Invariants and Labels in Lie–Poisson Systems

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Abstract

Reduction is a process that uses symmetry to lower the order of a Hamiltonian system. The new variables in the reduced picture are often not canonical: there are no clear variables representing positions and momenta, and the Poisson bracket obtained is not of the canonical type. Specifically, we give two examples that give rise to brackets of the noncanonical Lie–Poisson form: the rigid body and the two-dimensional ideal fluid. From these simple cases, we then use the *semidirect product* extension of algebras to describe more complex physical systems. The Casimir invariants in these systems are examined, and some are shown to be linked to the recovery of information about the configuration of the system. We discuss a case in which the extension is not a semidirect product, namely compressible reduced MHD, and find for this case that the Casimir invariants lend partial information about the configuration of the system.

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This paper explores the Casimir invariants of Lie–Poisson brackets, which generate the dynamics of some discrete and continuous Hamiltonian systems. Lie–Poisson brackets are a type of noncanonical Poisson bracket and are ubiquitous in the reduction of canonical Hamiltonian systems with symmetry. Casimir invariants are constants of motion for all Hamiltonians; they are associated with the degeneracy of noncanonical Poisson brackets. Finite-dimensional examples of systems described by Lie–Poisson brackets include the heavy top and the moment reduction of the Kida vortex, while infinite-dimensional examples include the 2–D ideal fluid, reduced magnetohydrodynamics (MHD), and the 1–D Vlasov equation. (See Ref. 1 and references therein for a full review.) The Casimir invariants determine the manifold on which the system is kinematically constrained to evolve. Understanding the nature of these constraints is thus of paramount importance.

In Section I we examine specific Lie–Poisson brackets, namely, those that arise from the reduction to Eulerian variables of a Lagrangian system with relabeling symmetry. We make use of two prototypical examples, the rigid body (finite-dimensional) and the 2–D ideal fluid (infinite-dimensional), and we interpret their Casimir invariants. This is done to motivate the introduction of such brackets and to show their physical relevance. In Section II we turn to building Lie–Poisson brackets directly from Lie algebras by the procedure of extension. We introduce the semidirect product extension and illustrate it with two physical examples: the heavy top and low-beta reduced MHD. Finally in Section III we look at a nonsemidirect example, compressible reduced MHD, and discuss work in progress.

I. LIE–POISSON BRACKETS AND REDUCTION

For our purposes, a reduction is a mapping of the dynamical variables of a system to a smaller set of variables, such that the transformed Hamiltonian and bracket depend only on the smaller set of variables. (See for example Ref. 2 for a detailed treatment.) The simplest example of a reduction is the case in which a cyclic variable is eliminated, but more generally a reduction exists as a consequence of an underlying symmetry of the system. We present two examples of reduction.

A. Reduction of the Free Rigid Body

The Hamiltonian for the free rigid body is an unwieldy function of three Euler angles ϕ, ψ, θ and their conjugate momenta p_ϕ, p_ψ, p_θ. The motion is described by Hamilton's equations using the canonical bracket

$$\{f\,,g\}_{\mathrm{can}} = \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial p_\phi} - \frac{\partial g}{\partial \phi}\frac{\partial f}{\partial p_\phi} + \frac{\partial f}{\partial \psi}\frac{\partial g}{\partial p_\psi} - \frac{\partial g}{\partial \psi}\frac{\partial f}{\partial p_\psi} + \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial p_\theta} - \frac{\partial g}{\partial \theta}\frac{\partial f}{\partial p_\theta}.$$

Here we have 3 degrees of freedom (6 coordinates), the configuration space is the rotation group $SO(3)$, and the phase space is its contangent bundle $T^*SO(3)$.

A reduction is possible for this system. In terms of angular momenta ℓ_i about the principal axes, we have

$$H(\phi, \psi, \theta, p_\phi, p_\psi, p_\theta) \;\longrightarrow\; H(\ell_1, \ell_2, \ell_3) = \sum_{i=1}^{3} \frac{\ell_i^2}{2I_i}\,.$$

Under this (noncanonical) mapping, the bracket obtains the Lie–Poisson form

$$\{f\,,g\} = -\ell \cdot \frac{\partial f}{\partial \ell} \times \frac{\partial g}{\partial \ell}\,.$$

The equations of motion generated by this bracket and $H(\ell_1, \ell_2, \ell_3)$ are Euler's equations for the rigid body,

$$\dot{\ell}_i = \frac{I_j - I_k}{I_j\, I_k}\, \ell_j\, \ell_k\,,$$

where i, j, k are cyclic permutations of $1, 2, 3$. The energy is conserved, and so is the quantity

$$C(\ell) = \sum_{i=1}^{3} \ell_i^2\,,$$

which commutes with any function of ℓ. Such a functions are called Casimir invariants (or Casimirs for short). Casimirs are conserved quantities for any Hamiltonian, so they tell us about the topology of the manifold on which the motion takes place. For the simple case of the rigid body, the motion takes place on the two-sphere, S^2 (not in physical space, but in angular momentum space). The symmetry that permits the reduction is the invariance of the equations of motion for $(\phi, \psi, \theta, p_\phi, p_\psi, p_\theta)$ under rotations (elements of $SO(3)$). This symmetry amounts to the freedom of choosing axes from which the Euler angles are measured. In that sense it is a relabeling symmetry, since the choice of axes amounts to making "marks," or labels, on the rigid body.

We shall say that the original system is a Lagrangian description (by analogy with the fluid case below) because at any time the exact configuration of the system (including orientation) is known, whereas the reduced system is Eulerian because only the angular momentum of the body is known.

B. Reduction of the 2–D Ideal Fluid

As our prototype for reduction in an infinite dimensional system we take an ideal 2–D fluid confined to some domain, D. The Lagrangian description involves fluid elements labeled by some coordinate a, which is usually taken to be the initial position of the fluid elements. These labels are analogous to the choice of axes in the rigid body example above (the "marks" on the rigid body). The Hamiltonian functional is

$$H[q;\pi] = \int_D \left(\frac{\pi^2}{2\rho_0} - p(a,t) \left(\left| \frac{\partial q}{\partial a} \right| - 1 \right) \right)\, d^2a,$$

where $q(a,t)$ is the position of the fluid element labeled by a and $\pi(a,t)$ is its momentum. The Jacobian of the transformation from the labels a to the position of the fluid elements at a later time is $|\partial q/\partial a|$. The density ρ_0 is taken to be constant, and the pressure $p(a,t)$ appears here as a Lagrange multiplier that enforces the incompressibility condition, $|\partial q/\partial a| = 1$ (see Ref. 3). This Hamiltonian together with the canonical bracket

$$\{F\,,G\}_{\mathrm{can}} = \int_D \left(\frac{\delta F}{\delta q}\frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q}\frac{\delta F}{\delta \pi} \right)\, d^2a$$

generates the equations of motion for a fluid in Lagrangian variables. The information about the position of every fluid element at any time is contained in the model. The dynamical evolution of the system is independent of the particular choice of labels for the fluid elements. This *relabeling* symmetry of the initial condition labels, a, suggests a reduction.

We introduce the streamfunction ϕ defined by $v(\mathbf{x},t) = (-\partial_y\,\phi, \partial_x\,\phi)$, so that $\nabla\cdot v = 0$ is automatically satisfied, and the vorticity $\omega(\mathbf{x},t) = \nabla^2\phi$. The mapping from the Lagrangian momentum to the Eulerian velocity field is

$$\rho_0 v(\mathbf{x},t) = \int_D \pi(a,t)\,\delta(\mathbf{x} - q(a,t))\,d^2a\,.$$

We take $\rho_0 = 1$ for simplicity. Taking the curl of v and dotting with $\hat{z}$ gives

$$\omega(\mathbf{x},t) = \hat{z}\cdot\nabla\times v(\mathbf{x},t) = \int_D \epsilon_{ij}\,\pi_i(a,t)\,\partial_j\delta(\mathbf{x} - q(a,t))\,d^2a\,,$$

where repeated indices are summed, $\partial_i := \partial/\partial x_i$, and the antisymmetric symbol ϵ_{ij} is defined by $\epsilon_{12} = 1$. The variation of ω is

$$\delta\omega(\mathbf{x},t) = \int_D \left(\epsilon_{ij}\,\delta\pi_i(a,t)\,\partial_j\delta(\mathbf{x} - q(a,t)) - \epsilon_{ij}\,\pi_i(a,t)\,\partial_j\partial_k\delta(\mathbf{x} - q(a,t))\,\delta q_k(a,t)\right)\,d^2a\,,$$

which we can insert into

$$\delta F = \int_D \frac{\delta F}{\delta\omega}\,\delta\omega\,d^2x = \int_D \left(\frac{\delta F}{\delta\pi}\,\delta\pi + \frac{\delta F}{\delta q}\,\delta q\right)\,d^2a$$

to find

$$\begin{aligned}
\frac{\delta F}{\delta\pi} &= \hat{z}\times\nabla\left(\frac{\delta F}{\delta\omega}\right)\bigg|_{\mathbf{x}=q(a,t)}\,,\\
\frac{\delta F}{\delta q} &= -\int_D \epsilon_{ij}\,\pi_i(a,t)\,\partial_j\nabla\delta(\mathbf{x} - q(a,t))\,\frac{\delta F}{\delta\omega}\,d^2x.
\end{aligned}$$

We then insert these two expressions into the canonical bracket. After some manipulation involving integration by parts (we assume boundary terms vanish) we obtain the Lie–Poisson bracket

$$\{F\,,G\} = \int_D \omega\,\left[\frac{\delta F}{\delta\omega}\,,\frac{\delta G}{\delta\omega}\right]\,d^2x\,,$$

where

$$[f\,,g] := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\,.$$

Note that the incompressibility of the fluid (the fact that the Jacobian $|\partial q/\partial a|$ is unity) was not used in the derivation of this bracket. However, in order to write the Hamiltonian in terms of ω one must introduce the streamfunction ϕ, which is possible only if $\nabla \cdot v = 0$. The equation of motion generated by the bracket and the transformed Hamiltonian

$$H[\omega] = -\frac{1}{2} \int_D \phi\, \omega \; d^2x = \frac{1}{2} \int_D |\nabla \phi|^2 \; d^2x$$

is just Euler's equation for an the ideal fluid

$$\dot{\omega}(\mathbf{x}) = -[\phi \,, \omega] \,.$$

This has a Casimir given by

$$C[\omega] = \int_D f\left(\omega(\mathbf{x})\right) \, d^2x,$$

where f is an arbitrary function. The interpretation of this invariant is given in detail in Ref. 4. It implies the preservation of contours of ω, so that the value ω_0 on a contour labels that contour for all times. This is a consequence of the dissipationless and divergence-free nature of the system. Substituting $f(\omega) = \omega^n$ we also see that all the moments of vorticity are conserved. By choosing $f(\omega) = \theta(\omega(\mathbf{x}) - \omega_0)$, a heavyside function, it follows that the area inside of any ω-contour is conserved.

II. EXTENSIONS AND THE SEMIDIRECT PRODUCT

We now investigate systems involving Lie algebras by *extension*, a procedure for combining two or more Lie algebras to make a new Lie algebra. There are a myriad of ways to extend algebras, and we will only touch on a few here. All the extensions discussed here have their equivalent for Lie groups, but we choose the algebra approach here because it leads more directly to a Lie–Poisson bracket. In this section we let $\mathfrak{h}$ be an extension of the Lie algebra $\mathfrak{g}$ by the algebra $\mathfrak{v}$. The elements of $\mathfrak{h}$ are written as 2-tuples, (ξ, η), where $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{v}$.

The simplest extension is the *direct product* of Lie algebras. Let ξ and ξ' be elements of a Lie algebra $\mathfrak{g}$ and η and η' be elements of a vector space $\mathfrak{v}$ (which is an Abelian Lie algebra under addition). The direct product of these two algebras is an algebra $\mathfrak{h}$ with bracket

$$[\,(\xi\,,\eta)\,,(\xi'\,,\eta')\,] := ([\,\xi\,,\xi'\,]\;,\;[\,\eta\,,\eta'\,])\,.$$

Given the same $\mathfrak{g}$ and $\mathfrak{v}$ as above there is a less trivial way to make a new Lie algebra called the *semidirect product* with an operation defined by

$$[\,(\xi\,,\eta)\,,(\xi'\,,\eta')\,] := ([\,\xi\,,\xi'\,]\;,\;[\,\xi\,,\eta'\,] + [\,\eta\,,\xi'\,])\,.$$

A simple example of a semidirect product structure is when $\mathfrak{g}$ is the Lie algebra $so(3)$ associated with the rotation group $SO(3)$ and $\mathfrak{v}$ is $\mathbb{R}^3$. Their semidirect product is the algebra of the 6-parameter Galilean group of rotations and translations. Both the elements of $\mathfrak{g}$ and $\mathfrak{v}$ can then be represented by vectors in $\mathbb{R}^3$, with bracket $[\,\xi\,,\eta\,] = \xi \times \eta$, the cross product of vectors. Since $\mathfrak{h}$ is itself a Lie algebra, it can be extended again as needed to make an n-fold extension (an algebra of n-tuples).

We can build Lie–Poisson brackets from these algebras by extension.[6] For an n-fold extension $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$, we define

$$\{F\,,G\} := \pm\left\langle \mu\,,\,\left[\frac{\delta F}{\delta\mu}\,,\frac{\delta G}{\delta\mu}\right]\right\rangle,$$

where $\mu \in \mathfrak{h}^*$, the dual of $\mathfrak{h}$ under the pairing $\langle\,,\rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$. The dynamical variables of the system are the elements of the n-tuple $\mu = \mu(t)$. These elements may be fields or variables, so the Lie–Poisson bracket derived from an algebra by extension generates the dynamics for a system involving several dynamical quantities. The functions (or functionals) F and G are maps from $\mathfrak{h}^*$ to $\mathbb{R}$. Here $\delta/\delta\mu$ is a derivative or a functional derivative, depending on the dimensionality of the algebra (finite or infinite). For $n = 1$, $\mathfrak{h} = so(3)$, we have $\mu := \ell$ and we recover the bracket for the free rigid body (Section I A). The overall sign of the Lie–Poisson bracket has to do with left- or right-invariance of vector fields and will not be discussed here (See Ref. 2).

Using this procedure to make a Lie–Poisson bracket from a direct product of algebras leads to a sum of n independent brackets. This will not interest us further, since the coupling between dynamical variables can only come from the Hamiltonian. However there are interesting physical examples of a direct product structure. (This is the case for the model in Ref. 5, although a coordinate transformation is needed to exhibit the structure.)

We illustrate the process of building a Lie–Poisson bracket from a semidirect product of algebras by two examples, which are extensions of the rigid body and ideal fluid examples of Section I.

A. The Heavy Top

The Lie–Poisson bracket for the semidirect product of the rotation group $SO(3)$ and the vector space $\mathbb{R}^3$ is

$$\{f\,,g\} = -\ell\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \ell}\right) - \alpha\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \alpha}+\frac{\partial f}{\partial \alpha}\times\frac{\partial g}{\partial \ell}\right)$$

where α denotes a 3-vector. By using

$$H(\ell,\alpha) = \sum_{i=1}^{3}\frac{\ell_i^2}{2I_i} + \alpha\cdot\mathbf{c}$$

where $\mathbf{c}$ is a vector representing the position of the center-of-mass, we get the prototypical example of a semidirect product system, the heavy rigid body (in the body frame):

$$\dot\ell_i = \frac{I_j - I_k}{I_j\,I_k}\,\ell_j\,\ell_k + \alpha_j\,c_k - \alpha_k\,c_j\,, \qquad \dot\alpha_i = \frac{\ell_k\,\alpha_j}{I_k} - \frac{\ell_j\,\alpha_k}{I_j}\,,$$

where i, j, k are cyclic permutations of $1, 2, 3$. The vector α rotates rigidly with the body, which is always true for a Hamiltonian quadratic in ℓ. The Casimirs for this bracket are

$$C_1 = \alpha^2\,, \quad C_2 = \ell\cdot\alpha\;.$$

Looking at the bracket as derived by reduction of the heavy top in Euler angles (as we did in Section I A, but here with gravity), the Casimir C_2 expresses conservation of p_ϕ, since ϕ is cyclic. Knowing α does not lead to a determination of the orientation of the rigid body: there is still a symmetry of rotation about α. Taking the semidirect product has led to the recovery of some of the Lagrangian (configuration) information.

B. Low-beta Reduced MHD

The semidirect product bracket for two fields ω and ψ is

$$\{F,G\} = \int_D \left\{ \omega \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right] + \psi \left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \psi} \right] + \left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta \omega} \right] \right) \right\} d^2x \,.$$

If $\omega = \nabla^2\phi$, where ϕ is the electric potential, ψ is the magnetic flux, and $J = \nabla^2\psi$ is the current, then the Hamiltonian

$$H[\omega;\psi] = \frac{1}{2}\int_D \left(|\nabla\phi|^2 + |\nabla\psi|^2 \right) d^2x$$

with the above bracket gives us

$$\dot\omega = [\omega,\phi] + [\psi,J] \ , \qquad \dot\psi = [\psi,\phi] \ ,$$

a model for low-beta reduced MHD.[7] (Ref. 8 contains a system with a similar structure, but for waves in a density-stratified fluid.)

The bracket has two Casimir invariants,

$$C_1[\psi] = \int_D f(\psi)\, d^2x, \qquad C_2[\omega;\psi] = \int_D \omega\, g(\psi)\, d^2x.$$

The first has the form of the Casimir for 2–D Euler of Section I B and has the same interpretation. To understand the second one we let $g(\psi) = \theta(\psi - \psi_0)$, a heavyside function. In this case we have

$$C_2[\omega;\psi] = \int_{\Psi_0} \omega\, d^2x,$$

where Ψ_0 represents the (not necessarily connected) region of D enclosed by the contour $\psi = \psi_0$, and $\partial\Psi_0$ is its boundary. The contour $\partial\Psi_0$ moves with the fluid, so this just expresses Kelvin's circulation theorem: the circulation around a closed material loop is conserved.

This theorem is true for any ψ-contour, therefore it holds in the region between two contours $\psi = \psi_0$ and $\psi = \psi_0 + \delta$. Letting $\delta \to 0$ we see that the two contours delineate a "line" of fluid elements with value ψ_0 of the magnetic flux. Knowledge of the value of ψ on a fluid element thus only determines which contour it is on, but not its location on the contour. Therefore, there is still a relabeling symmetry: the fluid elements can be shifted around the contour without changing the Casimirs C_1 and C_2. As with the heavy top, the semidirect product has led to the recovery of some, but not all, of the Lagrangian information.

C. Putting Labels on a Rigid-body

Remember that taking a semidirect product restricted the symmetry group of the body to rotations about α. If we take another semidirect product to get

$$\{f\,,g\} = -\ell\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \ell}\right) - \alpha\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \alpha}+\frac{\partial f}{\partial \alpha}\times\frac{\partial g}{\partial \ell}\right) - \beta\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \beta}+\frac{\partial f}{\partial \beta}\times\frac{\partial g}{\partial \ell}\right)$$

where β is a 3-vector, we have a bracket that can model a rigid body with two forces acting on it, for example a charged, rigid insulator in an electric field. The new bracket has Casimirs

$$C_1 = \alpha^2\,, \quad C_2 = \beta^2, \quad C_3 = \alpha\cdot\beta\,.$$

The angular momentum ℓ has disappeared from the Casimirs. This is because knowing α and β completely specifies the orientation of the rigid body (unless the two are colinear). In other words, by taking semidirect products we have reintroduced the Lagrangian information into the bracket. Note that taking more than two semidirect products is redundant as far as the Lagrangian information is concerned: knowing the orientation of more than two vectors does not add new information. This is reflected by the fact that the number of variables minus the number of Casimirs is six for two or more "advected" quantities. This is the dimension of $T^*SO(3)$, the original phase space (before reduction).

D. Advection in an Ideal Fluid

We now take a second semidirect product for the ideal fluid, say low-beta MHD with a second advected quantity, the pressure p. In that case we get a model for high-beta reduced MHD.[9] The Casimir is

$$C[\psi;p] = \int_D f(\psi,p)\,d^2x, \quad f \text{ arbitrary.}$$

This Casimir amounts to being able to label two contours. Locally this permits a unique labeling of the fluid elements as long as p and ψ are not constant in some region. However, globally there is some ambiguity, because contours can cross in several places. Thus, in the infinite-dimensional case the semidirect product is not equivalent to recovering the full Lagrangian information, unless the contours do not close, are monotonic, and nonparallel ($\nabla\psi\times\nabla p$ does not vanish). A third advected quantity will in general break this degeneracy. Note that if the advected quantities label the fluid elements unambiguously at $t=0$ then they will do so for all times.

III. BEYOND THE SEMIDIRECT PRODUCT: COCYCLES

In general there are other ways to extend Lie algebras besides the semidirect product. One example is the model derived in Refs. 10 and 11 for 2–D compressible reduced MHD. The model has four fields, and is obtained from an expansion in the inverse aspect ratio of the tokamak. The Hamiltonian is

$$H[\omega, v, p, \phi] = \frac{1}{2}\int_D \left(|\nabla\phi|^2 + v^2 + \frac{(p-2\beta\, x)^2}{\beta} + |\nabla\psi|^2 \right)\, d^2x,$$

where v is the parallel velocity, p is the pressure, and β is a parameter that measures compressibility. The bracket is

$$\begin{aligned}\{A\,,B\} = \int_D d^2x\, \Bigg\{ &\omega \left[\frac{\delta A}{\delta\omega}\,, \frac{\delta B}{\delta\omega} \right] \\ &+ v \left(\left[\frac{\delta A}{\delta\omega}\,, \frac{\delta B}{\delta v} \right] + \left[\frac{\delta A}{\delta v}\,, \frac{\delta B}{\delta\omega} \right] \right) + p \left(\left[\frac{\delta A}{\delta\omega}\,, \frac{\delta B}{\delta p} \right] + \left[\frac{\delta A}{\delta p}\,, \frac{\delta B}{\delta\omega} \right] \right) \\ &+ \psi \left(\left[\frac{\delta A}{\delta\omega}\,, \frac{\delta B}{\delta\psi} \right] + \left[\frac{\delta A}{\delta\psi}\,, \frac{\delta B}{\delta\omega} \right] - \beta \left[\frac{\delta A}{\delta p}\,, \frac{\delta B}{\delta v} \right] - \beta \left[\frac{\delta A}{\delta v}\,, \frac{\delta B}{\delta p} \right] \right) \Bigg\}\,.\end{aligned}$$

The term proportional to β is an obstruction to the semidirect product structure, and it cannot be removed by a coordinate transformation.

The theory that deals with the classification of extensions is Lie algebra cohomology. In general the way to extend a bracket is by adding a nontrivial *cocycle*. Though a priori there are an infinite number of ways to make an extension, for low dimensions, after allowing for coordinate transformations, very few possibilities remain; we have classified these in Ref. 12. We have also found all the Casimir invariants for the low-dimensional brackets (five fields or less).

The Casimirs of the above bracket are

$$\begin{aligned} C_1[\psi] &= \int_D f(\psi)\, d^2x, & C_2[p;\psi] &= \int_D p\, g(\psi)\, d^2x, \\ C_3[v;\psi] &= \int_D v\, h(\psi)\, d^2x, & C_4[\omega, v, p, \psi] &= \int_D \left(\omega\, k(\psi) + \frac{v\, p}{\beta}\, k'(\psi) \right)\, d^2x. \end{aligned}$$

Finding the invariant C_4 directly from the equations of motion would be tedious, but is straightforward from the bracket. These Casimirs do not allow a labeling of the fluid elements. The meaning of invariants of the form of C_1, C_2, and C_3 was discussed in Sections I B and II B: the total magnetic flux, pressure, and parallel velocity inside of any ψ-contour are preserved. To understand C_4 we use the fact that $\omega = \nabla^2\phi$ and then integrate by parts to obtain

$$C_4[\omega, v, p, \psi] = \int_D \left(-\nabla\phi \cdot \nabla\psi + \frac{v\, p}{\beta} \right)\, k'(\psi)\, d^2x.$$

The quantity in parentheses is thus invariant inside of any ψ-contour. It can be shown that this is a remnant of the conservation in the full MHD model of the cross helicity,

$$V = \int_D \mathbf{v} \cdot \mathbf{B}\, d^2x\,,$$

at second order in the inverse aspect ratio, while C_3 is a consequence of preservation of this quantity at first order. Here $\mathbf{B}$ is the magnetic field. As for C_1 and C_2 they are, respectively, the first and second order remnants of the preservation of helicity,

$$W = \int_D \mathbf{A} \cdot \mathbf{B}\, d^2x,$$

where $\mathbf{A}$ is the magnetic vector potential.

IV. CONCLUSIONS

We gave an introduction to the reduction of physical systems based on their symmetries. The prototypical examples were shown, the rigid body and the 2–D ideal fluid. For these two cases some information about the configuration of the system was lost after reduction, correponding to the symmetry used to reduce the system.

The semidirect product allowed us to build larger brackets from a "base" algebra in a systematic manner. We were thus able to describe the heavy top and low-beta reduced MHD. Examining the invariants, we concluded that the semidirect product had recovered some or all of the Lagrangian information.

For general extensions (not necessarily semidirect) things are different: the Lagrangian information is not necessarily a consequence of the Casimirs. However, for compressible reduced MHD the Casimirs represent constraints that are remnants of invariants of the full MHD equations from which the model is derived asymptotically.

As mentioned in Section III, when considering a general extension, all brackets can be reduced to a small number of normal forms, at least for low-dimensional extensions. It will be interesting to see if physical systems can be found that are realized by these brackets, both in the finite and infinite degree-of-freedom cases. We are currently investigating a toy model that we call the Leibniz top, which is one of the simplest non-semidirect system one can build. It is a straightforward generalization of the Lagrange top (a heavy top with $I_1 = I_2$). The Lagrange top is integrable, and we have found that so is the Leibniz top.

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