Locality in Free String Field Theory

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Abstract

Free string field operators are constructed for the open bosonic string in the light cone gauge in any dimension. These are naturally localized by the center of mass coordinate. Relative to this localization they are shown to have a causal commutator provided there are no tachyons. For the critical string in d=26 the result still holds if the tachyon is suppressed. We also show a causal commutator relative to the "string light cone".

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1 Introduction

We consider the general question of discovering the extent to which string field theory is local. The question for interacting strings is completely unsettled, since the theory itself is unsettled. One expects such a theory to have a quantized gravitational field, hence a fluctuating metric and light cone, and so the very formulation of the question is likely to be problematic.

On the other hand, for free strings in a fixed Minkowski metric one can at least pose the question. In this paper we find that under certain circumstances there are local string field operators.

The circumstances are as follows. We consider open bosonic strings in the light cone gauge in any dimension spacetime. We assume that there are no tachyons, or that they have been suppressed. The field operator is a function $\Phi(X)$ of parametrized strings $X = X^{\mu}(\sigma)$. The result is that there is a vanishing commutator

$$[\Phi(X), \Phi(Y)] = 0 \tag{1}$$

whenever the center of mass coordinates are space-like separated, i.e. whenever

$$\left(\int_0^{\pi} X(\sigma)d\sigma - \int_0^{\pi} Y(\sigma)d\sigma\right)^2 > 0 \tag{2}$$

The result is perhaps stronger than one might have expected since (2) does not rule out that a point on X be timelike separated from a point on Y.

The result is not really new, but it seems worthwhile to give a careful mathematical treatment. Indeed we take this opportunity to give careful treatment of the whole light cone gauge theory.

An even stronger result has been obtained by Martenic [9] and Lowe [7]. They find that the commutator vanishes whenever

$$\int_0^{\pi} (X(\sigma) - Y(\sigma))^2 \, d\sigma > 0 \tag{3}$$

a configuration defining the "string light cone". We also give a precise version of this result.

One can also state the result in terms of Fourier components

$$X^{\mu}(\sigma) = x_0^{\mu} + \sum_{n=1}^{\infty} \sqrt{2}\, x_n^{\mu} \cos n\sigma \tag{4}$$

Then the field is a function $\Phi = \Phi(x_0, x_1, x_2, ...)$ of these components, the region (2) is $(x_0 - y_0)^2 > 0$, and the region (3) is

$$(x_0 - y_0)^2 + 2\sum_{n=1}^{\infty}(x_n - y_n)^2 > 0 \tag{5}$$

In the light cone gauge $(x_n - y_n)^2 \geq 0$ for $n \geq 1$ so the second region is larger. The vanishing commutator for $(x_0 - y_0)^2 > 0$ embodies a limitation on how fast string disturbances propagate through spacetime. The vanishing commutator outside the string light cone incorporates a limitation on how fast the various modes can grow.

Throughout the paper we employ a canonical quantization procedure to pass from classical systems to quantum systems. This quantization procedure is meant only to motivate the quantum theory, and should not be construed as a derivation of the quantum theory. The justification of the quantum theory should be its consistency with general principles, its naturality, and ultimately its success as a model for the real world. Thus the quantization procedure is not subject to the same standard of rigor as the development of the quantum theory on its own terms. Nevertheless we we will try to be careful.

The organization of the paper is as follows. We first discuss quantization of particles in light cone coordinates and show it is equivalent to quantization in standard coordinates. Then we quantize point fields in light cone coordinates and show that the result is equivalent to quantization in standard coordinates. Next we discuss single string quantization in the light cone gauge. Finally we define string fields in the light cone gauge and prove the locality results. Each step builds on the preceding steps. In an appendix we discuss topics which supplement the main development. These are: (1.) the characteristic Cauchy problem for the Klein-Gordon equation, (2.) canonical field quantization in light cone coordinates, (3.) the existence of the light cone gauge for strings.

2 Point particles and fields

2.1 classical particles

Start with d dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$. Points are $x = (x^0, x^1, ...x^{d-1})$ or $x = (x^0, \mathbf{x})$ and $x^2 = x \cdot x = \eta_{\mu\nu} x^\mu x^\nu$ or $x^2 = -(x^0)^2 + \mathbf{x}^2$

The worldline $x(\tau)$ of a particle of mass squared $\mu \geq 0$ obeys $d^2x/d\tau^2 = 0$ with the constraint $(dx/d\tau)^2 + \mu = 0$. This can be written in the form

$$\begin{aligned} dx^\mu/d\tau &= p^\mu \\ dp^\mu/d\tau &= 0 \end{aligned} \tag{6}$$

and constraint becomes

$$p^2 + \mu = 0 \tag{7}$$

The constraint says that the momentum p lies on the mass shell.

We restrict to forward directed curves satisfying $dx^0/d\tau > 0$. These have positive energy $p^0 > 0$. The constraint $-(p^0)^2 + (\mathbf{p})^2 + \mu = 0$ is solved by $p^0 = \omega_\mu(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2 + \mu}$. To put it another way we work in terms of coordinates $\mathbf{p} = (p^1, ..., p^d)$ for the mass shell. Since p^0 is a constant, one can solve $dx^0/d\tau = p^0$ by $x^0 = p^0\tau$. The remaining variables are $\mathbf{x}, \mathbf{p}$ and if we parametrize in terms of x^0 instead of τ they satisfy

$$\begin{aligned} dx^j/dx^0 &= p^j/\omega(\mathbf{p}) \\ dp^j/dx^0 &= 0 \end{aligned} \tag{8}$$

This is a Hamiltonian system with Hamiltonian $\omega_\mu(\mathbf{p})$.

Now consider light cone coordinates on $\mathbb{R}^d$ given by $(p^+, p^-, \tilde{p})$ where

$$p^{\pm} = \frac{1}{\sqrt{2}}(p^0 \pm p^{d-1}) \tag{9}$$

$$\tilde{p} = (p^1, ...p^{d-2}) \tag{10}$$

In these coordinates the inner product is $p \cdot a = -p^+a^- - p^-a^+ + \tilde{p} \cdot \tilde{a}$. The equations (6) become

$$\begin{aligned} dx^{\pm}/d\tau = p^{\pm} \qquad & dp^{\pm}/d\tau = 0 \\ dx^k/d\tau = p^k \qquad & dp^k/d\tau = 0 \end{aligned} \tag{11}$$

for $1 \leq k \leq d-2$. The mass shell condition is

$$-2p^+p^- + \tilde{p}^2 + \mu = 0 \tag{12}$$

Excluding the case $\mu = 0, \tilde{p} = 0$ we have $p^+ \neq 0$ and can solve the constraint by $p^- = (\tilde{p}^2 + \mu)/2p^+$. Thus we take $p^+, \tilde{p}$ as coordinates for the mass shell. Note that this includes both positive energy ($p^{\pm} > 0$) and negative energy ($p^{\pm} < 0$). Since p^+ is constant we may solve $dx^+/d\tau = p^+$ by $x^+ = p^+\tau$. With x^+, p^- determined the remaining variables are $x^-, p^+, \tilde{x}, \tilde{p}$ and if we reparametrize in terms of x^+ they satisfy

$$\begin{aligned} dx^-/dx^+ = p^-/p^+ \qquad & dp^+/dx^+ = 0 \\ dx^k/dx^+ = p^k/p^+ \qquad & dp^k/dx^+ = 0 \end{aligned} \tag{13}$$

The x^- equation can be written $dx^-/dx^+ = -\partial p^-/\partial p^+$ and the x^k equations can be written $dx^k/dx^+ = \partial p^-/\partial p^k$. Thus we have a Hamiltonian system with canonical variables $(p^+, x^-), (x^k, p^k)$ and Hamiltonian

$$h_\mu = p^- = \frac{\tilde{p}^2 + \mu}{2p^+}, \tag{14}$$

2.2 quantum particles

The standard quantization of the single particle starts with the Hamiltonian system (8). We look for symmetric operators x^i, p^j with $1 \leq i, j \leq d-1$ so that $[x_i, p_j] = i\delta^{ij}$, form the Hamiltonian $\omega_\mu(\mathbf{p})$ and solve the dynamical equations by

$x^i(x^0) = \exp(i\omega_\mu x^0)x^i \exp(-i\omega_\mu x^0)$, etc. Start with the Hilbert space $L^2(\mathbb{R}^{d-1}, d\mathbf{p})$. Let p^j be multiplication by the j^{th} coordinate take $x^j = i\partial/\partial p^j$. Then $\omega_\mu(\mathbf{p})$ is also a multiplication operator.

Changing to a relativistic normalization we introduce the Hilbert space

$$\mathcal{H}_\mu = L^2(\mathbb{R}^{d-1}, \frac{d\mathbf{p}}{2\omega_\mu(\mathbf{p})}) \tag{15}$$

The operator of multiplication by $(2\omega_\mu(\mathbf{p}))^{1/2}$ is unitary from the old space to the new space, and we use it to transform the operators. Then $p^j, \omega_\mu(\mathbf{p})$ are unaffected and x^j becomes the Newton - Wigner coordinate operator

$$x^j = i\frac{\partial}{\partial p^j} - \frac{ip^j}{2(\mathbf{p}^2 + \mu)} \tag{16}$$

The choice of the Lorentz invariant measure in (15) means we have a simple representation of the Poincare group on $\mathcal{H}_\mu$, which however we do not need. We do take note of the representation of the translation subgroup which is generated by the momenta and so with $p^0 = \omega_\mu(\mathbf{p})$ has the form

$$u(a) = \exp(-ip \cdot a) = \exp(ip^0 a^0 - i\mathbf{p} \cdot \mathbf{a}) \tag{17}$$

Now consider quantization in light cone coordinates. We seek operators x^-, p^+, x^k, p^k for $1 \leq k \leq d-2$ satisfying $[p^+, x^-] = i, [x^j, p^k] = i\delta^{jk}$, then form the Hamiltonian h_μ, and generate x^+ evolution by the operator $e^{-ih_\mu x^+}$. For the Hilbert space start with $L^2(\mathbb{R}^{d-1}, dp^+ d\tilde{p})$, take p^+, p^k and h_μ to be multiplication operators and then take $x^- = -i\partial/\partial p^+$ and $x^k = i\partial/\partial p^k$. Next restrict to positive energy by restricting p^+ to $\mathbb{R}^+ = (0, \infty)$. The Hilbert space is then $L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2}, dp^+ d\tilde{p})$ and the operators are as before.

A word on domains is in order here. To form functions of these operators they should be self-adjoint and not just symmetric. This is no problem for the operators on $L^2(\mathbb{R}^{d-1})$, they are essentially self-adjoint on the Schwartz space $\mathcal{S}(\mathbb{R}^{d-1})$ of smooth rapidly decreasing functions, and satisfy the commutation relations there. However when we cut down to $L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2})$ there is a problem with $x^- = -i\partial/\partial p^+$. It is not essentially self-adjoint on its natural minimal domain $C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-2})$ and one must specify boundary conditions at $p^+ = 0$ to obtain a self-adjoint extension. However there is no natural choice, so we stick with the minimal symmetric operator.

Changing to a relativistic normalization we define Hilbert space

$$\mathcal{H}_+ = L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2}, \frac{dp^+ d\tilde{p}}{2p^+}), \tag{18}$$

The operator of multiplication by $(2p^+)^{1/2}$ is unitary from the old Hilbert space to the new Hilbert space and we use it to transform the operators. The operators p^+, p^k and

h_μ are unaffected and the coordinate operators become

$$
\begin{aligned}
x^- &= -i\frac{\partial}{\partial p^+} + \frac{i}{2p^+} \\
x^k &= i\frac{\partial}{\partial p^k}
\end{aligned} \tag{19}
$$

again with the minimal domain for x^-.

Spacetime translations are now written with $p^- = h_\mu$

$$
u'(a) = \exp(ip^- a^+ + ip^+ a^- - i\tilde{p}\cdot\tilde{a}) \tag{20}
$$

Next we establish an equivalence between the standard quantization and the light cone quantization. Our two choices of coordinates are related by holding $\tilde{p}$ fixed and exchanging p^+ and p^{d-1} The mappings, which are inverse to one another, are

$$
\begin{aligned}
p^{d-1} &= \frac{1}{\sqrt{2}}(p^+ - \frac{\tilde{p}^2+\mu}{2p^+}) \\
p^+ &= \frac{1}{\sqrt{2}}(\omega_\mu(\tilde{p},p^{d-1}) + p^{d-1})
\end{aligned} \tag{21}
$$

These are diffeomorphisms between $\mathbb{R}^+$ and $\mathbb{R}$ unless both $\mu = 0$ and $\tilde{p} = 0$.

Correspondingly we change coordinates in functions with an operator v defined by

$$
(v\psi)(\tilde{p},p^{d-1}) = \psi\left(\frac{1}{\sqrt{2}}(\omega_\mu(\tilde{p},p^{d-1}) + p^{d-1}),\tilde{p}\right) \tag{22}
$$

Proposition 1 *v is a unitary operator from $\mathcal{H}_+$ to $\mathcal{H}_\mu$ and satisfies*

$$
v\ u'(a)\ v^{-1} = u(a) \tag{23}
$$

Proof. The operator is norm preserving since for $\psi \in \mathcal{H}_+$

$$
\begin{aligned}
\|v\psi\|^2 &= \int |\psi\left(\frac{1}{\sqrt{2}}(\omega_\mu(\tilde{p},p^{d-1}) + p^{d-1}),\tilde{p}\right)|^2 \frac{d\tilde{p}dp^{d-1}}{2\omega_\mu(\tilde{p},p^{d-1})} \\
&= \int\int_0^\infty |\psi(p^+,\tilde{p})|^2 \frac{dp^+ d\tilde{p}}{2p^+} \\
&= \|\psi\|^2
\end{aligned} \tag{24}
$$

and hence determines a unitary operator . Here we have made the change of variables $p^{d-1} \to p^+$ and used $\omega \to (p^+ + (\tilde{p}^2+\mu)/2p^+)/\sqrt{2}$ and

$$
\frac{\partial p^{d-1}}{\partial p^+} = \frac{1}{\sqrt{2}}(1 + \frac{\tilde{p}^2+\mu}{2(p^+)^2}) \tag{25}
$$

The identity (23) is easily checked.

Remark. The theorem says that the operator v induces the expected transformations on the momenta. We do not claim that v connects the coordinates.

2.3 quantum fields

First some notation. For any Hilbert space $\mathcal{H}$, let $\mathcal{H}_n$ be the n-fold symmetric tensor product, and let $\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}\mathcal{H}_n$ be the Fock space. Let $a^*(g), a(g)$ be the usual creation and annihilation operators on the Fock space, respectively linear and anti-linear in $g \in \mathcal{H}$, and adjoint to each other. As a domain we usually take states with a finite number of entries denoted $\mathcal{F}_0(\mathcal{H})$. Any unitary operator U on $\mathcal{H}$ induces a unitary $\Gamma(U)$ on $\mathcal{F}(\mathcal{H})$. Then $\Gamma(U)a^*(g)\Gamma(U)^{-1} = a^*(Ug)$ and $\Gamma(U)a(g)\Gamma(U)^{-1} = a(Ug)$

For a standard multiparticle theory take the single particle Hilbert space $\mathcal{H}_\mu$, and form the Fock space $\mathcal{F}(\mathcal{H}_\mu)$. The single particle time evolution $e^{-i\omega_\mu x^0}$ becomes for many particles $\Gamma(e^{-i\omega_\mu x^0})$. More generally the single particle spacetime translation operator $u(a)$ becomes for many particles

$$U(a) \equiv \Gamma(u(a)) \tag{26}$$

The single particle wavefunction $e^{-i\omega x^0}\psi$, after Fourier transformation satisfies the Klein Gordon equation $(\Box - \mu)\psi = 0$ where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Correspondingly for quantum fields we want solutions ϕ of the Klein-Gordon equation $(\Box - \mu)\phi = 0$ which we write in Hamiltonian form as

$$\begin{aligned} \partial\phi/\partial x^0 &= \pi \\ \partial\pi/\partial x^0 &= -(-\Delta + \mu)\phi \end{aligned} \tag{27}$$

We seek operator-valued solutions which satisfy the canonical commutation relations at equal times , and for which time evolution is unitarily implemented with positive energy.

Regard the field as a distribution in space. For real $g \in C_0^\infty(\mathbb{R}^{d-1})$ we define $\phi(x^0, g) = \int \phi(x^0, \mathbf{x})g(\mathbf{x})d\mathbf{x}$ on the multiparticle Hilbert space $\mathcal{F}(\mathcal{H}_\mu)$ by

$$\begin{aligned} \phi(x^0, g) &= a^*(e^{i\omega_\mu x^0}\tilde{g}) + a(e^{i\omega_\mu x^0}\tilde{g}) \\ \pi(x^0, g) &= ia^*(\omega_\mu e^{i\omega_\mu x^0}\tilde{g}) - ia(\omega_\mu e^{i\omega_\mu x^0}\tilde{g}) \end{aligned} \tag{28}$$

where $\tilde{g}$ is the Fourier transform. Smearing in space and time, for real $f \in C_0^\infty(\mathbb{R}^d)$ we define $\phi(f) = \int \phi(x)f(x)dx$ by

$$\phi(f) = a(\Pi_\mu f) + a^*(\Pi_\mu f) \tag{29}$$

where Π_μ is restriction of the Fourier transform $\tilde{f}$ to the mass shell:

$$\begin{aligned} (\Pi_\mu f)(\mathbf{p}) &= (2\pi)^{-(d-1)/2}\int e^{i\omega_\mu(\mathbf{p})x^0 - i\mathbf{p}\cdot\mathbf{x}} f(x^0, \mathbf{x})dx^0 d\mathbf{x} \\ &= \sqrt{2\pi}\tilde{f}(-\omega_\mu(\mathbf{p}), \mathbf{p}) \end{aligned} \tag{30}$$

The field $\phi(f)$ is well-defined since $\Pi_\mu : C_0^\infty \to \mathcal{H}_\mu$. Also $u(a)\Pi_\mu f = \Pi_\mu(f(\cdot - a))$ which implies the translation covariance

$$U(a)\phi(f)U(a)^{-1} = \phi(f(\cdot - a)) \tag{31}$$

To exhibit the locality of the fields we introduce the retarded and advanced fundamental solutions $E^{\pm}_{\mu}$ for $\Box - \mu$. For $f \in C_0^{\infty}(\mathbb{R}^d)$ we define $u = E^{\pm} f$ to be the solution of $(\Box - \mu)u = f$ which vanishes in the distant past/future. Explicitly

$$(E^{\pm}_{\mu} f)(x) = \frac{-1}{(2\pi)^{d/2}} \int_{\Gamma_{\pm} \times \mathbb{R}^{d-1}} \frac{e^{ip\cdot x}}{p^2 + \mu} \tilde{f}(p) dp \tag{32}$$

The p^0 contour $\Gamma_{\pm}$ is a positive/negative imaginary translation of the real axis. The expression is independent of the contour since $\tilde{f}(p)$ is entire and rapidly decreasing in real directions. The support of $E^{\pm}_{\mu} f$ is contained in the future/past of the support of f .

Now we compute the commutator. With $E_{\mu} = E^{+}_{\mu} - E^{-}_{\mu}$ we have

$$[\phi(f_1), \phi(f_2)] = 2iIm(\Pi_{\mu} f_1, \Pi_{\mu} f_2) = i < f_1, E f_2 > \tag{33}$$

where $< f_1, f_2 >= \int f_1(x) f_2(x) dx$. The last step follows by working backwards and identifying the p^0 integral as an integrals around $p^0 = \pm\omega_{\mu}(\mathbf{p})$. Because of the support properties of $E^{\pm}$ the commutator vanishes if f, g have spacelike separated supports.

Now we turn to fields in light cone coordinates. Let $\Box' = -2\partial_+\partial_- + \Delta$ be the D'Alembertian in light cone coordinates. The single particle wave function $e^{-ih_{\mu}x^+}\psi$ after Fourier transformation satisfies $(\Box' - \mu)\psi = 0$. Correspondingly we look for a quantum field operator $\phi'(x^+, x^-, \tilde{x})$ satisfying $(\Box' - \mu)\phi' = 0$ and also some commutation relations. The canonical commutation relations in terms of

$$\phi'(x^+, g) = \int \phi'(x^+, x^-, \tilde{x}) g(x^-, \tilde{x}) dx^- d\tilde{x} \tag{34}$$

are

$$[\phi'(0, \frac{\partial g_1}{\partial x^-}), \phi'(0, \frac{\partial g_2}{\partial x^-})] = \frac{i}{2} \int g_1(x^-, \tilde{x}) \frac{\partial g_2}{\partial x^-}(x^-, \tilde{x}) dx^- d\tilde{x} \tag{35}$$

We explain in Appendix A.2 why this is the appropriate choice.

To fulfill these conditions we form the multiparticle light cone Hilbert space $\mathcal{F}(\mathcal{H}_+)$, and define on this space for real $g \in C_0^{\infty}(\mathbb{R}^{d-1})$:

$$\phi'(x^+, g) = a^*(e^{ih_{\mu}x^+} g^{\#}) + a(e^{ih_{\mu}x^+} g^{\#}) \tag{36}$$

Here $g^{\#}$ is defined on $\mathbb{R}^+ \times \mathbb{R}^{d-2}$ by

$$g^{\#}(p^+, \tilde{p}) = \tilde{g}(-p^+, \tilde{p}) \tag{37}$$

and $\tilde{g}(p^+, \tilde{p})$ is the Fourier transform of $g(x^-, \tilde{x})$.

Proposition 2 *The field $\phi'(x^+, g)$ is well defined for real $g \in C_0^{\infty}(\mathbb{R}^{d-1})$ provided $\tilde{g}(0, \tilde{p}) = 0$. It satisfies the field equation $(\Box' - \mu)\phi' = 0$ in the sense of distributions. The fields $\phi'(0, \partial g/\partial x^-)$ are well-defined and have the commutator (35).*

Proof. The function $\tilde{g}$ is in Schwartz space and if $\tilde{g}(0,\tilde{p})=0$ it satisfies $|\tilde{g}(p^+,\tilde{p})| = \mathcal{O}(|p^+|)$ near the origin. These facts are sufficient to guarantee $g^\# \in \mathcal{H}_+$. Thus $\phi'(x^+,g)$ is well defined for $\tilde{g}(0,\tilde{p})=0$ and $\phi'(x^+,\partial g/\partial x^-)$ is well defined for any g.

The field equation is the statement that

$$2\frac{\partial}{\partial x^+}\phi'(x^+,\frac{\partial g}{\partial x^-}) + \phi'(x^+,(\Delta-\mu)g) = 0 \tag{38}$$

This is easily checked.

For the commutator we have

$$\begin{aligned}
[\phi'(0,\frac{\partial g_1}{\partial x^-}),\phi'(0,\frac{\partial g_2}{\partial x^-})] &= (p^+g_1^\#,p^+g_2^\#)_{\mathcal{H}_+} - (p^+g_2^\#,p^+g_1^\#)_{\mathcal{H}_+} \\
&= -\frac{1}{2}\int \overline{\tilde{g}_1(p^+,\tilde{p})}p^+\tilde{g}_2(p^+,\tilde{p})dp^+d\tilde{p} \\
&= \frac{i}{2}\int g_1\frac{\partial g_2}{\partial x^-}dx^-d\tilde{x}
\end{aligned} \tag{39}$$

In the second step we have used $g^\#(p^+,\tilde{p}) = \tilde{g}(-p^+,\tilde{p}) = \overline{\tilde{g}(p^+,-\tilde{p})}$ to combine two integrals over $p^+>0$ into an integral over the whole line. This completes the proof.

Now we smear in all the coordinates and consider

$$\phi'(f) = \int \phi'(x^+,x^-,\tilde{x})f(x^+,x^-,\tilde{x})dx^+dx^-d\tilde{x}$$

The expression (36) becomes the operator on $\mathcal{F}(\mathcal{H}_+)$

$$\phi'(f) = a^*(\mathcal{P}_\mu f) + a(\mathcal{P}_\mu f) \tag{40}$$

where $\mathcal{P}_\mu$ is the restriction operator in the new coordinates:

$$(\mathcal{P}_\mu f)(p^+,\tilde{p}) = \sqrt{2\pi}\tilde{f}(-\frac{\tilde{p}^2+\mu}{2p^+},-p^+,\tilde{p}) \tag{41}$$

Theorem 1 *For any real $f\in C_0^\infty(\mathbb{R}^d)$:*

1. *$\phi'(f)$ is well defined for $d\geq 3,\mu\geq 0$ (and $d=2$ if $\mu>0$).*
2. *$\phi'(f)$ satisfies the field equation $\phi'((\Box'-\mu)f)=0$*
3. *With spacetime translations represented by $U'(a)=\Gamma(u'(a))$ we have the covariance*
$$U'(a)\phi'(f)U'(a)^{-1} = \phi'(f(\cdot - a)) \tag{42}$$
4. *The commutator $[\phi'(f_1),\phi'(f_2)]$ vanishes if f_1,f_2 have space-like separated supports.*

Proof. To show the field is well-defined we need to know that $\mathcal{P}_\mu f$ is in $\mathcal{H}_+$. Since $\tilde{f}$ is in Schwartz space we have that for any N there is a Schwartz norm $\|f\|_{\mathcal{S}}$ and a constant C such that

$$|\tilde{f}(p^-, p^+, \tilde{p})| \leq C\|f\|_{\mathcal{S}}(1 + |p^-| + |p^+| + |\tilde{p}|)^{-N} \tag{43}$$

We can assume that the norm has the form the form

$$\|f\|^2_{\mathcal{S}} = \sum_{\alpha,\beta} \int |(x^\alpha \partial^\beta f)(x)|^2 dx \tag{44}$$

with a finite sum over multi-indexes α, β. For $|p^+| \geq 1$ we drop the $|p^-|$ and get

$$|(\mathcal{P}_\mu f)(p^+, \tilde{p})| \leq C\|f\|_{\mathcal{S}}(1 + |p^+| + |\tilde{p}|)^{-N} \tag{45}$$

For $|p^+| \leq 1$ first bound $\mathcal{P}_\mu f$ by $(1+|p^-|)^{-1/8}(1+|\tilde{p}|)^{-N+1}$ and then use $|p^-| \geq |\tilde{p}|^2/2|p^+|$ to get

$$|(\mathcal{P}_\mu f)(p^+, \tilde{p})| \leq C\|f\|_{\mathcal{S}}|p^+|^{1/8}|\tilde{p}|^{-1/4}(1 + |\tilde{p}|)^{-N+1} \tag{46}$$

Combining these we have for a constant C independent of $\mu \geq 0$

$$\int\int_0^\infty |(\mathcal{P}_\mu f)(p^+, \tilde{p})|^2 \frac{dp^+ d\tilde{p}}{2p^+} \leq C\|f\|^2_{\mathcal{S}} < \infty \tag{47}$$

which shows $\mathcal{P}_\mu f$ is in $\mathcal{H}_+$. (If $d = 2$, then $\tilde{p}$ does not exist, and for a lower bound on p^- we use $|p^-| \geq \mu/2|p^+|$.).

The field equation follows from $\mathcal{P}_\mu((\Box' - \mu)f) = 0$.

The covariance follows from $u'(a)\mathcal{P}_\mu f = \mathcal{P}_\mu(f(\cdot - a))$.

The commutator is evaluated by returning to standard coordinates. Let

$$\hat{f}(x) = f(\frac{x^0 + x^{d-1}}{\sqrt{2}}, \frac{x^0 - x^{d-1}}{\sqrt{2}}, \tilde{x}) \tag{48}$$

be the test function in standard coordinates. One can check the identity

$$v\ \mathcal{P}_\mu f = \Pi_\mu \hat{f} \tag{49}$$

Since v is unitary the commutator is evaluated as

$$\begin{aligned} [\phi'(f_1), \phi'(f_2)] &= 2iIm(\mathcal{P}_\mu f_1, \mathcal{P}_\mu f_2)_{\mathcal{H}_+} \\ &= 2iIm(\Pi_\mu \hat{f}_1, \Pi_\mu \hat{f}_2)_{\mathcal{H}_\mu} \\ &= i < \hat{f}_1, E_\mu \hat{f}_2 > \end{aligned} \tag{50}$$

If $-2(x^+ - y^+)(x^- - y^-) + (\tilde{x} - \tilde{y})^2 > 0$ for all $x \in suppf_1, y \in suppf_2$, then $-(x^0 - y^0)^2 + (\mathbf{x} - \mathbf{y})^2 > 0$ for all $x \in supp\hat{f}_1, y \in supp\hat{f}_2$ and the commutator vanishes by the standard result. This completes the proof.

The next result shows the complete equivalence of the light cone field and the standard field.

Theorem 2 *Let $V : \mathcal{F}(\mathcal{H}_+) \to \mathcal{F}(\mathcal{H}_\mu)$ be the unitary operator $V = \Gamma(v)$. Then*

$$\begin{aligned} V\phi'(f)V^{-1} &= \phi(\hat{f}) \\ VU'(a)V^{-1} &= U(a) \end{aligned} \tag{51}$$

Proof. The first identity follows from (49) and the second identity follows from (23).

3 Strings

Our discussion of strings is more or less standard, see [2], [3], [6]. We start with a discussion of the classical free relativistic string. This is described by a Hamiltonian system and some constraints. We use the constraints to eliminate redundant variables and obtain a reduced Hamiltonian system with independent variables. This is then quantized.

3.1 classical strings

A single string is specified by the world sheet $X : [0, \pi] \times \mathbb{R} \to \mathbb{R}^d$ which is forward directed in the sense that $\partial X^0/\partial\tau > 0$. In the conformal gauge the string satisfies the wave equation

$$(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2})X^\mu(\tau, \sigma) = 0 \tag{52}$$

and the constraint

$$(\frac{\partial X}{\partial\tau} \pm \frac{\partial X}{\partial\sigma})^2 = 0 \tag{53}$$

The constraint need only be satisfied at $\tau = 0$; it is preserved by time evolution. For the open string we also require Neumann boundary conditions

$$\frac{\partial X}{\partial\sigma}(\tau, 0) = \frac{\partial X}{\partial\sigma}(\tau, \pi) = 0 \tag{54}$$

We write the equation as a first order system

$$\begin{aligned} \partial X^\mu/\partial\tau &= \Pi^\mu \\ \partial\Pi^\mu/\partial\tau &= \frac{\partial^2 X^\mu}{\partial\sigma^2} \end{aligned} \tag{55}$$

with the constraint

$$(\Pi \pm \frac{\partial X}{\partial\sigma})^2 = 0 \tag{56}$$

We are especially interested in the center of mass variables

$$\begin{aligned} x^\mu(\tau) &= \frac{1}{\pi}\int_0^\pi X^\mu(\tau, \sigma)d\sigma \\ p^\mu(\tau) &= \frac{1}{\pi}\int_0^\pi \Pi^\mu(\tau, \sigma)d\sigma \end{aligned} \tag{57}$$

which satisfy $dx^\mu/d\tau = p^\mu$ and $dp^\mu/d\tau = 0$.

Our system of equations (55), (56) is invariant under conformal diffeomorphisms on two dimensional Minkowski space. For a generic class of solutions, which entail $p^+ > 0$, we can use this fact to reparametrize so that:

$$X^+(\tau,\sigma) = p^+\tau \tag{58}$$

a condition which defines the light cone gauge. We discuss this step in detail in Appendix A.3. Hereafter we assume that (58) holds, and hence also that $\Pi^+(\tau,\sigma) = p^+$

Now the constraint says

$$2p^+(\Pi^- \pm \frac{\partial X^-}{\partial\sigma}) = \sum_{k=1}^{d-2}(\Pi^k \pm \frac{\partial X^k}{\partial\sigma})^2 \tag{59}$$

or equivalently

$$\begin{aligned} 2p^+\Pi^- &= \sum_{k=1}^{d-2}(\Pi^k)^2 + (\frac{\partial X^k}{\partial\sigma})^2 \\ 2p^+\frac{\partial X^-}{\partial\sigma} &= 2\sum_{k=1}^{d-2}\Pi^k\frac{\partial X^k}{\partial\sigma} \end{aligned} \tag{60}$$

Next we expand X^μ, Π^μ in eigenfunctions of $(-\partial^2/\partial\sigma^2)$ with Neumann boundary conditions, i.e. in a cosine series. We have

$$\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu(\tau) + \sum_{n=1}^{\infty}\sqrt{2}x_n^\mu(\tau)\cos n\sigma \\ \Pi^\mu(\tau,\sigma) &= p^\mu(\tau) + \sum_{n=1}^{\infty}\sqrt{2}p_n^\mu(\tau)\cos n\sigma \end{aligned} \tag{61}$$

where

$$\begin{aligned} x_n^\mu(\tau) &= \frac{\sqrt{2}}{\pi}\int_0^\pi X^\mu(\tau,\sigma)\cos n\sigma d\sigma \\ p_n^\mu(\tau) &= \frac{\sqrt{2}}{\pi}\int_0^\pi \Pi^\mu(\tau,\sigma)\cos n\sigma d\sigma \end{aligned} \tag{62}$$

The gauge condition says that $x^+ = p^+\tau$ and that $x_n^+ = 0$ and hence $p_n^+ = dx_n^+/d\tau = 0$. The constraints express p^-, p_n^-, x_n^- as functions of $p^+, \tilde{p}, \tilde{p}_n, \tilde{x}_n$ where $\tilde{p} = (p^1, ..., p^{d-2})$, etc. In particular we have

$$p^- = \frac{1}{2p^+}(\tilde{p}^2 + \sum_{n=1}^{\infty}(\tilde{p}_n^2 + n^2\tilde{x}_n^2)) \tag{63}$$

The remaining variables are $x^-, p^+, \tilde{x}, \tilde{p}, \tilde{x}_n, \tilde{p}_n$ and they satisfy

$$\begin{array}{ll} dx^-/d\tau = p^- & dp^+/d\tau = 0 \\ dx^k/d\tau = p^k & dp^k/d\tau = 0 \\ dx_n^k/d\tau = p_n^k & dp_n^k/d\tau = -n^2 x_n^k \end{array} \tag{64}$$

This is not a Hamiltonian system, but it is if we drop (p^+, x^-) and take a Hamiltonian

$$\hat{H} = p^+ p^- = \frac{1}{2}(\tilde{p}^2 + \sum_{n=1}^{\infty}(\tilde{p}_n^2 + n^2 \tilde{x}_n^2)) \tag{65}$$

Now changing variables from τ to $x^+ = p^+\tau$ we get

$$\begin{array}{ll} dx^-/dx^+ = p^-/p^+ & dp^+/dx^+ = 0 \\ dx^k/dx^+ = p^k/p^+ & dp^k/dx^+ = 0 \\ dx_\ell^k/dx^+ = p_n^k/p^+ & dp_n^k/dx^+ = (-n^2/p^+)x_n^k \end{array} \tag{66}$$

This is a Hamiltonian system with canonical variables $(p^+, x^-), (x^k, p^k), (x_n^k, p_n^k)$ and Hamiltonian $H = p^-$ given by (63).

3.2 quantum strings

Now we quantize the above system. We seek operators $x^-, p^+, x^k, p^k, x_n^k, p_n^k$ for $1 \leq k \leq d-2,\ 1 \leq n$ which at equal times satisfy the canonical commutation relations: [1]

$$\begin{aligned} [p^+, x^-] &= i \\ [x^j, p^k] &= i\delta_{jk} \\ [x_n^j, p_m^k] &= i\delta_{jk}\delta_{nm} \end{aligned} \tag{67}$$

Furthermore they should satisfy the dynamical equations (66) and x^+-evolution should be implemented with a Hamiltonian which has spectrum bounded below.

We start by considering only x^k, p^k, x_n^k, p_n^k and solving the equations (64). Just as an N-component scalar field on $\mathbb{R}^d$ has a simple definition on the Fock space over $L^2(\mathbb{R}^d, \mathbb{C}^N)$ or its Fourier transform, so the transverse modes of a string in $\mathbb{R}^d$ parametrized by $[0,\pi]$ have a simple definition on the Fock space over $L^2([0,\pi], \mathbb{C}^{d-2})$ or its cosine transform $\ell^2(\mathbb{N}, \mathbb{C}^{d-2})$. This is the same as $\ell^2(\mathbb{N} \times (1, ..., d-2))$ and we call it simply ℓ^2 so that

$$\ell^2 = \{\{\psi_n^k\} : \sum_{k=1}^{d-2}\sum_{n=1}^{\infty}|\psi_n^k|^2 < \infty\} \tag{68}$$

[1] Since we are considering a reduced Hamiltonian system one might think that quantization should be based on Dirac brackets rather than Poisson brackets as above. These turn out to be the same [5].

We have excluded the zero modes, which are treated separately. Thus the Hilbert space is not just the Fock space $\mathcal{F}(\ell^2)$ but rather

$$L^2(\mathbb{R}^{d-2}) \otimes \mathcal{F}(\ell^2) \tag{69}$$

In this space let $p^k = p^k \otimes I$ be the multiplication operator and let $x^k = i\partial/\partial p^k \otimes I$, just as for the single particle. Also let e_n^k be the orthonormal basis in ℓ^2 consisting of the characteristic functions of the points (n,k). Let $a_n^k = I \otimes a(e_n^k)$ and let $(a_n^k)^* = I \otimes (a(e_n^k))^*$ and define $x_n^k = (2n)^{-1/2}((a_n^k)^* + a_n^k)$ and $p_n^k = i(n/2)^{1/2}((a_n^k)^* - a_n^k)$. These satisfy the commutation relations, and the solution to (64) with these operators at $\tau = 0$ is

$$\begin{aligned}
x^k(\tau) &= x^k + p^k\tau \\
p^k(\tau) &= p^k \\
x_n^k(\tau) &= (2n)^{-1/2}(e^{in\tau}(a_n^k)^* + e^{-in\tau}a_n^k) \\
p_n^k(\tau) &= i(n/2)^{1/2}(e^{in\tau}(a_n^k)^* - e^{-in\tau}a_n^k)
\end{aligned} \tag{70}$$

We can recover the full string operator $X^k(\tau,\sigma)$ as in (61). In the literature these formulas are usually written in terms of $\alpha_n^k = -i\sqrt{n}a_n^k$ and $\alpha_{-n}^k = i\sqrt{n}(a_n^k)^*$.

Now τ-evolution is written $x_n^k(\tau) = e^{i\hat{H}\tau}x_n^k e^{-i\hat{H}\tau}$, etc. where the evolution operator is $e^{-i\hat{H}\tau} = e^{-iat}(e^{-i(\tilde{p})^2t/2} \otimes \Gamma(e^{-int}))$ for any constant a. The generator is

$$\hat{H} = \frac{\tilde{p}^2}{2} + N - a \tag{71}$$

Here $N = I \otimes N$ is defined by $N = d\Gamma(n) \equiv id/dt\ \Gamma(e^{-int})|_{t=0}$. It is called the number operator, although the terminology is different from field theory where $d\Gamma(1)$ would be called the number operator. It can also be written

$$N = \sum_{k,n} n(a_n^k)^*(a_n^k) = \frac{1}{2}\sum_{k,n} :(p_n^k)^2 + n^2(x_n^k)^2: \tag{72}$$

Thus the quantum $\hat{H}$ is obtained from the classical $\hat{H}$ by Wick ordering and (possibly) adjusting by a constant.

Now add the last pair of variables p^+, x^-. Taking our cue from the particle quantization we tensor in $L^2(\mathbb{R}^+, dp^+/2p^+)$ and define

$$\begin{aligned}
\mathcal{H} &= L^2(\mathbb{R}^+, \frac{dp^+}{2p^+}) \otimes L^2(\mathbb{R}^{d-2}, d\tilde{p}) \otimes \mathcal{F}(\ell^2) \\
&= L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2}, \frac{dp^+ d\tilde{p}}{2p^+}) \otimes \mathcal{F}(\ell^2) \\
&= L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2}, \mathcal{F}(\ell^2), \frac{dp^+ d\tilde{p}}{2p^+})
\end{aligned} \tag{73}$$

Here we have listed some naturally isomorphic Hilbert spaces. The last is the square integrable $\mathcal{F}(\ell^2)$-valued functions on $\mathbb{R}^+ \times \mathbb{R}^{d-2}$.

Now let p^+ be the multiplication operator and $x^- = -i\partial/\partial p^+ + i/2p^+$ just as in (19). Classically p^- is $\hat{H}/p^+$, and we take this as the quantum definition as well so

$$p^- = \frac{\hat{H}}{p^+} = \frac{\tilde{p}^2 + 2(N-a)}{2p^+} \tag{74}$$

All these operators can be thought of as acting on $C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-2}, \mathcal{F}_0(\ell^2))$, a dense domain in $\mathcal{H}$. For τ evolution we take $x^-(\tau) = x^- + p^-\tau$ and $p^+(\tau) = p^+$ and then (64) is solved.

Now we replace τ by x^+/p^+ to get a solution of (66). We have

$$\begin{aligned}
x^-(x^+) &= x^- + (p^-/p^+)x^+ \\
p^+(x^+) &= p^+ \\
x^k(x^+) &= x^k + (p^k/p^+)x^+ \\
p^k(x^+) &= p^k \\
x_n^k(x^+) &= (2n)^{-1/2}(e^{inx^+/p^+}(a_n^k)^* + e^{-inx^+/p^+}a_n^k) \\
p_n^k(x^+) &= i(n/2)^{1/2}(e^{inx^+/p^+}(a_n^k)^* - e^{-inx^+/p^+}a_n^k)
\end{aligned} \tag{75}$$

The x^+-evolution is implemented by $e^{-i\hat{H}x^+/p^+}$ which is written e^{-iHx^+} where

$$H = \frac{\hat{H}}{p^+} = p^- \tag{76}$$

Spacetime translations only change the center of mass, or to put it another way, the total momentum is the center of mass momentum. So spacetime translations are represented by the unitary operators

$$U(a) = \exp(ip^-a^+ + ip^+a^- - i\tilde{p}\cdot\tilde{a}) \tag{77}$$

with p^- as above. The mass operator is

$$M^2 = 2p^+p^- - \tilde{p}^2 = 2(N-a) \tag{78}$$

and the spectrum is $\{-2a, -2a+2, -2a+4, ...\}$ with increasing multiplicity.

Then there is the question of whether there is a unitary representation of the Poincare group with the the translation subgroup as above. Here there is a well-known anomaly in the commutator of the generators of the Lorentz transformation. The anomaly vanishes if and only if $d = 26$ and $a = 1$. One expects that in this case the unitary representation of the Poincare group exists, although there is apparently no rigorous proof. With $a = 1$ the spectrum starts at -2 which is the tachyon.

3.3 string wave function

The string wave function ψ is in $\mathcal{H} = L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2}, \mathcal{F}(\ell^2), dp^+ d\mathbf{p}/2p^+)$. But the center of mass coordinates x^k, x^- have a momentum space representation, and the the mode coordinates x_n^k have a harmonic oscillator representation. We also want to change to a coordinate representation in which these are multiplication operators.

First we take the inverse Fourier transform so that x^-, x^k are multiplication operators. At the same time we pass to the Schrodinger picture and incorporate the x^+ evolution. Thus we define

$$\psi(x^+, x^-, \tilde{x}) = (2\pi)^{-(d-1)/2} \int \int_0^\infty e^{-ip^- x^+ - ip^+ x^- + i\tilde{p}\tilde{x}} \psi(p^+, \tilde{p}) \frac{dp^+ d\tilde{p}}{(2p^+)^{1/2}} \tag{79}$$

where p^- is defined by (74). If we assume that $\psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{d-2}, \mathcal{F}_0(\ell^2))$, then $\psi(x^+, x^-, \tilde{x})$ is a smooth $\mathcal{F}(\ell^2)$ valued function on $\mathbb{R}^d$. As such it satisfies the equation

$$(\Box' - 2(N-a))\psi = 0 \tag{80}$$

which generalizes the Klein Gordon equation.

On the other hand we can represent all the x_n^k as multiplication operators as follows. (See for example [10], p228). Consider the Gaussian random process indexed by (n, k) with covariance $(2n)^{-1}$. This consists of a probability measure space (Q, μ) and Gaussian random variables x_n^k such that $\int x_n^k x_{n'}^{k'} d\mu = (2n)^{-1}\delta_{nn'}\delta_{kk'}$. The space Q can be thought of as a space of sequences $\{x_n^k\}$ and the functions x_n^k as the coordinates. There is a unitary operator S from $L^2(Q, d\mu)$ to $\mathcal{F}(\ell^2)$ such that $S \cdot 1 = \Omega_0 =$ the Fock vacuum, and $S^{-1} x_n^k S = x_n^k$, where on the left $x_n^k = (2n)^{-1/2}((a_n^k)^* + a_n^k)$ and on the right it is multiplication by x_n^k. Using S, states and operators in $\mathcal{F}(\ell^2)$ can be represented in $L^2(Q, d\mu)$. The operators p_n^k, N become differential operators, about which more later.

If we make both changes at once the wave functions are a functions from $\mathbb{R}^d$ to $L^2(Q, d\mu)$. Hence they are functions on $\mathbb{R}^d \times Q$ and can be written

$$\psi(x, \tilde{x}_1, \tilde{x}_2, ...) \tag{81}$$

where $x = (x^+, x^-, \tilde{x})$.

4 String fields

4.1 center of mass localization

First consider the multi-string quantum mechanics. The single string Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2}, \mathcal{F}(\ell^2), dp^+ d\tilde{p}/2p^+)$ and x^+-evolutions is given by e^{-iHx^+} where $H = (\tilde{p}^2 + 2(N-a))/2p^+$. For many strings we take the Fock space $\mathcal{F}(\mathcal{H})$ and the evolution operator is $\Gamma(e^{-iHx^+})$. More generally $\mathcal{U}(a) = \Gamma(U(a))$ gives a representation of the translation group on the Fock space.

We want to define a field operator on this Hilbert space, and we proceed by analogy with point field theory. The field operator should be a operator version of real-valued wave functions $\psi(x)$ as in (79). Real means real under the isomorphism $\mathcal{F}(\ell^2) \cong L^2(Q, d\mu)$. The wave function can be regarded as a $\mathcal{F}(\ell^2)$-valued distribution in the sense that

$$< F, \psi >= \int (F(x), \psi(x))_{\mathcal{F}} \, dx \tag{82}$$

is a continuous function on real-valued $F \in C_0^\infty(\mathbb{R}^d, \mathcal{F}_0(\ell^2))$. Correspondingly we ask for symmetric operators $\Phi(F)$ on $\mathcal{F}(\mathcal{H})$ defined, linear, and continuous in real $F \in C_0^\infty(\mathbb{R}^d, \mathcal{F}_0(\ell^2))$. The field operator should satisfy the same equation as the wave function, that is $(\Box' - 2(N-a))\Phi = 0$ This should be understood in the sense of distributions, namely

$$\Phi((\Box' - 2(N-a))F) = 0 \tag{83}$$

To fulfill these requirements follow (36) and define for real $G \in C_0^\infty(\mathbb{R}^{d-1}, \mathcal{F}_0(\ell^2))$ an operator $\Phi(x^+, G)$ on $\mathcal{F}(\mathcal{H})$ by

$$\Phi(x^+, G) = a^*(e^{iHx^+} G^\#) + a(e^{iHx^+} G^\#) \tag{84}$$

where $G^\#(p^+, \tilde{p}) = \tilde{G}(-p^+, \tilde{p})$. Just as for point fields, this can be motivated by a canonical quantization procedure for solutions of $(\Box' - 2(N-a))\Phi = 0$. Also as for point fields the operator is well defined under certain restrictions on G, but we do not tarry on this point.

Now smearing in x^+ as well, we define for real $F \in C_0^\infty(\mathbb{R}^d, \mathcal{F}_0(\ell^2))$

$$\Phi(F) = a^*(\mathcal{P}F) + a(\mathcal{P}F) \tag{85}$$

where

$$(\mathcal{P}F)(p^+, \tilde{p}) = \int \exp(iHx^+)\tilde{F}(x^+, -p^+, \tilde{p}) dx^+ \tag{86}$$

and where $\tilde{F}$ denotes the Fourier transform in the last $d-1$ variables. Just as for point fields we have

Theorem 3 *For real $F \in C_0^\infty(\mathbb{R}^d, \mathcal{F}_0(\ell^2))$:*

1. *$\Phi(F)$ is well defined for $d \geq 3, a \leq 0$ (and $d = 2$ if $a < 0$).*
2. *$\Phi(F)$ satisfies the field equation $\Phi((\Box' - 2(N-a))F) = 0$*
3. *Under spacetime translations: $\mathcal{U}(a)\Phi(F)\mathcal{U}(a)^{-1} = \Phi(F(\cdot - a))$*

Proof. For the proof it is convenient to choose an explicit orthonormal basis for $\mathcal{F}(\ell^2)$. The basis is the occupation number basis and is indexed by sequences of non-negative integers $\alpha = \{\alpha_{n,k}\}$ with $1 \leq n, 1 \leq k \leq d-2$ and finitely many non-zero entries. The basis elements are

$$\chi_\alpha = \prod_{n.k} \frac{[(a_n^k)^*]^{\alpha_{nk}}}{\sqrt{\alpha_{nk}!}} \Omega_0 \tag{87}$$

We have

$$N\chi_\alpha = (\sum_{nk} n\ \alpha_{nk})\chi_\alpha \tag{88}$$

For $\psi \in \mathcal{F}(\ell^2)$ let $\psi_\alpha = (\chi_\alpha, \psi)$ be the components in these basis elements. Similarly $\mathcal{F}(\ell^2)$-valued functions $F(x)$ have components $F_\alpha(x) = (\chi_\alpha, F(x))$.

Now for $\mathcal{P}F(p^+, \tilde{p})$ we can compute the components as

$$\begin{aligned} (\mathcal{P}F)_\alpha(p^+, \tilde{p}) &= \sqrt{2\pi}\tilde{F}_\alpha(-\frac{\tilde{p}^2 + \mu(\alpha)}{2p^+}, -p^+, \tilde{p}) \\ &= (\mathcal{P}_{\mu(\alpha)} F_\alpha)(p^+, \tilde{p}) \end{aligned} \tag{89}$$

where

$$\mu(\alpha) = 2((\sum_{nk} n\ \alpha_{nk}) - a) \tag{90}$$

To show that $\Phi(F)$ is well defined we must show that $\mathcal{P}F$ is in $\mathcal{H}$. To see this we expand $\mathcal{P}F$ in the above basis and use (47) noting that $a \leq 0$ implies $\mu(\alpha) \geq 0$ (and for $d = 2$ that $a < 0$ implies $\mu(\alpha) > 0$). We obtain for the $\mathcal{H}$ norm:

$$\begin{aligned} & \int\int_0^\infty \|(\mathcal{P}F)(p^+, \tilde{p})\|^2 \frac{dp^+ d\tilde{p}}{2p^+} \\ =& \sum_\alpha \int\int_0^\infty |(\mathcal{P}_{\mu(\alpha)}F_\alpha)(p^+, \tilde{p})|^2 \frac{dp^+ d\tilde{p}}{2p^+} \\ \leq& C\sum_\alpha \|F_\alpha\|_{\mathcal{S}}^2 \\ =& C\|F\|_{\mathcal{S}}^2 < \infty \end{aligned} \tag{91}$$

Here in the last step we refer to the $\mathcal{F}(\ell^2)$-valued version of the Schwartz norm (43) and use $x^\beta(\partial^\gamma F_\alpha)(x) = (x^\beta \partial^\gamma F)_\alpha(x)$.

The field equation is satisfied since $\mathcal{P}(\Box' - 2(N-a))F = 0$ and the covariance follows from $U(a)\mathcal{P}F = \mathcal{P}(F(\cdot - a))$. Each of these statements can be reduced to the point field case by expanding in the basis. This completes the proof.

Now we define advanced and retarded fundamental solutions for the standard coordinate operator $\Box - 2(N-a)$. As for the point field these are defined for $F \in C_0^\infty(\mathbb{R}^d, \mathcal{F}_0(\ell^2))$ by

$$(E^\pm F)(x) = \frac{-1}{(2\pi)^{d/2}} \int_{\Gamma_\pm \times \mathbb{R}^{d-1}} \frac{e^{ipx}}{p^2 + 2(N-a)} \tilde{F}(p) dp \tag{92}$$

We have that

$$(E^{\pm}F)_\alpha = E^{\pm}_{\mu(\alpha)}F_\alpha \tag{93}$$

where $E^{\pm}_{\mu}$ is defined in (32). This shows that the support of $E^{\pm}F$ is contained in the future/past of the support of F. Define as before $E = E^+ - E^-$.

If $F_1, F_2 \in C_0^\infty(\mathbb{R}^d, \mathcal{F}_0(\ell^2))$ are test functions in light cone coordinates , let $\hat{F}_1, \hat{F}_2$ be the expression in standard coordinates as in (48). The main result is:

Theorem 4 *(Locality) For F_1, F_2 as above*

$$[\Phi(F_1), \Phi(F_2)] = < \hat{F}_1, E\hat{F}_2 > \tag{94}$$

The commutator vanishes if F_1, F_2 have space-like separated supports.

Proof. If F is real, then $F_\alpha = (\chi_\alpha, F)$ is real since in the $L^2(Q, d\mu)$ representation F is real by definition and the χ_α correspond to Hermite polynomials which are real. Then by our earlier result (50)

$$\begin{aligned}
[\Phi(F_1), \Phi(F_2)] &= 2iIm(\mathcal{P}F_1, \mathcal{P}F_2)_{\mathcal{H}} \\
&= \sum_\alpha 2iIm(\mathcal{P}_{\mu(\alpha)}F_{1\alpha}, \mathcal{P}_{\mu(\alpha)}F_{2\alpha})_{\mathcal{H}^+} \\
&= \sum_\alpha < \hat{F}_{1\alpha}, E_{\mu(\alpha)}\hat{F}_{2\alpha} > \\
&= < \hat{F}_1, E\hat{F}_2 > .
\end{aligned} \tag{95}$$

If $supp\hat{F}_1$ and $supp\hat{F}_2$ are space-like separated then $supp\hat{F}_{1\alpha}$ and $supp\hat{F}_{2\alpha}$ are space-like separated (since $supp\hat{F}_\alpha \subset supp\hat{F}$) , hence $< \hat{F}_{1\alpha}, E_{\mu(\alpha)}\hat{F}_{2\alpha} >= 0$ and the commutator vanishes.

Remarks.

1. The result still holds for the critical theory $d = 26, a = 1$ provided the tachyon is suppressed, i.e. provided $F_{1,0} = F_{2,0} = 0$. In this case $\alpha = 0$ does not contribute and $\mu(\alpha) \geq 0$ still holds.

   If we combine the locality result with a treatment of Lorentz covariance for $d = 26, a = 1$, it may be possible to show that the field operators $\Phi(F)$ generate local C^* algebras satisfying all the Haag-Kastler axioms.

2. We expect that the locality result would hold for any free string theory in the light cone gauge, i.e. closed strings , superstrings, etc. For critical superstrings in d= 10 the tachyon is absent from the start and one would not have to remove it [7]

   There is some speculation about interacting strings in [8].

3. One can also formulate the question in a covariant gauge. However the covariant string field operator then contains certain projection operators which are difficult to analyze, and so there is not result as yet. See also [4]

4.2 string light cone

Now we go for the stronger result of [9]. In the Hilbert space ℓ^2 we consider the finite dimensional subspaces ℓ^2_N which are spanned by the vectors e^k_n with $1 \leq k \leq d-2$ and $1 \leq n \leq N$. Correspondingly the Fock space $\mathcal{F}(\ell^2)$ has the subspace $\mathcal{F}(\ell^2_N)$. We are going take our test functions (and hence fields) to be $\mathcal{F}(\ell^2_N)$-valued for some arbitrary N. Since the union of the spaces $\mathcal{F}(\ell^2_N)$ is dense in $\mathcal{F}(\ell^2)$ we do not expect to miss anything important with this restriction

There is a unitary operator S from $L^2(\mathbb{R}^{N(d-2)}, d\mu_N)$ to $\mathcal{F}(\ell^2_N)$ where $d\mu_N$ is the Gaussian measure

$$d\mu_N(\tilde{x}^1, ..., \tilde{x}^N) = \prod_{n=1}^{N} (n/\pi)^{(d-2)/2} e^{-n\ \tilde{x}_n^2}\ d\tilde{x}_n \tag{96}$$

The operator is determined by its action on polynomials which is

$$S\left(P(\tilde{x}_1, ..., \tilde{x}_N)\right) = P(\tilde{x}_1, ..., \tilde{x}_N)\Omega_0 \tag{97}$$

where on the right $\tilde{x}_n$ is the Fock space operator. This is the finite variable version of the previously defined S.

Under the inverse mapping operators on $\mathcal{F}(\ell^2_N)$ transform as

$$\begin{aligned} S^{-1}x^k_n\ S &= x^k_n \\ S^{-1}p^k_n\ S &= -i\frac{\partial}{\partial x^k_n} + inx^k_n \\ S^{-1}N\ S &= L \end{aligned} \tag{98}$$

where we define the differential operator

$$L = \frac{1}{2}(\sum_{k=1}^{d-2}\sum_{n=1}^{N} -(\frac{\partial}{\partial x^k_n})^2 + 2nx^k_n\frac{\partial}{\partial x^k_n}) \tag{99}$$

The result for the string light cone is:

Theorem 5 *Let $F_1, F_2 \in C_0^\infty(\mathbb{R}^d, \mathcal{F}(\ell^2_N))$ and suppose that the transforms $S^{-1}\hat{F}_1$, $S^{-1}\hat{F}_2$ are in $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^{N(d-2)})$ and have spacelike separated supports with respect to the Minkowski metric on $\mathbb{R}^d \times \mathbb{R}^{N(d-2)}$. Then*

$$[\Phi(F_1), \Phi(F_2)] = 0 \tag{100}$$

Proof. The commutator is evaluated as

$$\int ((S^{-1}\hat{F}_1)(x), (S^{-1}E\hat{F}_2)(x))dx$$

$$= \int (S^{-1}\hat{F}_1)(x, \tilde{x}_1, ..., \tilde{x}_N)(S^{-1}E\hat{F}_2)(x, \tilde{x}_1, ..., \tilde{x}_N)dx \; d\mu_N(\tilde{x}_1...\tilde{x}_N) \tag{101}$$

Now both operators N, L are essentially self-adjoint on the polynomial domains. One can show that the full domain $D(L)$ includes $C_0^\infty(\mathbb{R}^{N(d-2)})$, and so $D(N)$ includes $S[C_0^\infty(\mathbb{R}^{N(d-2)})]$.

Now suppose $H \in C_0^\infty(\mathbb{R}^d, \mathcal{F}(\ell_N^2))$ is such that $H(x) \in S[C_0^\infty(\mathbb{R}^{N(d-2)})]$ for all x, as for $\hat{F}_1, \hat{F}_2$ in the hypotheses the theorem. Then $H(x) \in D(N)$ and one can also deduce that $(E^\pm H)(x) \in D(N)$. It follows that $(S^{-1}E^\pm H)(x) \in D(L)$. Similarly it is in $D(L^r)$ for any r and from this we can deduce that it is a smooth function.

Now we can compute

$$\begin{aligned} &(\Box - 2(L-a))S^{-1}E^\pm H \\ &= S^{-1}(\Box - 2(N-a))E^\pm H \\ &= S^{-1}H \end{aligned} \tag{102}$$

Furthermore $S^{-1}E^\pm H$ vanishes in the distant past. It follows that

$$S^{-1}E^\pm \hat{H} = \mathcal{E}^\pm(S^{-1}H) \tag{103}$$

where $\mathcal{E}^\pm$ are the fundamental retarded and advanced solutions for the operator

$$\Box - 2(L-a) = -(\frac{\partial}{\partial x^0})^2 + \sum_k (\frac{\partial}{\partial x^k})^2 + \sum_{n,k} (\frac{\partial}{\partial x_n^k})^2 - 2nx_n^k \frac{\partial}{\partial x_n^k} + 2a \tag{104}$$

This is the wave operator in $\mathbb{R}^d \times \mathbb{R}^{N(d-2)}$ perturbed by some lower order terms. But it is well known that lower order terms do not affect the domain of dependence or the domain of influence. Thus the support of $\mathcal{E}^\pm(S^{-1}H)$ is contained in the future/past of the support $S^{-1}H$ as defined by the metric

$$-(dx^0)^2 + \sum_{k=1}^{d}(dx^k)^2 + \sum_{k=1}^{d-2}\sum_{n=1}^{N}(dx_n^k)^2 \tag{105}$$

Returning now to (101) we have that the support of $S^{-1}E^\pm\hat{F}_2 = \mathcal{E}^\pm(S^{-1}\hat{F}_2)$ is contained in the future/past of the support of $S^{-1}\hat{F}_2$. Hence it vanishes on the support of $S^{-1}\hat{F}_1$ by our assumption, and so the commutator vanishes.

A Appendix

A.1 a characteristic Cauchy problem

One does not generally expect a characteristic Cauchy problem to be well-posed. However if one restricts the class of solutions, one can obtain existence and uniqueness theorems. Here we show how this obtains for the Klein-Gordon equation $(\Box - \mu)u = 0$ in $\mathbb{R}^d$ based on developments in the text. We restrict the discussion to the case $\mu > 0$.

We consider the class of regular solutions of the Klein Gordon equation, defined to be smooth functions $u(x^0, \mathbf{x})$ with data $u(0, \mathbf{x}) = f(\mathbf{x})$ and $(\partial u/\partial x^0)(0, \mathbf{x}) = g(\mathbf{x})$ in the Schwartz space $\mathcal{S}(\mathbb{R}^{d-1})$. Any such solution can be written as a sum of positive and negative frequency parts:

$$\begin{aligned} u(x^0, \mathbf{x}) &= (2\pi)^{-(d-1)/2} \int e^{-i\omega(\mathbf{p})x^0 + i\mathbf{p}\cdot\mathbf{x}} h_+(\mathbf{p}) \frac{d\mathbf{p}}{2\omega(\mathbf{p})} \\ &+ (2\pi)^{-(d-1)/2} \int e^{+i\omega(\mathbf{p})x^0 + i\mathbf{p}\cdot\mathbf{x}} h_-(\mathbf{p}) \frac{d\mathbf{p}}{2\omega(\mathbf{p})} \end{aligned} \tag{106}$$

Here $h_\pm$ are given by

$$h_\pm(\mathbf{p}) = \omega(\mathbf{p})\tilde{f}(\mathbf{p}) \pm i\tilde{g}(\mathbf{p}) \tag{107}$$

and are also in $\mathcal{S}(\mathbb{R}^{d-1})$

We also consider such solutions in light cone coordinates defined by

$$\breve{u}(x^+, x^-, \tilde{x}) = u\left(\frac{1}{\sqrt{2}}(x^+ + x^-), \tilde{x}, \frac{1}{\sqrt{2}}(x^+ - x^-)\right) \tag{108}$$

We are interested in the Cauchy problem for the light-like (characteristic) surface $x^0 = -x^{d-1}$ or $x^+ = 0$. Does the restriction $U(x^-, \tilde{x}) \equiv \breve{u}(0, x^-, \tilde{x})$ uniquely determine a solution? First we have the following:

Lemma 1 *Let u be a regular solution of $(\Box - \mu)u = 0$. Then the restriction $U(x^-, \tilde{x})$ to $x^+ = 0$ is in $\mathcal{S}(\mathbb{R}^{d-1})$. Furthermore for any multi-index α and any r there is a constant C such that the Fourier transform satisfies*

$$|\partial^\alpha \tilde{U}(p^+, \tilde{p})| \leq C|p^+|^r \tag{109}$$

Proof. We have for $U(x^-, \tilde{x})$ the expression

$$\begin{aligned} &(2\pi)^{-(d-1)/2} \int \exp(-i(\frac{\omega + p^{d-1}}{\sqrt{2}})x^-) e^{i\tilde{p}\cdot\tilde{x}} h_+(\tilde{p}, p^{d-1}) \frac{d\tilde{p}dp^{d-1}}{2\omega(\tilde{p}, p^{d-1})} \\ + \quad &(2\pi)^{-(d-1)/2} \int \exp(i(\frac{\omega + p^{d-1}}{\sqrt{2}})x^-) e^{i\tilde{p}\cdot\tilde{x}} h_-(\tilde{p}, -p^{d-1}) \frac{d\tilde{p}dp^{d-1}}{2\omega(\tilde{p}, p^{d-1})} \end{aligned} \tag{110}$$

Now make the change of variables $p^{d-1} \to p^+$ as in (21) and we find

$$\begin{aligned} &(2\pi)^{-(d-1)/2} \int \int_0^\infty e^{-ip^+x^- + i\tilde{p}\cdot\tilde{x}} h_+(\tilde{p}, \frac{1}{\sqrt{2}}(p^+ - \frac{\tilde{p}^2 + \mu}{2p^+})) \frac{dp^+ d\tilde{p}}{2p^+} \\ + \quad &(2\pi)^{-(d-1)/2} \int \int_0^\infty e^{ip^+x^- + i\tilde{p}\cdot\tilde{x}} h_-(\tilde{p}, -\frac{1}{\sqrt{2}}(p^+ - \frac{\tilde{p}^2 + \mu}{2p^+})) \frac{dp^+ d\tilde{p}}{2p^+} \end{aligned} \tag{111}$$

¿From this we identify

$$\tilde{U}(p^+,\tilde{p}) = \begin{cases} -h_+(\tilde{p}, -\frac{1}{\sqrt{2}}(p^+ - \frac{\tilde{p}^2+\mu}{2p^+}))/2p^+ & p^+ > 0 \\ 0 & p^+ = 0 \\ h_-(\tilde{p}, -\frac{1}{\sqrt{2}}(p^+ - \frac{\tilde{p}^2+\mu}{2p^+}))/2p^+ & p^+ < 0 \end{cases} \tag{112}$$

We show that $\tilde{U}$ is rapidly decreasing at infinity, and is rapidly vanishing as $p^+ \to 0$. Since $h_\pm$ are Schwartz functions we have

$$|\tilde{U}(p^+,\tilde{p})| \leq C(1 + |\tilde{p}| + |p^+ - \frac{\tilde{p}^2+\mu}{2p^+}|)^{-N}|p^+|^{-1} \tag{113}$$

for any N. If $\tilde{p}^2 + \mu \leq (p^+)^2$ we have $|p^+ - \frac{\tilde{p}^2+\mu}{2p^+}| \geq |p^+|/2$ and the rapid decrease follows from (113). If $\mu/4 \leq (p^+)^2 \leq \tilde{p}^2 + \mu$ we drop this factor and get $|\tilde{U}(p^+,\tilde{p})| \leq C(1+|\tilde{p}|)^{-N}$ which gives the rapid decrease in this region. Finally if $(p^+)^2 \leq \mu/4$ then $|p^+ - \frac{\tilde{p}^2+\mu}{2p^+}| \geq \mu/4|p^+|$ and we obtain

$$|\tilde{U}(p^+,\tilde{p})| \leq C(1+|\tilde{p}|)^{-N/2}|p^+|^{N/2-1} \tag{114}$$

This gives the rapid decrease and also establishes the bound (109) for $\alpha = 0$.

The same bounds hold for all partial derivatives of $\tilde{U}$ for $p^+ \neq 0$. In particular they all converge to zero as $p^+ \to 0$. From this one can deduce that $\tilde{U}$ is smooth. Hence $\tilde{U}$ is in $\mathcal{S}(\mathbb{R}^{d-1})$ and so is U.

This result tells us how to pose the existence and uniqueness result.

Theorem 6 *Let $g(x^-,\tilde{x}) \in \mathcal{S}(\mathbb{R}^{d-1})$ have a Fourier transform $\tilde{g}(p^+,\tilde{p})$ which satisfies (109). Then there is a unique regular solution of $(\Box - \mu)u = 0$ such that the restriction to $x^+ = 0$ satisfies $U(x^-,\tilde{x}) = g(x^-,\tilde{x})$*

Proof. Let u be such a solution with $g = 0$. Then $h^\pm = 0$ by (112) and so $u = 0$. This proves the uniqueness.

For existence we define $h_\pm$ by

$$h_\pm(\tilde{p}, p^{d-1}) = \mp 2p^+\tilde{g}(p^+,\tilde{p}) \tag{115}$$

evaluated at

$$p^+ = \frac{1}{\sqrt{2}}(\pm\omega(\tilde{p}, p^{d-1}) - p^{d-1}) \tag{116}$$

This is a smooth function and we have

$$|h_\pm(\tilde{p}, p^{d-1})| \leq C(1 + |\omega(\tilde{p}, p^{d-1}) \mp p^{d-1}| + |\tilde{p}|)^{-N} \tag{117}$$

To see it is rapidly decreasing consider several cases. For $\mp p^{d-1} \geq 0$ the above bound suffices, and if $-1 \leq \mp p^{d-1} \leq 0$ just the $(1+|\tilde{p}|)^{-N}$ suffices. Finally if $\mp p^{d-1} < -1$ we use $|\tilde{g}(p^+, \tilde{p})| \leq C|p^+|^r$ to obtain

$$\begin{aligned} |h_\pm(\tilde{p}, p^{d-1})| &\leq C|\omega(\tilde{p}, p^{d-1}) \mp p^{d-1}|^{r+1} \\ &\leq C|\frac{\tilde{p}^2+\mu}{2p^{d-1}}|^{r+1} \end{aligned} \tag{118}$$

A convex combination of this bound and a $(1+|\tilde{p}|)^{-N}$ bound yields the rapid decrease in this region. The same bounds hold for all the partial derivatives of $h_\pm$. Hence $h_\pm \in \mathcal{S}(\mathbb{R}^{d-1})$.

Now u defined by (106) is a regular solution, and we we have chosen $h_\pm$ so that $\tilde{U}$ computed in (112) is just $\tilde{g}$. Hence $U = g$.

A.2 canonical quantization for fields

We quantize the space of solutions of the Klein-Gordon equation $(\Box - \mu)u = 0$. There is a natural symplectic form on this space given by

$$\sigma(u_1, u_2) = \int_\Sigma (u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n}) d\sigma \tag{119}$$

where Σ is a space-like hypersurface with normal n_μ and and forward normal derivative $\partial/\partial n = n^\mu \partial/\partial x^\mu$. The form is independent of the choice of the surface. Quantization can be regarded as asking for a field operator ϕ such that

$$[\sigma(\phi, u_1), \sigma(\phi, u_2)] = i\sigma(u_1, u_2) \tag{120}$$

for any solutions u_1, u_2. This formulation generalizes nicely to manifolds, see for example [1]

In the standard formulation one takes the surface Σ to be $x^0 = 0$ in which case $\partial/\partial n = \partial/\partial x^0$. Solutions $u(x)$ are identified with their data $u(0, \mathbf{x}) = f(\mathbf{x})$ and $(\partial u/\partial x^0)(0, \mathbf{x}) = g(\mathbf{x})$ and the form becomes

$$\sigma(f_1, g_1; f_2, g_2) = \int (f_1(\mathbf{x})g_2(\mathbf{x}) - g_1(\mathbf{x})f_2(\mathbf{x}))d\mathbf{x} \tag{121}$$

For quantization one takes $\sigma(\phi, \pi; f, g) = \phi(g) - \pi(f)$ with the standard commutator $[\phi(f), \pi(g)] = i\int f(\mathbf{x})g(\mathbf{x})d\mathbf{x}$, and obtains the announced

$$[\sigma(\phi, \pi; f_1, g_1), \sigma(\phi, \pi; f_2, g_2)] = i\sigma(f_1, g_1; f_2, g_2) \tag{122}$$

For the light cone formulation the surface Σ is $x^+ = 0$. This is lightlike, but can be regarded as a limit of spacelike hypersurfaces. Solutions are written in light cone coordinates $\check{u}(x^+, x^-, \tilde{x})$ and are identified with their restriction $\check{u}(0, x^-, \tilde{x}) =$

$f(x^-, \tilde{x})$ to this surface. As we have seen in Appendix A.1, this is justified under certain circumstances. The forward derivative is $\partial/\partial n = \partial/\partial x^-$, and the symplectic form becomes after an integration by parts

$$\sigma(f_1, f_2) = 2\int f_1(x^-, \tilde{x})\frac{\partial f_2}{\partial x^-}(x^-, \tilde{x})dx^- d\tilde{x} \tag{123}$$

There is now no natural split into coordinates and momenta.

For quantization we seek for a field operator $\phi(x^-, \tilde{x})$ such that the smeared version $\sigma(\phi, f) = 2\phi(\partial f/\partial x^-)$ satisfies (120), that is so that

$$[\phi(\frac{\partial f_1}{\partial x^-}), \phi(\frac{\partial f_2}{\partial x^-})] = \frac{i}{2}\sigma(f_1, f_2) \tag{124}$$

A.3 existence of the light cone gauge

Let $X^\mu(\tau, \sigma)$ be an open string. Thus it satisfies the wave equation $\partial^2 X^\mu/\partial\tau^2 - \partial^2 X^\mu/\partial\sigma^2 = 0$ and the constraint $(\partial X/\partial\tau \pm \partial X/\partial\sigma)^2 = 0$ and the forward moving condition $\partial X^0/\partial\tau > 0$. We assume $X^\mu(\tau, \sigma)$ is defined on the whole plane and impose Neumann boundary conditions by requiring that as a function of σ it is even and periodic with period 2π. We investigate the circumstances under which one can find a conformal change of coordinates so that in the new coordinates $X^+(\tau, \sigma) = p^+\tau$.

We work in coordinates $u = \tau + \sigma, v = \tau - \sigma$ in terms of which the equation is $\partial^2 X^\mu/\partial u\partial v = 0$ and the constraint is $(\partial X/\partial u)^2 = (\partial X/\partial v)^2 = 0$. The general solution of the equation has the form $X^\mu(u, v) = f^\mu(u) + g^\mu(v)$. or in the original coordinates $X^\mu(\tau, \sigma) = f^\mu(\tau + \sigma) + g^\mu(\tau - \sigma)$. The Neumann boundary conditions are equivalent to the condition that $f^\mu = g^\mu$ up to a constant (and we may as well take them equal), and that f' is periodic with period 2π. The latter condition implies that

$$f^\mu(u) = a^\mu u + h^\mu(u) \tag{125}$$

where h^μ is periodic with period 2π and

$$a^\mu = (2\pi)^{-1}\int_{-\pi}^{\pi}(f^\mu)'(s)ds = (2\pi)^{-1}(f^\mu(\pi) - f^\mu(-\pi)) \tag{126}$$

A short calculation shows that $2a^\mu = p^\mu$.

The constraint equation becomes $(f')^2 = 0$ and the forward moving condition becomes $(f^0)' > 0$. Thus we have

$$(f^0)' = \left(\sum_{k=1}^{d-1}((f^k)')^2\right)^{1/2} \tag{127}$$

It follows that

$$(f^0)' \geq |(f^{d-1})'| \tag{128}$$

and hence that

$$(f^+)' \geq 0 \tag{129}$$

We make the further mild restriction that

$$(f^+)' > 0 \tag{130}$$

This is a condition on initial data. It implies that $a^+ > 0$.

Now we can state:

Theorem 7 *Let X be an open string so that*

$$X^\mu(\tau,\sigma) = f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma) \tag{131}$$

with f' smooth, f' periodic, $(f')^2 = 0$, and $(f^0)' > 0$. If also $(f^+)' > 0$ then there is a conformal diffeomorphism $(\tau,\sigma) \to (\tilde{\tau},\tilde{\sigma})$ such that in the new coordinates

$$\tilde{X}^\mu(\tilde{\tau},\tilde{\sigma}) = \tilde{f}^\mu(\tilde{\tau}+\tilde{\sigma}) + \tilde{f}^\mu(\tilde{\tau}-\tilde{\sigma}) \tag{132}$$

where $\tilde{f}$ satisfies the same conditions and in addition

$$\tilde{X}^+(\tilde{\tau},\tilde{\sigma}) = p^+\tilde{\tau} \tag{133}$$

Proof. The condition (130) and the periodicity imply that $(f^+)' > \epsilon > 0$, and it follows that $\tilde{u} = \alpha(u) \equiv (a^+)^{-1} f^+(u)$ is a diffeomorphism on $\mathbb{R}$. With this coordinate change we define

$$\tilde{f}^\mu(\tilde{u}) = f^\mu(\alpha^{-1}(\tilde{u})) = f^\mu((f^+)^{-1}(a^+\tilde{u})) \tag{134}$$

Then $\tilde{f}^+(\tilde{u}) = a^+\tilde{u}$

Now $\tilde{u} = \alpha(u), \tilde{v} = \alpha(v)$ is a conformal diffeomorphism on $\mathbb{R}^2$ with the Minkowski metric $dudv$. Then $X^\mu(u,v) = f^\mu(u) + f^\mu(v)$ becomes in the new coordinates

$$\tilde{X}^\mu(\tilde{u},\tilde{v}) = \tilde{f}^\mu(\tilde{u}) + \tilde{f}^\mu(\tilde{v}) \tag{135}$$

Hence

$$\tilde{X}^+(\tilde{u},\tilde{v}) = a^+(\tilde{u}+\tilde{v}) \tag{136}$$

and introducing $\tilde{\tau} = (\tilde{u}+\tilde{v})/2$ and $\tilde{\sigma} = (\tilde{u}-\tilde{v})/2$ we get that

$$\tilde{X}^+(\tilde{\tau},\tilde{\sigma}) = 2a^+\tilde{\tau} = p^+\tilde{\tau} \tag{137}$$

as desired.

It remains to show that $\tilde{f}$ has same properties as f. To see that $\tilde{f}'$ is periodic note that $\alpha(u+2\pi) = \alpha(u)+2\pi$ hence $\alpha^{-1}(\tilde{u}+2\pi) = \alpha^{-1}(\tilde{u})+2\pi$ and the periodicity follows from

$$(\tilde{f}^\mu)'(\tilde{u}) = \frac{(f^\mu)'(\alpha^{-1}(\tilde{u}))}{\alpha'(\alpha^{-1}(\tilde{u}))} \tag{138}$$

The other conditions follow as well.

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