The Fuzzy Supersphere

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Abstract

We introduce the fuzzy supersphere as sequence of finite-dimensional, noncommutative $\mathbb{Z}_2$-graded algebras tending in a suitable limit to a dense subalgebra of the $\mathbb{Z}_2$-graded algebra of $\mathcal{H}^\infty$-functions on the $(2|2)$-dimensional supersphere. Noncommutative analogues of the body map (to the (fuzzy) sphere) and the super-deRham complex are introduced. In particular we reproduce the equality of the super-deRham cohomology of the supersphere and the ordinary deRham cohomology of its body on the "fuzzy level".

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1 Introduction

Thinking about space and time noncommutative geometry [7] offers an enormous general framework for physical model building, because one can get rid of the concept of points. The basic idea of noncommutative geometry is to formulate first notions on differentiable manifolds in terms of their commutative $\mathbb{C}$-algebras of differentiable, complex-valued functions in order to generalize subsequently that notions, which do not depend on the commutativity to abstract, not necessarily commutative algebras. So in general one will lose the notion of points (corresponding in ordinary geometry to the spectrum of the commutative algebra of functions) and the role of general, noncommutative manifolds is played by abstract algebras. Of course there is no canonical way how to associate a noncommutative algebra with some

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mathematical or physical model of spacetime, phase space or some more "exotic" objects and one can contrive lots of different procedures. Beside that of fuzzy manifolds [39],[24],[23],[26], which is intimately related to quantization and which we will follow in the sequel, let us also mention (without being complete) a similar approach for the Minkowski space [12], quantum group motivated approaches (see for example [40]) and approaches based on posets (see for example [1]). Fuzzy manifolds are not only $\mathbb{C}$-algebras but whole sequences (or more precisely directed systems) of noncommutative $\mathbb{C}$-algebras, which approximate in a very specific way the corresponding ordinary manifolds. Each of the $\mathbb{C}$-algebras of such a sequence can be interpreted as description of the corresponding ordinary manifold on which localization is possible only up to a minimal length. By employing the tools of noncommutative geometry and of matrix geometry [13] in particular, it was possible to introduce on a specific fuzzy manifold, namely the fuzzy sphere, (sections of) vector bundles, a differential calculus, an integral, etc. and to use this for the formulation of field theoretical models [39],[38],[21],[22],[25],[34]. When these models are quantized a very interesting feature shows up: They are finite and the fuzziness plays the role of a non-perturbative regulator, which does not break the characteristic symmetries of the corresponding continuum theories.
Although supermanifolds are to some extend "baby-noncommutative geometries" they are treated and interpreted in the spirit of classical differential and algebraic geometry. Localization is (depending on the approach to supermanifolds) more or less present and the term "super" should rather be seen as additional structure. Noncommutative generalizations should be described by $\mathbb{Z}_2$-graded algebras over a ring, that depends on the class of supermanifolds under consideration. We want to mention, that there exist already several articles and books in the literature dealing with various aspects of $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras, supersymmetry and noncommutative geometry. Without being complete let us just mention [31], where notions as cyclic cohomology and Fredholm modules are treated in the $\mathbb{Z}_2$-graded setting, [14],[33], where a possibility of generalizing matrix geometry to the $\mathbb{Z}_2$-graded framework is presented and [32], where the concept of a spectral triple is extended to algebras which contain bosonic and fermionic degrees of freedom.
Our aim in this article is to develop first a "fuzzy variant" of the $(2|2)$-dimensional supersphere and subsequently an analogue to the super-deRham complex on each of the resulting $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras. Motivated by the wish to find an adequate language for the description of the spinor bundle on the fuzzy sphere one of the authors together with C.Klimčik and P.Prešnajder already solved the first problem [21]. Here we want to embed the description given there a little bit more in the language of supermanifolds in the sense of A.Rogers [47],[48] as well as that of graded manifolds [35],[36] in order to have a good guideline for the development of a $\mathbb{Z}_2$-graded differential calculus later on.

More precisely the article is organized as follows. In the first two chapters we describe the $(2|2)$-dimensional supersphere as $\mathcal{H}^\infty$-deWitt super- respectively graded manifold and establish on its $\mathbb{Z}_2$-graded algebra of complex-Grassmann-valued functions additional structures such as a Fréchet topology, an indefinite scalar product and a "completely reducible" grade star representation of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. The article is written

as self-contained as possible, but for the basics of the theory of supermanifolds we refer the reader to the above cited original articles as well as to the excellent book [2].
In chapter 4 we endow each element of a specific sequence of submodules of the $\mathfrak{osp}(1|2)$-module of complex-Grassmann-valued $\mathcal{H}^\infty$-functions on the $(2|2)$-dimensional supersphere with a new $\mathbb{Z}_2$-graded product and define by this the fuzzy supersphere. The graded-commutative limit of these products is proven. The body of the $(2|2)$-dimensional supersphere is the ordinary 2-sphere and chapter 5 is devoted to extend the corresponding body map to the fuzzy setting. In chapter 6 we define on each of the "truncated superspheres" (in the sense explained above) an analogue to the super-deRham complex by transposing the idea of derivation-based differential calculi [13],[16] and its extension used for the definition of a differential calculus on the fuzzy sphere [39],[37] to the $\mathbb{Z}_2$-graded setting. The resulting complex is nothing else but the complex of Lie superalgebras with coefficients in a non-trivial $\mathfrak{osp}(1|2)$-module and it is infinite-dimensional, as it is usual for supermanifolds. The latter fact shows in particular, that the complex is completely different to that proposed in [14],[33]. Our construction is natural in the sense, that the noncommutative body map extends - as it is the case for graded manifolds - to a cochain map from the algebra of superforms on the truncated supersphere to the algebra of forms on its body.
The final chapter is devoted to the calculation of the cohomology corresponding with the differential complex. In particular we reproduce the equality of the super-deRham cohomology of the supersphere and the ordinary deRham cohomology of its body on the "fuzzy level".

Let us finally mention some conventions which are used throughout the article. When we speak of algebras we always mean associative algebras with identity; algebra homomorphisms always preserve the identity. Left or right modules over an algebra are always assumed to be unital.
There will appear lots of $\mathbb{Z}_2$-graded objects. If the object is denoted by $\mathcal{O}$ its even part is denoted by $\mathcal{O}_{\overline{0}}$, its odd part by $\mathcal{O}_{\overline{1}}$. If e is some homogeneous element of such an object its degree will be denoted by $\overline{e}$. Speaking of grading in the context of an ungraded object we mean, that the object is endowed with its trivial graduation. Moreover a left or right $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded, graded-commutative algebra is always assumed to be given its canonical $\mathbb{Z}_2$-graded bimodule structure.
There is a one-to-one correspondence between representations of a Lie (super)algebra and its universal enveloping algebra: We do not distinguish between these representations and pass freely from the language of representations to the language of modules and vice versa.

2 The (2|2)-dimensional supersphere

In this preliminary section we introduce the $(2|2)$-dimensional supersphere as $\mathcal{H}^\infty$-deWitt supermanifold (and by this also as graded manifold) and characterize the $\mathbb{Z}_2$-graded algebra of $\mathcal{H}^\infty$-functions on it as a suitable quotient of the algebra of $\mathcal{H}^\infty$-functions on the $(3|2)$-dimensional vector superspace. Moreover we endow this algebra with additional structures, such as a Fréchet topology and a indefinite scalar product, which we will need later on for the formulation and proof of the "supercommutative limit" of the fuzzy supersphere.

For $L \in \mathbb{N}$ let B_L denote the Grassmann algebra over $\mathbb{R}^L$ and C_L the Grassmann algebra over $\mathbb{C}^L$. We view both of them as $\mathbb{Z}_2$-graded algebras. Furthermore we introduce the set

$$\mathfrak{I}_L := \{M = (i_1, \cdots, i_p) |\ i_1, \cdots, i_p = 1, \cdots, L;\ 1 \leq p \leq L \text{ with } i_1 < \cdots < i_p\} \cup \{\emptyset\}\ . \tag{1}$$

If $\{e_i\}_{i=1,\cdots,L}$ is a basis of $\mathbb{R}^L$ (or $\mathbb{C}^L$) then a homogeneous basis of B_L (or C_L) is formed by the elements

$$\begin{aligned} e_M &:= e_{i_1} e_{i_2} \cdots e_{i_p} \quad \text{with} \quad M = (i_1, i_2, \cdots, i_p) \in \mathfrak{I}_L \\ e_\emptyset &:= 1 \end{aligned} \tag{2}$$

and

$$\|y\| \equiv \left\| \sum_{M \in \mathfrak{I}_L} y_M e_M \right\| := \sum_{M \in \mathfrak{I}_L} |y_M| \tag{3}$$

for every $y \in B_L(C_L)$, defines a norm on B_L (or C_L). The $\mathbb{R}$-linear map $B_L \longrightarrow C_L$, defined by identifying the basis elements (2) of B_L with the corresponding basis elements of C_L, is an injective isometric homomorphism of $\mathbb{Z}_2$-graded $\mathbb{R}$-Banach algebras, which we will use to interpret B_L as closed and graded $\mathbb{R}$-subalgebra of C_L. Consequently the elements of B_L are those elements of C_L, which are invariant with respect to complex conjugation $* : C_L \longrightarrow C_L$, defined by

$$y^* \equiv \left(\sum_{M \in \mathfrak{I}_n} y_M e_M \right)^* := \sum_{M \in \mathfrak{I}_n} y_M^* e_M\,. \tag{4}$$

The direct sum of $B_{L,\overline{0}}$-modules

$$B_L^{r|s} := B_{L,\overline{0}} \oplus \overset{r}{\cdots} \oplus B_{L,\overline{0}} \oplus B_{L,\overline{1}} \oplus \overset{s}{\cdots} \oplus B_{L,\overline{1}}\,, \qquad r, s \in \mathbb{N}_0\,, \tag{5}$$

together with the $\mathbb{Z}_2$-grading

$$\begin{aligned} B_{L,\overline{0}}^{r|s} &:= B_{L,\overline{0}} \oplus \overset{r}{\cdots} \oplus B_{L,\overline{0}} \equiv B_L^{r|0} \\ B_{L,\overline{1}}^{r|s} &:= B_{L,\overline{1}} \oplus \overset{s}{\cdots} \oplus B_{L,\overline{1}} \equiv B_L^{0|s} \end{aligned} \tag{6}$$

and the norm

$$\left\|(x^k, \theta^\alpha)\right\| := \sum_{k=1}^{r} \left\|x^k\right\| + \sum_{\alpha=r+1}^{r+s} \|\theta^\alpha\|\ , \qquad (x^k, \theta^\alpha) \in B_L^{r|s}\,, \tag{7}$$

is called $(r|s)$-dimensional real vector superspace.
The body (or augmentation) map on C_L will be denoted by ϵ; the body map $B_L^{r|s} \longrightarrow \mathbb{R}^r$, $(x^k, \theta^\alpha) \mapsto (\epsilon(x^k))$ by $\Phi_{r|s}$. Beside the topology induced by the norm (7), the so-called fine topology, there is another important topology on the vector superspace: the coarse or deWitt-topology. By definition it is the coarsest topology on $B_L^{r|s}$ such that the body map $\Phi_{r|s}$ is continuous.

For an arbitrary $\rho \in \mathbb{R}^+$ we define the (2|2)-dimensional supersphere S_ρ of "radius" ρ as the closed topological subspace (with respect to the fine topology) of all points of $B_L^{3|2}$ fulfilling

$$P_\rho(x^k, \theta^\alpha) := \sum_{i=1}^{3} (x^i)^2 + 2\theta^4\theta^5 - \rho^2 = 0\,. \tag{8}$$

S_ρ can be endowed with the structure of a $(2|2)$-dimensional $\mathcal{H}^\infty$-deWitt supermanifold (see also [3],[5],[9],[53]). In order to do so one introduces the "north" and "south pole-ϵ-fibers"

$$F_\pm := \Big\{(x^k,\theta^\alpha) \in S_\rho |\ \Phi_{3|2}(x^k,\theta^\alpha) = (0,0,\pm\rho)\Big\} \tag{9}$$

as well as their open complements

$$U_\pm := S_\rho \backslash F_\pm \tag{10}$$

in S_ρ. Then the superstereographic projections

$$\begin{aligned} h_\pm : U_\pm &\longrightarrow B_L^{2|2} \\ (x^k,\theta^\alpha) &\mapsto \frac{\rho}{\rho \mp x^3}(x^1,x^2,\theta^4,\theta^5) \end{aligned} \tag{11}$$

form a $\mathcal{H}^\infty$-deWitt atlas on S_ρ, because the transition functions

$$h_\pm \circ h_\mp^{-1}(y^j,\eta^\alpha) = \frac{\rho^2}{(y^1)^2+(y^2)^2+2\eta^3\eta^4}(y^j,\eta^\alpha) \tag{12}$$

are $\mathcal{H}^\infty$-functions $B_L^{2|2}\backslash\Phi_{2|2}^{-1}(0,0) \longrightarrow B_L^{2|2}\backslash\Phi_{2|2}^{-1}(0,0)$ on the one hand and $h_\pm(U_\pm)$ are coarse open on the other hand. The corresponding ($\mathcal{C}^\infty$-)body manifold can be identified canonically with the 2-dimensional sphere S_ρ of radius ρ embedded in $\mathbb{R}^3$ and the body projection Φ_S is then simply given by

$$\Phi_S = \Phi_{3|2}\Big|_{S_\rho}. \tag{13}$$

Let us denote the sheaf of real- and complex Grassmann-valued $\mathcal{H}^\infty$-functions on a $\mathcal{H}^\infty$-supermanifold X by $\mathcal{H}^\infty(-,B_L)$ and $\mathcal{H}^\infty(-,C_L)$, respectively. We view $\mathcal{H}^\infty(-,B_L)$ as subsheaf of $\mathcal{H}^\infty(-,C_L)$ in the natural way. That is, for every open subset U of the supermanifold X we define complex conjugation $* : \mathcal{H}^\infty(U,C_L) \longrightarrow \mathcal{H}^\infty(U,C_L)$ pointwise and characterize $\mathcal{H}^\infty(U,B_L)$ as the graded $\mathbb{R}$-subalgebra of $*$-invariant elements. It should be noted, that, if X is a $\mathcal{H}^\infty$-deWitt supermanifold with body X and body projection Φ_X, the pair $(\mathrm{X},\Phi_{X*}\mathcal{H}^\infty(-,B_L))$, where $\Phi_{X*}\mathcal{H}^\infty(-,B_L)$ denotes the direct image of $\mathcal{H}^\infty(-,B_L)$ by Φ_X, is a graded manifold [2]. Consequently the map β_X from the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra $\mathcal{H}^\infty(X,C_L)$ to the (trivially $\mathbb{Z}_2$-graded) $\mathbb{C}$-algebra $\mathcal{C}^\infty(\mathrm{X},\mathbb{C})$ of smooth complex-valued functions on X, defined by

$$\left(\beta_X(f)\right)\left(\Phi_X(x)\right) := \epsilon \circ f(x) \tag{14}$$

for all $x \in X$, is a surjective homomorphism of graded algebras.

The set

$$\mathcal{I}_{S_\rho} := \left\{\ f \in \mathcal{H}^\infty(B_L^{3|2},C_L)\ \middle|\ f\Big|_{S_\rho} = 0\right\} \tag{15}$$

of all complex-Grassmann-valued functions on $B_L^{3|2}$ vanishing on the supersphere is a graded ideal in $\mathcal{H}^\infty(B_L^{3|2},C_L)$. We will identify $\mathcal{H}^\infty(S_\rho,C_L)$ and $\mathcal{H}^\infty(B_L^{3|2},C_L)/\mathcal{I}_{S_\rho}$ according to the first part of the following result.

Lemma 1 *The map* $\chi : \mathcal{H}^\infty(B_L^{3|2},C_L)/\mathcal{I}_{S_\rho} \longrightarrow \mathcal{H}^\infty(S_\rho,C_L)$*, defined by*

$$\chi(\boldsymbol{f}) := f\Big|_{S_\rho}, \quad f \in \boldsymbol{f}, \tag{16}$$

is an isomorphism of $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras. Moreover, for every $f \in \mathcal{I}_{S_\rho}$ there exists a $\mathcal{H}^\infty$-function $g \in \mathcal{H}^\infty(B_L^{3|2}, C_L)$, such that

$$f = P_\rho g \tag{17}$$

is fulfilled.

Proof: Analogous to (9),(10) one introduces "north" and "south pole-ϵ-fibers" $\tilde{F}_\pm := \Phi_{3|2}^{-1}(0,0,\mathbb{R}_0^\pm)$ as well as their open complements $\tilde{U}_\pm := B_L^{3|2} \setminus \tilde{F}_\pm$ in $B_L^{3|2}$. Then the maps

$$\begin{aligned} \tilde{h}_\pm \quad : \quad & \tilde{U}_\pm \longrightarrow \Phi_{3|2}^{-1}(\mathbb{R}^2 \times \mathbb{R}^+) \\ & (x^k, \theta^\alpha) \mapsto \frac{\sqrt{\sum_{i=1}^3 (x^i)^2 + 2\theta^4\theta^5}}{\sqrt{\sum_{i=1}^3 (x^i)^2 + 2\theta^4\theta^5} \mp x^3} \cdot \\ & \qquad \cdot (x^1, x^2, \left(\sqrt{\sum_{i=1}^3 (x^i)^2 + 2\theta^4\theta^5} \mp x^3\right) \sqrt{\sum_{i=1}^3 (x^i)^2 + 2\theta^4\theta^5}, \theta^4, \theta^5)\,, \end{aligned} \tag{18}$$

where the square root is defined via the Z-expansion of the ordinary square root, are "subsupermanifold charts" of the vector superspace, which one can use to conclude $f|_{S_\rho} \in \mathcal{H}^\infty(S_\rho, C_L)$. Obviously χ is an injective, even homomorphism of $\mathbb{Z}_2$-graded algebras and the surjectivity of χ follows from the existence of $\mathcal{H}^\infty$-partitions of the unity on coarse open coverings of $B_L^{3|2}$ [2].

Using once again such a coarse $\mathcal{H}^\infty$-partitions of the unity one can conclude, that it is enough to show (17) on $\tilde{U}_\pm$. The next step is to note, that the condition $f|_{\tilde{U}_\pm \cap S_\rho} = 0$ means $f \circ \tilde{h}_\pm^{-1}|_{y^3=\rho^2} = 0$ for the functions $f \circ \tilde{h}_\pm^{-1} \in \mathcal{H}^\infty(\Phi_{3|2}^{-1}(\mathbb{R}^2 \times \mathbb{R}^+), C_L)$ and that it is enough to prove

$$f \circ \tilde{h}_\pm^{-1} = \left(y^3 - \rho^2\right) g_\pm\,, \quad g_\pm \in \mathcal{H}^\infty(\Phi_{3|2}^{-1}(\mathbb{R}^2 \times \mathbb{R}^+), C_L)\,.$$

But if one exploits the properties of the "superfield" and Z-expansion one reduces the above problem to the same problem on the "$\mathcal{C}^\infty$-level", which can be solved in the usual way (see [2],[6]) via integration. □

Analogous to the case of the supersphere we can identify the $\mathbb{C}$-algebra $\mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$ with the $\mathbb{C}$-algebra $\mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})$ factored by the ideal $\mathcal{I}_{\mathrm{S}_\rho}$ of all $\mathcal{C}^\infty$-functions on $\mathbb{R}^3$ vanishing on the ordinary 2-sphere S_ρ. Because of (13) and

$$\beta_{3|2}\left(\mathcal{I}_{S_\rho}\right) \subseteq \mathcal{I}_{\mathrm{S}_\rho}\,, \tag{19}$$

the "algebraic body map" $\beta_{S_\rho} : \mathcal{H}^\infty(S_\rho, C_L) \longrightarrow \mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$ is simply determined by

$$\beta_{S_\rho}(\boldsymbol{f}) = \mathbf{f}\,, \tag{20}$$

where we introduced the notation $\mathbf{f}$ for the equivalence class of $\beta_{3|2}(f)$ for some $f \in \boldsymbol{f}$.

In the spirit of the theory of graded manifolds we introduce the set $\mathfrak{V}^g(X)$ of complex global supervector fields on a $\mathcal{H}^\infty$-deWitt supermanifold X as the set of graded derivations $\mathfrak{Der}_{\mathbb{C}}^g(\mathcal{H}^\infty(X, C_L))$ of the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra $\mathcal{H}^\infty(X, C_L)$. $\mathfrak{V}^g(X)$ forms in a natural way a

$\mathbb{C}$-Lie superalgebra as well as a $\mathbb{Z}_2$-graded $\mathcal{H}^\infty(X, C_L)$-module. In addition one should note, that there is a surjective Lie algebra homomorphism $\tilde{\beta}_X$ from $\mathfrak{V}^g(X)_{\overline{0}}$ to the $\mathbb{C}$-Lie algebra $\mathfrak{V}(\mathrm{X})$ of complex global vector fields on the body manifold X given by

$$\tilde{\beta}_X(D)\beta_X(f) := \beta_X(Df) \tag{21}$$

for all $D \in \mathfrak{V}^g(X)_{\overline{0}}$ and all $f \in \mathcal{H}^\infty(X, C_L)$.
Now let us in particular denote the partial derivatives of $\mathcal{H}^\infty$-functions on the vector superspace $B_L^{3|2}$ by $\partial/\partial x^k$ in the case of even coordinates and by $\partial/\partial\theta^\alpha$ in the case of odd coordinates. If $N \in \mathbb{N}_0^3$ is a multiindex and $K \subseteq B_L^{3|2}$ is compact then

$$|f|_{K,n}^{3|2} := \max_{\substack{(x^k,\theta^\alpha)\in K \\ |N|\le n, M\in\mathfrak{I}_2}} \left\| \left(\left(\frac{\partial}{\partial x}\right)^N \left(\frac{\partial}{\partial\theta}\right)^M f \right)(x^k, \theta^\alpha) \right\| \tag{22}$$

for all $f \in \mathcal{H}^\infty(B_L^{3|2}, C_L)$, where $|N|$ denotes the length of the multiindex N and we used the standard notation for partial derivatives of higher order, defines a seminorm on $\mathcal{H}^\infty(B_L^{3|2}, C_L)$. The family of all seminorms for all compact subsets $K \subseteq B_L^{3|2}$ and all natural numbers $n \in \mathbb{N}_0$ induce a locally convex topology on $\mathcal{H}^\infty(B_L^{3|2}, C_L)$ such that it becomes a $\mathbb{Z}_2$-graded $\mathbb{C}$-Fréchet algebra [2]. For our later considerations it is important to note, that the subset $\mathcal{P}(B_L^{3|2}) \subseteq \mathcal{H}^\infty(B_L^{3|2}, C_L)$ of all polynomials in the coordinate projections with complex coefficients form a dense graded subalgebra of $\mathcal{H}^\infty(B_L^{3|2}, C_L)$ [2],[35],[54].
The graded ideal $\mathcal{I}_{S_\rho}$ is closed in $\mathcal{H}^\infty(B_L^{3|2}, C_L)$ and consequently $\mathcal{H}^\infty(S_\rho, C_L)$, endowed with the quotient topology, is also a $\mathbb{Z}_2$-graded $\mathbb{C}$-Fréchet algebra. The topology is again induced by a family of seminorms, which are given explicitely by

$$|\boldsymbol{f}|_{K,n}^{S} := \inf_{f\in\boldsymbol{f}} |f|_{K,n}^{3|2}\ , \qquad \boldsymbol{f} \in \mathcal{H}^\infty(S_\rho, C_L)\,. \tag{23}$$

The set of all equivalence classes $\boldsymbol{f} \in \mathcal{H}^\infty(S_\rho, C_L)$ having a polynomial representant forms a dense graded subalgebra of $\mathcal{H}^\infty(S_\rho, C_L)$, which we denote by $\mathcal{P}(S_\rho)$.
$\mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})$ and $\mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$ can be endowed with a $\mathbb{C}$-Fréchet algebra structure in an analogous way. Then the "algebraic body maps" $\beta_{3|2}$ and β_{S_ρ} are continuous and the subalgebra $\mathcal{P}(\mathbb{R}^3)$ of polynomials in the coordinate projections with complex coefficients as well as its image $\mathcal{P}(S_\rho)$ in $\mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$ form dense subalgebras in $\mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})$ and $\mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$, respectively.

Every $f \in \mathcal{H}^\infty(B_L^{3|2}, C_L)$ has by definition a unique "superfield expansion"

$$f = f_\emptyset + f_4\theta^4 + f_5\theta^5 + f_{45}\theta^4\theta^5 \equiv \sum_{M\in\mathfrak{I}_2} f_M\theta^M\,. \tag{24}$$

Inspired by the correspondence of delta functions and Fourier transformation and the rules of Berezin integration [6],[11] we define

$$\int dx d\theta\, f(x^k,\theta^\alpha)\,\delta\left(\sum_{i=1}^{3}(x^i)^2 + 2\theta^4\theta^5 - \rho^2\right) := \tag{25}$$
$$= \frac{1}{2}\int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi \left\{ \frac{\partial \mathrm{f}_\emptyset}{\partial\rho}(\rho,\vartheta,\varphi) + \frac{1}{\rho}\mathrm{f}_\emptyset(\rho,\vartheta,\varphi) - \rho\mathrm{f}_{45}(\rho,\vartheta,\varphi) \right\}\,,$$

for all $f \in \mathcal{H}^\infty(B_L^{3|2}, C_L)$, where $\mathrm{f}_M \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})$ are the images of $f_M \in \mathcal{H}^\infty(B_L^{3|2}, C_L)$ with respect to the map (14) expressed in spherical coordinates. (25) induces a continuous, even $\mathbb{C}$-linear map $\mathcal{H}^\infty(B_L^{3|2}, C_L) \longrightarrow \mathbb{C}$, which vanishes on $\mathcal{I}_{S_\rho}$ according to the second part of lemma 1. Consequently the map $I : \mathcal{H}^\infty(S_\rho, C_L) \longrightarrow \mathbb{C}$,

$$I(\boldsymbol{f}) := \int dx d\theta f(x^k, \theta^\alpha) \delta \left(\sum_{i=1}^{3} (x^i)^2 + 2\theta^4\theta^5 - \rho^2 \right) , \qquad f \in \boldsymbol{f} , \tag{26}$$

is well-defined and again continuous, even and $\mathbb{C}$-linear. In order to introduce an indefinite scalar product we define a second "involution" (besides $\cdot^*$) $\cdot^\times : \mathcal{H}^\infty(B_L^{3|2}, C_L) \longrightarrow \mathcal{H}^\infty(B_L^{3|2}, C_L)$ via

$$f^\times \equiv \left(\sum_{M \in \mathfrak{I}_2} f_M \theta^M \right)^\times := f_\emptyset^* + f_4^* \theta^5 - f_5^* \theta^4 + f_{45}^* \theta^4 \theta^5 . \tag{27}$$

Apparently (27) is antilinear and fulfills

$$\begin{aligned} (fg)^\times &= (-1)^{\overline{f}\overline{g}} g^\times f^\times \\ (f^\times)^\times &= (-1)^{\overline{f}} f \end{aligned} \tag{28}$$

for homogeneous $f, g \in \mathcal{H}^\infty(B_L^{3|2}, C_L)$. Because $\cdot^\times$ leaves $\mathcal{I}_{S_\rho}$ invariant, $\cdot^\times$ induces the same kind of "involution" on $\mathcal{H}^\infty(S_\rho, C_L)$, which we again denote by $\cdot^\times$. Using (26) as well as $\cdot^\times$ we can define an even sesquilinear form on $\mathcal{H}^\infty(S_\rho, C_L)$ by

$$\langle \boldsymbol{f} \,|\, \boldsymbol{g} \rangle := \frac{\rho}{2\pi} I(\boldsymbol{f}^\times \boldsymbol{g}) , \tag{29}$$

where the normalization has been chosen such that $\langle \mathbf{1} \,|\, \mathbf{1} \rangle = 1$. It is continuous in both entries, non-degenerate and fulfills

$$\langle \boldsymbol{f} \,|\, \boldsymbol{g} \rangle = \langle \boldsymbol{g} \,|\, \boldsymbol{f} \rangle^* . \tag{30}$$

3 Action of the (1|2)-dimensional orthosymplectic Lie superalgebra

Beside the algebra of functions on a manifold itself there is another important ingredient for the definition of fuzzy manifolds [24],[26],[39]: The action of a Lie group respectively a Lie algebra on the algebra of functions. There are natural graded representations of the (1|2)-dimensional orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ on the $\mathbb{Z}_2$-algebras of $\mathcal{H}^\infty$-functions on the (3|2)-dimensional vector superspace as well as on the (2|2)-dimensional supersphere. They can be seen as the super-generalizations of the actions of the ordinary angular momentum on the algebras of $\mathcal{C}^\infty$-functions on the respective bodies [10]. These graded representations are reducible. We give their reduction into irreducible subspaces and introduce super-analogues of the spherical harmonics. It should be remarked, that the above mentioned reduction can be found mutatis mutandis also in [21] and that superspherical harmonics were studied in a different framework in [10].

Before analyzing the above mentioned infinite-dimensional graded representations of the orthosymplectic Lie superalgebra let us first review some well-known facts of this Lie superalgebra and its finite-dimensional, irreducible graded representations in order to fix notations and conventions.
The (complex) orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ is the $(3|2)$-dimensional $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space spanned by three even basis elements $J_k, k = 1, 2, 3$, and two odd basis elements $J_\alpha, \alpha = 4, 5$, together with the graded Lie bracket defined by

$$\begin{aligned} [J_i, J_j]_g &:= i\sum_{k=1}^{3} \varepsilon_{ijk} J_k \\ [J_i, J_\alpha]_g &:= \frac{1}{2}\sum_{\beta=4}^{5} (\sigma_i)_{\beta\alpha} J_\beta \\ [J_\alpha, J_\beta]_g &:= \frac{1}{2}\sum_{i=1}^{3} (i\sigma_2\sigma_i)_{\alpha\beta} J_i\,, \end{aligned} \tag{31}$$

where σ_k are the Pauli matrices and ε_{ijk} is 3-dimensional permutation symbol. It is a basic classical simple Lie superalgebra of type I [8], whose even part is isomorphic to the 2-dimensional complex special linear Lie algebra $\mathfrak{sl}(2)$ according to (31). The triangular decomposition

$$\mathfrak{osp}(1|2) = \mathfrak{N}^- \oplus \mathfrak{H} \oplus \mathfrak{N}^+ \tag{32}$$

of $\mathfrak{osp}(1|2)$, where $\mathfrak{H}$ is the Cartan subalgebra and $\mathfrak{N}^\pm$ are the nilpotent graded Lie subsuperalgebras of $\mathfrak{osp}(1|2)$ corresponding with the positive respectively negative roots, is chosen - as usual [4],[43] - according to

$$\begin{aligned} \mathfrak{H} &:= \mathrm{span}_{\mathbb{C}}\,\{J_3\} \\ \mathfrak{N}^- &:= \mathrm{span}_{\mathbb{C}}\,\{J_-, J_5\} \\ \mathfrak{N}^+ &:= \mathrm{span}_{\mathbb{C}}\,\{J_+, J_4\} \end{aligned} \tag{33}$$

with

$$J_\pm := J_1 \pm iJ_2\,. \tag{34}$$

Furthermore there exist two essentially different grade adjoint operations $\cdot^{\ddagger_\lambda}, \lambda = 0, 1$, on $\mathfrak{osp}(1|2)$ [4],[42],[43], which correspond, when restricted to the even part of $\mathfrak{osp}(1|2)$, with the compact real form of $\mathfrak{osp}(1|2)_{\overline{0}}$. Explicitely they are given by

$$\begin{aligned} J_i^{\ddagger_\lambda} &:= J_i \\ J_4^{\ddagger_\lambda} &:= (-1)^\lambda J_5 \\ J_5^{\ddagger_\lambda} &:= (-1)^{\lambda+1} J_4\,. \end{aligned} \tag{35}$$

All irreducible graded representations of $\mathfrak{osp}(1|2)$ on finite-dimensional $\mathbb{Z}_2$-graded $\mathbb{C}$-vector spaces are highest weight modules specified by a highest "weight" $j \in \frac{1}{2}\mathbb{N}_0$, the so-called superspin, and a definite degree $\overline{\varsigma} \in \mathbb{Z}_2$ of the unique, 1-dimensional highest weight space. Between two irreducible, finite-dimensional graded representations of $\mathfrak{osp}(1|2)$ with the same highest weight j and the same degree $\overline{\varsigma}$ of the highest weight space there exists an isomorphism of graded $\mathfrak{osp}(1|2)$-representations, determined by the requirement, that the highest weight vectors are mapped onto each other [29],[30].

For fixed $j \in \frac{1}{2}\mathbb{N}_0$ and $\overline{\varsigma} \in \mathbb{Z}_2$ let us denote the corresponding irreducible graded representation by $\cdot^{(j)}$, the finite-dimensional representation space by $V(j,\overline{\varsigma})$ and by $e^{(j)}_{j,j,\overline{\varsigma}} \in V(j,\overline{\varsigma})_{\overline{\varsigma}}$ the highest weight vector, specified (up to a complex constant) via

$$\begin{aligned} J_3^{(j)} e^{(j)}_{j,j,\overline{\varsigma}} &= j e^{(j)}_{j,j,\overline{\varsigma}} \\ J_+^{(j)} e^{(j)}_{j,j,\overline{\varsigma}} &= J_4^{(j)} e^{(j)}_{j,j,\overline{\varsigma}} = 0 \end{aligned} \tag{36}$$

according to (32),(33). Then the elements

$$e^{(j)}_{l,m,\overline{\varsigma}+\overline{\mu}} := \sqrt{\frac{4^{2(j-l)}\Gamma(l+m+1)}{\Gamma(2j+1)\Gamma(l-m+1)}}\, J_-^{(j)\,l-m} J_5^{(j)\,\mu} e^{(j)}_{j,j,\overline{\varsigma}} \tag{37}$$

of $V(j,\overline{\varsigma})_{\overline{\varsigma}+\overline{\mu}}$, where the (non-negative) number l is given by $l := j - \frac{1}{2}\mu$ and the numbers μ and m run through $\{0,1\}$ respectively $\{-l,-l+1,\cdots,l-1,l\}$, form a homogeneous basis of $V(j,\overline{\varsigma})$. Using the graded commutation relations (31) one can deduce the action

$$\begin{aligned} J_3^{(j)} e^{(j)}_{l,m,\overline{\varsigma}+\overline{\mu}} &= m\, e^{(j)}_{l,m,\overline{\varsigma}+\overline{\mu}} \\ J_+^{(j)} e^{(j)}_{l,m,\overline{\varsigma}+\overline{\mu}} &= \sqrt{(l-m)(l+m+1)}\, e^{(j)}_{l,m+1,\overline{\varsigma}+\overline{\mu}} \\ J_-^{(j)} e^{(j)}_{l,m,\overline{\varsigma}+\overline{\mu}} &= \sqrt{(l+m)(l-m+1)}\, e^{(j)}_{l,m-1,\overline{\varsigma}+\overline{\mu}} \\ J_4^{(j)} e^{(j)}_{j,m,\overline{\varsigma}} &= -\frac{1}{2}\sqrt{j-m}\, e^{(j)}_{j-\frac{1}{2},m+\frac{1}{2},\overline{\varsigma}+\overline{1}} \\ J_4^{(j)} e^{(j)}_{j-\frac{1}{2},m,\overline{\varsigma}+\overline{1}} &= -\frac{1}{2}\sqrt{j+m+\frac{1}{2}}\, e^{(j)}_{j,m+\frac{1}{2},\overline{\varsigma}} \\ J_5^{(j)} e^{(j)}_{j,m,\overline{\varsigma}} &= \frac{1}{2}\sqrt{j+m}\, e^{(j)}_{j-\frac{1}{2},m-\frac{1}{2},\overline{\varsigma}+\overline{1}} \\ J_5^{(j)} e^{(j)}_{j-\frac{1}{2},m,\overline{\varsigma}+\overline{1}} &= -\frac{1}{2}\sqrt{j-m+\frac{1}{2}}\, e^{(j)}_{j,m-\frac{1}{2},\overline{\varsigma}} \end{aligned} \tag{38}$$

of the homogeneous basis $\{J_3, J_\pm, J_\alpha\}$ of $\mathfrak{osp}(1|2)$ on the homogeneous basis elements (37). (37) and (38) show in particular, that the even and the odd subspace of $V(j,\overline{\varsigma})$ correspond to irreducible $\mathfrak{sl}(2)$-modules with highest weights $l = j$ and $l = j - \frac{1}{2}$ and

$$\begin{aligned} &\dim_{\mathbb{C}} V(j,\overline{\varsigma})_{\overline{\varsigma}} = 2j+1 \\ &\dim_{\mathbb{C}} V(j,\overline{\varsigma})_{\overline{\varsigma}+\overline{1}} = 2j \\ &\dim_{\mathbb{C}} V(j,\overline{\varsigma}) = 4j+1\,. \end{aligned} \tag{39}$$

Now let us introduce the graded derivations

$$\begin{aligned} J_i^{(3|2)} &:= -i \sum_{j,k=1}^{3} \varepsilon_{ijk} x^j \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{\alpha,\beta=4}^{5} (\sigma_i)_{\alpha\beta}\, \theta^\alpha \frac{\partial}{\partial \theta^\beta}\,, \qquad i = 1,2,3\,, \\ J_\alpha^{(3|2)} &:= \frac{1}{2} \sum_{k=1}^{3} \sum_{\beta=4}^{5} \left\{ i x^k\, (\sigma_2\sigma_k)_{\alpha\beta} \frac{\partial}{\partial \theta^\beta} - \theta^\beta\, (\sigma_k)_{\beta\alpha} \frac{\partial}{\partial x^k} \right\}, \qquad \alpha = 4,5\,, \end{aligned} \tag{40}$$

of $\mathcal{H}^\infty(B_L^{3|2}, C_L)$. The corresponding $\mathbb{C}$-linear map $\cdot^{(3|2)} : \mathfrak{osp}(1|2) \longrightarrow \mathfrak{V}^g(B_L^{3|2})$ is a graded representation of $\mathfrak{osp}(1|2)$, that leaves $\mathcal{I}_{S_\rho}$ (according to lemma 1) invariant. Consequently, defining the action via the action on representants, $\cdot^{(3|2)}$ induces a graded representation $\cdot^{(S)} : \mathfrak{osp}(1|2) \longrightarrow \mathfrak{V}^g(S_\rho)$ and the latter is a "grade star representation with respect to $\cdot^{\ddagger_1}$", that is,

$$\left\langle \boldsymbol{f} \,\middle|\, J_A^{(S)} \boldsymbol{g} \right\rangle = (-1)^{\overline{J_A}\,\overline{\boldsymbol{f}}} \left\langle J_A^{\ddagger_1 (S)} \boldsymbol{f} \,\middle|\, \boldsymbol{g} \right\rangle \tag{41}$$

is fulfilled for all homogeneous $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{H}^\infty(S_\rho, C_L)$ and $A = 1, \cdots, 5$.

$\cdot^{(3|2)}$ as well as $\cdot^{(S)}$ are reducible. In order to find its decomposition into irreducible subspaces we note, that the even polynomials

$$Y^{s,j}_{j,j,\overline{0}} := \frac{\sqrt{\Gamma(2j+1)}}{2^j \Gamma(j+1) \rho^{j+2s}} \left(\sum_{i=1}^{3} (x^k)^2 + 2\theta^4\theta^5 \right)^s \left(x^1 + i x^2 \right)^j , \tag{42}$$

for all $j, s \in \mathbb{N}_0$, and the odd polynomials

$$Y^{s,j}_{j,j,\overline{1}} := \frac{\sqrt{\Gamma(2j+1)}}{2^{j-\frac{1}{2}} \Gamma(j+\frac{1}{2}) \rho^{j+\frac{3}{2}+2s}} \left(\sum_{i=1}^{3} (x^k)^2 + 2\theta^4\theta^5 \right)^s \left(x^1 + i x^2 \right)^{j-\frac{1}{2}} \left(x^3\theta^4 + (x^1 + i x^2)\theta^5 \right) , \tag{43}$$

for all $j \in \mathbb{N}_0 + \frac{1}{2}$, $s \in \mathbb{N}_0$, are highest weight vectors of $\cdot^{(3|2)}$ with highest weight j. Such highest weight vectors with different s but the same j are mapped under the canonical projection onto the same element of $\mathcal{P}(S_\rho)$, which we will denote by $\boldsymbol{Y}^j_{j,j,\overline{2j}}$. The normalization in (42) and (43) is chosen such that

$$\left\langle \boldsymbol{Y}^j_{j,j,\overline{2j}} \,\middle|\, \boldsymbol{Y}^j_{j,j,\overline{2j}} \right\rangle = 1 \tag{44}$$

is fulfilled. As we will see, the graded $\mathfrak{osp}(1|2)$-submodules

$$\begin{aligned} V^{s,j} &:= U(\mathfrak{osp}(1|2))^{(3|2)} Y^{s,j}_{j,j,\overline{2j}} \\ \boldsymbol{V}^j &:= U(\mathfrak{osp}(1|2))^{(S)} \boldsymbol{Y}^j_{j,j,\overline{2j}} , \end{aligned} \tag{45}$$

where $U(\mathfrak{osp}(1|2))$ denotes the enveloping algebra of $\mathfrak{osp}(1|2)$, constitute the direct summands of the graded representations $\cdot^{(3|2)}$ and $\cdot^{(S)}$.

Lemma 2 *The restriction of $\cdot^{(3|2)}$ to $V^{s,j}$ is an irreducible graded representation with highest weight vector $Y^{s,j}_{j,j,\overline{2j}}$, highest weight j and degree $\overline{2j}$ of the highest weight vector. Moreover the dense graded subalgebra $\mathcal{P}(B_L^{3|2}) \subseteq \mathcal{H}^\infty(B_L^{3|2}, C_L)$ can be decomposed as $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-module according to*

$$\mathcal{P}(B_L^{3|2}) = \bigoplus_{\substack{s \in \mathbb{N}_0 \\ j \in \frac{1}{2}\mathbb{N}_0}} V^{s,j} . \tag{46}$$

Proof: Let us assume, that the restriction of $\cdot^{(3|2)}$ to $V^{s,j}$ is reducible. Then there exists a graded subrepresentation of $\cdot^{(3|2)}$ on a graded vector subspace $W \subset V^{s,j}$ and a non-trivial homogeneous $w \in W \setminus V^{s,j}$. Because $\cdot^{(3|2)}$ is standard-cyclic, w is given either by

$$w = \sum_{q=0}^{N_j} \alpha_q J_-^{(3|2)\,q} Y^{s,j}_{j,j,\overline{2j}} , \qquad \alpha_q \in \mathbb{C}, \alpha_{N_j} \neq 0 ,$$

or by

$$w = \sum_{q=0}^{N_j} \alpha_q J_-^{(3|2)\,q} J_5^{(3|2)} Y^{s,j}_{j,j,\overline{2j}}\,, \qquad \alpha_q \in \mathbb{C}, \alpha_{N_j} \neq 0\,.$$

Using the explicite expressions (40),(42) and (43) one finds by induction $N_j \leq 2j$ and

$$J_+^{(3|2)\,N_j} w = \gamma_j Y^{s,j}_{j,j,\overline{2j}}\,, \qquad \gamma_j \in \mathbb{C}\setminus\{0\}\,,$$

in the first case and $N_j \leq 2j-1$ and

$$J_+^{(3|2)\,N_j} J_4^{(3|2)} w = \gamma_j Y^{s,j}_{j,j,\overline{2j}}\,, \qquad \gamma_j \in \mathbb{C}\setminus\{0\}\,,$$

in the second case. In any case one has $Y^{s,j}_{j,j,\overline{2j}} \in W$ implying the contradiction $W = V^{s,j}$.
Apparently $\cdot^{(3|2)}$ does not only leave $\mathcal{P}(B_L^{3|2})$ invariant but also the graded vector subspaces $\mathcal{P}^n(B_L^{3|2}) \subseteq \mathcal{P}(B_L^{3|2})$ of polynomials in the coordinate projections with complex coefficients of degree $n \in \mathbb{N}_0$. Consequently we have $V^{s,j} \subseteq \mathcal{P}^{j+\frac{3}{2}\overline{2j}+2s}(B_L^{3|2})$, where the summand $\frac{3}{2}\overline{2j}$ is of course to be understood as 0, if $j \in \mathbb{N}_0$, and as $\frac{3}{2}$, if $j \in \mathbb{N}_0 + \frac{1}{2}$. In order to show (46) we only have to prove

$$\mathcal{P}^n(B_L^{3|2}) = \bigoplus_{j+\frac{3}{2}\overline{2j}+2s=n} V^{s,j} \tag{47}$$

for all $n \in \mathbb{N}_0$. Using the fact, that highest weight vectors of $V^{s,j}$ and $V^{s',j'}$, with $s \neq s', j \neq j'$, such that $j + \frac{3}{2}\overline{2j} + 2s = j' + \frac{3}{2}\overline{2j}' + 2s' = n$, have to be linear independent because they are highest weight vectors to different highest weights, one finds, that the intersection of $V^{s,j}$ with the sum of all other vector spaces $V^{s',j'}$ with $s \neq s', j \neq j'$ such that $j + \frac{3}{2}\overline{2j} + 2s = j' + \frac{3}{2}\overline{2j}' + 2s' = n$ is $\{0\}$. Consequently $\oplus_{j+\frac{3}{2}\overline{2j}+2s=n} V^{s,j}$ is a well-defined graded vector subspace of $\mathcal{P}^n(B_L^{3|2})$. In order to show equality it is enough to show the equality of dimensions. But using (39) we find $2n^2+2n+1$ for $\dim_{\mathbb{C}}\left(\oplus_{j+\frac{3}{2}\overline{2j}+2s=n} V^{s,j}\right)$ by induction, which is exactly $\dim_{\mathbb{C}}\mathcal{P}^n(B_L^{3|2})$. □

Although the above result about the representation $\cdot^{(3|2)}$ is not of primary interest for itself it is of central importance for deducing the following reduction of $\cdot^{(S)}$.

Proposition 1 *The restriction of $\cdot^{(S)}$ to $\boldsymbol{V}^j$ is an irreducible graded representation with highest weight vector $\boldsymbol{Y}^j_{j,j,\overline{2j}}$, highest weight j and degree $\overline{2j}$ of the highest weight vector. Moreover the dense graded subalgebra $\mathcal{P}(S_\rho) \subseteq \mathcal{H}^\infty(S_\rho, C_L)$ can be decomposed as $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-module according to*

$$\mathcal{P}(S_\rho) = \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \boldsymbol{V}^j\,. \tag{48}$$

Proof : The first statement is a consequence of the facts, that $\boldsymbol{Y}^j_{j,j,\overline{2j}}$ is non-trivial and that the canonical projection $\mathcal{H}^\infty(B_L^{3|2}, C_L) \longrightarrow \mathcal{H}^\infty(S_\rho, C_L)$ is an surjective, even homomorphism of $\mathfrak{osp}(1|2)$-modules.
Because of lemma 2 we only have to show, that the intersection of $\boldsymbol{V}^j$ and the sum of all other vector subspaces $\boldsymbol{V}^{j'}$ is trivial, in order to show (48). But this is an easy consequence

of the fact, that $\boldsymbol{V}^j$ and $\boldsymbol{V}^{j'}$ are irreducible $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules with different highest weights. $\square$

According to proposition 1 and (37) the elements of $\boldsymbol{V}^j \subseteq \mathcal{P}(S_\rho) \subseteq \mathcal{H}^\infty(S_\rho, C_L)$

$$\boldsymbol{Y}^j_{l,m,\overline{2j}+\overline{\mu}} := \sqrt{\frac{4^{2(j-l)}\Gamma(l+m+1)}{\Gamma(2j+1)\Gamma(l-m+1)}}\, \boldsymbol{J}_-^{(S)\,l-m} \boldsymbol{J}_5^{(S)\,\mu} \boldsymbol{Y}^j_{j,j,\overline{2j}}\,, \tag{49}$$

with $l := j - \frac{1}{2}\mu$ and $\mu \in \{0,1\}$, $m \in \{-l, -l+1, \cdots, l-1, l\}$, form a homogeneous basis of $\boldsymbol{V}^j$ for all $j \in \frac{1}{2}\mathbb{N}_0$. We will call them superspherical harmonics. Of course, the action of $\mathfrak{osp}(1|2)$ on the superspherical harmonics is given by (38) and using (41) and (44) we find, that they are (pseudo)orthonormalized according to

$$\left\langle \boldsymbol{Y}^j_{l,m,\overline{2j}+\overline{\mu}} \middle| \boldsymbol{Y}^{j'}_{l',m',\overline{2j'}+\overline{\mu'}} \right\rangle = (-1)^{\delta_{\overline{2j}\,\overline{1}}\delta_{\mu 1}} \delta_{jj'}\delta_{\mu\mu'}\delta_{mm'}\,. \tag{50}$$

4 Truncation and the fuzzy supersphere

We truncate the direct sum (48) of graded subrepresentations of $\mathfrak{osp}(1|2)$ on the $\mathbb{Z}_2$-graded algebra of complex-Grassmann-valued $\mathcal{H}^\infty$-functions on the $(2|2)$-dimensional supersphere at each integer superspin and establish a new, $\mathbb{Z}_2$-graded associative product on each of these truncated sums. In this way we get the fuzzy supersphere, that is a whole sequence of finite-dimensional, noncommutative $\mathbb{Z}_2$-graded algebras possessing a "graded-commutative limit". The procedure described above was studied in a similar way in [21]. Here we have slightly different conventions and we will discuss the graded-commutative limit in greater detail, which includes the introduction of noncommutative superspherical harmonics in particular.

Let us first formulate our aim, that is the basic idea of the construction of fuzzy manifolds applied to our case. For every $q \in \frac{1}{2}\mathbb{N}$ let us introduce the truncated direct sum

$$\mathcal{H}_q := \bigoplus_{\substack{j \in \frac{1}{2}\mathbb{N}_0 \\ j \le q}} \boldsymbol{V}^j \tag{51}$$

of $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules. For some infinite subset $\mathbb{S} \subseteq \frac{1}{2}\mathbb{N}_0$ and every $q \in \mathbb{S}$ we want to find $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras $\mathcal{A}_q$, which are at the same time $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules together with even $\mathfrak{osp}(1|2)$-module isomorphisms $\psi_q : \mathcal{H}_q \longrightarrow \mathcal{A}_q$. Denoting by $\iota_{q'q}, q, q' \in \mathbb{S}, q \le q'$, the canonical injections $\mathcal{H}_q \longrightarrow \mathcal{H}_{q'}$ we can introduce even, injective $\mathfrak{osp}(1|2)$-module homomorphisms $\eta_{q'q} : \mathcal{A}_q \longrightarrow \mathcal{A}_{q'}$ by

$$\eta_{q'q} := \psi_{q'} \circ \iota_{q'q} \circ \psi_q^{-1}\,, \tag{52}$$

which fulfill

$$\eta_{q''q'} \circ \eta_{q'q} = \eta_{q''q} \tag{53}$$

for all $q, q', q'' \in \mathbb{S}, q \le q' \le q''$. Consequently $(\mathcal{H}_q, \iota_{q'q})$ and $(\mathcal{A}_q, \eta_{q'q})$ are isomorphic directed systems of $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules and their direct limits can be identified with $\mathcal{P}(S_\rho)$ (as $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules). The corresponding homomorphisms $\mathcal{H}_q \longrightarrow \mathcal{P}(S_\rho)$ and

$\mathcal{A}_q \longrightarrow \mathcal{P}(S_\rho)$ of $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules are denoted by ι_q and η_q, respectively.

Now let $\boldsymbol{f}$ and $\boldsymbol{f}\,'$ be elements of $\mathcal{P}(S_\rho)$. Then there will be a number $p \in \mathbb{S}$ such that (omitting the canonical injections ι_p and ι_{qp} as we will always do in the sequel) $\boldsymbol{f}, \boldsymbol{f}\,' \in \mathcal{H}_q$ for all $q \geq p$. We cannot form products $\boldsymbol{f}\boldsymbol{f}\,'$ in the truncated sums $\mathcal{H}_q$, but we can form $\psi_q(\boldsymbol{f})\psi_q(\boldsymbol{f}\,')$ in the isomorphic objects $\mathcal{A}_q$. A priori these products are not connected with the product $\boldsymbol{f}\boldsymbol{f}\,' \in \mathcal{H}^\infty(S_\rho, C_L)$ in any way, but we can map them into the direct limit according to

$$\left(\boldsymbol{f}\boldsymbol{f}\,'\right)_q := \eta_q\left(\psi_q(\boldsymbol{f})\psi_q(\boldsymbol{f}\,')\right) = \psi_q^{-1}\left(\psi_q(\boldsymbol{f})\psi_q(\boldsymbol{f}\,')\right) \in \mathcal{P}(S_\rho) \subseteq \mathcal{H}^\infty(S_\rho, C_L) \tag{54}$$

and "compare" $(\boldsymbol{f}\boldsymbol{f}\,')_q$ with $\boldsymbol{f}\boldsymbol{f}\,'$. More exactly $\{(\boldsymbol{f}\boldsymbol{f}\,')_q\}_{p\leq q, q\in\mathbb{S}}$ is a sequence in $\mathcal{H}^\infty(S_\rho, C_L)$, whose convergence to $\boldsymbol{f}\boldsymbol{f}\,'$ (with respect to the Fréchet topology) can be investigated.

Definition 1 *The directed system $(\mathcal{A}_q, \eta_{q'q})$ is said to possess a graded-commutative limit if*

$$\lim_{q\to\infty}\left(\boldsymbol{f}\boldsymbol{f}\,'\right)_q = \boldsymbol{f}\boldsymbol{f}\,' \tag{55}$$

is fulfilled for all $\boldsymbol{f}, \boldsymbol{f}\,' \in \mathcal{P}(S_\rho)$.

Because of the algebraic structure of each $\mathcal{A}_q$, we can view each $\mathcal{A}_q$ as "noncommutative $\mathcal{H}^\infty$-supermanifold"; the existence of the graded-commutative limit guarantees the relation with the $(2|2)$-dimensional supersphere. We will show in the sequel, that a directed system with graded-commutative limit really exists.

We choose $\mathbb{S} = \mathbb{N}$. For all $q \in \mathbb{N}$ one finds by induction

$$\begin{aligned} \dim_{\mathbb{C}}\mathcal{H}_{q,\bar{0}} &= q^2 + (q+1)^2 \\ \dim_{\mathbb{C}}\mathcal{H}_{q,\bar{1}} &= 2q\,(q+1) \\ \dim_{\mathbb{C}}\mathcal{H}_q &= (2q+1)^2\ , \end{aligned} \tag{56}$$

from which we can conclude, that there will exist isomorphisms of $\mathbb{Z}_2$-graded $\mathbb{C}$-vector spaces between $\mathcal{H}_q$ and the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2}, \bar{1}))$ of all endomorphisms of the $\mathbb{Z}_2$-graded representation space of the irreducible graded $\mathfrak{osp}(1|2)$-representation with highest weight $\frac{q}{2}$ and odd highest weight vector.

On each of the $\mathbb{Z}_2$-graded representation spaces $V(\frac{q}{2}, \bar{1})$ we can introduce a scalar product by

$$\left\langle e^{(\frac{q}{2})}_{l,m,\bar{1}+\overline{\mu}} \middle| e^{(\frac{q}{2})}_{l',m',\bar{1}+\overline{\mu'}} \right\rangle = \delta_{\mu\mu'}\delta_{mm'} \tag{57}$$

for all $\mu, \mu' \in \{0, 1\}$, $m, m' \in \{-l, \cdots, l\}$, such that $V(\frac{q}{2}, \bar{1})$ becomes a $\mathbb{Z}_2$-graded Hilbert space. With respect to this scalar product and the grade adjoint operation $\cdot^{\ddagger_0}$, the irreducible graded representation $\cdot^{(\frac{q}{2})}$ is a grade star representation [4],[43]. Employing the superadjoint operation $\cdot^\ddagger$ with respect to this scalar product as well as the supertrace Tr_s we can define a sesquilinear form $\langle\cdot\,|\cdot\rangle_{\frac{q}{2}} : \mathrm{End}_{\mathbb{C}}(V(\frac{q}{2}, \bar{1})) \times \mathrm{End}_{\mathbb{C}}(V(\frac{q}{2}, \bar{1})) \longrightarrow \mathbb{C}$ via

$$\langle f\,|g\rangle_{\frac{q}{2}} := -\mathrm{Tr}_s\left(f^\ddagger g\right) = \sum_{\mu=0}^{1}\sum_{m=-l}^{l}(-1)^\mu\left\langle e^{(\frac{q}{2})}_{l,m,\bar{1}+\overline{\mu}} \middle| f^\ddagger g\left(e^{(\frac{q}{2})}_{l',m',\bar{1}+\overline{\mu'}}\right)\right\rangle\ . \tag{58}$$

The normalization has been chosen such that $\langle \mathrm{Id}_{V(q/2,\overline{1})} | \mathrm{Id}_{V(q/2,\overline{1})} \rangle_{\frac{q}{2}} = 1$. $\langle \cdot | \cdot \rangle_{\frac{q}{2}}$ is even, non-degenerate and fulfills

$$\langle f \,| g \rangle_{\frac{q}{2}} = \langle g \,| f \rangle^*_{\frac{q}{2}} \ . \tag{59}$$

Beside this indefinite scalar product, which has exactly the same properties as (29), we can also establish a $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-module structure on $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$ in a natural way by defining $\mathrm{ad}^{(\frac{q}{2})} : \mathfrak{osp}(1|2) \longrightarrow \mathfrak{Der}^g_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(V(K,\overline{1}))) \subseteq \mathfrak{pl}(\mathrm{End}_{\mathbb{C}}(V(K,\overline{1})))$ via

$$\mathrm{ad}^{(\frac{q}{2})} J(f) := \left[J^{(\frac{q}{2})}, f \right]_g \tag{60}$$

for all $J \in \mathfrak{osp}(1|2)$, $f \in \mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$. Here we denoted by $\mathfrak{pl}(\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1})))$ the general linear Lie superalgebra of the $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$ and by $[\cdot,\cdot]_g$ its graded Lie bracket. Using the fact, that $\cdot^{(\frac{q}{2})}$ is a grade star representation with respect to $\cdot^{\ddagger_0}$, we can deduce, that $\mathrm{ad}^{(\frac{q}{2})}$ is a graded representation with respect to $\cdot^{\ddagger_1}$.
The graded representations $\mathrm{ad}^{(\frac{q}{2})}$ are reducible for all $q \in \mathbb{N}$. In order to find their reduction into irreducible subspaces we proceed as we did for the infinite-dimensional case. Employing (38) and the graded Leibniz rule we can check, that the even endomorphisms

$$Y^{(\frac{q}{2})j}_{j,j,\overline{0}} := \sqrt{\frac{2^j(2j-1)!!\Gamma(q-j+1)}{\Gamma(j+1)\Gamma(q+j+1)}} J^{(\frac{q}{2})j}_{+} \,, \tag{61}$$

for all $j \in \mathbb{N}_0, j \leq q$, as well as the odd endomorphisms

$$Y^{(\frac{q}{2})j}_{j,j,\overline{1}} := \frac{1}{q+\frac{1}{2}} \sqrt{\frac{2^{j+\frac{7}{2}}(2j)!!\Gamma(q-j+\frac{1}{2})}{\Gamma(j+\frac{1}{2})\Gamma(q+j+\frac{3}{2})}} J^{(\frac{q}{2})j-\frac{1}{2}}_{+} \left[\left(J^{(\frac{q}{2})}_3 - \frac{3}{4} \right) J^{(\frac{q}{2})}_4 + J^{(\frac{q}{2})}_{+} J^{(\frac{q}{2})}_5 \right] \,, \tag{62}$$

for all $j \in \mathbb{N}_0 + \frac{1}{2}, j \leq q$, are highest weight vectors of $\mathrm{ad}^{(\frac{q}{2})}$ with highest weight j and normalized according to

$$\left\langle Y^{(\frac{q}{2})j}_{j,j,\overline{2j}} \middle| Y^{(\frac{q}{2})j}_{j,j,\overline{2j}} \right\rangle_{\frac{q}{2}} = 1 \,. \tag{63}$$

The corresponding graded $\mathfrak{osp}(1|2)$-submodules,

$$V^{(\frac{q}{2})j} := \mathrm{ad}^{(\frac{q}{2})} \left(U(\mathfrak{osp}(1|2)) \right) Y^{(\frac{q}{2})j}_{j,j,\overline{2j}} \,, \tag{64}$$

are the direct summands we are looking for.

Proposition 2 *The restriction of* $\mathrm{ad}^{(\frac{q}{2})}$ *to* $V^{(\frac{q}{2})j}$ *is an irreducible graded representation of* $\mathfrak{osp}(1|2)$ *with highest weight vector* $Y^{(\frac{q}{2})j}_{j,j,\overline{2j}}$*, highest weight* j *and degree* $\overline{2j}$ *of the highest weight vector. Moreover* $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$ *can be decomposed as* $\mathbb{Z}_2$*-graded* $\mathfrak{osp}(1|2)$*-module according to*

$$\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1})) = \bigoplus_{\substack{j \in \frac{1}{2}\mathbb{N}_0 \\ j \leq q}} V^{(\frac{q}{2})j} \,. \tag{65}$$

Proof : The proof of irreducibility can be taken over literally from lemma 2: The numbers N_j can again be determined by induction (with the same result as in lemma 2) but now by using the explicite expressions (61),(62). The fact, that we are considering only graded highest weight modules with different highest weights guarantees that the direct sum in (65) is well-defined and the equality with $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$ results from our dimensional considerations (56). $\square$

Consequently for every $q \in \mathbb{N}$ and $j \leq q$ the elements

$$Y^{(\frac{q}{2})\,j}_{l,m,\overline{2j}+\overline{\mu}} := \sqrt{\frac{4^{2(j-l)}\Gamma(l+m+1)}{\Gamma(2j+1)\Gamma(l-m+1)}}\left(\mathrm{ad}^{(\frac{q}{2})}J_{-}\right)^{l-m}\left(\mathrm{ad}^{(\frac{q}{2})}J_{5}\right)^{\mu} Y^{(\frac{q}{2})\,j}_{j,j,\overline{2j}} \tag{66}$$

of $V^{(\frac{q}{2})\,j}$, with $l := j - \frac{1}{2}\mu$ and $\mu \in \{0,1\}$, $m \in \{-l,-l+1,\cdots,l-1,l\}$, form a homogeneous basis of $V^{(\frac{q}{2})\,j}$. Because of the isomorphisms (68) defined below, we will call these elements of $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$ noncommutative superspherical harmonics and an analogous calculation which yielded (50) shows, that they are again (pseudo)orthonormalized according to

$$\left\langle Y^{(\frac{q}{2})\,j}_{l,m,\overline{2j}+\overline{\mu}}\middle| Y^{(\frac{q}{2})\,j'}_{l',m',\overline{2j'}+\overline{\mu'}}\right\rangle_{\frac{q}{2}} = (-1)^{\delta_{\overline{2j}\,\overline{1}}\delta_{\mu 1}}\delta_{jj'}\delta_{\mu\mu'}\delta_{mm'}\,. \tag{67}$$

In consideration of proposition 2 we can introduce lots of even $\mathfrak{osp}(1|2)$-module isomorphisms $\psi_q : \mathcal{H}_q \longrightarrow \mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$ and we choose especially for every $q \in \mathbb{N}$ the $\mathbb{C}$-linear map defined by

$$\psi_q(\boldsymbol{Y}^{j}_{l,m,\overline{2j}+\overline{\mu}}) := Y^{(\frac{q}{2})\,j}_{l,m,\overline{2j}+\overline{\mu}} \tag{68}$$

for all $j \leq q, \mu \in \{0,1\}$ and $m \in \{-l,-l+1,\cdots,l-1,l\}$, because the corresponding directed system $(\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1})),\eta_{q'q})$ has the desired property.

Proposition 3 *The directed system* $(\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1})),\eta_{q'q})$ *corresponding with (68) possesses a graded-commutative limit.*

Proof : Because the topology on $\mathcal{H}^{\infty}(S_\rho, C_L)$ is induced by seminorms (23), superspherical harmonics - ordinary and noncommutative as well - are (pseudo)orthonormalized and $\cdot^{(S)}$ and $\mathrm{ad}^{(\frac{q}{2})}$ are grade star representations of the same type, it is enough to show

$$\lim_{q\to\infty}\left\langle Y^{(\frac{q}{2})\,j}_{l,m,\overline{2j}+\overline{\mu}}\middle| Y^{(\frac{q}{2})\,j'}_{l',m',\overline{2j'}+\overline{\mu'}} Y^{(\frac{q}{2})\,j''}_{l'',m'',\overline{2j''}+\overline{\mu''}}\right\rangle_{\frac{q}{2}} = \left\langle \boldsymbol{Y}^{j}_{l,m,\overline{2j}+\overline{\mu}}\middle| \boldsymbol{Y}^{j'}_{l',m',\overline{2j'}+\overline{\mu'}}\boldsymbol{Y}^{j''}_{l'',m'',\overline{2j''}+\overline{\mu''}}\right\rangle \tag{69}$$

for all $j',j'' \in \frac{1}{2}\mathbb{N}_0, \mu',\mu'' \in \{0,1\}, m' \in \{-l',\cdots,l'\}, m' \in \{-l',\cdots,l'\}$ and $j \in \frac{1}{2}\mathbb{N}_0, \mu \in \{0,1\}, m \in \{-l,\cdots,l\}$ with $|j'-j''| \leq j \leq j'+j'', m = m'+m'', \overline{\mu} = \overline{2j+2j'+2j''+\mu'+\mu''}$. Now we can interpret $\boldsymbol{Y}^{j'}_{l',m',\overline{2j'}+\overline{\mu'}}$ and $Y^{(\frac{q}{2})\,j'}_{l',m',\overline{2j'}+\overline{\mu'}}$ as multiplication operators in $\mathcal{P}(S_\rho)$ and $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2},\overline{1}))$, respectively. Then the sets $\{\ \boldsymbol{Y}^{j'}_{l',m',\overline{2j'}+\overline{\mu'}} \mid \mu' = 0,1; m' = -l,\cdots,l\ \}$ and $\{\ Y^{(\frac{q}{2})\,j'}_{l',m',\overline{2j'}+\overline{\mu'}} \mid \mu' = 0,1; m' = -l,\cdots,l\ \}$ are irreducible $\mathfrak{osp}(1|2)$-tensor operators and we can apply the $\mathfrak{osp}(1|2)$-Wigner-Eckart theorem [41],[55] to conclude, that we can restrict our attention to the cases $\mu' = \mu'' = 0, m' = j', m'' = j''$.
We find

$$\boldsymbol{Y}^{j'}_{j',j',\overline{2j'}}\boldsymbol{Y}^{j''}_{j'',j'',\overline{2j''}} = c_{j'j''}\boldsymbol{Y}^{j'+j''}_{j'+j'',j'+j'',\overline{2(j'+j'')}}, \qquad j',j'' \in \frac{1}{2}\mathbb{N}_0,$$

as well as

$$Y^{(\frac{q}{2})\,j'}_{j',j',\overline{2j'}}Y^{(\frac{q}{2})\,j''}_{j'',j'',\overline{2j''}} = c^{q}_{j'j''}Y^{(\frac{q}{2})\,j'+j''}_{j'+j'',j'+j'',\overline{2(j'+j'')}}, \qquad j',j'' \in \frac{1}{2}\mathbb{N}_0,$$

where the coefficients $c_{j'j''}, c^q_{j'j''} \in \mathbb{R}$, which are also determined by the explicite expressions (42), (43) and (61),(62), fulfill

$$\lim_{q\to\infty} c^q_{j'j''} = c_{j'j''}$$

for all $j', j'' \in \frac{1}{2}\mathbb{N}_0$. But because of (50) and (67) this proves the proposition. $\square$

Consequently we have succeeded in finding a sequence of $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras tending in the limit described above to the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra of $\mathcal{H}^\infty$-functions $\mathcal{H}^\infty(S_\rho, C_L)$ on the $(2|2)$-dimensional supersphere S_ρ. Transposed to the language of noncommutative geometry this means, that we have approximated the $(2|2)$-dimensional supersphere by a sequence of noncommutative supermanifolds. Because our construction is exactly the same as the one of ordinary fuzzy manifolds [24],[26],[39] we will call the directed system $(\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2}, \overline{1})), \eta_{q'q})$ ((2|2)-dimensional) fuzzy supersphere. Each one of the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras and graded $\mathfrak{osp}(1|2)$-modules $\mathrm{End}_{\mathbb{C}}(V(\frac{q}{2}, \overline{1}))$ we will call truncated ((2|2)-dimensional) supersphere and we introduce the shorter notation $\mathcal{S}_{\rho,q}$ for it.
It is worthwhile to mention the following nice property of the isomorphisms ψ_q and the truncated superspheres $\mathcal{S}_{\rho,q}$: For all $q \in \mathbb{N}$ we find

$$\begin{aligned}
\psi_q(\boldsymbol{x}^k) &= \frac{2\rho}{\sqrt{q(q+1)}} J_k^{(\frac{q}{2})} =: X_k^{(\frac{q}{2})}, \qquad k = 1,2,3, \\
\psi_q(\boldsymbol{\theta}^\alpha) &= \frac{2\rho}{\sqrt{q(q+1)}} J_\alpha^{(\frac{q}{2})} =: \Theta_\alpha^{(\frac{q}{2})}, \qquad \alpha = 4,5,
\end{aligned} \tag{70}$$

and consequently

$$\begin{aligned}
&\sum_{k=1}^{3} \psi_q(\boldsymbol{x}^k)^2 + \psi_q(\boldsymbol{\theta}^4)\psi_q(\boldsymbol{\theta}^5) - \psi_q(\boldsymbol{\theta}^5)\psi_q(\boldsymbol{\theta}^4) \equiv \\
&\quad \equiv \sum_{k=1}^{3} \left(X_k^{(\frac{q}{2})}\right)^2 + \Theta_4^{(\frac{q}{2})}\Theta_5^{(\frac{q}{2})} - \Theta_5^{(\frac{q}{2})}\Theta_4^{(\frac{q}{2})} = \rho^2 \mathrm{Id}_{V(\frac{q}{2},\overline{1})},
\end{aligned} \tag{71}$$

because the left hand side is $4\rho^2/(q(q+1))$ times the representation of the standard second-order Casimir operator [43] of $\mathfrak{osp}(1|2)$. That is, the defining relation of the $(2|2)$-dimensional supersphere is fulfilled on each truncated supersphere.

5 The noncommutative body map

The existence of a body and an "algebraic body map" is of great structural importance in the theory of supermanifolds respectively graded manifolds and one should have an fuzzy analogue. We will see, that it is very natural to interpret the fuzzy sphere as "noncommutative body" of the fuzzy supersphere. The corresponding noncommutative body map will be a surjective homomorphism of directed systems, but none of the single maps will be an algebra homomorphism (which would be impossible as map between simple algebras of different dimensions). The algebra homomorphism property will be recovered in the "graded-commutative limit".

Let us first describe the "noncommutative body" of the fuzzy supersphere, that is the fuzzy sphere, in an adequate language (see also [39],[37],[22],[21]). The $\mathbb{C}$-linear map $\cdot^{(3)} : \mathfrak{sl}(2) \longrightarrow \mathfrak{V}(\mathbb{R}^3)$, defined by

$$\mathrm{J}_i^{(3)} := -i \sum_{j,k=1}^{3} \varepsilon_{ijk} \mathrm{x}^j \partial_k \,, \qquad i = 1,2,3, \tag{72}$$

is a representation of $\mathfrak{sl}(2)$. It leaves $\mathcal{I}_{\mathrm{S}_\rho}$ invariant and induces a representation $\cdot^{(\mathrm{S})} : \mathfrak{sl}(2) \longrightarrow \mathfrak{V}(\mathrm{S}_\rho)$, which is a star representation with respect to the adjoint operation $\cdot^\dagger$ corresponding with the compact real form of $\mathfrak{sl}(2)$ and the scalar product $\langle \cdot \,|\, \cdot \rangle : \mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C}) \times \mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C}) \longrightarrow \mathbb{C}$, defined by

$$\langle \mathbf{f} \,|\, \mathbf{g} \rangle := \frac{1}{4\pi} \int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi \, \mathrm{f}(\rho,\vartheta,\varphi)^* \mathrm{g}(\rho,\vartheta,\varphi) \,, \tag{73}$$

where f, g are representants of $\mathbf{f}$ and $\mathbf{g}$, expressed in spherical coordinates. The normalized (highest weight) spherical harmonics

$$\mathbf{Y}_j^j := \frac{\sqrt{\Gamma(2j+2)}}{2^j \Gamma(j+1) \rho^j} \left(\mathbf{x}^1 + i \mathbf{x}^2 \right)^j \,, \qquad j \in \mathbb{N}_0 \,, \tag{74}$$

are highest weight vectors of $\cdot^{(\mathrm{S})}$ with highest weight j, the restriction of $\cdot^{(\mathrm{S})}$ to the $\mathfrak{sl}(2)$-submodules

$$\mathbf{V}^j := U(\mathfrak{sl}(2))^{(\mathrm{S})} \mathbf{Y}_j^j \tag{75}$$

is irreducible and the dense graded subalgebra $\mathcal{P}(\mathrm{S}_\rho) \subseteq \mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$ can be decomposed as $\mathfrak{sl}(2)$-module according to

$$\mathcal{P}(\mathrm{S}_\rho) = \bigoplus_{j \in \mathbb{N}_0} \mathbf{V}^j \,. \tag{76}$$

The (ordinary) spherical harmonics, given by

$$\mathbf{Y}_m^j := \sqrt{\frac{\Gamma(j+m+1)}{\Gamma(2j+1)\Gamma(j-m+1)}} \, \mathrm{J}_-^{(\mathrm{S})\, j-m} \mathbf{Y}_j^j \,, \tag{77}$$

where $m \in \{-j, -j+1, \cdots, j\}$, are orthonormal and they form a basis of $\mathbf{V}^j$ for every fixed $j \in \mathbb{N}_0$.

As in the case of the supersphere one establishes on each truncated direct sum

$$\mathcal{C}_q := \bigoplus_{\substack{j \in \mathbb{N}_0 \\ j \leq q}} \mathbf{V}^j \,, \qquad q \in \mathbb{N}_0 \,, \tag{78}$$

of $\mathfrak{sl}(2)$-modules a (noncommutative) associative product. In order to do so, let us denote by $\cdot^{(\frac{q}{2})} : \mathfrak{sl}(2) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$ the irreducible $\mathfrak{sl}(2)$-representation with highest weight $\frac{q}{2}$, by $\{\mathrm{e}_m^{(\frac{q}{2})}\}_{m \in \{-\frac{q}{2}, \cdots, \frac{q}{2}\}}$ the canonical basis of $\mathrm{V}(\frac{q}{2})$ corresponding with the weight space decomposition and by $\langle \cdot \,|\, \cdot \rangle$ the scalar product on $\mathrm{V}(\frac{q}{2})$ fixed by the requirement, that $\{\mathrm{e}_m^{(\frac{q}{2})}\}_{m \in \{-\frac{q}{2}, \cdots, \frac{q}{2}\}}$ becomes an orthonormal basis. Then

$$\langle \mathrm{f} \,|\, \mathrm{g} \rangle_{\frac{q}{2}} := \frac{1}{q+1} \mathrm{Tr} \left(\mathrm{f}^\dagger \mathrm{g} \right) = \frac{1}{q+1} \sum_{m=-\frac{q}{2}}^{m=\frac{q}{2}} \left\langle \mathrm{e}_m^{(\frac{q}{2})} \middle| \mathrm{f}^\dagger \mathrm{g} \, \mathrm{e}_m^{(\frac{q}{2})} \right\rangle \tag{79}$$

is a scalar product on $\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$. Moreover, denoting by $\mathfrak{Der}_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})))$ the $\mathbb{C}$-Lie algebra of derivations of the $\mathbb{C}$-algebra $\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$, the representation $\mathrm{ad}^{(\frac{q}{2})} : \mathfrak{sl}(2) \longrightarrow \mathfrak{Der}_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})))$, defined via

$$\mathrm{ad}^{(\frac{q}{2})}\mathrm{J}(\mathrm{f}) := \left[\mathrm{J}^{(\frac{q}{2})}, \mathrm{f}\right] \tag{80}$$

for all $\mathrm{J} \in \mathfrak{sl}(2), \mathrm{f} \in \mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$, becomes a star representation with respect to (79).
For all $q \in \mathbb{N}_0$ and all $j \in \mathbb{N}_0, j \leq q$ the normalized endomorphisms

$$\mathrm{Y}_j^{(\frac{q}{2})\,j} := \sqrt{\frac{2^j(2j+1)!!(q+1)\Gamma(q-j+1)}{\Gamma(j+1)\Gamma(q+j+2)}}\,\mathbf{J}_+^{(\frac{q}{2})\,j} \tag{81}$$

are highest weight vectors of the representations $\mathrm{ad}^{(\frac{q}{2})}$, the corresponding $\mathfrak{sl}(2)$-submodules

$$\mathrm{V}^{(\frac{q}{2})\,j} := \mathrm{ad}^{(\frac{q}{2})}\left(U(\mathfrak{sl}(2))\right)\mathrm{Y}_j^{(\frac{q}{2})\,j} \tag{82}$$

are irreducible and $\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$ can be decomposed as $\mathfrak{sl}(2)$-module according to

$$\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})) := \bigoplus_{\substack{j\in\mathbb{N}_0\\ j\leq q}} \mathrm{V}^{(\frac{q}{2})\,j}\,. \tag{83}$$

For every $q \in \mathbb{N}$ we can introduce noncommutative spherical harmonics by

$$\mathrm{Y}_m^{(\frac{q}{2})\,j} := \sqrt{\frac{\Gamma(j+m+1)}{\Gamma(2j+1)\Gamma(j-m+1)}}\left(\mathrm{ad}^{(\frac{q}{2})}\mathrm{J}_-\right)^{j-m}\mathrm{Y}_j^{(\frac{q}{2})\,j}\,, \tag{84}$$

where $j \in \mathbb{N}_0, j \leq q$ and $m \in \{-j, \cdots, j\}$, and they form an orthonormal basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$.
Now we can define isometric isomorphisms $\hat{\psi}_q : \mathcal{C}_q \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$ of $\mathfrak{sl}(2)$-modules by

$$\hat{\psi}_q(\mathbf{Y}_m^j) := \mathrm{Y}_m^{(\frac{q}{2})\,j}\,, \qquad j \in \mathbb{N}_0, j \leq q, m \in \{-j, \cdots, j\}\,, \tag{85}$$

as well as injective $\mathfrak{sl}(2)$-module homomorphism $\hat{\eta}_{q'q} : \mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q'}{2})), q \leq q'$, via

$$\hat{\eta}_{q'q} := \hat{\psi}_{q'} \circ \hat{\iota}_{q'q} \circ \hat{\psi}_q^{-1}\,, \tag{86}$$

where $\hat{\iota}_{q'q}$ denote the canonical injections $\mathcal{C}_q \longrightarrow \mathcal{C}_{q'}$. Then $(\mathcal{C}_q, \hat{\iota}_{q'q})$ and $(\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})), \hat{\eta}_{q'q})$ are isomorphic directed systems of $\mathfrak{sl}(2)$-modules and their direct limits can be identified with $\mathcal{P}(\mathrm{S}_\rho)$. Moreover the directed system $(\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})), \hat{\eta}_{q'q})$ possesses a commutative limit: That is, for two arbitrary $\mathbf{f}, \mathbf{f}' \in \mathcal{P}(\mathrm{S}_\rho)$ the sequence $\{(\mathbf{ff}')_q\}_{p\leq q, q\in\mathbb{N}}$ with

$$(\mathbf{ff}')_q := \hat{\eta}_q\left(\hat{\psi}_q(\mathbf{f})\hat{\psi}_q(\mathbf{f}')\right) = \hat{\psi}_q^{-1}\left(\hat{\psi}_q(\mathbf{f})\hat{\psi}_q(\mathbf{f}')\right)\,, \tag{87}$$

where $\hat{\eta}_q$ are the "limit homomorphisms" $\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})) \longrightarrow \mathcal{P}(\mathrm{S}_\rho)$ and p is some number, such that $\mathbf{f}, \mathbf{f}' \in \mathcal{C}_p$, converges to $\mathbf{ff}' \in \mathcal{P}(\mathrm{S}_\rho)$ with respect to the Fréchet topology of $\mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$.
The directed system $(\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2})), \hat{\eta}_{q'q})$ is called (2-dimensional) fuzzy sphere; each one of the $\mathbb{C}$-algebras and $\mathfrak{sl}(2)$-modules $\mathrm{End}_{\mathbb{C}}(\mathrm{V}(\frac{q}{2}))$ is called truncated (2-dimensional) sphere and we denote it by $\mathbb{S}_{\rho,q}$. Similar to the truncated supersphere one finds for all $q \in \mathbb{N}$

$$\hat{\psi}_q(\mathbf{x}^k) = \frac{2\rho}{\sqrt{q(q+2)}}\mathrm{J}_k^{(\frac{q}{2})} =: \mathrm{X}_k^{(\frac{q}{2})}\,, \qquad k = 1, 2, 3, \tag{88}$$

and

$$\sum_{k=1}^{3} \hat{\psi}_q(\mathbf{x}^k)^2 = \sum_{k=1}^{3} \left(\mathrm{X}_k^{(\frac{q}{2})}\right)^2 = \rho^2 \mathrm{Id}_{V(\frac{q}{2})} . \tag{89}$$

In order to introduce a natural noncommutative analogue to the body map $\beta_{S_\rho} : \mathcal{H}^\infty(S_\rho, C_L) \longrightarrow \mathcal{C}^\infty(\mathrm{S}_\rho, \mathbb{C})$ we note, that β_{S_ρ} is a $\mathfrak{sl}(2)$-module homomorphism because of (20) and (21). Moreover the restriction of β_{S_ρ} to $\mathcal{P}(S_\rho)$ is (respectively induces) a surjective homomorphism

$$\beta_{S_\rho}|_{\mathcal{P}(S_\rho)} : \mathcal{P}(S_\rho) \longrightarrow \mathcal{P}(\mathrm{S}_\rho) \tag{90}$$

of $\mathfrak{sl}(2)$-modules and $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras and we have explicitely

$$\beta_{S_\rho}(\boldsymbol{Y}^j_{l,m,\overline{2j}+\overline{\varsigma}}) = \begin{cases} \dfrac{1}{\sqrt{j+1}} \mathbf{Y}^j_m & , j \in \mathbb{N}_0 \text{ and } \varsigma = 0 \\ 0 & , j \in \mathbb{N}_0 + \dfrac{1}{2} \text{ or } \varsigma = 1 \end{cases} \tag{91}$$

for all $m \in \{-l, \cdots, l\}$. Consequently

$$\beta_{\mathcal{S}_{\rho,q}} := \hat{\psi}_q \circ \beta_{S_\rho} \circ \psi_q^{-1} : \mathcal{S}_{\rho,q} \longrightarrow \mathbb{S}_{\rho,q} \tag{92}$$

is a well-defined surjective $\mathfrak{sl}(2)$-module homomorphism for all $q \in \mathbb{N}$ and we will call it (noncommutative) body map of the truncated ((2|2)-dimensional) supersphere. Beside the nice feature

$$\begin{aligned} \beta_{\mathcal{S}_{\rho,q}}(X_k^{(\frac{q}{2})}) &= \mathrm{X}_k^{(\frac{q}{2})}, \qquad k = 1,2,3, \\ \beta_{\mathcal{S}_{\rho,q}}(\Theta_\alpha^{(\frac{q}{2})}) &= 0, \qquad \alpha = 4,5, \end{aligned} \tag{93}$$

there are some other immediate consequences, which suggest this interpretation.

Proposition 4 *($\beta_{\mathcal{S}_{\rho,q}}$) is a surjective homomorphism $(\mathcal{S}_{\rho,q}, \eta_{q'q}) \longrightarrow (\mathbb{S}_{\rho,q}, \hat{\eta}_{q'q})$ of directed systems and $\varinjlim \beta_{\mathcal{S}_{\rho,q}}$ is simply given by $\beta_{S_\rho}|_{\mathcal{P}(S_\rho)}$. Moreover*

$$\lim_{q \longrightarrow \infty} \hat{\eta}_q \left(\beta_{\mathcal{S}_{\rho,q}} \left(\psi_q(\boldsymbol{f})\psi_q(\boldsymbol{f}')\right)\right) = \lim_{q \longrightarrow \infty} \left(\beta_{S_\rho}(\boldsymbol{f})\beta_{S_\rho}(\boldsymbol{f}')\right)_q \tag{94}$$

is fulfilled for all $\boldsymbol{f}, \boldsymbol{f}' \in \mathcal{P}(S_\rho)$.

Proof : For all $q, q' \in \mathbb{N}, q \le q'$ we have $\beta_{\mathcal{S}_{\rho,q'}} \circ \eta_{q'q} = \hat{\eta}_{q'q} \circ \beta_{\mathcal{S}_{\rho,q}}$ as well as $\beta_{S_\rho}|_{\mathcal{P}(S_\rho)} \circ \eta_q = \hat{\eta}_q \circ \beta_{\mathcal{S}_{\rho,q}}$ by construction, which proves the first part. The "homomorphism property in the limit" follows according to

$$\begin{aligned} \lim_{q \longrightarrow \infty} \hat{\eta}_q \left(\beta_{\mathcal{S}_{\rho,q}} \left(\psi_q(\boldsymbol{f})\psi_q(\boldsymbol{f}')\right)\right) &= \lim_{q \longrightarrow \infty} \beta_{S_\rho}\left((\boldsymbol{f}\boldsymbol{f}')_q\right) = \beta_{S_\rho}(\boldsymbol{f}\boldsymbol{f}') = \\ &= \beta_{S_\rho}(\boldsymbol{f})\beta_{S_\rho}(\boldsymbol{f}') = \lim_{q \longrightarrow \infty} \left(\beta_{S_\rho}(\boldsymbol{f})\beta_{S_\rho}(\boldsymbol{f}')\right)_q \end{aligned}$$

where we used the first part of the proposition, the (graded-)commutative limit of the fuzzy (super)sphere and the continuity of β_{S_ρ}. $\square$

6 A graded differential calculus on the fuzzy supersphere

Let X be a $\mathcal{H}^\infty$-deWitt supermanifold. We introduced the elements of the $\mathbb{C}$-Lie superalgebra and $\mathbb{Z}_2$-graded $\mathcal{H}^\infty(X, C_L)$-module $\mathfrak{V}^g(X)$ of complex, global supervector fields on X as graded derivations of the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra $\mathcal{H}^\infty(X, C_L)$. The latter concept can be generalized to arbitrary $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras without any change, when we replace "$\mathbb{Z}_2$-graded $\mathcal{H}^\infty(X, C_L)$-module" by the formulation "$\mathbb{Z}_2$-graded module over the graded center $\mathcal{Z}^g(\mathcal{H}^\infty(X, C_L))$ of the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra $\mathcal{H}^\infty(X, C_L)$", which is equivalent in the graded-commutative case. Moreover, again viewing X as graded manifold, $\mathbb{N}_0$-homogeneous, complex, global superdifferential forms are by definition graded-alternating $\mathcal{H}^\infty(X, C_L)$-multilinear, or equivalently $\mathcal{Z}^g(\mathcal{H}^\infty(S_\rho, C_L))$-multilinear maps from $\mathfrak{V}^g(X)$ to the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra of $\mathcal{H}^\infty$-functions $\mathcal{H}^\infty(X, C_L)$ [2],[35]. Adopting the graded center-formulation we can generalize the notion of superdifferential forms to arbitrary $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras and take it as starting point for the development of a graded differential calculus. This is the basic idea of the construction of derivation-based differential calculi [13],[15],[17],[16], transferred to the $\mathbb{Z}_2$-graded case.
This idea can even be generalized by taking into account only a subset of all graded derivations, which is at the same time a $\mathbb{C}$-Lie subsuperalgebra as well as a graded submodule over the graded center of the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra under consideration. We will employ, as it was done in the ungraded "fuzzy case" [39],[37],[24], a specific variant of this generalization to develop an analogue of the super-deRham complex and the graded Cartan calculus on each of the truncated superspheres. A very natural feature of this approach will be, that the non-commutative body map extends - as in the case of graded manifolds - to a cochain map from the differential complex on the truncated supersphere to the one on its body.

According to the general principles formulated above the first thing we need to know is the graded center $\mathcal{Z}^g(\mathcal{S}_{\rho,q})$ of the truncated superspheres $\mathcal{S}_{\rho,q}, q \in \mathbb{N}$. Analogous to the ungraded case (see for example [45]) we find, that $\mathcal{S}_{\rho,q}$ is graded-central,

$$\mathcal{Z}^g(\mathcal{S}_{\rho,q}) = \mathcal{Z}^g(\mathcal{S}_{\rho,q})_{\overline{0}} = \mathbb{C}\mathrm{Id}_{V(\frac{q}{2},\overline{1})} . \tag{95}$$

Consequently one can choose (according to the argumentation above) in principle every $\mathbb{C}$-Lie subsuperalgebra of $\mathfrak{Der}^g_{\mathbb{C}}(\mathcal{S}_{\rho,q})$ as $\mathbb{C}$-Lie superalgebra of supervector fields $\mathfrak{V}^g(\mathcal{S}_{\rho,q})$ on each truncated supersphere. But there is a natural choice given by the action of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ on $\mathcal{S}_{\rho,q}$,

$$\mathfrak{V}^g(\mathcal{S}_{\rho,q}) := \mathrm{ad}^{(\frac{q}{2})}\left(\mathfrak{osp}(1|2)\right) , \quad q \in \mathbb{N}. \tag{96}$$

The graded representations $\mathrm{ad}^{(\frac{q}{2})}$ are faithful by (65), such that there are natural isomorphisms

$$\mathfrak{V}^g(\mathcal{S}_{\rho,q}) \cong \mathfrak{V}^g(\mathcal{S}_{\rho,q'}) \cong \mathfrak{osp}(1|2) , \quad q, q' \in \mathbb{N}, \tag{97}$$

of $\mathbb{C}$-Lie superalgebras and this fact will "control the growth of the graded differential calculus with respect to q".
An additional justification of the choice (96) stems from a translation of the $\mathfrak{sl}(2)$-module homomorphism property of $\beta_{\mathcal{S}_{\rho,q}}$: We can define a map $\tilde{\beta}_{\mathcal{S}_{\rho,q}}$ from $\mathfrak{V}^g(\mathcal{S}_{\rho,q})_{\overline{0}}$ to the $\mathbb{C}$-Lie

algebra $\mathfrak{V}(\mathbb{S}_{\rho,q}) := \mathrm{ad}^{(\frac{q}{2})}(\mathfrak{sl}(2)) \cong \mathfrak{sl}(2)$ of (complex) vector fields on the truncated sphere $\mathbb{S}_{\rho,q}$ [39],[37] via

$$\tilde{\beta}_{\mathcal{S}_{\rho,q}}(D)\beta_{\mathcal{S}_{\rho,q}}(f) := \beta_{\mathcal{S}_{\rho,q}}(Df)\,, \qquad f \in \mathcal{S}_{\rho,q}, \tag{98}$$

analogous to (21). Then $\tilde{\beta}_{\mathcal{S}_{\rho,q}}$ is a surjective (in fact a bijective) Lie algebra homomorphism, which maps $\mathrm{ad}^{(\frac{q}{2})}J \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})_{\overline{0}}$ to

$$\tilde{\beta}_{\mathcal{S}_{\rho,q}}(\mathrm{ad}^{(\frac{q}{2})}J) = \mathrm{ad}^{(\frac{q}{2})}\mathrm{J}\,, \tag{99}$$

where $\mathrm{J} \in \mathfrak{sl}(2)$ corresponds with $J \in \mathfrak{osp}(1|2)_{\overline{0}}$ via the natural isomorphism.

For every natural number $p \in \mathbb{N}$ let us denote by $\mathrm{Hom}^p_{\mathbb{C}}(\mathfrak{V}^g(\mathcal{S}_{\rho,q}); \mathcal{S}_{\rho,q})$ the $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space of all p-linear maps $\mathfrak{V}^g(\mathcal{S}_{\rho,q}) \times \overset{p}{\cdots} \times \mathfrak{V}^g(\mathcal{S}_{\rho,q}) \longrightarrow \mathcal{S}_{\rho,q}$ and by $\mathfrak{S}_p$ the symmetric group on p letters. Introducing the commutation factor $\gamma_p : \mathfrak{S}_p \times \mathbb{Z}_2 \times \overset{p}{\cdots} \times \mathbb{Z}_2 \longrightarrow \{\pm 1\}$ via

$$\gamma_p(\sigma; \overline{i}_1, \cdots, \overline{i}_p) := \prod_{\substack{r,s=1,\cdots,p;\, r<s \\ \sigma^{-1}(r) > \sigma^{-1}(s)}} (-1)^{\overline{i}_r \overline{i}_s}\,, \tag{100}$$

we can define a representation π of $\mathfrak{S}_p$ on $\mathrm{Hom}^p_{\mathbb{C}}(\mathfrak{V}^g(\mathcal{S}_{\rho,q}); \mathcal{S}_{\rho,q})$ by

$$(\pi_\sigma \omega)(D_1, \cdots, D_p) := \gamma_p(\sigma; \overline{D}_1, \cdots, \overline{D}_p)\omega(D_{\sigma(1)}, \cdots, D_{\sigma(p)}) \tag{101}$$

for all $\omega \in \mathrm{Hom}^p_{\mathbb{C}}(\mathfrak{V}^g(\mathcal{S}_{\rho,q}); \mathcal{S}_{\rho,q}), \sigma \in \mathfrak{S}_p$ and all homogeneous $D_1, \cdots, D_p \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ [50]. Now by definition a p-linear map $\omega \in \mathrm{Hom}^p_{\mathbb{C}}(\mathfrak{V}^g(\mathcal{S}_{\rho,q}); \mathcal{S}_{\rho,q})$ is called graded-alternating if

$$\pi_\sigma \omega = \mathrm{sgn}\sigma\, \omega \tag{102}$$

is fulfilled for all $\sigma \in \mathfrak{S}_p$ and, according to the discussion at the beginning of the section, we should interpret them as p-superforms on the truncated supersphere $\mathcal{S}_{\rho,q}$. The set of all p-superforms on the truncated supersphere $\mathcal{S}_{\rho,q}$ forms a graded vector subspace of $\mathrm{Hom}^p_{\mathbb{C}}(\mathfrak{V}^g(\mathcal{S}_{\rho,q}); \mathcal{S}_{\rho,q})$ and it will be denoted by $\Omega^{g,p}(\mathcal{S}_{\rho,q})$.
A general superform on the truncated supersphere $\mathcal{S}_{\rho,q}$ is an element of the direct sum

$$\Omega^g(\mathcal{S}_{\rho,q}) := \bigoplus_{p \in \mathbb{N}_0} \Omega^{g,p}(\mathcal{S}_{\rho,q})\,, \tag{103}$$

where we set $\Omega^{g,0}(\mathcal{S}_{\rho,q}) := \mathcal{S}_{\rho,q}$. Employing the multiplicative structure of $\mathcal{S}_{\rho,q}$ we can proceed exactly as in the case of graded manifolds [2],[35] (respectively graded Lie-Cartan pairs [27],[28],[46]) to introduce a graded wedge product on $\Omega^g(\mathcal{S}_{\rho,q})$. So we define first for all $p, p' \in \mathbb{N}_0, \overline{i}, \overline{i'} \in \mathbb{Z}_2$ a bilinear map $\wedge : \Omega^{g,p}(\mathcal{S}_{\rho,q})_{\overline{i}} \times \Omega^{g,p'}(\mathcal{S}_{\rho,q})_{\overline{i'}} \longrightarrow \Omega^{g,p+p'}(\mathcal{S}_{\rho,q})_{\overline{i}+\overline{i'}}$ by

$$\begin{aligned}(\omega \wedge \omega')(D_1, \cdots, D_{p+p'}) := \frac{1}{p!p'!} \sum_{\sigma \in \mathfrak{S}_{p+p'}} \mathrm{sgn}\sigma\, \gamma_{p+p'}(\sigma; \overline{D}_1, \cdots, \overline{D}_{p+p'}) \cdot \\ \cdot (-1)^{\overline{i'} \sum_{l=1}^{p} \overline{D}_{\sigma(l)}} \omega(D_{\sigma(1)}, \cdots, D_{\sigma(p)}) \omega'(D_{\sigma(p+1)}, \cdots, D_{\sigma(p+p')})\end{aligned} \tag{104}$$

for all homogeneous $D_1, \cdots, D_{p+p'} \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ and extend these by bilinearity to $\Omega^g(\mathcal{S}_{\rho,q})$. With respect to it $\Omega^g(\mathcal{S}_{\rho,q})$ becomes a $\mathbb{N}_0 \times \mathbb{Z}_2$-bigraded $\mathbb{C}$-algebra.

Having built up the algebra of superforms on each truncated supersphere we can introduce the graded Cartan calculus as one does it for ordinary graded manifolds [35],[2]. As far as we are only interested in the linear structure of $\Omega^g(\mathcal{S}_{\rho,q})$ we are doing nothing else than Lie superalgebra cohomology of $\mathfrak{V}^g(\mathcal{S}_{\rho,q})$ with values in $\mathcal{S}_{\rho,q}$ [18],[52]. So let us follow the excellent article [52] and apply it to our case.

One first extends the action of the $\mathbb{C}$-Lie superalgebra $\mathfrak{V}^g(\mathcal{S}_{\rho,q})$ to $\Omega^g(\mathcal{S}_{\rho,q})$: For every homogeneous $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ one introduces a $\mathbb{C}$-linear map $L_D : \Omega^g(\mathcal{S}_{\rho,q}) \longrightarrow \Omega^g(\mathcal{S}_{\rho,q})$ by defining its action on bihomogeneous $\omega \in \Omega^{g,p}(\mathcal{S}_{\rho,q}), p \in \mathbb{N}_0$, according to

$$\begin{aligned}(L_D\omega)(D_1,\cdots,D_p) \quad := \quad & D(\omega(D_1,\cdots,D_p)) - \\ & -\sum_{l=1}^{p}(-1)^{\overline{D}(\overline{\omega}+\sum_{l'=1}^{l-1}\overline{D}_{l'})}\omega(D_1,\cdots,[D,D_l]_g,\cdots,D_p)\end{aligned} \tag{105}$$

for all homogeneous $D_1,\cdots,D_p \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$. L_D is a bihomogeneous endomorphism of the bigraded $\mathbb{C}$-vector space $\Omega^g(\mathcal{S}_{\rho,q})$ of bidegree $(0,\overline{D})$. A general graded derivation $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ can be uniquely decomposed into its homogeneous components $D_i, i = 0,1$, and $L_D := L_{D_0} + L_{D_1} \in \mathrm{End}_{\mathbb{C}}(\Omega^g(\mathcal{S}_{\rho,q}))$ is well-defined. Moreover, the map $L : \mathfrak{V}^g(\mathcal{S}_{\rho,q}) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Omega^g(\mathcal{S}_{\rho,q}))$, given by $D \mapsto L_D$, is a graded representation of the $\mathbb{C}$-Lie superalgebra $\mathfrak{V}^g(\mathcal{S}_{\rho,q})$ on $\Omega^g(\mathcal{S}_{\rho,q})$.

Besides this extension of $\mathfrak{V}^g(\mathcal{S}_{\rho,q})$ it is useful to introduce for all $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ a $\mathbb{C}$-linear map $\imath_D : \Omega^g(\mathcal{S}_{\rho,q}) \longrightarrow \Omega^g(\mathcal{S}_{\rho,q})$ by

$$\imath_D f := 0 \tag{106}$$

for all $f \in \mathcal{S}_{\rho,q}$ and by

$$(\imath_D\omega)(D_2,\cdots,D_p) := \omega(D,D_2,\cdots,D_p) \tag{107}$$

for all $\omega \in \Omega^{g,p}(\mathcal{S}_{\rho,q}), p \in \mathbb{N}$ and $D_2,\cdots,D_p \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$. For every $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ it is a $\mathbb{N}_0$-homogeneous of degree -1 and the map $\imath : \mathfrak{V}^g(\mathcal{S}_{\rho,q}) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Omega^g(\mathcal{S}_{\rho,q})), D \mapsto \imath_D$ is $\mathbb{C}$-linear and $\mathbb{Z}_2$-even. In addition the relations

$$\imath_D \circ \imath_{D'} + (-1)^{\overline{D}\overline{D'}}\imath_{D'} \circ \imath_D = 0 \tag{108}$$

and

$$(L_D \circ \imath_{D'} - \imath_{D'} \circ L_D)\,\omega = (-1)^{\overline{D}\overline{\omega}}\imath_{[D,D']_g}\omega \tag{109}$$

are fulfilled for all homogeneous $D, D' \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ and all $\mathbb{Z}_2$-homogeneous $\omega \in \Omega^g(\mathcal{S}_{\rho,q})$.

The Lie superalgebra cochain map is most elegantly introduced as $\mathbb{C}$-linear map $d : \Omega^g(\mathcal{S}_{\rho,q}) \longrightarrow \Omega^g(\mathcal{S}_{\rho,q})$, whose action on $\mathbb{Z}_2$-homogeneous 0-superforms $f \in \mathcal{S}_{\rho,q}$ is given by

$$df(D) := (-1)^{\overline{f}\overline{D}}Df \tag{110}$$

for all homogeneous $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ and whose action on $\mathbb{Z}_2$-homogeneous p-superforms $\omega \in \Omega^{g,p}(\mathcal{S}_{\rho,q}), p \in \mathbb{N}$, is defined inductively via

$$\imath_D(d\omega) := (-1)^{\overline{D}\overline{\omega}}L_D\omega - d(\imath_D\omega) \tag{111}$$

for all homogeneous $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$. Then d is a bihomogeneous endomorphism of bidegree $(1,\overline{0})$ fulfilling the cochain condition

$$d \circ d = 0 \tag{112}$$

and commuting with L_D for all $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$. Explicitely one finds

$$d\omega(D_0,\cdots,D_p) = \sum_{l=0}^{p}(-1)^{l+\overline{D}_l(\overline{\omega}+\sum_{l'=0}^{l-1}\overline{D}_{l'})} L_{D_l}\left(\omega(D_0,\cdots,\overset{\vee}{D}_l,\cdots,D_p)\right) + \\ + \sum_{0\le l<l'\le p}(-1)^{l'+\overline{D}_{l'}\sum_{l''=l+1}^{l'-1}\overline{D}_{l''}}\omega(D_0,\cdots,D_{l-1},[D_l,D_{l'}]_g,\cdots,\overset{\vee}{D}_{l'},\cdots,D_p) \tag{113}$$

for homogeneous $D_0,D_1,\cdots,D_p \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ and bihomogeneous $\omega \in \Omega^g(\mathcal{S}_{\rho,q})$ of bidegree $(p,\overline{\omega})$ ($\vee$ denotes omission).

The preceeding discussion can be summarized as follows: As far as one considers only the linear structure of $\Omega^g(\mathcal{S}_{\rho,q})$ the endomorphisms $L_D, \imath_D$ and d fulfill exactly the same relations as Lie derivative, inner product and exterior derivative in the case of graded manifolds. The latter observation stays also true if one considers the graded wedge product (104) on $\Omega^g(\mathcal{S}_{\rho,q})$.

Proposition 5 *The relations*

$$\begin{aligned} L_D\,(\omega\wedge\omega') &= (L_D\omega)\wedge\omega' + (-1)^{\overline{D}\overline{\omega}}\omega\wedge L_D\omega' \\ \imath_D\,(\omega\wedge\omega') &= (-1)^{\overline{D}\overline{\omega'}}(\imath_D\omega)\wedge\omega' + (-1)^p\omega\wedge\imath_D\omega' \\ d\,(\omega\wedge\omega') &= (d\omega)\wedge\omega' + (-1)^p\omega\wedge d\omega' \end{aligned} \tag{114}$$

are fulfilled for all homogeneous $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$ and all bihomogeneous $\omega,\omega' \in \Omega^g(\mathcal{S}_{\rho,q})$ of bidegree $(p,\overline{\omega})$ and $(p',\overline{\omega'})$, respectively.

Proof: This can be shown exactly as in the case of graded manifolds. That is, one starts with a direct proof of the second relation and proofs the other equations inductively using the relations (109) and (111). □

Because of the analogy to the case of graded manifolds (see [35]) we will call the endomorphisms $L_D, \imath_D$ and d of $\Omega^g(\mathcal{S}_{\rho,q})$ Lie derivative and interior product (with respect to the supervector field $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$) as well as exterior derivative.

With respect to the product $\wedge$ the $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $\Omega^{g,p}(\mathcal{S}_{\rho,q}), p \in \mathbb{N}_0$, of p-superforms forms a $\mathbb{Z}_2$-graded $\mathcal{S}_{\rho,q}$-bimodule. In the special case $\Omega^{g,0}(\mathcal{S}_{\rho,q}) \equiv \mathcal{S}_{\rho,q}$ this $\mathbb{Z}_2$-graded (left as well as right) module is graded-free with homogeneous basis $\{\mathrm{Id}_{V(\frac{q}{2},\overline{1})}\}$, of course. In order to investigate the other $\mathbb{Z}_2$-graded $\mathcal{S}_{\rho,q}$-bimodules $\Omega^{g,p}(\mathcal{S}_{\rho,q}), p \in \mathbb{N}$, a little bit closer, let $\{\ E_k \in \mathfrak{osp}(1|2)_{\overline{0}}, E_\alpha \in \mathfrak{osp}(1|2)_{\overline{1}} \mid k = 1,2,3; \alpha = 4,5\ \}$ be some homogeneous basis of $\mathfrak{osp}(1|2)$ and

$$\partial_{q,A} := \mathrm{ad}^{(\frac{q}{2})}E_A\,, \qquad A = 1,\cdots,5, \tag{115}$$

the elements of the corresponding homogeneous basis of $\mathfrak{V}^g(\mathcal{S}_{\rho,q})$. Denoting by λ_q^A the elements of the dual basis to $\{\partial_{q,A}\}$ we can introduce homogeneous 1-superforms $\lambda_q^A \in \Omega^{g,1}(\mathcal{S}_{\rho,q}), A = 1,\cdots,5$, via

$$\lambda_q^A(D) := \lambda_q^A(D)\mathrm{Id}_{V(\frac{q}{2},\overline{1})} \tag{116}$$

for all $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$. By applying both sides on supervector fields we find

$$\begin{aligned} \lambda_q^A\wedge\lambda_q^B &= -(-1)^{\overline{\lambda}_q^A\overline{\lambda}_q^B}\lambda_q^B\wedge\lambda_q^A \\ f\wedge\lambda_q^{A_1}\wedge\cdots\wedge\lambda_q^{A_p} &= (-1)^{\overline{f}\sum_{l=1}^{p}\overline{\lambda}_q^{A_l}}\lambda_q^{A_1}\wedge\cdots\wedge\lambda_q^{A_p}\wedge f \end{aligned} \tag{117}$$

for all homogeneous $f \in \mathcal{S}_{\rho,q}$ and all $A, B, A_1, \cdots, A_p = 1, \cdots, 5, p \in \mathbb{N}$, as well as

$$d\lambda_q^A = \frac{1}{2} \sum_{B,C=1}^{5} c_{BC}^A \lambda_q^C \wedge \lambda_q^B , \tag{118}$$

where c_{BC}^A denote the $\mathfrak{osp}(1|2)$-structure constants corresponding with the basis $\{E_A\}$. Introducing the graded vector subspace $\Omega_{\mathcal{Z}^g}^{g,p}(\mathcal{S}_{\rho,q}), p \in \mathbb{N}$, of all p-superforms with values in the graded center $\mathcal{Z}^g(\mathcal{S}_{\rho,q})$ of $\mathcal{S}_{\rho,q}$, we can conclude from (117) (or from the usual isomorphisms between graded-alternating maps and the graded exterior algebra [2],[50],[51],[52]), that

$$\left\{ \lambda_q^{A_1} \wedge \cdots \wedge \lambda_q^{A_p} \mid (A_1, \cdots, A_p) \in \mathfrak{I}_p^{(3|2)} \right\} \tag{119}$$

with

$$\begin{aligned} \mathfrak{I}_p^{(3|2)} := \{ \ & (A_1, \cdots, A_{p'}, A_{p'+1}, \cdots, A_p) \mid 0 \le p' \le p; A_1, \cdots, A_{p'} = 1, 2, 3; \\ & A_{p'+1}, \cdots, A_p = 4, 5; A_1 < A_2 < \cdots < A_{p'} < A_{p'+1} \le \cdots \le A_{p-1} \le A_p \ \} \end{aligned} \tag{120}$$

forms a homogeneous basis of $\Omega_{\mathcal{Z}^g}^{g,p}(\mathcal{S}_{\rho,q})$. Moreover the bigraded subalgebra of $\Omega^{g,p}(\mathcal{S}_{\rho,q})$,

$$\Omega_{\mathcal{Z}^g}^{g}(\mathcal{S}_{\rho,q}) := \bigoplus_{p \in \mathbb{N}_0} \Omega_{\mathcal{Z}^g}^{g,p}(\mathcal{S}_{\rho,q}) \tag{121}$$

with $\Omega_{\mathcal{Z}^g}^{g,0}(\mathcal{S}_{\rho,q}) := \mathcal{Z}^g(\mathcal{S}_{\rho,q})$, is stable under the Lie derivative, the interior product as well as under exterior differentiation according to (118).
¿From the preceding discussion we can conclude in particular, that the $\mathbb{Z}_2$-graded $\mathcal{S}_{\rho,q}$-modules $\Omega^{g,p}(\mathcal{S}_{\rho,q}), p \in \mathbb{N}$, are graded-free (as left and as right modules) and a homogeneous basis is given by (119). Consequently every $\omega \in \Omega^{g,p}(\mathcal{S}_{\rho,q})$ can be written as

$$\omega = \sum_{(A_1,\cdots,A_p) \in \mathfrak{I}_p^{(3|2)}} \omega_{A_1 \cdots A_p} \wedge \lambda_q^{A_1} \wedge \cdots \wedge \lambda_q^{A_p} \tag{122}$$

and the unique coefficients $\omega_{A_1 \cdots A_p} \in \mathcal{S}_{\rho,q}$ are given by

$$\omega_{A_1 \cdots A_p} = (-1)^{\frac{1}{2} p''(p''-1)} \frac{1}{\prod_{A=1}^{5} N_A!} \omega(\partial_{q,A_1}, \cdots, \partial_{q,A_p}) , \tag{123}$$

where p'' is the number of entries in $(A_1, \cdots, A_p)$ greater than 3 and N_A is the number of entries in $(A_1, \cdots, A_p)$ being equal A.

Similar to the case of matrix geometry ([13],[37]) it is possible to introduce via

$$\Lambda_q := \sum_{A=1}^{5} E_A^{(\frac{q}{2})} \wedge \lambda_q^A \tag{124}$$

an even 1-superform on each truncated supersphere $\mathcal{S}_{\rho,q}, q \in \mathbb{N}_0$, which is invariant and fulfills super-version of the Maurer-Cartan equation.

Proposition 6 *The definition of Λ_q is independent of the choice of the homogeneous basis of $\mathfrak{osp}(1|2)$. Λ_q is invariant and up to complex multiples it is the only invariant 1-superform on $\mathcal{S}_{\rho,q}, q \in \mathbb{N}_0$. Moreover, its exterior differential fulfills*

$$d\Lambda_q = \Lambda_q \wedge \Lambda_q \tag{125}$$

and the exterior differential of each $f \in \Omega^{g,0}(\mathcal{S}_{\rho,\mathcal{S}_{\rho,q}})$ can be written according to

$$df = [\Lambda_q, f]_g \equiv \Lambda_q \wedge f - f \wedge \Lambda_q \,. \tag{126}$$

Proof : Beside the uniqueness statement only simple calculations are involved. In order to see, that Λ_q is the only invariant 1-superform on $\mathcal{S}_{\rho,q}$ it is important to note, that $\Omega^{g,1}_{\mathcal{Z}^g}(\mathcal{S}_{\rho,q})$ is an irreducible, $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-module with highest weight 1 and that the $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-module $\Omega^{g,1}(\mathcal{S}_{\rho,q})$ is isomorphic to the tensor product of the $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules $\mathcal{S}_{\rho,q}$ and $\Omega^{g,1}_{\mathcal{Z}^g}(\mathcal{S}_{\rho,q})$. Then the uniqueness of Λ_q follows from proposition 2 and the "Clebsch-Gordan decomposition" of tensor products of (irreducible) $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules [4]. □

If X is a $\mathcal{H}^\infty$-deWitt supermanifold with body X the body map β_X extends to an algebra homomorphism and cochain map from the super-deRham complex of X to the ordinary deRham complex on the body manifold X [35]. Because the noncommutative body map $\beta_{\mathcal{S}_{\rho,q}} : \mathcal{S}_{\rho,q} \longrightarrow \mathbb{S}_{\rho,q}, q \in \mathbb{N}$, is no algebra homomorphism, some extension from the algebra of superforms on the truncated supersphere to the algebra of forms on the truncated sphere cannot be an algebra homomorphism. But by translating the construction of $\mathcal{H}^\infty$-deWitt supermanifolds (respectively graded manifolds) we will introduce a cochain map, which we can interpret as noncommutative analogue to the extension of β_X in the graded-commutative setting. Let us give first of all definition and basic results of the Cartan calculus (see [13],[39],[37]) on the truncated sphere $\mathbb{S}_{\rho,q}, q \in \mathbb{N}$, as far as they are relevant for the subsequent discussion of the body map of superforms on the truncated supersphere $\mathcal{S}_{\rho,q}$.
For every $p \in \mathbb{N}$ a p-form on the truncated sphere $\mathbb{S}_{\rho,q}$ is a p-linear, alternating map $\mathfrak{V}(\mathbb{S}_{\rho,q}) \times \overset{p}{\cdots} \times \mathfrak{V}(\mathbb{S}_{\rho,q}) \longrightarrow \mathbb{S}_{\rho,q}$ and we denote by $\Omega^p(\mathbb{S}_{\rho,q})$ the $\mathbb{C}$-vector space of all p-forms. A general form on the truncated sphere $\mathbb{S}_{\rho,q}$ is an element of the direct sum

$$\Omega(\mathbb{S}_{\rho,q}) := \bigoplus_{p \in \mathbb{N}_0} \Omega^p(\mathbb{S}_{\rho,q}) \tag{127}$$

with $\Omega^0(\mathbb{S}_{\rho,q}) := \mathbb{S}_{\rho,q}$. $\Omega(\mathbb{S}_{\rho,q})$ becomes a $\mathbb{N}_0$-graded $\mathbb{C}$-algebra, if one introduces for all $p, p' \in \mathbb{N}_0$ a bilinear map $\hat{\wedge} : \Omega^p(\mathbb{S}_{\rho,q}) \times \Omega^{p'}(\mathbb{S}_{\rho,q}) \longrightarrow \Omega^{p+p'}(\mathbb{S}_{\rho,q})$ by

$$(\hat{\omega} \hat{\wedge} \hat{\omega}') (\mathrm{D}_1, \cdots, \mathrm{D}_{p+p'}) := \frac{1}{p!p'!} \sum_{\sigma \in \mathfrak{S}_{p+p'}} \operatorname{sgn}\sigma \, \hat{\omega}(\mathrm{D}_{\sigma(1)}, \cdots, \mathrm{D}_{\sigma(p)}) \hat{\omega}'(\mathrm{D}_{\sigma(p+1)}, \cdots, \mathrm{D}_{\sigma(p+p')}) \tag{128}$$

for all $\mathrm{D}_1, \cdots, \mathrm{D}_{p+p'} \in \mathfrak{V}(\mathbb{S}_{\rho,q})$, and extends these by bilinearity. Analogous to the graded case the set $\Omega^p_{\mathcal{Z}}(\mathbb{S}_{\rho,q}), p \in \mathbb{N}$, of p-forms with values in the center $\mathcal{Z}(\mathbb{S}_{\rho,q}) = \mathbb{C}\mathrm{Id}_{V(\frac{q}{2})}$ of $\mathbb{S}_{\rho,q}$ forms a vector subspace of $\Omega^p(\mathbb{S}_{\rho,q})$ and $\Omega_{\mathcal{Z}}(\mathbb{S}_{\rho,q}) := \oplus_{p \in \mathbb{N}_0} \Omega^p_{\mathcal{Z}}(\mathbb{S}_{\rho,q}), \Omega^0_{\mathcal{Z}}(\mathbb{S}_{\rho,q}) := \mathbb{S}_{\rho,q}$, is a graded subalgebra of $\Omega(\mathbb{S}_{\rho,q})$.
Exterior differential d, Lie derivative L_D as well as interior product $\imath_\mathrm{D}$ (with respect to a vector field $\mathrm{D} \in \mathfrak{V}(\mathbb{S}_{\rho,q})$) are defined exactly as in the case of the truncated supersphere, if one views $\mathfrak{V}(\mathbb{S}_{\rho,q})$ and $\Omega(\mathbb{S}_{\rho,q})$ as trivially $\mathbb{Z}_2$-graded.
For some basis $\{\mathrm{E}_k | k = 1, 2, 3\}$ of $\mathfrak{sl}(2)$ one can introduce a basis $\{\hat{\partial}_{q,k} | k = 1, 2, 3\}$ of $\mathfrak{V}(\mathbb{S}_{\rho,q})$ as well as 1-forms $\hat{\lambda}^k_q \in \Omega^1_{\mathcal{Z}}(\mathbb{S}_{\rho,q})$ analogous to (115) and (116). The latter 1-forms anticommute

and one finds, that the set of p-forms

$$\left\{ \hat{\lambda}_q^{k_1} \hat{\wedge} \cdots \hat{\wedge} \hat{\lambda}_q^{k_p} \,|(k_1, \cdots, k_p) \in \mathfrak{I}_p^{(3)} \right\} \tag{129}$$

with

$$\mathfrak{I}_p^{(3)} := \{ \ (k_1, \cdots, k_p) \mid k_1, \cdots, k_p = 1,2,3; k_1 < \cdots < k_p \ \} \tag{130}$$

forms for all $p \in \mathbb{N}$ (trivially for $p > 3$, of course) a basis of the (left and right) $\mathbb{S}_{\rho,q}$-module $\Omega^p(\mathbb{S}_{\rho,q})$ as well as of the $\mathbb{C}$-vector space $\Omega^p_{\mathcal{Z}}(\mathbb{S}_{\rho,q})$.

In analogy to the case of graded manifolds [35] we can introduce now maps $\beta^{(p)}_{\mathcal{S}_{\rho,q}} : \Omega^{g,p}(\mathcal{S}_{\rho,q}) \longrightarrow \Omega^p(\mathbb{S}_{\rho,q}), p \in \mathbb{N}_0$, via

$$\left(\beta^{(p)}_{\mathcal{S}_{\rho,q}}(\omega)\right) \left(\tilde{\beta}_{\mathcal{S}_{\rho,q}}(D_1), \cdots, \tilde{\beta}_{\mathcal{S}_{\rho,q}}(D_p)\right) := \beta_{\mathcal{S}_{\rho,q}}\left(D_1, \cdots, D_p\right) \tag{131}$$

for all $D_1, \cdots, D_p \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})$. In the case $p = 0$ (131) is to be understood as $\beta^{(0)}_{\mathcal{S}_{\rho,q}} = \beta_{\mathcal{S}_{\rho,q}}$, of course. For each $p \in \mathbb{N}_0$ it is an even $\mathbb{C}$-linear map and we can define its $\mathbb{C}$-linear extension $\Omega^g(\mathcal{S}_{\rho,q}) \longrightarrow \Omega(\mathbb{S}_{\rho,q})$, which we again denote by $\beta_{\mathcal{S}_{\rho,q}}$.

Proposition 7 *$\beta_{\mathcal{S}_{\rho,q}}$ is a surjective $\mathfrak{sl}(2)$-module homomorphism and cochain map, whose restriction to $\Omega^g_{\mathcal{Z}^g}(\mathcal{S}_{\rho,q})$ is an algebra homomorphism onto $\Omega_{\mathcal{Z}}(\mathbb{S}_{\rho,q})$.*

Proof : Using the $\mathfrak{sl}(2)$-module homomorphism property of the body map and the fact, that $\tilde{\beta}_{\mathcal{S}_{\rho,q}}$ is a Lie algebra isomorphism one finds immediately

$$\begin{aligned} &\mathrm{d} \circ \beta_{\mathcal{S}_{\rho,q}} = \beta_{\mathcal{S}_{\rho,q}} \circ d \\ &\mathrm{L}_{\tilde{\beta}_{\mathcal{S}_{\rho,q}}(D)} \circ \beta_{\mathcal{S}_{\rho,q}} = \beta_{\mathcal{S}_{\rho,q}} \circ L_D \end{aligned} \tag{132}$$

for all $D \in \mathfrak{V}^g(\mathcal{S}_{\rho,q})_{\overline{0}}$. The action of the extension of the body map on a p-superform $\omega \in \Omega^{g,p}(\mathcal{S}_{\rho,q})$ can be described alternatively by

$$\begin{aligned} \beta_{\mathcal{S}_{\rho,q}}(\omega) \equiv \beta_{\mathcal{S}_{\rho,q}} &\left(\sum_{(A_1,\cdots,A_p)\in \mathfrak{I}_p^{(3|2)}} \omega_{A_1\cdots A_p} \wedge \lambda_q^{A_1} \wedge \cdots \wedge \lambda_q^{A_p} \right) = \\ = &\sum_{(A_1,\cdots,A_p)\in \mathfrak{I}_p^{(3)}} \beta_{\mathcal{S}_{\rho,q}}\left(\omega_{A_1\cdots A_p}\right) \hat{\wedge} \hat{\lambda}_q^{A_1} \hat{\wedge} \cdots \hat{\wedge} \hat{\lambda}_q^{A_p}, \end{aligned} \tag{133}$$

where $\lambda_q^k \in \Omega^{g,1}(\mathcal{S}_{\rho,q})$ and $\hat{\lambda}_q^k \in \Omega^1(\mathbb{S}_{\rho,q})$ correspond to bases in $\mathfrak{osp}(1|2)_{\overline{0}}$ and $\mathfrak{sl}(2)$, which are related by the canonical isomorphism. From (132) we can conclude in particular, that $\beta_{\mathcal{S}_{\rho,q}}$ is surjective and that its restriction to $\Omega^g_{\mathcal{Z}^g}(\mathcal{S}_{\rho,q})$ is an algebra homomorphism. □

The interpretation of the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras and $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules $\mathcal{S}_{\rho,q}$ stems from the fact, that there are suitable $\mathfrak{osp}(1|2)$-module homomorphisms $\eta_{q'q} : \mathcal{S}_{\rho,q} \longrightarrow \mathcal{S}_{\rho,q'}, q, q' \in \mathbb{N}, q \leq q'$, constituting a directed system, whose direct limit can be identified with the graded subalgebra $\mathcal{P}(S_\rho) \subseteq \mathcal{H}^\infty(S_\rho, C_L)$ and which possesses a graded-commutative limit. We want to understand noncommutative p-superforms for $p \in \mathbb{N}$ in a similar way as noncommutative pendants to p-superforms on the $(2|2)$-dimensional supersphere S_ρ. Although we

reserve the precise treatment of this question to a subsequent paper we give some indications how this is done.
The first thing one has to find are maps $\eta^{(p)}_{q'q} : \Omega^{g,p}(\mathcal{S}_{\rho,q}) \longrightarrow \Omega^{g,p}(\mathcal{S}_{\rho,q'}), q, q' \in \mathbb{N}, q \leq q'$, such that $(\Omega^{g,p}(\mathcal{S}_{\rho,q}), \eta^{(p)}_{q'q}), p \in \mathbb{N}_0$, become directed systems, which are "compatible with the Cartan calculus". A natural choice for these maps (especially in consideration of a "graded-commutative limit" of superforms) is

$$\eta^{(p)}_{q'q}(\omega) \equiv \eta^{(p)}_{q'q}\left(\sum_{(A_1,\cdots,A_p)\in\mathfrak{I}^{(3|2)}_p} \omega_{A_1\cdots A_p} \wedge \lambda^{A_1}_q \wedge \cdots \wedge \lambda^{A_p}_q\right) := \\ := \sum_{(A_1,\cdots,A_p)\in\mathfrak{I}^{(3)}_p} \eta_{q'q}\left(\omega_{A_1\cdots A_p}\right) \hat{\wedge}\hat{\lambda}^{A_1}_{q'} \hat{\wedge} \cdots \hat{\wedge}\hat{\lambda}^{A_p}_{q'} . \tag{134}$$

Then $(\Omega^{g,p}(\mathcal{S}_{\rho,q}), \eta^{(p)}_{q'q}), p \in \mathbb{N}_0$, are directed systems of $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules, whose direct limits should be connected with the algebra of superforms on the (2|2)-dimensional supersphere. This problem is in fact in exactly the same way "singular" as in the case of the ordinary fuzzy sphere [39],[37]. It is more natural to interpret the elements of the direct limits as superforms on the supergroup UOSP(1|2) (see [3],[4], for example), which is a superfibre bundle over the (2|2)-dimensional supersphere.

7 Cohomological considerations

According to a general theorem of [35] the super-deRham cohomology of the (2|2)-dimensional supersphere is isomorphic to the deRham cohomology of the 2-dimensional sphere. The isomorphism is induced by extension of the body map (in the sense described in the preceding chapter) to the algebra of superforms on the (2|2)-dimensional supersphere. As we will see exactly the same is true on every truncated supersphere.
The cohomology

$$H(\mathcal{S}_{\rho,q}) \equiv \bigoplus_{p\in\mathbb{N}_0} H^p(\mathcal{S}_{\rho,q}) := \frac{\ker d}{\operatorname{im} d} \tag{135}$$

of the complex $(\Omega^g(\mathcal{S}_{\rho,q}), d)$ is our substitute for the (complexified) super-deRham cohomology on the truncated supersphere $\mathcal{S}_{\rho,q}, q \in \mathbb{N}$, as well as the cohomology

$$H(\mathbb{S}_{\rho,q}) \equiv \bigoplus_{p\in\mathbb{N}_0} H^p(\mathbb{S}_{\rho,q}) := \frac{\ker \mathrm{d}}{\operatorname{im} \mathrm{d}} \tag{136}$$

of the complex $(\Omega(\mathbb{S}_{\rho,q}), \mathrm{d})$ is the noncommutative pendant to the (complexified) deRham cohomology of the sphere. By construction $H(\mathcal{S}_{\rho,q})$ is the Lie superalgebra cohomology of $\mathfrak{V}^g(\mathcal{S}_{\rho,q})$ with coefficients in $\mathcal{S}_{\rho,q}$, while $H(\mathbb{S}_{\rho,q})$ is the Lie algebra cohomology of $\mathfrak{V}(\mathbb{S}_{\rho,q})$ with coefficients in $\mathbb{S}_{\rho,q}$.
The cochain map $\beta_{\mathcal{S}_{\rho,q}}$ induces as usual via

$$H(\beta_{\mathcal{S}_{\rho,q}}) : H(\mathcal{S}_{\rho,q}) \longrightarrow H(\mathbb{S}_{\rho,q}) \\ [\omega] \mapsto [\beta_{\mathcal{S}_{\rho,q}}(\omega)] \tag{137}$$

a homomorphism of $\mathbb{N}_0$-graded $\mathbb{C}$-vector spaces. Because $\beta_{\mathcal{S}_{\rho,q}}$ is surjective and even, the same is true for $H(\beta_{\mathcal{S}_{\rho,q}})$.

Proposition 8 *$H(\beta_{\mathcal{S}_{\rho,q}})$ is an isomorphism of bigraded $\mathbb{C}$-vector spaces and we have explicitely*

$$H^p(\mathcal{S}_{\rho,q}) \cong H^p(\mathbb{S}_{\rho,q}) \cong \begin{cases} \mathbb{C}\,, & p = 0,3 \\ \{0\}\,, & p \in \mathbb{N}_0 \setminus \{0,3\}\,. \end{cases} \tag{138}$$

Proof : Because the (graded) representations $\mathrm{ad}^{(\frac{q}{2})}$ are faithful, $H^p(\mathcal{S}_{\rho,q})$ is isomorphic to the Lie superalgebra cohomology $H(\mathfrak{osp}(1|2);\mathcal{S}_{\rho,q})$ of $\mathfrak{osp}(1|2)$ with coefficients in $\mathcal{S}_{\rho,q}$ as well as $H^p(\mathbb{S}_{\rho,q})$ is isomorphic to the Lie algebra cohomology $H(\mathfrak{sl}(2);\mathbb{S}_{\rho,q})$ of $\mathfrak{sl}(2)$ with coefficients in $\mathbb{S}_{\rho,q}$. Moreover, using the direct sum decompositions (65) and (83), we find [52]

$$H^p(\mathcal{S}_{\rho,q}) \cong \bigoplus_{\substack{j\in\frac{1}{2}\mathbb{N}_0 \\ j\leq q}} H(\mathfrak{osp}(1|2); V^{(\frac{q}{2})\,j})$$

and

$$H^p(\mathbb{S}_{\rho,q}) \cong \bigoplus_{\substack{j\in\mathbb{N}_0 \\ j\leq q}} H(\mathfrak{sl}(2); \mathrm{V}^{(\frac{q}{2})\,j})\,.$$

The standard second-order Casimir operator of $\mathfrak{osp}(1|2)$, restricted to $V^{(\frac{q}{2})\,j}, j \neq 0$, has a nonvanishing eigenvalue and there are no nontrivial $\mathfrak{osp}(1|2)$-invariant elements in $V^{(\frac{q}{2})\,j}, j \neq 0$, such that we can conclude [52]

$$H^p(\mathcal{S}_{\rho,q}) \cong H(\mathfrak{osp}(1|2); V^{(\frac{q}{2})\,0}) \cong H(\mathfrak{osp}(1|2);\mathbb{C})\,,$$

where $H(\mathfrak{osp}(1|2);\mathbb{C})$ denotes the Lie superalgebra cohomology of $\mathfrak{osp}(1|2)$ with trivial coefficients. The same argument leads in the case of the truncated sphere to

$$H^p(\mathbb{S}_{\rho,q}) \cong H(\mathfrak{sl}(2);\mathbb{C})\,.$$

But these cohomologies with trivial coefficients are known to be given by (138) (see [18],[19] for the case of $\mathfrak{osp}(1|2)$ and [20] for $\mathfrak{sl}(2)$). $\square$

One should note, that (138) is not exactly the super-deRham cohomology (deRham cohomology) of the (2|2)-dimensional supersphere (2-dimensional sphere). The explicite result is rather a cohomological verification of the remarks we made at the end of the preceding chapter about the singular character of the identification of the noncommutative "limit superforms" with superforms on the (2|2)-dimensional supersphere.

8 Concluding Remarks

In close analogy to the construction of the fuzzy sphere and its derivation-based differential calculus we have introduced the fuzzy supersphere together with a differential calculus, which is based on the Lie superalgebra $\mathfrak{osp}(1|2)$ acting on each of the truncated superspheres via

graded derivations. The natural interpretation of the fuzzy sphere as noncommutative body of the fuzzy supersphere guaranteed the existence of the usual relations between $\mathcal{H}^\infty$-deWitt supermanifolds and their bodies. In particular the noncommutative body projection induced in the same way as in the theory of graded manifolds the isomorphism between the ((graded) derivation-based) cohomologies of the fuzzy supersphere and its body.
The present work can be seen as first step towards the development of the differential geometry and the formulation of (quantum) field theoretical models on a non-trivial, (fuzzy) noncommutative supermanifold. From the point of view of pure supergeometry the introduction of metric and supervector bundle concepts as well as the investigation of their "limits" are the next challenging tasks. Passing from mathematics to (quantum) physical model building we will have to undertake a "categorial jump" from $\mathbb{Z}_2$-graded algebras of $\mathcal{H}^\infty$-functions and their noncommutative approximations to larger classes of "functions". In our opinion the most promising program in that direction is a G-extension, graded-commutative and noncommutative as well.

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