - E - i\zeta)^{-1}\delta_n\| \le \liminf_{\ell\to\infty} \|(h_{I_\ell}(E) - E)^{-1}\delta_n\|.$$

The final technical result we need is

Proposition 3.5 *Let $E_0 \notin [-4,4]$ be given. Then*

$$\lim_{E\to E_0} \sup_{I\subset\mathbf{Z}} \|h_I(E) - h_I(E_0)\| = 0.$$

Since this last proposition was not discussed in [JM], we sketch its proof.
Proof: It follows from Lemma 5.2 in [JM] (see also [Ka], Section 1.4.3 and Lemma 7.1 in [SS]) that

$$\sup_{I\subset\mathbf{Z}} \|h_I(E) - h_I(E_0)\| \le \sum_{n\in\mathbf{Z}} |j(n,E) - j(n,E_0)|.$$

Note that

$$j(n,E) - j(n,E_0) = \int_{\mathbf{T}} e^{-in\phi}[\lambda(\phi,E) - \lambda(\phi,E_0)]d\phi.$$

If $n \neq 0$, integrating by parts twice we arrive at the estimate

$$|j(n,E) - j(n,E_0)| \le O(|E-E_0|)/n^2.$$

The result follows. □

3.2 Appriori estimates

The results of the previous section have to be complemented with an appropriate version of Kolmogorov's lemma (Proposition 2.4 in [JM]) for technique of [JM] to work. We however cannot randomize energies E if h_0 depends on E, and this part of the argument will be distinctly different from the one in [JM]. The technical results which will be used instead of Proposition 2.4 of [JM] are described in this section.

Let H_0 be a symmetric matrix (operator) on $\mathbf{R}^N$, and let $\xi_1, \ldots, \xi_N$ be independent random variables on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. We define random operators H_ω by the formula

$$H_\omega = H_0 + \sum_{i=1}^{N} \xi_i(\omega)(\delta_i, \cdot)\delta_i.$$

We denote by $h_{kl}^{(0)}$ the matrix elements of H_0 in the basis $\{\delta_i\}$. We make the following hypotheses:
(A1) For any k, $h_{kk}^{(0)} \neq 0$, and for any k and l, $h_{kk}^{(0)} h_{ll}^{(0)} - (h_{kl}^{(0)})^2 \neq 0$.
(A2) The random variables $\xi_i(\omega)$ have densities p_i which are uniformly bounded, i.e. for some $\sigma > 0$ and all i, $\|p_i\|_\infty < \sigma$.

We remark that Hypothesis (A1) is automatically satisfied if $H_0 > 0$ or $H_0 < 0$ (and this will be the case in our applications, recall the remark after (3.27)). We also remark that without loss of generality we can take for our probability space $\tilde{\Omega} = \mathbf{R}^N$. Then $\tilde{\mathcal{F}}$ is the Borel σ-algebra on $\mathbf{R}^N$, $d\tilde{P} = \prod p(x_i)dx_i$, and if $\omega = (x_1, \ldots, x_N)$ then $\xi_i(\omega) = x_i$. We denote $\mathbf{E}(f) = \int f(\omega) d\tilde{P}(\omega)$.

The first observation we need is

Lemma 3.6 *Assume that Hypothesis* (A2) *hold. Then* $0 \notin \sigma(H_\omega)$ $\tilde{P}$*-a.s.*

Remark. For this lemma to hold we only need that the random variables ξ_i have densities.
Proof: It suffices to show that $\det H_\omega \neq 0$ $\tilde{P}$-a.s. This can be shown by induction as follows. Statement is obvious if $N = 1$. If $N > 1$, expanding the determinant with respect to the last row we can write

$$\det H_\omega = \xi_N(\omega) R(\omega) + D(\omega).$$

Here, $R(\omega)$ is the determinant of the matrix obtained from H_ω by removing the last row and the last column. By induction hypothesis, $R(\omega) \neq 0$ $\tilde{P}$-a.s., and $\det H_\omega = 0$ implies $\xi_N(\omega) - D(\omega)/R(\omega) = 0$. Since ξ_N and R/D are independent random variables,

$$\tilde{P}\{\omega : \det H_\omega = 0\} \leq \sup_{a \in \mathbf{R}} \tilde{P}\{\omega : \xi_N(\omega) = a\} = 0. \ \Box$$

From this lemma it follows that H_ω^{-1} exists $\tilde{P}$-a.s. We denote $R_\omega(k, l) = (\delta_k, H_\omega^{-1}\delta_l)$. The principal result of this section is

Theorem 3.7 *Assume that Hypothesis* (A1) *and* (A2) *hold. Let* $0 < s < 1$ *be given. Then for any* k *and* l*,*

$$\mathbf{E}\left(|R_\omega(k, l)|^s\right) \leq C(s, \sigma),$$

where the constant $C(s, \sigma)$ *depends on* s *and* σ *only.*

Remark 1. The proof of this result is outlined in [M1]. For reader convenience, we present a detailed proof below.

Remark 2. For latter applications, it is critical that this theorem holds for random variables which are not necessarily identically distributed.

We will make use of the following consequence of Theorem 3.7.

Corollary 3.8 *Assume that Hypothesis* (A1) *and* (A2) *hold. Then there exist a constant* $C(\sigma)$*, which depends on* σ *only, so that for any* k *and* l*,*

$$\tilde{P}\{\omega : |R_\omega(k,l)| > M\} \leq C(\sigma)/M, \tag{3.29}$$

$$\tilde{P}\{\omega : \sum_{j=1}^{N} |R_\omega(k,j)|^2 > M\} \leq C(\sigma)N/M^{1/4}. \tag{3.30}$$

Proof: Relation (3.29) follows from Chebyshev's inequality. The prove (3.30), we note first that if $\{x_j\}_{k=1}^{N}$ is a positive sequence and $0 < s < 1$, then

$$\sum_{j=1}^{N} x_j^s \geq \left(\sum_{j=1}^{N} x_j\right)^s. \tag{3.31}$$

Thus, if $X_j(\omega) = |R_\omega(k,j)|$,

$$\begin{aligned}
\tilde{P}\{\omega : \sum_{j=1}^{N} X_j(\omega)^2 > M\} &= \tilde{P}\{\omega : \left(\sum_{j=1}^{N} X_j(\omega)^2\right)^{1/4} > M^{1/4}\} \\
&\leq \tilde{P}\{\omega : \sum_{j=1}^{N} X_k(\omega)^{1/2} > M^{1/4}\} \\
&\leq \frac{1}{M^{1/4}} \sum_{j=1}^{N} \mathbf{E}(X_j(\omega)^{1/2}) \\
&\leq C(\sigma)N/M^{1/4}.
\end{aligned}$$

In the first estimate we have used (3.31) and in the second Chebyshev's inequality. □.

The rest of this section is devoted to the proof of Theorem 3.7. We will need

Lemma 3.9 *Let* k *and* l *be given, and let*

$$\begin{aligned}
H_\omega^{(k)} &\equiv H_0 + \sum_{i \neq k} \xi_i(\omega)(\delta_i, \cdot)\delta_i = H_\omega - \xi_k(\omega)(\delta_k, \cdot)\delta_k \\
H_\omega^{(k,l)} &\equiv H_0 + \sum_{i \neq k,l} \xi_i(\omega)(\delta_i, \cdot)\delta_i = H_\omega - \xi_k(\omega)(\delta_k, \cdot)\delta_k - \xi_l(\omega)(\delta_l, \cdot)\delta_l.
\end{aligned}$$

Then $0 \notin \sigma(H_\omega^{(k)})$ *and* $0 \notin \sigma(H_\omega^{(k,l)})$ $\tilde{P}$*-a.s.*

Proof: Using Hypothesis (A1) and induction with respect to N, one argues in the same way as in the proof of Lemma 3.6. □

In the sequel we will consider separately the cases $k = l$ and $k \neq l$. The first case is simpler since an argument based on the rank one perturbation theory suffices. The second case requires an argument based on the rank two perturbation theory.

Case 1: $k = l$.

Let $\tilde{R}_\omega(i,j)$ be the matrix elements of $[H_\omega^{(k)}]^{-1}$. The identity

$$H_\omega^{-1} - [H_\omega^{(k)}]^{-1} = -\xi_k(\omega) H_\omega^{-1} [(\delta_k, \cdot)\delta_k] [H_\omega^{(k)}]^{-1},$$

leads to the formula

$$R_\omega(k,k) = \frac{\tilde{R}_\omega(k,k)}{1 + \xi_k(\omega)\tilde{R}_\omega(k,k)}.$$

Since $\xi_k(\omega)$ and $\tilde{R}_\omega(k,k)$ are independent random variables,

$$\mathbf{E}\left(|R_\omega(k,k)|^s\right) \leq \sup_{t \in \mathbf{R}} \int_{\mathbf{R}} \frac{1}{|x-t|^s} p_k(x) dx.$$

Since $0 < s < 1$ and $\|p_k\|_\infty \leq \sigma$, decomposing $\int_{\mathbf{R}} = \int_{[t-1,t+1]} + \int_{\mathbf{R}\setminus[t-1,t+1]}$, we easily estimate

$$\mathbf{E}\left(|R_\omega(k,k)|^s\right) \leq 1 + 2\sigma/(1-s).$$

This concludes the Case 1.

Case 2: $k \neq l$.

Let $\tilde{R}_\omega(i,j)$ be the matrix elements of $[H_\omega^{(k,l)}]^{-1}$. The identity

$$H_\omega^{-1} - [H_\omega^{(k,l)}]^{-1} = -\xi_k(\omega) H_\omega^{-1} [(\delta_k, \cdot)\delta_k] [H_\omega^{(k,l)}]^{-1} - \xi_l(\omega) H_\omega^{-1} [(\delta_l, \cdot)\delta_l] [H_\omega^{(k,l)}]^{-1},$$

yields that for any i, j,

$$R_\omega(i,j) = \tilde{R}_\omega(i,j) - \xi_k(\omega) R_\omega(i,k)\tilde{R}_\omega(k,j) - \xi_l(\omega) R_\omega(i,l)\tilde{R}_\omega(l,j).$$

Substituing $i = k, j = k$ and $i = k, j = l$ in this relation, we get after simple algebra

$$\begin{aligned} R_\omega(k,k)\left[1 + \xi_k(\omega)\tilde{R}_\omega(k,k)\right] + R(k,l)\xi_l(\omega)\tilde{R}_\omega(l,k) &= \tilde{R}_\omega(k,k), \\ R_\omega(k,k)\xi_k(\omega)\tilde{R}(k,l) + R_\omega(k,l)\left[1 + \xi_l(\omega)\tilde{R}_\omega(l,l)\right] &= \tilde{R}_\omega(k,l). \end{aligned} \tag{3.32}$$

The random variables $\xi_k(\omega)$ and $\xi_l(\omega)$ are independent from the random variables $\tilde{R}_\omega(k,k)$, $\tilde{R}_\omega(l,l)$ and $\tilde{R}_\omega(k,l)$, and it is a simple exercise to show that

$$(1 + \xi_k(\omega)\tilde{R}_\omega(k,k))(1 + \xi_l(\omega)\tilde{R}_\omega(l,l) - \xi_k(\omega)\xi_l(\omega)\tilde{R}_\omega(k,l)^2 \neq 0 \qquad \tilde{P} - a.s.$$

Then equations (3.32) yield that

$$R_\omega(k,l) = \frac{\tilde{R}_\omega(k,l)}{(1 + \xi_k(\omega)\tilde{R}_\omega(k,k))(1 + \xi_l(\omega)\tilde{R}_\omega(l,l)) - \xi_k(\omega)\xi_l(\omega)\tilde{R}_\omega(k,l)^2}.$$

We have used that $\tilde{R}_\omega(k,l) = \tilde{R}_\omega(l,k)$. Let

$$\Delta_\omega = R_\omega(k,k)R_\omega(l,l) - R_\omega(k,l)^2.$$

We will prove below that

$$\Delta_\omega \neq 0 \qquad \tilde{P} - a.s. \tag{3.33}$$

Assuming this, we finish the proof of Theorem 3.7.

If (3.33) holds, then

$$R_\omega(k,l) = \frac{\tilde{R}_\omega(k,l)/\Delta_\omega}{(\xi_k(\omega) + \tilde{R}_\omega(k,k)/\Delta_\omega)(\xi_l(\omega) + \tilde{R}_\omega(l,l)/\Delta_\omega) - \tilde{R}_\omega(k,l)^2/\Delta_\omega^2}.$$

Thus, we have a bound

$$\int_{\tilde{\Omega}} |R_\omega(k,l)|^s \, dP(\omega) \leq \sup_{a,b,c \in \mathbf{R}} \int_\Omega \left(\frac{|a|^s}{(\xi_k(\omega) + b)(\xi_l(\omega) + c) - a^2} \right) d\tilde{P}(\omega).$$

We proceed to estimate the left-hand side of this inequality. Let a, b, c be fixed. Without loss of generality we can assume that $a \neq 0$. We introduce a new random variables $\hat{\xi}_k(\omega) = \xi_k(\omega) + b$, $\hat{\xi}_l(\omega) = \xi_l(\omega) + c$. Clearly, $\hat{\xi}_k$ and $\hat{\xi}_l$ are are independent and their densities satisfy $\|\hat{p}_l\|_\infty \leq \sigma$, $\|\hat{p}_k\| \leq \sigma$. We introduce the following sets:

$$\Omega_1 = \{\omega : \hat{\xi}_k(\omega) \leq a/2, \, \hat{\xi}_l(\omega) \leq a/2\},$$

$$\Omega_2 = \{\omega : \hat{\xi}_k(\omega) > a/2\}, \qquad \Omega_3 = \{\omega : \hat{\xi}_l(\omega) > a/2\}.$$

Clearly, $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \tilde{\Omega}$. We now have

$$\begin{aligned}
\int_{\Omega_1} \frac{|a|^s}{|\hat{\xi}_k(\omega)\hat{\xi}_l(\omega) - a^2|^s} d\tilde{P}(\omega) &\leq (4/(3a))^s \, \tilde{P}(\Omega_1) \\
&\leq (4/(3a))^s \min\{1, a^2/4\} \\
&\leq (4/3)^s.
\end{aligned}$$

Furthermore.

$$\begin{aligned}
\int_{\Omega_2} \frac{|a|^s}{|\hat{\xi}_k(\omega)\hat{\xi}_l(\omega) - a^2|^s} d\tilde{P}(\omega) &= \int_{\mathbf{R}\setminus[-a/2,a/2]} \hat{p}_k(x)dx \int_{\mathbf{R}} \frac{|a|^s}{|xy - a^2|^s} \hat{p}_l(y)dy \\
&\leq \int_{\mathbf{R}\setminus[-a/2,a/2]} \frac{|a|^s}{|x|^s} \hat{p}_k(x)dx \int_{\mathbf{R}} \frac{1}{|y - a^2/x|^s} \hat{p}_l(y)dy \\
&\leq 2^s \left[1 + 2\sigma/(1-s)\right].
\end{aligned}$$

The estimation of $\int_{\Omega_3}$ is analogous to that of $\int_{\Omega_2}$. Thus, we arrive at the estimate

$$\mathbf{E}\left(|R_\omega(k,l)|^s\right) \equiv \int_{\tilde{\Omega}} |R_\omega(k,l)|^s \, d\tilde{P}(\omega) \leq (4/3)^s + 2^{s+1}(1 + 2\sigma/(1-s)), \tag{3.34}$$

and the theorem follows.

It remains to prove (3.33). If $N = 2$, $\Delta_\omega = \det(H_0)^{-1} \neq 0$ (recall (A1)). If $N > 2$, we will use that

$$\tilde{R}_\omega(k,k) = \frac{M_{kk}(\omega)}{\det H_\omega}, \qquad \tilde{R}_\omega(l,l) = \frac{M_{ll}(\omega)}{\det H_\omega}, \qquad \tilde{R}_\omega(k,l)^2 = \left(\frac{M_{lk}(\omega)}{\det H_\omega}\right)^2$$

where $M_{ij}(\omega)$ is the cofactor of the element $h_{ij}(\omega)$ of the matrix $H_\omega^{(k,l)}$ (all the matrices are computed in the standard basis $\{\delta_i\}$). Thus, $\Delta_\omega \neq 0$ $\tilde{P}$-a.s. if

$$M_{kk}(\omega)M_{ll}(\omega) - M_{lk}(\omega)^2 \neq 0 \qquad \tilde{P} - a.s.$$

Let $r \neq k, l$. $M_{kk}(\omega)$ is the determinant of the matrix obtained from $\tilde{H}_\omega$ by removing the k-th row and column. Expanding this determinant with respect to the row which contains ξ_r, we get that

$$M_{kk}(\omega) = \xi_r(\omega)M_{kk}^{(1)}(\omega) + E_{kk}(\omega).$$

The random variable $\xi_r(\omega)$ is independent of $M_{kk}^{(1)}(\omega)$ and $E_{kk}(\omega)$. Note also that $M_{kk}^{(1)}(\omega)$ is the determinant of the matrix obtained from $\tilde{H}_\omega$ by removing the k-th and the r-th row and column. Similarly, we have that

$$\begin{aligned} M_{ll}(\omega) &= \xi_r(\omega)M_{ll}^{(1)}(\omega) + E_{ll}(\omega) \\ M_{lk}(\omega)^2 &= \left[\xi_r(\omega)M_{lk}^{(1)}(\omega) + E_{lk}(\omega)\right]^2. \end{aligned}$$

Here $M_{ll}^{(1)}(\omega)$ is the determinant of the matrix obtained from $\tilde{H}_\omega$ by removing the l-th and the r-th row and columns, and $M_{lk}^{(1)}(\omega)$ is the determinant of the matrix obtained from $\tilde{H}_\omega$ by removing l-th row, k-th column, and r-th row and column. We then have that

$$M_{kk}(\omega)M_{ll}(\omega) - M_{lk}(\omega)^2 = \xi_r(\omega)^2 a(\omega) + \xi_r(\omega)b(\omega) + c(\omega),$$

where

$$a(\omega) = M_{kk}^{(1)}(\omega)M_{ll}^{(1)}(\omega) - M_{kl}^{(1)}(\omega)^2$$

and $\xi_r(\omega)$ is independent of $a(\omega)$, $b(\omega)$ and $c(\omega)$. If

$$a(\omega) \neq 0 \qquad \tilde{P} - a.s., \tag{3.35}$$

then

$$\tilde{P}\{\omega : \Delta_\omega = 0\} \leq \sup_{\substack{a,b,c \\ a\neq 0}} \tilde{P}\{\omega : \xi_r(\omega)^2 a + \xi_r(\omega)b + c = 0\} = 0.$$

To show that $a(\omega) \neq 0$, now pick $r' \neq r, l, k$, expand the determinants M_{kk}, M_{ll} and M_{lk} with respect to the r'-row of the matrix H_ω, and continue inductively. The algorithm terminates after $N - 2$ steps, and in the last step we get that

$$M_{kk}^{(n-2)}(\omega)M_{ll}^{(n-2)}(\omega) - M_{lk}^{(n-2)}(\omega)^2 = h_{kk}^{(0)}h_{ll}^{(0)} - (h_{lk}^{(0)})^2,$$

which is different from zero by Hypothesis (A1).

4 The main theorem

Let $h_0(E)$ be given (1.6) and V_ω be a random potential on $\mathbf{Z}$ such that $V_\omega(n)$ are independent, but *not* neccesarily identically distributed random variables on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. We assume that each random variable $V_\omega(n)$ has density $p_n(x)$. Furthermore, we assume
(A) There exist $\sigma > 0$ such that $\forall n$, $\|p_n\|_\infty < \sigma$.
Let

$$h_\omega(E) = h_0(E) + V_\omega(n).$$

This operator is in general different from $h_\omega(E)$ defined by (1.7).

We will freely use the notation of the previous sections. In this section we prove

Theorem 4.1 *Let $\mathcal{I} = [c, d]$ be an interval outside $[-4, 4]$ and $a \geq 2$ an integer. Assume that* (A) *holds and that there exists an integer $N \geq 0$ such that, $\forall n > 0$, the intervals*

$$\pm[a^{N+n} + 1, a^{N+n+1} - 1],$$

contain sub-intervals $I_{\pm n}$ of the length $l_{\pm n} \geq n$ such that $\tilde{P}$-a.s.

$$\sup_{E \in \mathcal{I}} \mathrm{dist}\{E, \sigma(h_{I_{\pm,n},\omega}(E)\} = \delta > 0. \tag{4.36}$$

Then for a.e. $(E, \omega) \in \mathcal{I} \times \tilde{\Omega}$ with respect to the measure $m \otimes \tilde{P}$, and for any $n \in \mathbf{Z}$,

$$\lim_{\zeta \to 0} \|(h_\omega(E) - E - i\zeta)^{-1}\delta_n\| < \infty. \tag{4.37}$$

Remark. We emphasize that the intervals $I_{\pm n}$ are deterministic.
Given the results of the previous section, the proof of Theorem 4.1 reduces to translating line by line the arguments of Section 3 in [JM]. To see how this translation is carried out, we will reproduce here a part of the argument.

The first observation we will need is that (4.36) and Proposition 3.2 yield that for any $E \in \mathcal{I}$ and $p, q \in I_{\pm n}$,

$$|R_{I_{\pm n},\omega}(p, q; E)| < C_{\delta,l}\langle p - q\rangle^{-l}, \tag{4.38}$$

where $C_{\delta,l}$ does not depend on $I_{\pm n}$ and ω. In this technical sense the intervals $I_{\pm n}$ are the intuitive "barriers" discussed in the introduction.

To simplify the notation, we will prove (4.37) only in the case $n = 0$. This is the case that we will use latter.

We begin by introducing several sequences of intervals. Let the I_n's be as in the theorem, $I_n \equiv [a_n, b_n]$, and let $l_n = |a_n - b_n| + 1$. Let $M_0 = [a_{-1}, b_1]$. For $n > 0$, we set $M_n = [a_n, b_{n+1}]$, and for $n < 0$, $M_n = [a_{n-1}, b_n]$. Let $\Delta_0 = [b_{-1}, a_1]$. For $n > 0$, we set $\Delta_n = [b_n, a_{n+1}]$, and for $n < 0$, $\Delta_n = [b_{n-1}, a_n]$. Note that for $n > 0$,

$$M_n = I_n \cup \Delta_n \cup I_{n+1}. \tag{4.39}$$

A similar relation for $n < 0$

Notation. In the sequel we will drop subscript ω whenever there is no danger of confusion. Thus, we write $h(E)$ for $h_\omega(E)$ etc.

We denote by $h_{M_n}(E)$ the restriction of $h(E)$ to M_n with the Dirichlet boundary condition. Let $R_{M_n}(E) = (h_{M_n}(E) - E)^{-1}$ be the resolvent of $h_{M_n}(E)$ and $R_{M_n}(p, q; E)$ its matrix elements. We first collect some appriori estimates on $R_{M_n}(E)$. Let

$$x_n^{(1)} = a_n,\ x_n^{(2)} = b_n,\ x_n^{(3)} = a_{n+1},\ x_n^{(4)} = b_{n+1}.$$

Recall that $\langle x \rangle = (1 + x^2)^{1/2}$. We denote by L_n the number of points in M_n, $L_n = \# M_n$.

Proposition 4.2 *Let $E \in \mathcal{I}$, $\gamma > 0$ and $l > 0$ be fixed. Then for every $\varepsilon > 0$ there is a measurable set $\tilde{\Omega}(\varepsilon) \subset \tilde{\Omega}$ such that:*
1. $\tilde{P}(\tilde{\Omega} \setminus \tilde{\Omega}(\varepsilon)) = 0$.
2. *For each $\omega \in \tilde{\Omega}(\varepsilon)$ there is a positive integer $n_{\omega,\varepsilon}$ such that for $|n| \geq n_{\omega,\varepsilon}$ the following estimates hold:*

$$\begin{aligned}
\max_{i,j} \left| R_{M_n}(x_n^{(i)} + p, x_n^{(j)} + q; E) \right| &\leq \varepsilon \langle n \rangle^{1+\gamma} \langle p \rangle^{1+\gamma} \langle q \rangle^{1+\gamma}, & (4.40)\\
\max_i \sum_{q \in M_n} \left| R_{M_n}(x_n^{(i)} + p, q; E) \right|^2 &\leq L_n^4 \langle n \rangle^{4(1+\gamma)} \langle p \rangle^{4(1+\gamma)}, & (4.41)\\
\max_{|p-q| > l_n/2} |R_{I_n}(p, q; E)| &< \varepsilon \langle p - q \rangle^{-l}. & (4.42)
\end{aligned}$$

Given Theorem 3.7 and Corollary 3.8, the proof of (4.40) and (4.41) reduces to a simple application of Borel-Cantelli lemma. Note that Hypothesis (A) and the remark after (3.27) imply that all the conditions of Theorem 3.7 are satisfied. The estimate (4.42) follows from Proposition 3.2.

Note that $n_{\omega,\varepsilon}$ is not specified uniquely. To avoid some ambiguites, for given $\varepsilon > 0$ and $\omega \in \tilde{\Omega}(\varepsilon)$ we define $n_{\omega,\varepsilon}$ as the smallest positive integer such that (4.40)-(4.42) hold for all $|n| \geq n_{\omega,\varepsilon}$.

Proposition 4.2 gives information on the resolvent matrix elements of R_{M_n} starting with a sufficiently large index n which depends on ω. To circumvent some difficulties which arise from this ω-dependence, we introduce the sets

$$\tilde{\Omega}_{k,\varepsilon} = \bigcup_{j=0}^{k} \{\omega : \ \omega \in \tilde{\Omega}(\varepsilon) \text{ and } n_{\omega,\varepsilon} = j\}.$$

Since $R_{M_n}(s,t;E)$ are measurable functions of ω, the sets $\tilde{\Omega}_{k,\varepsilon}$ are measurable. Clearly, if $i > k$ then $\tilde{\Omega}_{k,\varepsilon} \subset \tilde{\Omega}_{i,\varepsilon}$ Furthermore, it follows from Proposition 4.2 that for each $\varepsilon > 0$, $\cup_{k\geq 0}\tilde{\Omega}_{k,\varepsilon}$ is of full measure in $\tilde{\Omega}$. Note that some of the sets $\tilde{\Omega}_{k,\varepsilon}$ might be empty. However, for each $\varepsilon > 0$ there is $k(\varepsilon) > 0$ such that $\tilde{\Omega}_{k,\varepsilon} \neq \emptyset$ if $k > k(\varepsilon)$. Let C_l be the constant from Proposition 3.3 and let

$$L = \max\{|c|,|d|\} + j_0(E) + C_l/\varepsilon,$$

(recall that $\mathcal{I} = [c,d],)$. For given k and ε, we introduce an auxiliary potential $V_{k,\varepsilon}$ by the formula

$$V_{k,\varepsilon}(n) = \begin{cases} L & \text{if } n \in M_s,\ |s| \leq k, \\ V(n) & \text{if } n \in M_s,\ |s| > k. \end{cases}$$

The reasons for introducing this auxiliary potential are the following:

a) If $\omega \in \tilde{\Omega}_{k,\varepsilon}$ and V is replaced by $V_{k,\varepsilon}$ then the inequalites (4.40) and (4.41) hold for all n.

b) If $|n| \leq k$ then it follows from Proposition 2.3 and the choice of L that the inequality (4.42) holds for all $p, q \in I_n$.

Let

$$J_\ell \equiv \bigcup_{j,|j|\leq \ell} M_j.$$

We denote by $h_{\ell,k,\varepsilon}(E)$ the operator $h_0(E) + V_{k,\varepsilon}$ restricted to J_ℓ with the Dirichlet boundary condition. We will prove below the following result.

Proposition 4.3 *Let $E \in \mathcal{I}$ be given. Then there exists $\varepsilon_0 > 0$ such that for $k > k(\varepsilon_0)$, $\omega \in \tilde{\Omega}_{k,\varepsilon_0}$, and $i \in \cup_{s=-k}^{k} M_s$,*

$$\limsup_{\ell\to\infty} \sum_{n\in J_\ell} \left|(\delta_i, (h_{\ell,k,\varepsilon_0}(E) - E)^{-1}\delta_n)\right|^2 < \infty.$$

Let us show how Relation (4.37) (for $n = 0$) follows from this proposition. Denote for the moment by R_{k,ε_0} the resolvent of the operator $h_0(E) + V_{k,\varepsilon_0}$. It then follows from Propositions 3.4 and 4.3 that for $\omega \in \tilde{\Omega}_{k,\varepsilon_0}$ and $i \in \cup_{s=-k}^{k} M_s$,

$$\lim_{\zeta\to 0} \sum_{n\in\mathbf{Z}} |R_{k,\varepsilon_0}(i,n;E+i\zeta)|^2 \leq C_{E,i,k,\varepsilon_0} < \infty. \tag{4.43}$$

Furthermore, it follows from the resolvent identity that

$$R(0,n;E+i\zeta) = R_{k,\varepsilon_0}(0,n;E+i\zeta) + \sum_{i\in M_s,|s|\leq k} (L - V(i))R(0,i;E+i\zeta)R_{k,\varepsilon_0}(i,n;E+i\zeta).$$

Note that for a given ω,

$$\limsup_{\zeta\to 0} |R(0,i;E+i\zeta)| < \infty,$$

for a.e. $E \in \mathbf{R}$ (recall the remark after the proof of Lemma 2.1). Thus, for a.e. $E \in \mathcal{I}$ and a.e. $\omega \in \tilde{\Omega}_{k,\epsilon_0}$

$$|R(0,n;E+i\zeta)|^2 \leq C_{E,\omega} \sum_{i\in M_s, |s|\leq k} |R_{k,\varepsilon_0}(i,n;E+i\zeta)|^2 .$$

This inequality and (4.43) yield Relation (4.37) for $n=0$.

The proof of Proposition 4.3 follows closely the proof of Proposition 3.3 in [JM]. We just sketch the main steps.

Notation. In the sequel we will denote by the same letter C various constants which depend *only* on $C_{\delta,\ell}$ in 4.38. The values of these constants may change from estimate to estimate. Furthermore, we will drop the subscripts k and ε in the sequel whenever there is no danger of confusion. For example, we denote by $R_\ell(n,m;z)$ the matrix elements of the resolvent $(h_{\ell,k,\varepsilon}(E)-z)^{-1}$, etc.

We will discuss Proposition 4.3 only in the case where $i=0$. A similar argument applies to the other values of i.

Let $\ell>0$ be given. Let us recall the construction of the iterative expansion of the matrix resolvent element $R_\ell(0,n;z)$ with respect to R_{M_s}. Let τ be any path in the expansion (3.26) which connects 0 and n, $\tau=(0,n_1,n_2,\dots,n_k,n)$. To such a path we associate a sequence of bonds $(b_1,\dots,b_l)$ and a sequence of blocks $(M_{s_1},\dots,M_{s_l})$ in the following way. Let n_{k_1} be the first of the n_l's which is not in the block M_0. Then let $b_1=(n_{k_1-1},n_{k_1})$. We denote the block to which n_{k_1} belongs by M_{s_1}. Let n_{k_2} be the first of the n_l's, for $l>k_1$, which is not in M_{s_1}, and let $b_2=(n_{k_2-1},n_{k_2})$. We denote the block to which n_{k_2} belongs by M_{s_2}. If $n_{k_2}\in M_s\cap M_t$ then, by definition, $k_2=\min\{s,t\}$ if $s,t\geq 0$, and $k_2=\max\{s,t\}$ if $s,t\leq 0$. We now continue inductively. It is helpful to invoke the following picture. The path τ starts in the block M_0, and wanders for some time within this block. It then leaves M_0 and jumps to a different block M_{s_1}. In the bond b_1 we record the site $n_{k_1-1}\in M_0$ at which the path takes off, and the site $n_{k_1}\in M_{s_1}$ at which it lands. The path now wanders through M_{s_1} and then jumps to M_{s_2}, etc. The last bond $b_l=(n_{k_{l-1}},n_{k_l})$ corresponds to the last entry into the block $M_{s_l}\equiv M_{n_0}$ which contains n. Since neighboring blocks intersect, the paths can land at the site which belongs simultaneously to two blocks; in this case, by definition, we say that the path landed in the block which is closer to 0. Clearly, the sequences $\{b_i\}$ and $\{M_{s_i}\}$ are not uniquely determined by the path τ: great many paths τ will determine the same sequences of blocks. Note that $\{b_i\}$, however, uniquely determines $\{M_{s_i}\}$. Let $\mathcal{B}$ be the set of all sequences of bonds $\tau_b=\{b_i\}$ obtained in the above way.

Regrouping the elements in the expansion (3.26) we get

$$R_\ell(0,n;z)=\delta_{0n}/(V(0)-z)+\sum_{\tau_b\in\mathcal{B}} R_{M_0}(0,n_{k_1-1};z)j(n_{k_1-1}-n_{k_1})R_{M_{s_1}}(n_{k_1},n_{k_2-1};z)\dots$$

$$\dots R_{M_{s_{l-1}}}(n_{k_{l-1}},n_{k_l-1};z)j(n_{k_l-1}-n_{k_l})R_{M_{n_0}}(n_{k_l},n;z).$$

At this point, of course, this relation holds only for $\mathrm{Im}(z) > j_0(E)$. However, if z is arbitrary and the series on the right hand side converges absolutely then its sum is $R_l(0, n; z)$. To show this, for $z \in \mathbf{C}$ we define

$$\mathcal{R}_\ell(0, n; z) = \delta_{0n}/(V(0) - z) + \sum_{\tau_b \in \mathcal{B}} R_{M_0}(0, n_{k_1-1}; z) j(n_{k_1-1} - n_{k_1}) R_{M_{s_1}}(n_{k_1}, n_{k_2-1}; z) \dots$$

$$\dots R_{M_{s_{l-1}}}(n_{k_{l-1}}, n_{k_l-1}; z) j(n_{k_l-1} - n_{k_l}) R_{M_{n_0}}(n_{k_l}, n; z). \tag{4.44}$$

whenever the sum converges absolutely. We then have

Proposition 4.4 *If $z \in \mathbf{C}$ and if $\mathcal{R}(0, n; z)$ is defined for all $n \in J_\ell$, then $z \notin \sigma(h_\ell(z))$ and $R_\ell(0, n; z) = \mathcal{R}_\ell(0, n; z)$.*

The proof is the same as in [JM]. In the sequel, we will apply this proposition in the case $z = E \in \mathbf{R}$.

At this point one proceeds to prove the following statement. Let $E \in \mathcal{I}$ be given. Then there exists $\varepsilon_0 > 0$ such that for $k > k(\varepsilon_0)$ and $\omega \in \tilde{\Omega}_{k,\varepsilon_0}$, the formal series (4.44) converges absolutely and

$$\sum_{m \in J_\ell} |\mathcal{R}_\ell(0, m; E)|^2 \leq C < \infty$$

where the constant C does not depend on ℓ. Proposition 4.3 then follows from Proposition 4.4.

Let us consider a typical term in the formal expansion (4.44):

$$R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; E) j(n_{k_i-1} - n_{k_i}) R_{M_{s_i}}(n_{k_i}, n_{k_{i+1}-1}; E).$$

We fix $\varepsilon > 0$ and $k > k(\varepsilon)$, and proceed to obtain a suitable estimate on

$$R_{M_{s_{i-1}}}(n_{k_{i-1}}, n_{k_i-1}; E) j(n_{k_i-1} - n_{k_i})$$

We now use (4.39) and the path expansion of Section 3 to decompose $R_{M_{s_{i-1}}}$ in a such a way that the estimate (4.38) could be taken into the account. The rest of the arguments is virtually identical to the arguments in [JM] and we leave details as an exercise to the reader. We note that since $j(n, E)$ is decaying exponentially, the estimates of [JM] could be substantially improved. Also, the argument of [JM] (see the remarks at the end of Section 3 in [JM]) yields the estimate

$$\sup_{0<\zeta<1} |(\delta_0, (h_\omega(E) - E - i\zeta)^{-1}\delta_n)| < C_{\omega,E,k}(1 + |n|)^{-k}.$$

for any $k > 0$. This estimate and Simon-Wolff theorem [SW] will yield the decay of eigenfunctions described in Remark 1 after Theorem 1.1. We expect that this result is not optimal, and we will not discuss it any further.

5 Probabilistic reduction

In this section we construct a partition of the probability space $(\Omega, \mathcal{F}, P)$ associated to the model (1.1). This partition, combined with the results of the Sections 2 and some additional technical results described in Section 6, will allow us to reduce the proof of Theorem 1.2 to Theorem 4.1.

We first recall the definition of $(\Omega, \mathcal{F}, P)$ (for details see e.g. [CFKS]). The set Ω is given by

$$\Omega = \mathbf{R}^{\mathbf{Z}} = \times_{\mathbf{Z}} \mathbf{R}.$$

Each $\omega \in \Omega$ can be identified with the real sequence $\{\omega_i\}_{i\in\mathbf{Z}}$. The sigma algebra $\mathcal{F}$ is generated by cillinder sets $\{\omega : \omega_{i_1} \in B_1, \dots, \omega_{i_n} \in B_n\}$, where $B_1, \dots, B_n$ are Borel subsets of $\mathbf{R}$. If $d\mu = p(x)dx$, the probability measure P is given by $P = \times_{\mathbf{Z}}\mu$.

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two given disjoint open intervals and let $\mathcal{J}_0 = \mathbf{R} \setminus (\mathcal{J}_1 \cup \mathcal{J}_2)$. We will assume that

$$p_0 = \int_{\mathcal{J}_0} p(x)dx > 0, \qquad p_1 = \int_{\mathcal{J}_1} p(x)dx > 0, \qquad p_2 = \int_{\mathcal{J}_2} p(x)dx > 0.$$

Clearly, $p_0 + p_1 + p_2 = 1$. To each $\omega \in \Omega$ we associate a sequence $s(\omega) = \{s_i\}$ of 0's, 1's and 2's as follows:

$$s_i = \begin{cases} 0 & \text{if } \omega_i \in \mathcal{J}_0, \\ 1 & \text{if } \omega_i \in \mathcal{J}_1, \\ 2 & \text{if } \omega_i \in \mathcal{J}_2. \end{cases}$$

The sequence $s(\omega)$ is the *skeleton* of the event ω. We denote by $\mathcal{S}$ the set $\{s(\omega) : \omega \in \Omega\}$. Let $\mathcal{T}$ be the sigma algebra on $\mathcal{S}$ generated by the cylinder sets, and π a measure defined by $\pi = \times_{\mathbf{Z}} b$, where $b\{0\} = p_0$, $b\{1\} = p_1$ and $b\{2\} = p_2$. Note that if $T : \Omega \mapsto \mathcal{S}$ is defined by $T(\omega) = s(\omega)$, then T is a measurable transformation and for any measurable set $F \subset \mathcal{S}$, $P(T^{-1}(F)) = \pi(F)$.

For any $s \in \mathcal{S}$ let $\Omega_s = \{\omega : s(\omega) = s\}$. Each Ω_s is a measurable subset of Ω, $\Omega_s \cap \Omega_{s'} = \emptyset$ if $s \neq s'$ and $\Omega = \cup_{s\in\mathcal{S}} \Omega_s$. We remark that for any s, $P(\Omega_s) = 0$. Note that each Ω_s has the form

$$\Omega_s = \times_{i\in\mathbf{Z}} \mathcal{J}_{s_i}.$$

Let s be given, and let $\mu^{(i)}$ be a probability measure on $\mathbf{R}$ with the density

$$p^{(i)}(x) \equiv p(x)\chi_{\mathcal{J}_{s_i}}(x)dx / \int_{\mathcal{J}_{s_i}} p(x)dx,$$

(in the sequel χ_A stands for the characteristic function of the set A). Note that

$$\|p^{(i)}\|_\infty \leq \|p\|_\infty / \min\{p_0, p_1, p_2\}. \tag{5.45}$$

Let P_s be a probability measure on Ω defined by

$$P_s = \times_{i\in\mathbf{Z}} \mu^{(i)}.$$

Note that the measure $\mu^{(i)}$ is supported on $\mathcal{J}_{s_i}$ and that P_s is supported on Ω_s. In this way we obtain, for each $s \in \mathcal{S}$, a new probability space $(\Omega, \mathcal{F}, P_s)$. Note also that

$$P(A/s) \equiv P_s(A \cap \Omega_s),$$

is the usual conditional probability of event A given s.

Lemma 5.1 *For any $A \in \mathcal{F}$, the function $P(A/\cdot) : \mathcal{S} \mapsto \mathbf{R}$ is π-measurable, and*

$$P(A) = \int_{\mathcal{S}} P(A/s) d\pi(s).$$

Proof: If A is a cylinder set, the proof reduces to a simple computation. The general case follows by limiting argument. □

We will also make use of

Lemma 5.2 *Let $C \subset \mathcal{I} \times \Omega$ be a measurable set and*

$$C_E = \{\omega : \ (E, \omega) \in C\}. \tag{5.46}$$

Then $f(E, s) \equiv P(C_E/s)$ is a measurable function on $\mathcal{I} \otimes \mathcal{S}$.

Proof: If $C = B \times A$ then the previous lemma yields that $P(C_E/s) = \chi_B(E)P(A/s)$ is a measurable function on $\mathcal{I} \times \mathcal{S}$. The general case follows by limiting argument. □

The stage is now set for our probabilistic reduction. Let $\mathcal{I}$ be a closed interval outside $[-4, 4]$ and

$$\mathcal{C} = \{(E, \omega) \in \mathcal{I} \times \Omega : \ \lim_{\zeta \to 0} \|(h_\omega(E) - E - i\zeta)^{-1}\delta_0\| < \infty\}.$$

It is not difficult to show that for fixed ζ the function $\|(h_\omega(E) - E - i\zeta)^{-1}\delta_0\|$ is measurable on $\mathcal{I} \times \Omega$. Therefore, the set $\mathcal{C}$ is measurable. According to Theorem 1.2, Theorem 1.1 holds if $m \otimes P$ measure of the set $\mathcal{C}$ is equal to $|\mathcal{I}|$, the length of the interval $\mathcal{I}$. Let $\mathcal{C}_E$ be given by (5.46). Lemmas 5.1 and 5.2 together with Fubini's theorem yield that

$$\int_{\mathcal{C}} dE \otimes dP = \int_{\mathcal{I}} P(\mathcal{C}_E) dE = \int_{\mathcal{I}} dE \int_{\mathcal{S}} P(\mathcal{C}_E/s) d\pi(s) = \int_{\mathcal{S}} d\pi(s) \int_{\mathcal{I}} P(\mathcal{C}_E/s) dE,$$

and finally, that

$$\int_{\mathcal{C}} dE \otimes dP = \int_{\mathcal{S}} d\pi(s) \int_{\mathcal{C}} dE \otimes dP_s.$$

We summarize:

Theorem 5.3 *Theorem 1.2 holds if for π-almost all s and for a.e. $(E, \omega) \in \mathcal{I} \otimes \Omega$ with respect to $m \otimes P_s$ we have that*

$$\lim_{\zeta \to 0} \|(h_\omega(E) - E - i\zeta)^{-1}\delta_0\| < \infty.$$

For the obvious reasons, we will refer to this result as the probabilistic reduction.

We finish this section with a probabilistic estimate which will allow us to construct long periodic approximations of the random potential V_ω.

Proposition 5.4 *Let $p > 0$ and a be given integers such that $p_1 a > 1$ and $p_2 a > 1$. Then π-a.s. there exists an integer $n(s)$ such that the intervals*

$$\pm[(pa)^{n(s)+n} + 1, (pa)^{n(s)+n+1} - 1], \qquad n > 0,$$

contain sub-intervals $I_{\pm n}$ of the length $4np$ such that

$$s_i = \begin{cases} 1 & \text{if } i \in I_{\pm n},\ i \equiv 0 \ mod\ p, \\ 2 & \text{if } i \in I_{\pm n},\ i \not\equiv 0 \ mod\ p. \end{cases}$$

Proof: Let $pa = b$. For any positive n let

$$I_n^{(k)} = [(b^n + 8(k-1)pn + 1, b^n + 8(k-1/2)pn + 1], \qquad I_{-n}^{(k)} = I_n^{(k)},$$

where $1 \le k \le [b(b^n - 1)/8np] - 1$ and $[\cdot]$ is the greatest integer part. Let

$$A_{n,k} = \{s : \ s_i = 1 \text{ if } i \in I_{\pm n}^{(k)},\ i \equiv 0 \operatorname{mod} p, \text{ and } s_i = 2 \text{ if } i \in I_{\pm n}^{(k)},\ i \not\equiv 0 \operatorname{mod} p\}.$$

Let $r = \min\{p_1, p_2\}$. One easily shows that $\pi(A_{n,k}) \ge r^{8np}$. Let B_n be the event that no $A_{n,k}$ take place, $B_n = \mathcal{S} \setminus (\cup_k A_{n,k})$. It follows that

$$\pi(B_n) \le (1 - r^{8np})^{[b(b^n-1)/8np]-1} = O(2^{-(ra)^n}).$$

If $ra > 1$, $\sum_n \pi(B_n) < \infty$ and Borel-Cantelli lemma yields that π-a.s. only finitely many events B_n take place. □

6 Periodic approximations and gaps

In this section we collect a few additional results from [JM] which will be used in the next section to verify the hypothesis of Theorem 4.1.

Let $p > 0$ be a positive integer, $\varepsilon > 0$ a positive parameter, and $V_{\varepsilon,p}$ a periodic potential of the form

$$V_{\varepsilon,p}(n) = \begin{cases} \varepsilon & \text{if } n \equiv 0 \text{ mod } p, \\ 0 & \text{if } n \not\equiv 0 \text{ mod } p. \end{cases} \tag{6.47}$$

We set $h_{\varepsilon,p} = h_0(E) + V_{\varepsilon,p}$.

The operator $h_0(E)$ in the Fourier representation acts as the operator of multiplication by the function $\hat{j}(\phi, E) = \lambda(\phi, E) + 2\cos\phi$, which, for fixed E, is even, analytic and strictly monotone on the interval $[0, \pi]$. Thus, Hypothesis (H) of Section 4 in [JM] is satisfied. We will use the shorthand $e_{k,p}(E) = \hat{j}(k\pi/p, E)$. Theorem 4.1 of [JM] applied to $h_{\varepsilon,p}(E)$ states the following.

Theorem 6.1 *Let $|E| > 4$ be given, and assume that $(\theta_1, \theta_2) \subset \sigma(h_0(E))$. Then there exist $\varepsilon_0(E) > 0$ and $p_0(E) > 0$ such that for $0 < \varepsilon < \varepsilon_0(E)$, $p > p_0(E)$, and $e_{k,p}(E) \in (\theta_1, \theta_2)$,*

$$\sigma(h_{\varepsilon,p}(E)) \cap (e_{k,p}(E), e_{k,p}(E) + \delta_{\varepsilon,k,p}(E)) = \emptyset,$$

for some $\delta_{\varepsilon,p}(E) > 0$

We will also need a technical result from [JM] (Proposition 5.1) which asserts that the conclusions of Theorem 6.1 are essentially unaffected by Dirichlet decoupling. Let again $|E| > 4$ be given, and let V_p be a periodic potential with the period p. Let $h_p(E) = h_0(E) + V_p$. For any positive integer L, we define the operator $h_p^L(E)$ as the restriction of $h_p(E)$ to the interval $[-2pL, 2pL]$ with the Dirichlet boundary condition. We then have

Proposition 6.2 *Let (a, b) be an interval such that $0 \notin (a, b)$ and $\sigma(h_p(E)) \cap (a, b) = \emptyset$. Let $\epsilon > 0$ and $\delta > 0$ be given small numbers. Then there exists finitely many points $r_1, \ldots, r_{k_{\epsilon,\delta}}$ in $(a + \epsilon, b - \epsilon)$ and a positive number $L_{\varepsilon,\delta}$ such that for $L > L_{\varepsilon,\delta}$,*

$$\sigma(h_p^L(E)) \cap (a + \epsilon, b - \epsilon) \subset \cup_{l=1}^{k_{\epsilon,\delta}} [r_l - \delta, r_l + \delta].$$

The points r_l and the numbers $L_{\epsilon,\delta}$ and $k_{\epsilon,\delta}$ depend only on ϵ, δ and E. Furthermore, $\sup_{\delta>0} k_{\epsilon,\delta} \leq k_\epsilon < \infty$, where k_ϵ depends only on ϵ and E.

We will also make use of the following technical results.

Lemma 6.3 *Assume that Hypothesis* (H1) *hold. Let $E_0 \in \mathcal{S}(\mathcal{V})$, $E_0 \notin [-4, 4]$, be given. Then there is $\phi_0 \in [-\pi, \pi]$ and $a_0 \in \mathcal{V}$ such that*

$$\hat{\jmath}(\phi_0, E_0) + a_0 = E_0. \tag{6.48}$$

Furthermore, there is a discrete set $\mathcal{B} \subset \mathbf{R} \setminus [-4, 4]$ such that if $E_0 \notin \mathcal{B}$ then a_0 and ϕ_0 can be chosen so that a_0 is an interior point of $\mathcal{V}$ and that $\hat{\jmath}(\phi_0, E_0)$ is an interior point of $\sigma(h_0(E_0))$.

Remark. This is the only place where we will use Hypothesis (H1).
Proof: $\lambda(\phi, E_0)$ is the solution of the equation

$$\lambda(\phi, E_0) + 1/\lambda(\phi, E_0) + 2\cos\phi = E_0, \tag{6.49}$$

which satisfies $|\lambda(\phi, E_0)| < 1$. If $E_0 \in \mathcal{S}(\mathcal{V})$, then (recall (1.2)) there exists $\phi_0 \in [-\pi, \pi]$ and $a_0 \in \mathcal{V}$, $|a_0| \geq 1$, such that $2\cos\phi_0 + a_0 + 1/a_0 = E_0$. It now follows from (6.49) that

$$a_0 + 1/a_0 = \lambda(\phi_0, E_0) + 1/\lambda(\phi_0, E_0).$$

Since the function $x + 1/x$ is strictly monotone on $[1, \infty)$, we have that $a_0 = 1/\lambda(\phi_0, E_0)$. Substituting back in (6.49) we derive (6.48).

To prove the second part part of the lemma, note first that the function $\hat{j}(\phi, E_0)$ has two extreme points, at $\phi = 0$ and $\phi = \pi$. Clearly, by wigling ϕ_0 and a_0 in (6.49) a little, one can always achieve that a_0 is an interior point of $\mathcal{V}$ and that $\hat{j}(\phi_0, E_0)$ is an interior point of $\sigma(h_0(E_0))$ except possibly in singular cases where $a_0 \in \partial\mathcal{V}$ and $\phi_0 = 0$ or π. Let

$$\mathcal{B} = \{E : \hat{j}(0, E) + a = E \text{ or } \hat{j}(\pi, E) + a = E \text{ for some } a \in \partial\mathcal{V}\}. \tag{6.50}$$

Since $\hat{j}(0, E)$ and $\hat{j}(\pi, E)$ are analytic functions in $\mathbf{C}\setminus[-4, 4]$ and the set $\partial\mathcal{V}$ is discrete, we derive that $\mathcal{B}$ is a discrete set as well. □

7 Proof of Theorem 1.1

Let $\mathcal{B}$ be given by (6.50), and let E_0 be such that $|E_0| > 4$, $E_0 \in \mathcal{S}(\mathcal{V})$, and $E_0 \notin \mathcal{B}$. We will show that there exist an open interval $\mathcal{I}_0 \ni E_0$ such that for a.e. $(E, \omega) \in \mathcal{I}_0 \times \Omega$ with respect to $m \otimes P$,

$$\lim_{\zeta \to 0} \|(h_\omega(E) - E - i\zeta)^{-1}\delta_0\| < \infty. \tag{7.51}$$

It then follows from Theorem 1.2 that $\Sigma_c \cap \mathcal{I}_0 = \emptyset$. Since $\mathcal{B}$ is discrete and E_0 is arbitrary, Theorem 1.1 follows.

It follows from Lemma 6.3 that there exist $a_0 \in \mathcal{V}$ and ϕ_0 such that $\hat{j}(\phi_0, E_0) + a_0 = E_0$. Furthermore, a_0 and ϕ_0 can be chosen so that a_0 belongs to the interior of $\mathcal{V}$ and $\hat{j}(\phi_0, E_0)$ to the interior of $\sigma(h_0(E_0))$. Let $\theta > 0$ be such that $(a_0 - \theta, a_0 + \theta) \in \mathcal{V}$, and θ_1, θ_2 such that $\hat{j}(\phi_0, E_0) \in (\theta_1, \theta_2) \subset \sigma(h_0(E_0))$. Choose $\varepsilon_0 > 0$ and $p_0 > 0$ such that Theorem 6.1 holds. Pick $p > p_0$ and k such that

$$|\hat{j}(k\pi/p, E_0) - \hat{j}(\phi_0, E_0)| < \theta/4,$$

and that $k\pi/p \in (\theta_1, \theta_2)$. Choose $\varepsilon > 0$ such that $\varepsilon < \min\{\varepsilon_0, \theta/4\}$, and let $V_{\varepsilon,p}$ be the periodic potential (6.47). We now use Proposition 6.2: For any $\epsilon > 0$ and $\delta > 0$ we can find $L_{\epsilon,\delta}$ such that for $L > L_{\epsilon,\delta}$ the spectrum of the operator $h^L_{\varepsilon,p}(E_0)$ (the restriction of $h_0(E_0) + V_{\varepsilon,p}$ to $[-2pL, 2pL]$ with the Dirichlet boundary condition) satisfies

$$\sigma(h^L_{\varepsilon,p}(E_0)) \cap (a + \epsilon, b - \epsilon) \subset \cup_{l=1}^{k_{\epsilon,\delta}}[r_l - \delta, r_l + \delta],$$

where $a = \hat{j}(k\pi/p, E_0)$, $b = \hat{j}(k\pi/p, E_0) + \delta_{\varepsilon,k,p}(E_0)$. Choose now ϵ, δ and $x_0 \in (-\theta/4, \theta/4)$ s.t. $x_0 + E_0 - a_0 \in (a + \epsilon, b - \epsilon)$, $x_0 + E_0 - a_0 \notin \cup_{l=1}^{k_{\epsilon,\delta}}[r_l - \delta, r_l + \delta]$. This is certainly possible since $\sup_{\delta>0} k_{\epsilon,\delta} \le k_\epsilon < \infty$. It follows that

$$\inf_{L > L_{\epsilon,\delta}} \text{dist}\{\sigma(h^L_{\varepsilon,p}(E_0)) + a_0 - x_0, E_0\} > 0.$$

Furthermore, it follows from Proposition 3.5 that

$$\lim_{E\to E_0}\sup_{L>0}\|h^L_{\varepsilon,p}(E)-h^L_{\varepsilon,p}(E_0)\|=0.$$

A simple perturbation argument (see Lemma 5.3 in [JM]) yields that there exist an open interval $\mathcal{I}_0 \ni E_0$ and $\gamma>0$ such that for any $E\in\mathcal{I}_0$ and $x\in(a_0-x_0-\gamma, a_0-x_0+\gamma)$,

$$\inf_{L>L_{\epsilon,\delta}} \mathrm{dist}\{\sigma(h^L_{\varepsilon,p}(E))+x, E\}>0.$$

This result can be rephrased as follows: For any $L>L_{\epsilon,\delta}$, and any potential V on $I=[-2pL,2pL]$ which satisfies

$$\begin{aligned} V(n)\in\mathcal{J}_1 &\equiv (a_0-x_0+\varepsilon-\gamma, a_0-x_0+\varepsilon+\gamma) \ \text{ if } n\equiv 0 \bmod p,\\ V(n)\in\mathcal{J}_2 &\equiv (a_0-x_0-\gamma, a_0-x_0+\gamma) \ \text{ if } n\not\equiv 0 \bmod p,\end{aligned}$$

we have that

$$\sup_{E\in\mathcal{I}_0}\mathrm{dist}\{\sigma(h_I(E)),E\}=\alpha>0, \tag{7.52}$$

where α does not depend on L. We of course can choose γ such that $\mathcal{J}_1\cap\mathcal{J}_2=\emptyset$. From the construction, a_0-x_0 and $a_0-x_0+\varepsilon$ belong to $(a_0-\theta/2, a_0+\theta/2)$. Thus,

$$p_1=\int_{\mathcal{J}_1}p(x)dx>0, \qquad p_2=\int_{\mathcal{J}_2}p(x)dx>0.$$

Also, by possibly reducing γ, we may assume that $p_0=1-p_1-p_2>0$.

We are now ready to apply the probabilistic reduction of Section 5. Let $s\in\mathcal{S}$ be an event for which the conclusions of Proposition 5.4 hold. According to Theorem 5.3 to establish (7.51) it suffices to show that for each such s, the relation

$$\lim_{\zeta\downarrow 0}\|(h_\omega(E)-E-i\zeta)^{-1}\delta_0\|<\infty,$$

holds for a.e. $(E,\omega)\in\mathcal{I}_0\times\Omega$ with respect to the measure $m\otimes P_s$. We are now in position to use Theorem 4.1. Consider the random Schrödinger operator $h_0(E)+V_\omega$ on $(\Omega,\mathcal{F},P_s)$. It follows from (5.45) that Hypothesis (A) of Theorem 4.1 is satisfied. Estimate (7.52), Proposition 5.4 and translation invariance yield that all the other conditions of Theorem 4.1 are satisfied, and the result follows.

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