STABILITY PROPERTIES IN HAMILTONIAN PERTURBATIONS OF RESONANT PDE'S WITH SYMMETRY: THE CASE OF NLS[1]

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Abstract. In this paper we present some stability results for Hamiltonian PDE's which have an exact symmetry, and are perturbations of a linear resonant system. We also give an application to some nonlinear Schrödinger equations constructing infinitely many families of elliptic periodical orbits, one of which is orbitally stable, and infinitely many families of "approximatively invariant" 2-dimensional tori, which are stable over very long times.

1. Introduction

In this paper, we will study, in the spirit of Nekhoroshev's theorem [1,2], the dynamics of Hamiltonian PDE's which have an exact symmetry, and are perturbations of a linear resonant system. The point is that when an exact symmetry is present one can pass to the reduced system (Marsden–Weinstein reduction procedure [3,4]) to study the dynamics of the system under consideration. In particular it is well known that critical points of the reduced hamiltonian are invariant manifolds of the original system, and that extrema of the reduced hamiltonian are (orbitally) stable invariant manifolds of the system (see e.g. [5]; for applications in PDE's see e.g. [6]). Moreover, if the original system is a perturbation of a linear resonant system we will show that one can construct a resonant normal form [7], which can be used (see [8, 9]) to find "approximatively invariant manifolds" which

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are stable over times exponentially with a suitable small parameter ϵ; more precisely, for any $b \geq 2$, we will prove that solutions starting $\mathcal{O}(\epsilon^b)$ close to such manifold remain $\mathcal{O}(\epsilon^{b/2})$ close to it up to times of order $\epsilon^{b-2}\exp(a/\epsilon^2)$ $(a > 0)$.

Here, to illustrate in a precise way the results that can be obtained by the above method (and the difficulties one meets in actually applying it) we will concentrate on the nonlinear Schrödinger equation with vanishing boundary conditions on a segment, namely

$$iu_t = -u_{xx} - u\varphi(|u|^2) \ , \quad u(0,t) = u(\pi,t) = 0 \ , \tag{1.1}$$

where φ is a function analytic in a neighbourhood of the origin such that $\varphi(0) = 0$, $\varphi'(0) \neq 0$. In particular we will construct infinitely many families of (small amplitude) periodical orbits one of which is also proved to be orbitally stable, and infinitely many families of two dimensional "approximatively invariant" tori which are stable over times exponentially long with the inverse of the square of the H^1 norm of the initial datum.

We recall that very much is already known for small amplitude solutions of eq. (1.1). In particularly some Nekhoroshev type results for (1.1) were already obtained in [10,11,12]; the tori we construct in the present paper are stable for times much longer than those considered in such papers. A discussion of the relation of the present results with those of the above quoted papers, together with a description of the picture of the dynamics arising from these works (and from [13]) will be given in sect. 5.

2. The nonlinear Shrödinger equation

Consider NLS (1.1) on $[0,\pi]$ with vanishing boundary conditions. Since we are interested in small amplitude solutions it is useful to make the rescaling $u = \epsilon u'$, and moreover to separate the main part of the perturbation so that the equation assumes the form

$$iu_t + u_{xx} = \epsilon^2 \pi u\,|u|^2 + \epsilon^4 u\psi(|u|^2) \ , \tag{2.1}$$

where we omitted the primes, denoted

$$\psi(x) := \frac{1}{\epsilon^4}[\varphi(\epsilon^2 x) - \epsilon^2 \pi x]$$

and assumed (without lack of generality) that $\varphi'(0) = \pi$. It is well known that NLS is a Hamiltonian system e.g. on the phase space $\mathcal{P} := H_0^1([0,\pi], \mathbf{C})$ endowed by the symplectic form Ω and by the scalar product $\langle .,. \rangle$ defined by

$$\Omega(u,v) := \mathrm{Re} \int_0^\pi uiv^* dx \ , \quad \langle u,v \rangle := \frac{1}{2} \mathrm{Re} \int_0^\pi u_x^* v_x dx \ ; \tag{2.2}$$

the hamiltonian being given by

$$H(u) := \frac{1}{2} \int_0^\pi |u_x|^2 \, dx + \epsilon^2 \frac{\pi}{4} \int_0^\pi |u|^4 dx + \epsilon^4 \int_0^\pi \Psi(|u|^2) dx \ , \tag{2.3}$$

where Ψ is a primitive of ψ satisfying $\Psi(0) = \Psi'(0) = \Psi''(0) = 0$. To introduce canonically conjugated variables consider the space Fourier expansion of u, namely

$$u(x) = \sum_{k \geq 1} u_k \sqrt{\frac{2}{\pi}} \sin kx \ , \tag{2.4}$$

and put

$$p_k := \mathrm{Re} u_k \ , \quad q_k := \mathrm{Im} u_k \ , \quad I_k := \frac{|u_k|^2}{2} = \frac{p_k^2 + q_k^2}{2} \ , \tag{2.5}$$

then (p_k, q_k) are canonically conjugated variables and I_k are the linear actions. We will use the following notations

$$h_0(u) := \frac{1}{2} \int_0^\pi |u_x|^2 \, dx = \sum_{k \geq 1} k^2 \frac{p_k^2 + q_k^2}{2} \ ,$$
$$f(u) := \frac{\pi}{4} \int_0^\pi |u|^4 \, dx \ , \quad f_1(u) := \int_0^\pi \Psi(|u|^2) dx,$$

so that (2.3) can be rewritten as

$$H(u) = h_0(u) + \epsilon^2 f(u) + \epsilon^4 f_1(u) \ . \tag{2.6}$$

We will also denote by

$$N(u) := \frac{1}{2} \int_0^\pi |u|^2 dx = \sum_{k \geq 1} \frac{p_k^2 + q_k^2}{2} \ ,$$

the constant of motion associated to the phase shift invariance of (1.1).

3. Periodic orbits

As anticipated above, the main idea is to use the group action $S^1 \times \mathcal{P} \ni (\alpha, u) \mapsto e^{i\alpha}u$ to reduce the system, and then to study the equilibria of the reduced system. The procedure is quite standard, but, as an introduction to the developments of the next section, we think it is useful to sketch it at an elementary (and formal) level.

Introduce (formal) action angle coordinates given by (I_k, ϕ_k) (cf. eq. (2.5)), pass to new action J defined by

$$J_1 := \sum_{k\geq 1} I_k \ ; \quad J_2 := \sum_{k\geq 1} k^2 I_k \ ; \quad J_k := I_k \ , \quad k \geq 3 \ ,$$

and denote by $\psi_1, \psi_2, ..., \psi_\infty$ the corresponding new angles. Since $N = J_1$ is a constant of motion, the hamiltonian is independent of ψ_1, and so one can consider J_1 as a parameter (i.e. pass to the reduced system). To find the critical points of the reduced system we want to exploit the fact that H is a perturbation of $h_0(\equiv J_2)$, by looking for critical points of h_0, and then continuing them to critical points of H; however, it is clear that the calculations has to be developed in the original (p, q) variables[1] exploiting the fact that the critical points of the reduced hamiltonian are indeed critical points of the restriction of the original hamiltonian to the surface of constant L^2. However, due to the existence of a symmetry which is common to h_0 and N, these critical points consist of a whole orbit of the symmetry group, and so they are always degenerate, and the implicit function theorem does not apply. So, we first introduce a Poincaré section P and apply the implicit function theorem to the restriction of H to the intersection of P with a level surface of N. This is possible provided the critical points are nondegenerate in the directions transversal to the orbit of the symmetry group (for a detailed description of the continuation procedure see [8] lemma 8.4).

To give a precise statement consider the surface

$$\mathcal{N}_a := \{u \in \mathcal{P} \ : \ \|u\|_{L^2} = a\} \ ,$$

and, for any fixed integer $j \geq 1$, the family of closed curves defined by

$$\mathcal{C}_{j,\alpha} := \{u \in \mathcal{P} \ : \ |u_j| = \alpha \ , \ u_k = 0 \ , \forall k \neq j\} \ .$$

We directly state our result for small amplitude solutions of the original system (1.1).

[1] Clearly, the critical points coincide with the singularities of the coordinates (J, ψ); indeed at such points J_1 is not independent of J_2

Theorem 3.1. *Consider equation (1.1); for any integer $j \geq 1$ there exist positive constants ϵ_j, C_j and a family of periodic orbits with trajectories $\{\gamma_\epsilon^j\}_{0\leq\epsilon\leq\epsilon_j}$ with the following properties*

1) *The periodic solution with trajectory γ_ϵ^j lies on $\mathcal{N}_\epsilon$, has period T_j^ϵ which depends analytically on $\epsilon \in (0, \epsilon_*)$ and which, as $\epsilon \to 0$, converges to $2\pi/j^2$.*
2) *γ_ϵ^j is an analytic curve, moreover $u \in \gamma_\epsilon^j \Longrightarrow u \in C^\infty(0, \pi)$.*
3) *γ_ϵ^j is close to $\mathcal{C}_{j,\epsilon}$, precisely one has*

$$d_{H^1}(u, \gamma_\epsilon^j) \leq C_j \epsilon^3 \ , \quad \forall u \in \mathcal{C}_{j,\epsilon}$$

4) *γ_ϵ^1 is orbitally stable.*
5) *γ_ϵ^j is elliptic (i.e. it has vanishing Lyapunov exponents).*

Remark 3.2. *Statements 1,2,3, were already proved, by a slightly different method, in [13].*

Proof. We use the method of Lagrange multiplier to compute the critical points of $h_0\big|_{\mathcal{N}_1}$. For $X \in \mathcal{P}$ we have

$$\begin{aligned}\langle \lambda dN - dh_0, X\rangle &= \sum_{k\geq 1} X_{p_k} \frac{\partial}{\partial p_k}(\lambda dN - dh_0) + X_{q_k}\frac{\partial}{\partial q_k}(\lambda dN - dh_0) \\ &= \sum_{k\geq 1}(\lambda - k^2)(p_k X_{p_k} + q_k X_{q_k}) \ ,\end{aligned} \tag{3.1}$$

where X_{p_k} and X_{q_k} are the real and the imaginary part of the $k-$th Fourier coefficient of X. So, for any $j \geq 1$ one has that

$$\begin{cases} p_k = q_k = 0 \ , & \text{if } k \neq j \\ \dfrac{p_j^2 + q_j^2}{2} = 1 \end{cases} \quad , \qquad \lambda = \lambda_j = j^2 \tag{3.2}$$

is a solution of $\lambda dN - dh_0 = 0$, and the curve $\mathcal{C}_{j,1}$ is a critical set for $h_0\big|_{\mathcal{N}_1}$.

In order to continue these critical points to critical points of the total hamiltonian (2.3) we have to show that they are nondegenerate in the transversal directions. So, fix $u \in \mathcal{C}_{j,1}$ and compute the second differential of $h_0\big|_{\mathcal{N}_1}$ at u. For $X \in T_u\mathcal{N}_1$ one has

$$\left\langle d^2 h_0\big|_{\mathcal{N}_1}; X, X\right\rangle = \lambda_j \left\langle d^2 N; X, X\right\rangle - \left\langle d^2 h_0; X, X\right\rangle = \sum_{j\neq k}(\lambda_j - k^2)(X_{p_k}^2 + X_{q_k}^2) \ . \tag{3.3}$$

Let $P \subset \mathcal{P}$ be the hyperplane through u orthogonal to $\mathcal{C}_{j,1}$, then it is easy to see that the application

$$\begin{aligned} T_u\mathcal{N}_1 \cap P &\to (T_u\mathcal{N}_1 \cap P)^* \\ X &\mapsto \alpha_X := \langle d^2 h_0\big|_{\mathcal{N}_1}(u); X, .\rangle \end{aligned} \tag{3.4}$$

is an isomorphism.

So the implicit function theorem applied to $H\big|_{\mathcal{N}_1 \cap P}$ ensures that for small enough ϵ such a function has a critical point close to $\mathcal{C}_{j,1} \cap P$ which depends smoothly on ϵ^2. Going back to original variables (i.e. removing the scaling) (1) follows. To see (2) remark that γ_ϵ^j is the trajectory of the solutions of the hamiltonian vector field of N, which is a bounded linear vector field. Smoothness of the functions $u \in \gamma_\epsilon^j$ follows by a bootstrap argument (for the details see [8] lemma 8.2) which allows to show that u is in the domain of $(\partial_{xx})^k$ for any positive k. The stability property of γ_1 follows from the fact that, since the second differential (3.4) is definite for $j = 1$, such a curve is an extremum of the reduced hamiltonian.

To see 5) consider the reduced system obtained by exploiting the phase shift invariance: the curve γ_ϵ^j is a stationary point for its Hamiltonian. Moreover, the linearization of the system at such a stationary point is close to the linearization of the unperturbed system ($\epsilon = 0$) at the stationary point corresponding to the orbit γ_0^j. In turn the unperturbed linear system has the form $\langle z, Qz\rangle$, where Q is a linear opertor which, by (3.3), is diagonal and has eigenvalues $\{\lambda_j - k^2\}_{k\geq 1, j\neq k}$. Since such eigenvalues are simple and real, 5) follows by standard perturbation theory for linear Hamiltonian system. □

4. Exponentially stable two dimensional tori

Using the (formal) coordinates (J, ψ) the hamiltonian takes the form

$$J_2 + \epsilon^2 f(J, \psi_2, ..., \psi_\infty) + \epsilon^4 f_1(J, \psi_2, ..., \psi_\infty) \ . \tag{4.1}$$

The idea is now to make a resonant normal form in order to eliminate the dependence of the hamiltonian on ψ_2. By classical averaging theorem (see e.g. [14,15]), one can expect that it should be possible to find a canonical transformation $\mathcal{T}$ so that the hamiltonian is reduced to

$$H \circ \mathcal{T} = J_2 + \epsilon^2 \langle f\rangle(J, \psi_3, ..., \psi_\infty) + \epsilon^4 Z(J, \psi_3, ..., \psi_\infty) + e^{-C/\epsilon^2}\mathcal{R}(J, \psi_2, \psi_3, ..., \psi_\infty) \tag{4.2}$$

where $\langle f\rangle$ is the average of the hamiltonian with respect to ψ_2, Z is a part in normal form (i.e. it does not depend on ψ_2), and $\mathcal{R}$ is a remainder which gives an exponentially small correction to the dynamics.

To study the dynamics of (4.2) we begin by considering the simplified system in which the remainder $\mathcal{R}$ is neglected: its hamiltonian is independent of ψ_2, and (passing to the reduced system) is equivalent to

$$H_s = \epsilon^2\langle f\rangle(J,\psi_3,...,\psi_\infty) + \epsilon^4 Z(J,\psi_3,...,\psi_\infty) \tag{4.3}$$

Since (4.3) is a perturbation of $\epsilon^2\langle f\rangle$, which can be explicitly computed, the idea is again to compute the critical points of $\langle f\rangle$, and to continue them to critical points of the simplified hamiltonian.

Afterwards one has to take into account the remainder $\mathcal{R}$: it is clear that it can produce a drift which is only exponentially small. So one has that the invariant tori of the simplified system are only ensured to be approximatively invariant for the complete dynamics, and their stability properties can be ensured only for exponentially long times.

This is the result one can expect to obtain; however, as pointed out in the previous section there is the technical problem that one cannot use the variables J,ψ to study the dynamics. In the present case the problem is even more serious: ψ_2 is essentially the time along the flow generated by J_2, which in turn is the hamiltonian of the linear system $i\dot{u} = -u_{xx}$ whose solutions are only continuous in time. *So ψ_2 is a continuous nondifferentiable variable.* Remark that this has the intrinsic (i.e. coordinate independent) consequence that the group action of T^2 on $\mathcal{P}$ under which the simplified system is invariant is only continuous, so that the reduced system has in general a hamiltonian which is not differentiable.

To overcome the above problems we proceed in some steps. First we apply the method of [7,12] to put the system in normal form (without introducing the variables (J,ψ), and preserving the symmetry [16]).

In this way one obtains the following theorem in which we will denote by $B_R \subset \mathcal{P}$ the ball of radius R in the H^1 norm. Moreover we will denote by $C_1, C_2, C_3...$ positive constants independent of ϵ

Theorem 4.1. *Provided ϵ is small enough there exists an analytic canonical transformation $\mathcal{T}$ defined on B_2 with the following properties*

1. *$\mathcal{T}$ leaves invariant N, namely $N\circ\mathcal{T} = N$, and satisfies*

$$\sup_{B_2} \| \mathbb{1} - \mathcal{T}\| \leq C_1\epsilon^2 \ ; \tag{4.4}$$

the same estimate holds for $\mathcal{T}^{-1}$.

2. *$\mathcal{T}$ puts the hamiltonian in resonant normal form, namely one has*

$$H\circ\mathcal{T} = h_0 + \epsilon^2\langle f\rangle + \epsilon^4 Z + \mathcal{R} \tag{4.5}$$

where

2a. $\langle f\rangle$ *is the average of* f *with respect to the unperturbed flow, i.e.*

$$\langle f\rangle(u) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it\partial_{xx}}u)dt$$

and is explicitly given by

$$\langle f\rangle(u) := \frac{1}{2}\Big(N(u)\Big)^2 - \frac{1}{8}\sum_{k\geq 1}|u_k|^4 \ ; \tag{4.6}$$

2b. $Z = Z(u,\epsilon)$ *satisfies* $\{h_0, Z\} = 0$ *and* $\{N, Z\} = 0$*; it is analytic on* B_2*, together with its symplectic vector field, and is jointly* C^∞ *in* u *and* ϵ*. Moreover it is bounded on* B_2*, together with its symplectic vector field, by a constant independent of* ϵ

2c. $\mathcal{R}$ *is a remainder analytic on* B_2 *together with its symplectic gradient; it satisfies* $\{\mathcal{R}, N\} = 0$ *and the estimates*

$$\sup_{B_2}|\mathcal{R}| \leq C_2\epsilon^4 \exp\left[-\left(\frac{a}{\epsilon}\right)^2\right] \quad , \quad \sup_{B_2}\|Jd\mathcal{R}\| \leq C_3\epsilon^4 \exp\left[-\left(\frac{a}{\epsilon}\right)^2\right] \ . \tag{4.7}$$

Remark 4.2. *The classical algorithm gives a normal form* Z *which depends in a discontinuous way on* ϵ*; to obtain a smooth* Z *one has to use the technique developed in [8] for the proof of lemma 8.1.*

For the proof see [7,8,12].

Once the system has been put in normal form, following [8], we look for the critical points of $\langle f\rangle$ restricted to a surface of constant N and h_0. Such critical points, which are invariant under the flow generated by $h_0(= J_2)$ and $N(= J_1)$ turn out to be *smooth* tori, even if the flow generated by h_0 is only continuous (this is due to the fact that the perturbation has a smooth hamiltonian vector field; for details see [8]). So one can introduce smooth coordinates which are adapted to such tori, and use them to study the dynamics of the system. This was done in [8] for systems without symmetry and extended to the case of systems with symmetry in [9]; here we just give the statement of the corresponding lemma. To this end denote

$$S_{E,c} = h_0^{-1}(E)\cap\mathcal{N}_c, \qquad \mathcal{Z} := \langle f\rangle + \epsilon^2 Z$$

and assume that

(ND) For some fixed c the function $\langle f\rangle|_{S_{1,c}}$ has a nondegenerate extremal set $T_{1,c}$ which is a torus; precisely there exists a positive constant $C_\perp$ such that, for any $u \in T_{1,c}$ and any $X \in T_u S_{1,c}$ (tangent space to $S_{1,c}$ at u) orthogonal to $T_{1,c}$ one has

$$\pm \langle d^2 \langle f \rangle |_{S_{1,c}}; X, X \rangle \geq C_\perp \|X\|^2 \ ,$$

where the signum plus holds for minima and the minus for maxima.

Lemma 4.3. *There exists ϵ_* such that, for any positive $\epsilon < \epsilon_*$ there exists a torus $\mathcal{T}^\epsilon_{1,c}$ smoothly dependent on ϵ, and close to $T_{1,c}$; precisely, such that*

$$d(u, T_{1,c}) \leq C_4 \epsilon^2 \quad \forall u \in \mathcal{T}^\epsilon_{1,c} \ ,$$

which is a nondegenerate extremum of $\mathcal{Z}|_{S_{1,c}}$; moreover, fix $u \in \mathcal{T}^\epsilon_{1,c}$, and denote by P the codimension 2 hyperplane through u orthogonal to $T_u \mathcal{T}^\epsilon_{1,c}$; then there exists a coordinate system

$$\begin{gathered} \mathbb{R}^4 \times \mathcal{B} \supset \mathcal{V} \to \mathcal{P} \\ ((\phi_1, \phi_2, E, l), \xi) \mapsto u(\phi_1, \phi_2, E, l, \xi) \end{gathered}$$

where $\mathcal{B} := T_u(P \cap S_{1,c})$, such that the following properties hold
i) $u(\phi_1, \phi_2, E, l, \xi) \in S_{E,l}$
ii) $u(0, 0, 1, c, \xi) \in P$
iii) $u(\phi_1, \phi_2, 1, c, 0) \in \mathcal{T}^\epsilon_{1,c}$,
iv) for $u \in \mathcal{N}_c$ one has $d(u, \mathcal{T}^\epsilon_{1,c}) \leq C_5 |E - 1| + C_6 \|\xi\|$, where the norm is the natural norm of $\mathcal{B}$.

The proof is a combination of Marsden–Weinstein reduction theorem and of lemma 8.6 of [8], which in turn exploits just the smoothness of the periodic orbit of the system obtained by reduction with respect to (J_1, ψ_1) only; it will be given in detail in [9].

In the above coordinates the function

$$\mathcal{Z} = \mathcal{Z}(\phi_1, \phi_2, E, l, \xi) = \mathcal{Z}_{\phi_1, \phi_2, l}(E, \xi)$$

has a Taylor expansion in E, ξ, which at any point of $\mathcal{T}_{1,c}$ is of the form

$$\mathcal{Z}(E, \xi) = \mathcal{Z}(1, 0) + \frac{\partial \mathcal{Z}}{\partial E}(1, 0)(E - 1) + \frac{\partial^2 \mathcal{Z}}{\partial \xi^2}(1, 0)(\xi, \xi) + \mathcal{R}_3(E, \xi) \ , \tag{4.8}$$

where

$$\frac{\partial^2 \mathcal{Z}}{\partial \xi^2}(1, 0)$$

is the second differential of $\mathcal{Z}$ with respect to the variables ξ; since $(1, 0)$ is an extremum of $\mathcal{Z}$ (as a function of ξ), it is a (positive or negative) definite two form.

So it is easy to show that the distance of $u \in \mathcal{N}_c$ from $\mathcal{T}_{1,c}$ is controlled by the two quantities $\Delta\mathcal{Z} := |\mathcal{Z}(E,\xi) - \mathcal{Z}(1,0)|$ and $\Delta E := |E-1|$, whose evolution, by the normal form theorem, is only exponentially slow. Precisely we have the following

Theorem 4.4. *Under the same assumptions of 4.3 the torus $\mathcal{T}^{\epsilon}_{1,c}$ is approximatively invariant, and is stable over exponentially long times. Precisely, $\forall\, b > 2$, one has $d(u_0, \mathcal{T}^{\epsilon}_{1,c}) \le C_7\epsilon^b$ implies $d(u(t), \mathcal{T}^{\epsilon}_{1,c}) \le C_8\epsilon^{b/2}$ for all times less than*

$$C_9\epsilon^{b-2}\exp\left(\frac{a}{\epsilon}\right)^2$$

We are now going to explicitly compute the extrema of the average of the perturbation in the case of NLS, and go back to the original (nonrescaled) variables. In the statement of the result we will use the notations

$$\begin{aligned} &\bar{E}(k_1,k_2) = \frac{k_1^2+k_2^2}{2}, \quad \hat{c}(k_1,k_2) = \frac{k_1^2+k_2^2-2}{k_1^4+k_2^4-k_1^2-k_2^2}, \\ &I_\delta(k_1,k_2) = (\hat{c}+\delta, \bar{E}^{-1}-\delta)\ ; \end{aligned}$$

moreover, for any fixed couple of integers (k_1,k_2) we shall denote by $T^{k_1,k_2}_{\epsilon,c}$ the family of 2–dimensional tori given by

$$T^{k_1,k_2}_{\epsilon,c} = \{u : h_0(u) = \epsilon^2, N(u) = c\epsilon^2, u_j = 0 \quad \forall j \ne k_1,k_2\}.$$

Theorem 4.5. *Consider equation (1.1) with vanishing boundary conditions and fix a positive $K \ge 2$; then there exist positive constant ϵ_*, a and $\delta = \mathcal{O}(K^{-2})$ (depending only on K, and on the perturbation φ) such that for any couple of integers $k_1, k_2 < K$, $k_1 \ne k_2$ there exists a family of 2–dimensional tori $\left\{\mathcal{T}^{k_1,k_2}_{\epsilon,c}\right\}^{c\in I_\delta(k_1,k_2)}_{|\epsilon|\le\epsilon_*}$, which are approximatively invariant exponentially stable, and close to the tori $T^{k_1,k_2}_{\epsilon,c}$.*

More precisely one has

i) *$\mathcal{T}^{k_1,k_2}_{\epsilon,c}$ lies in $\mathcal{N}_{\epsilon\sqrt{c}}$ and*

$$d(u, \mathcal{T}^{k_1,k_2}_{\epsilon,c}) \le C_{10}\epsilon^2 \qquad \forall\, u \in T^{k_1,k_2}_{\epsilon,c}\ .$$

ii) *for any $b > 2$ and for any initial datum u_0 satisfying*

$$d(u_0, \mathcal{T}^{k_1,k_2}_{\epsilon,c}) \le C_{11}\epsilon^b$$

the corresponding solution $u(t)$ satisfies the estimate

$$d(u(t), \mathcal{T}^{k_1,k_2}_{\epsilon,c}) \le C_{12}\epsilon^{b/2}$$

for all the times

$$t \leq C_{13}\epsilon^{b-2} \exp(\frac{a}{\epsilon^2}) \ .$$

iii) For $(k_1, k_2) = (1, 2)$ *the above stability property holds also for* $c \in (\frac{13}{28} + \delta, 1 - \delta)$.

Proof. Recalling the expression (4.6) it is clear that the critical set of $\langle f \rangle \mid_{S_{1,c}}$ coincides with the critical set of the function $g \mid_{S_{1,c}}$, where $g = -\frac{1}{8} \sum_j (p_j^2 + q_j^2)^2$.

We use the method of Lagrange multipliers to look for the critical points. This gives

$$\begin{array}{l} \dfrac{\partial g}{\partial p_j}(p,q) + \lambda_1 \dfrac{\partial h_0}{\partial p_j}(p,q) + \lambda_2 \dfrac{\partial N}{\partial p_j}(p,q) = 0 \\ \dfrac{\partial g}{\partial q_j}(p,q) + \lambda_1 \dfrac{\partial h_0}{\partial q_j}(p,q) + \lambda_2 \dfrac{\partial N}{\partial q_j}(p,q) = 0 \end{array} \qquad \forall j \geq 1 \ .$$

An easy calculation shows that the solutions of this system are the points (p, q) such that

$$\begin{cases} p_j^2 + q_j^2 = 2\pi(\lambda_1 j^2 + 2\lambda_2) \text{ for } j \in \mathcal{I} \\ p_j = q_j = 0 \text{ elsewhere} \ , \end{cases} \tag{4.9}$$

where $\mathcal{I} \subset N \setminus \{0\}$ is any index set and $\lambda_1, \ \lambda_2$ have to be computed in such a way that $(p, q) \in S_{1,c}$. Therefore, to any set of indexes there corresponds a set of critical points. As these points must have finite values of energy and norm, we can immediately deduce that the cardinality of the set $\mathcal{I}$ must be finite: $\mathcal{I} := \{k_1 \dots k_n\}$. Therefore, the critical points of $\langle f \rangle \mid_{S_{1,c}}$ are constituted by finite dimensional tori.

One dimensional tori (corresponding to $\mathcal{I} = \{k\}$) coincide with the closed curves $\mathcal{C}_{k,c}$, which have already been studied in section 3. We want now to identify those sets $\mathcal{I}$ which can correspond to nondegenerate critical points (i.e. points that satisfy condition (ND)). Tori with dimension $n \geq 3$ are necessarily degenerate; thus we focus on the 2–dimensional tori $\mathcal{I} = \{k_1, k_2\}$, $(k_1 < k_2)$. The condition $(p, q) \in S_{1,c}$ gives

$$\begin{aligned} p_{k_1}^2 + q_{k_1}^2 &= \frac{1}{2}[c + \frac{2(c\bar{E} - 1)}{k_2^2 - k_1^2}], \\ p_{k_2}^2 + q_{k_2}^2 &= \frac{1}{2}[c - \frac{2(c\bar{E} - 1)}{k_2^2 - k_1^2}], \\ p_j = q_j &= 0 \quad \forall j \neq k_1, k_2 \ . \end{aligned} \tag{4.10}$$

Equations (4.10) are meaningful only when their r.h.s. are positive, and this gives the condition

$$\frac{1}{k_2^2} \leq c \leq \frac{1}{k_1^2}.$$

For values of c at the limits of this interval the torus shrinks to a closed curve (the k_1th or the k_2th mode).

This concludes our study of the first differential.

We want now to find among the critical tori those which are extremal. One finds that at a point (p,q) belonging to the (k_1,k_2)–torus the second differential takes the form

$$d^2(\langle f\rangle\mid_{S_{1,c}})\mid_{(p,q)}(X,X)=\sum_{j\neq k_1,k_2}(X_{p_j}^2+X_{q_j}^2)\cdot C_j(k_1,k_2)\ ,$$

with

$$C_j(k_1,k_2)=\frac{1}{4(k_2^2-k_1^2)^2}[c\cdot(k_2^2-k_1^2)^2+4(1-c\bar{E})(j^2-\bar{E})]\ .$$

The sign of $C_j(k_1,k_2)$ definitively coincides with the sign of $(1-c\bar{E})$; then if we look for minima it has to be $c<\bar{E}^{-1}$. For these values of c the coefficients C_j are clearly increasing in j, so we study the sign of C_1 and find

$$C_1(k_1,k_2)>0\Leftrightarrow c>\frac{k_2^2+k_1^2-2}{k_2^4+k_1^4-k_2^2-k_1^2}:=\hat{c}(k_1,k_2).$$

So we conclude that for $c\in(\hat{c},\bar{E}^{-1})$ the points of the (k_1,k_2) torus are minima.

In order to find maxima we assume $c>\bar{E}^{-1}$ and remark that $C_1(k_1,k_2)>0$ for any $k_1\geq 2$; if we choose $k_1=1$ then $C_2(k_1,k_2)>0$ for any $k_2\geq 3$. This means that only points belonging to the $(1,2)$ torus can be maxima; indeed, this is true until $1>c>13/28$, while when c reaches this value the coefficient $C_3(1,2)$ changes its sign and becomes positive. However it is easy to see that for $c\leq 13/28$ the maximum points do not disappear, but are situated on a n-dimensional torus whose dimension n increases for decreasing values of c.

Remark that the measure of the intervals $(\hat{c},\bar{E}^{-1})$ tends to 0 when k_1,k_2 tends to ∞; therefore we have to fix an integer K and work with $k_1,k_2<K$. Moreover, when c reaches the limits of the intervals $(\hat{c},\bar{E}^{-1})$ there is an eigenvalue of the second differential of $\langle f\rangle$ which changes its sign, and due to this degeneration it is no more possible to apply the implicit function theorem (more precisely, the constant ϵ^* appearing in the statement of theorem 4.5 tends to 0). We then introduce a small δ and work on the intervals $I_\delta(k_1,k_2)$ so that the constant ϵ^* is strictly positive for every value of c in these intervals.

Finally, one has to go back to the original variables; this has to be done carefully as this time we have to consider not only the rescaling but also the canonical transformation that puts the system in normal form. $\square$

5. Multiple time scales in NLS

First we emphasize that in the present paper we used only the phase shift invariance of (1.1) and the complete resonance of the linearized system. However, (1.1) can be view as a perturbation of the integrable Zakharov–Shabat nonlinear Schrödinger equation, and this allows to obtain many more informations on the dynamics. In particular collecting the results of the papers [13,10,11,12] one obtains the picture of the dynamics we are going to briefly sketch.

For times of order higher than any power of

$$\epsilon := \left[\frac{1}{2}\int_0^\pi |\partial_x u_0|^2 dx\right]^{1/2}$$

(where u_0 is the initial datum), but shorter than exponential, the dynamics corresponding to smooth enough initial data is essentially integrable in the sense that all the actions are approximate constant [12]. Afterwards, up to times of order $\exp\left(a/\epsilon^{1/2n}\right)$, all n–dimensional invariant tori of the linearized system are stable in H^1 norm [12]. Then, up to times of order $\exp\left(a/\epsilon^{1/n}\right)$, only diophantine[2] n dimensional tori are stable [11] (again in H^1 norm). We also recall that close to such diophantine tori of the linearized system there exist tori which are exactly invariant under the complete dynamics [13], and that one can expect such tori to be stable up to times of order $\mathcal{O}(\epsilon^{-M})$ (given a positive M), for higher Sobolev norms [10]. Afterwards only the particular two dimensional tori we found in the present paper remain stable, and this for times of order $\exp(a/\epsilon^2)$; finally, after the times considered in theorem 4.5 the only remaining stable objects are given by the periodic orbits found in theorem 3.1.

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