On quantum dynamical entropy

M. Fannes[1], P. Tuyls[2]

Instituut voor Theoretische Fysica
Katholieke Universiteit Leuven
Celestijnenlaan 200D
B-3001 Heverlee, Belgium

Abstract

In this paper, we are mainly concerned with the computation of the dynamical entropy for the non-commutative toral automorphisms and the non-commutative shifts on a spin chain. The entropy that we consider is based on coarse-grained models of the system. In contrast to the classical situation, successive observations of a quantum system can generate entropy by themselves. We pay special attention to the delicate problem of delimiting a natural class of models for computing the entropy. We present also examples that show the importance of such a choice and obtain as a byproduct some general properties of the entropy.

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[1] Onderzoeksleider FWO, Email: `mark.fannes@fys.kuleuven.ac.be`
[2] Onderzoeker FWO, Email: `pim.tuyls@fys.kuleuven.ac.be`

1. Introduction

Dynamical entropy is a useful tool for studying chaos in classical dynamical systems. It is a quantitative statistical measure for the amount of disorder produced by the dynamics in the initial state of the system. This is, for dynamical systems on a smooth manifold, the contents of Pésin's theorem [L] which states that the dynamical entropy equals the sum of the positive Lyapunov exponents. It measures therefore the rate at which small errors grow in time.

The notion of entropy in quantum dynamical systems is less clear and several approaches are considered. Introducing a suitable entropy as a measure of loss of information on the detailed structure of the initial state, offers certainly an interesting possibility. The time evolution of a truly quantum dynamical system is not the only source of entropy production, also the measuring process contributes because of the fundamental non-commutativity of successive observations of the system. A good notion of quantum dynamical entropy should therefore try to separate both.

There are nowadays several approaches to such a construction. The so-called CNT approach couples the quantum system with a classical one and computes the mutual entropy of this coupled system [CNT,ST,CS]. It satisfies the "natural" properties of a dynamical entropy but its computability in concrete models is not straightforward and it produces less satisfactory results in some of the basic examples [AN]. In the approach of Voiculescu [V], the dynamical entropy is computed as the growth under the time evolution of the dimension of a matrix algebra which approximates a set of observables. In this paper, we investigate the entropy as introduced in [AF]. The relation between this construction and the CNT entropy will be studied in a forthcoming paper.

The main idea behind the construction is to use non-commutative shifts on a spin chain as a symbolic model for the system. Such a model is obtained by successive coarse-grained observations. The largest mean entropy of these models is then the entropy of the dynamical system. An important point in the construction is to specify the allowable observations. Unlike the classical case, one can easily show that the entropy becomes infinite if one allows arbitrary observations. In this paper, we will consider the non-commutative torus with its usual dynamics, the so-called Arnold cat maps. The non-commutative torus possesses a natural differential structure. We will compute the entropy using coarse-grained observations constructed in terms of smooth observables and show that we recover the result of Pésin's: the entropy coincides with the non-negative Lyapunov exponent.

The coin tossing experiment is another basic example in the theory of dynamical systems and it has been investigated extensively in classical theories. It is modelled by the shift on a classical spin chain. It is well-known that its entropy equals the entropy density of the probability measure describing the experiment. A quantum version of this model is described in terms of the shift on a quantum spin chain. The dynamical entropy of this system can be easily computed when restricting attention to partitions belonging to the local algebras. The natural question arises whether one can allow more general partitions. Therefore we will enlarge the local algebra to an algebra of well-localized elements on a quantum spin chain. Again we can prove the continuity of the dynamical entropy with respect to allowing a larger class of partitions. It turns out that the entropy equals the sum of the entropy density of the state describing the system and the logarithm of the dimension of the one site algebras. This extra term is an immediate consequence of the non-commutativity of the successive measurements performed on the system.

Along the road we will show that a deterministic, finite-dimensional, quantum dynamical system has zero entropy whereas arbitrary coarse-grainings of infinite systems lead to infinite entropy. We also discuss the behaviour of entropy under rescaling of the time.

2. Preliminaries

We briefly recall the construction of the entropy as proposed in [AF]. A deterministic, discrete time, quantum dynamical system is formally described in terms of a unital C*-algebra $\mathfrak{A}$ of observables, a $*$-automorphism Θ of $\mathfrak{A}$ which gives the time evolution during a single time step and a state ω of $\mathfrak{A}$. We will not assume a priori that ω is invariant under the evolution: $\omega \circ \Theta = \omega$. Such a system will be statistically modelled by repeated coarse-grained observations defined by operational partitions of unity [Li]. These partitions are constructed in terms of collections of elements in the algebra. We will show in section 5 that one has to impose restrictions on the partitions, i.e. one should only consider observations that don't disturb the system too profoundly. Technically this is translated in the requirement that we consider only partitions defined by elements in a distinguished unital $*$-subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ that is invariant under the dynamics.

A partition of unity is a k-tuple $\mathcal{X} = (x_0, x_1, \ldots, x_{k-1})$ of elements of $\mathfrak{A}$ satisfying

$$\sum_{i=0}^{k-1} x_i^* x_i = \mathbb{1}.$$

We will call k the size of $\mathcal{X}$. The partition $\mathcal{X}$ is called bistochastic if it satisfies the additional requirement:

$$\sum_{i=0}^{k-1} x_i\, x_i^* = \mathbb{1}.$$

Coarse-graining is a map from the $k \times k$ matrices $\mathcal{M}_k$ to $\mathfrak{A}$:

$$[a_{ij}]_{ij} \mapsto \sum_{ij} a_{ij}\, x_i^* x_j .$$

This map is completely positive and unity preserving and it pulls back the state ω on $\mathfrak{A}$ in the $k \times k$ density matrix $\rho_\omega[\mathcal{X}]$, more explicitly:

$$\rho_\omega[\mathcal{X}]_{i,j} = \omega(x_j^* x_i).$$

In order to produce a symbolic model of our dynamical system, we consider the time evolution of the coarse-graining and refine it at each time step by composition. During one step of the evolution, the partition $\mathcal{X}$ evolves into the partition

$$\Theta(\mathcal{X}) := \big(\Theta(x_0), \Theta(x_1), \ldots, \Theta(x_{k-1})\big)$$

and the composition of the partitions $\mathcal{X}$ and $\mathcal{Y} = (y_0, y_1, \ldots, y_{\ell-1})$ of sizes k and ℓ produces the partition

$$\mathcal{X} \circ \mathcal{Y} := (x_0 y_0, x_0 y_1, \ldots, x_{k-1} y_{\ell-1})$$

of size $k\ell$. So, after $n-1$ time steps we have obtained a coarse-graining from $\mathcal{M}_k^{\otimes n}$ in $\mathfrak{A}$ and a corresponding density matrix $\rho_\omega^n[\mathcal{X}]$, associated to the partition $\Theta^{n-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}$. It is straightforward to check that the restriction of $\rho_\omega^n[\mathcal{X}]$ to the first $n-1$ factors in $\mathcal{M}_k^{\otimes n}$ coincides with $\rho_\omega^{n-1}[\mathcal{X}]$. Therefore, the $\rho_\omega^n[\mathcal{X}]$ form a compatible family of density matrices and they define a state $\rho_\omega^{\mathbf{N}}[\mathcal{X}]$ on the half-chain $\mathcal{M}_k^{\mathbf{N}}$ which is the uniform closure of $\bigcup_{n \in \mathbf{N}} \mathcal{M}_k^{\otimes n}$. The spin half-chain, the state $\rho_\omega^{\mathbf{N}}[\mathcal{X}]$, and the right-shift constitute the symbolic model of the dynamical system induced by the partition $\mathcal{X}$. It should be noted that a symbolic model is usually not shift-invariant unless

$$\omega\left(\sum_{i=0}^{k-1} x_i^* y\, x_i\right) = \omega(y), \qquad y \in \mathcal{M}_k^{\mathbf{N}}.$$

We now define the dynamical entropy $\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}]$ of the partition $\mathcal{X}$ as the von Neumann entropy density of the state $\rho_\omega^{\mathbf{N}}[\mathcal{X}]$ on the spin chain:

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] := \limsup_{m \to \infty} \frac{1}{m} \mathsf{S}\big(\rho_\omega^m[\mathcal{X}]\big) = \limsup_{m \to \infty} \mathsf{S}\big(\rho_\omega[\Theta^{m-1}(\mathcal{X}) \circ \cdots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]\big).$$

$\mathsf{S}(\rho) := \mathrm{Tr}\, \eta(\rho)$ with η the standard entropy function $x \mapsto -x \log x$ on $[0,1]$. The fact that we have to take a "lim sup" instead of a "lim" is due to the lack of translation invariance of the state $\rho_\omega^{\mathbf{N}}[\mathcal{X}]$. In specific models however, the state on the half-chain is often asymptotically shift-invariant and its mean entropy exists as a limit [T]. The dynamical entropy $\mathsf{h}_{(\mathfrak{A}_0,\Theta,\omega)}$ is now obtained as the supremum of $\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}]$ over all finite partitions in $\mathfrak{A}_0$:

$$\mathsf{h}_{(\mathfrak{A}_0,\Theta,\omega)} := \sup_{\mathcal{X} \subset \mathfrak{A}_0} \mathsf{h}_{(\Theta,\omega)}[\mathcal{X}].$$

3. Auxiliary results

In this section, we prove some technical properties of the entropy of a partition.

Lemma 3.1 *Let $\mathfrak{H}$ be a Hilbert space and $\mathcal{X} = (x_0, x_1, \ldots, x_{k-1}) \subset \mathfrak{B}(\mathfrak{H})$ a size k partition of unity. Let $\Omega \in \mathfrak{H}$ be a normalized vector and denote by $|\Omega\rangle\langle\Omega|$ the vector state $y \mapsto \langle\Omega, y\,\Omega\rangle$ on $\mathfrak{B}(\mathfrak{H})$. If $\Pi_{\mathcal{X}}$ is the unity preserving completely positive map*

$$\Pi_{\mathcal{X}} : \mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H}) : y \mapsto \sum_{i=0}^{k-1} x_i^* y x_i,$$

then

$$\mathsf{S}\big(\rho[\mathcal{X}]\big) = \mathsf{S}\big(|\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{X}}\big).$$

Proof: Let $\Omega_{\mathcal{X}}$ be the normalized vector in $\mathbf{C}^k \otimes \mathfrak{H}$ given by

$$\Omega_{\mathcal{X}} := \sum_{i=0}^{k-1} e_i \otimes x_i\Omega$$

where $\{e_0, e_1, \ldots, e_{k-1}\}$ is an orthonormal basis in $\mathbf{C}^k$. The restrictions of the vector state $|\Omega_{\mathcal{X}}\rangle\langle\Omega_{\mathcal{X}}|$ to the $k \times k$ matrices $\mathcal{M}_k$ and to $\mathfrak{B}(\mathfrak{H})$ have the same eigenvalues up to multiplicities of zero. By $\mathrm{Tr}_{\mathfrak{H}}$ and $\mathrm{Tr}_{\mathbf{C}^k}$, we denote the operations $\mathrm{id} \otimes \mathrm{Tr}$ and $\mathrm{Tr} \otimes \mathrm{id}$ on $\mathcal{M}_k \otimes \mathfrak{B}(\mathfrak{H})$ where the symbol Tr stands for the trace on the factor on which it acts. Obviously,

$$\mathrm{Tr}_{\mathfrak{H}}\, |\Omega_{\mathcal{X}}\rangle\langle\Omega_{\mathcal{X}}| = \sum_{i,j=0}^{k-1} \langle\Omega, x_j^* x_i \Omega\rangle\, |e_i\rangle\langle e_j| = \rho[\mathcal{X}]$$

and

$$\mathrm{Tr}_{\mathbf{C}^k}\, |\Omega_{\mathcal{X}}\rangle\langle\Omega_{\mathcal{X}}| = \sum_{i=0}^{k-1} |x_i\Omega\rangle\langle x_i\Omega| = |\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{X}}.$$

This proves the lemma.

■

Let $\mathfrak{A}$ be a C^*-algebra and $\mathcal{X} = (x_0, \ldots, x_{k-1})$ a k-tuple of elements in $\mathfrak{A}$. A completely positive map $\Gamma_{\mathcal{X}}$ of $\mathfrak{A}$ into itself is naturally associated with $\mathcal{X}$:

$$\Gamma_{\mathcal{X}} : \mathfrak{A} \to \mathfrak{A} : x \mapsto \Gamma_{\mathcal{X}}(a) := \sum_{i=0}^{k-1} x_i^* a\, x_i. \tag{1}$$

Lemma 3.2 *Let ρ be a density matrix on a Hilbert space $\mathfrak{H}$ and consider a size k partition $\mathcal{X} = (x_0, x_1, \ldots, x_{k-1})$ of unity in $\mathfrak{B}(\mathfrak{H})$. If $\rho[\mathcal{X}]$ is the $k \times k$ density matrix $[\rho(x_j^* x_i)]_{i,j}$, then*

$$\mathsf{S}\big(\rho[\mathcal{X}]\big) \leq \mathsf{S}(\rho) + \mathsf{S}(\sigma),$$

where $\sigma = \rho \circ \ \Gamma_{\mathcal{X}}$ with $\Gamma_{\mathcal{X}}$ the completely positive map generated by $\mathcal{X}$ as in (1) .

Proof: Consider the completely positive map $\Gamma^{\mathcal{X}}$ from $\mathcal{M}_k \otimes \mathfrak{B}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{H})$ associated to the partition $\mathcal{X}$:

$$\Gamma^{\mathcal{X}} : \mathcal{M}_k \otimes \mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H}) : [a_{i,j}]_{i,j} \mapsto \sum_{i,j} x_i^* a_{i,j}\, x_j.$$

As $\mathcal{X}$ is a partition of unity, $\Gamma^{\mathcal{X}}$ is unity preserving. It can be realized concretely as:

$$\Gamma^{\mathcal{X}}\big([a_{i,j}]_{i,j}\big) = (\, x_1^* \quad \ldots \quad x_n^* \,) \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Since $\Gamma^{\mathcal{X}}$ is pure and unity preserving, its adjoint is a homomorphism. Consequently, the density matrix of the state $\omega^{\mathcal{X}}$ on $\mathcal{M}_k \otimes \mathfrak{B}(\mathfrak{H})$ defined as

$$\omega^{\mathcal{X}}(A \otimes a) := \rho\big(\Gamma^{\mathcal{X}}(A \otimes a)\big)$$

has the same spectrum as ρ up to multiplicities of zero. The restriction of $\omega^{\mathcal{X}}$ to the factor $\mathcal{M}_k$ is precisely $\rho[\mathcal{X}]$ and the restriction σ of $\omega^{\mathcal{X}}$ to $\mathfrak{B}(\mathfrak{H})$ is $\rho \circ \Gamma_{\mathcal{X}}$ with $\Gamma_{\mathcal{X}}$ as in (1). Then, from the triangle inequality for the entropy [AL] we get:

$$\mathsf{S}(\rho[\mathcal{X}]) \leq \mathsf{S}(\rho) + \mathsf{S}(\sigma).$$

∎

In the next lemma, we prove a continuity property.

Lemma 3.3 *Let $\mathcal{X}^{(\alpha)} = \big(x_0^{(\alpha)}, x_1^{(\alpha)}, \ldots, x_{k-1}^{(\alpha)}\big)$ and $\mathcal{Y}^{(\alpha)} = \big(y_0^{(\alpha)}, y_1^{(\alpha)}, \ldots, y_{k-1}^{(\alpha)}\big)$, $\alpha = 1, 2, \ldots, n$ be two families of partitions of unity in a C^*-algebra $\mathfrak{A}$ and let ω be a state on $\mathfrak{A}$. Assume that these partitions satisfy:*

$$\big\| x_j^{(\alpha)} - y_j^{(\alpha)} \big\| \leq \epsilon_\alpha, \quad j = 0, 1, \ldots, k-1, \quad \textit{and} \quad 2k \sum_\alpha \epsilon_\alpha \leq \frac{1}{3},$$

then
$$\Big|\mathsf{S}\Big(\rho_\omega\big[\mathcal{X}^{(n)}\circ\mathcal{X}^{(n-1)}\circ\cdots\circ\mathcal{X}^{(1)}\big]\Big)-\mathsf{S}\Big(\rho_\omega\big[\mathcal{Y}^{(n)}\circ\mathcal{Y}^{(n-1)}\circ\cdots\circ\mathcal{Y}^{(1)}\big]\Big)\Big|$$
$$\leq 2k\log(2k^n)\sum_{\alpha=1}^n\epsilon_\alpha-2k\left(\sum_{\alpha=1}^n\epsilon_\alpha\right)\log\left(2k\sum_{\alpha=1}^n\epsilon_\alpha\right).$$

Proof: If ρ and σ are two density matrices on $\mathbf{C}^d$ with $\|\rho-\sigma\|_1\leq 1/3$, then the difference of the entropies of ρ and σ can be estimated by

$$\big|\mathsf{S}(\rho)-\mathsf{S}(\sigma)\big|\leq\|\rho-\sigma\|_1\log d+\eta(\|\rho-\sigma\|_1).$$

We now express the entropies of $\rho_\omega\big[\mathcal{X}^{(n)}\circ\mathcal{X}^{(n-1)}\circ\cdots\circ\mathcal{X}^{(1)}\big]$ and $\rho_\omega\big[\mathcal{Y}^{(n)}\circ\mathcal{Y}^{(n-1)}\circ\cdots\circ\mathcal{Y}^{(1)}\big]$ using lemma 3.1. Consider therefore the GNS representation of ω and identify $x\in\mathfrak{A}$ with its representation. The vector Ω will denote the cyclic vector in the GNS representation of ω. Both density matrices $|\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{X}^{(n)}\circ\ldots\circ\mathcal{X}^{(1)}}$ and $|\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{Y}^{(n)}\circ\ldots\circ\mathcal{Y}^{(1)}}$ are supported by a space of dimension at most k^n and in order to estimate their norm difference, we use $\|x_j^{(\alpha)}-y_j^{(\alpha)}\|\leq\epsilon_\alpha$ in a telescoping estimate. More precisely,

$$\begin{aligned}
&\big\|\Pi_{\mathcal{X}^{(2)}}\circ\Pi_{\mathcal{X}^{(1)}}-\Pi_{\mathcal{Y}^{(2)}}\circ\Pi_{\mathcal{Y}^{(1)}}\big\|\\
&\qquad=\big\|\big(\Pi_{\mathcal{X}^{(2)}}-\Pi_{\mathcal{Y}^{(2)}}\big)\circ\Pi_{\mathcal{X}^{(1)}}+\Pi_{\mathcal{Y}^{(2)}}\circ\big(\Pi_{\mathcal{X}^{(1)}}-\Pi_{\mathcal{Y}^{(1)}}\big)\big\|\\
&\qquad\leq\big\|\Pi_{\mathcal{X}^{(2)}}-\Pi_{\mathcal{Y}^{(2)}}\big\|+\big\|\Pi_{\mathcal{X}^{(1)}}-\Pi_{\mathcal{Y}^{(1)}}\big\|.
\end{aligned}$$

We can now estimate the difference of the entropies:

$$\begin{aligned}
&\Big|\mathsf{S}\Big(\rho_\omega\big[\mathcal{X}^{(n)}\circ\mathcal{X}^{(n-1)}\circ\cdots\mathcal{X}^{(1)}\big]\Big)-\mathsf{S}\Big(\rho_\omega\big[\mathcal{Y}^{(n)}\circ\mathcal{Y}^{(n-1)}\circ\cdots\mathcal{Y}^{(1)}\big]\Big)\Big|\\
&\quad=\Big|\mathsf{S}\Big(|\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{X}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{X}^{(1)}}\Big)-\mathsf{S}\Big(|\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{Y}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{Y}^{(1)}}\Big)\Big|\\
&\quad\leq\big\||\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{X}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{X}^{(1)}}-|\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{Y}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{Y}^{(1)}}\big\|_1\ \log(2k^n)\\
&\qquad\quad+\eta\Big(\big\||\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{X}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{X}^{(1)}}-|\Omega\rangle\langle\Omega|\circ\Pi_{\mathcal{Y}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{Y}^{(1)}}\big\|_1\Big)\\
&\quad\leq\big\|\Pi_{\mathcal{X}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{X}^{(1)}}-\Pi_{\mathcal{Y}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{Y}^{(1)}}\big\|\ \log(2k^n)\\
&\qquad\quad+\eta\Big(\big\|\Pi_{\mathcal{X}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{X}^{(1)}}-\Pi_{\mathcal{Y}^{(n)}}\circ\cdots\circ\Pi_{\mathcal{Y}^{(1)}}\big\|\Big)\\
&\quad\leq\log(2k^n)\sum_{\alpha=1}^n\|\Pi_{\mathcal{X}^{(\alpha)}}-\Pi_{\mathcal{Y}^{(\alpha)}}\|+\eta\Big(\sum_{\alpha=1}^n\|\Pi_{\mathcal{X}^{(\alpha)}}-\Pi_{\mathcal{Y}^{(\alpha)}}\|\Big)\\
&\quad\leq 2k\ \log(2k^n)\sum_{\alpha=1}^n\epsilon_\alpha+\eta\Big(2k\sum_{\alpha=1}^n\epsilon_\alpha\Big).
\end{aligned}$$

∎

4. The entropy of the Arnold cat map

The quantum Arnold cat map is a dynamical system that lives on the non-commutative torus. In order to introduce the system, we consider two unitaries u and v that satisfy the commutation relation:

$$u\, v = \mathrm{e}^{iq}\, v\, u,$$

$q \in [0, 2\pi[$ is a deformation parameter. It is helpful to introduce Weyl-like operators:

$$W(\eta) := \mathrm{e}^{-iqmn/2}\, u^m\, v^n \qquad \text{with } \eta = (m,n) \in \mathbf{Z}^2.$$

In this definition u^{-1} is understood to be u^* and the same for v. By the unitarity of u and v and the commutation relation of above, one easily checks that

$$W(\eta)^* = W(-\eta) \qquad \text{and} \qquad W(\eta_1)\, W(\eta_2) = \mathrm{e}^{iq\sigma(\eta_1,\eta_2)/2}\, W(\eta_1 + \eta_2)$$

with $\sigma(\eta_1, \eta_2) = m_1 n_2 - m_2 n_1$. The algebra $\mathfrak{A}_q$ of the q-deformed torus is the universal C*-algebra generated by the $W(\eta)$, $\eta \in \mathbf{Z}^2$. The structure of $\mathfrak{A}_q$ is highly dependent on the deformation parameter. For $q = 0$, $\mathfrak{A}_q$ is isomorphic to the algebra $\mathcal{C}(\mathbf{T}^2)$ of continuous complex-valued functions on the two-dimensional torus $\mathbf{T}^2 = (\mathbf{R}/2\pi)^2$, the element $W((m,n))$ being identified with the function

$$(\theta_1, \theta_2) \mapsto \mathrm{e}^{i(m\theta_1 + n\theta_2)}$$

in the usual parametrization of $\mathbf{T}^2$. More generally, if q is a rational multiple of 2π, then $\mathfrak{A}_q$ is isomorphic to the tensor product of a matrix algebra with $\mathcal{C}(\mathbf{T}^2)$. Otherwise it has a trivial center.

It will be convenient to introduce the notation $W(f)$ for the element

$$W(f) := \sum_{\eta \in \mathbf{Z}^2} f(\eta)\, W(\eta)$$

with $f \in \ell^1(\mathbf{Z}^2)$. This is a slight abuse of notation as we identify $\eta \in \mathbf{Z}^2$ with its characteristic function. The algebraic relations for the $W(\eta)$ become:

$$W(f)^* = W(f^\dagger) \qquad \text{and} \qquad W(f)\, W(g) = W(f *_q g)$$

with

$$f^\dagger(\eta) := \overline{f(-\eta)} \qquad \text{and} \qquad (f *_q g)(\eta) = \sum_{\eta'} \mathrm{e}^{iq\sigma(\eta',\eta)}\, f(\eta')\, g(\eta - \eta').$$

There is a natural $*$-subalgebra $\mathfrak{A}_q^\infty$ of smooth functions on the non-commutative torus given by

$$\mathfrak{A}_q^\infty = \{W(f) \mid f \in \mathfrak{S}(\mathbf{Z}^2)\}.$$

$\mathfrak{S}(\mathbf{Z}^2)$ is the space of sequences of rapid decrease: $f \in \mathfrak{S}$ if and only if

$$\lim_{\|\eta\| \to \infty} \|\eta\|^k \, |f(\eta)| = 0 \quad \text{for all } k \in \mathbf{N}.$$

In the non-deformed case $\mathfrak{A}_0^\infty$ is isomorphic to the space of infinitely differentiable functions on $\mathbf{T}^2$.

We will consider an automorphic dynamics Θ_T on $\mathfrak{A}_q$ defined in terms of a matrix $T \in \mathrm{SL}(2, \mathbf{Z})$ with $\mathrm{Tr}\, T > 2$ by the equation:

$$\Theta_T\big(W(\eta)\big) := W(T^{\,\mathrm{t}}\, \eta).$$

$T^{\,\mathrm{t}}$ is the transpose of T. This map extends to a $*$-automorphism of $\mathfrak{A}_q$ which leaves $\mathfrak{A}_q^\infty$ globally invariant and is therefore a diffeomorphism of the non-commutative torus.

There is a very strong notion of Lyapunov exponent at hand for this type of dynamical system in terms of horocyclic actions [ENTS]. More precisely, the space of outer $*$-derivations on $\mathfrak{A}_q$ that leave $\mathfrak{A}_q^\infty$ globally invariant is two-dimensional. An explicit basis for this space is given by the derivations δ_1 and δ_2 with

$$\delta_j\big(W(f)\big) := W(\mathrm{M}_j\, f) \qquad j = 1, 2, \quad f \in \mathfrak{S}(\mathbf{Z}^2).$$

The operators M_j are the coordinate multiplication operators on $\mathfrak{S}(\mathbf{Z}^2)$:

$$\big(\mathrm{M}_1\, f\big)(\eta) := m\, f(\eta) \qquad \text{and} \qquad \big(\mathrm{M}_2\, f\big)(\eta) := n\, f(\eta).$$

In the undeformed case, the δ_j generate the coordinate translations on $\mathbf{T}^2$. $\delta_\alpha := \alpha_1\delta_1 + \alpha_2\delta_2$ generates then a non-commutative shift of $\mathfrak{A}_q$ in the direction $\alpha := (\alpha_1, \alpha_2) \in \mathbf{R}^2$. A straightforward computation shows that

$$\delta_\alpha \circ \Theta_T = \Theta_T \circ \delta_{T\alpha}.$$

In particular, translations along the eigenvectors $e_\pm$ of T will be scaled during one step of the time evolution by factors $\lambda_\pm$ which are the eigenvalues of T. The Lyapunov exponents are then $\log|\lambda_\pm|$. As we assumed that $|\det T| = 1$ and $\mathrm{Tr}\, T > 2$, $0 < \lambda_- < 1 < \lambda_+$ and one of the Lyapunov exponents is strictly positive.

There exists a unique trace τ on $\mathfrak{A}_q$ that is explicitly given by $\tau\big(W(\eta)\big) := \delta_{\eta,0}$. Clearly, $\tau\big(W(f)^* W(g)\big) = \langle f, g\rangle$. The state τ is invariant under Θ_T and the system $(\mathfrak{A}_q, \Theta_T, \tau)$ exhibits excellent ergodic behaviour [BNS]. We will prove the following results:

Theorem 4.1 *Let λ^+ be the largest eigenvalue of the matrix $T \in SL(2, \mathbf{Z})$ and assume that* $\operatorname{Tr} T > 2$. *For any $0 \leq q < 2\pi$ and any state ω on $\mathfrak{A}_q$, we have that $\mathsf{h}_{(\mathfrak{A}^q_\infty, \Theta_T, \omega)} \leq \log \lambda^+$.*

Theorem 4.2 *Let λ^+ be the largest eigenvalue of the matrix $T \in SL(2, \mathbf{Z})$ and assume that $\operatorname{Tr} T > 2$. For any $0 \leq q < 2\pi$ and any state ω on $\mathfrak{A}_q$ that is normal with respect to τ we have that $\mathsf{h}_{(\mathfrak{A}^q_\infty, \Theta_T, \tau)} \geq \log \lambda^+$.*

Combining these results, we can state that

Theorem 4.3 *Let λ^+ be the largest eigenvalue of the matrix $T \in SL(2, \mathbf{Z})$ and assume that $\operatorname{Tr} T > 2$. For any $0 \leq q < 2\pi$, we have that*

$$\mathsf{h}_{(\mathfrak{A}^q_\infty, \Theta_T, \omega)} = \log \lambda^+,$$

where ω is any state on $\mathfrak{A}_q$ that is normal with respect to τ.

In order to prove the statements of above, we will first consider partitions in elements of finite support and then the extension to general partitions in $\mathfrak{A}^\infty_q$. We therefore introduce the algebra $\mathfrak{D}_q$ of strictly local elements on the non-commutative torus:

$$\mathfrak{D}^q = \{W(f) | f \in \ell^0(\mathbf{Z}^2)\}$$

The support $\operatorname{Supp}\big(W(f)\big)$ of an element $W(f) \in \mathfrak{D}_q$ is the support of f and the support $\operatorname{Supp}(\mathcal{X})$ of a partition $\mathcal{X} = \big(W(f_0), W(f_1), \ldots, W(f_{k-1})\big)$ in $\mathfrak{D}_q$ is

$$\operatorname{Supp}(\mathcal{X}) := \bigcup_{j=0}^{k-1} \operatorname{Supp}(f_j).$$

Lemma 4.1 *Consider a partition $\mathcal{X} = \big(W(f_0), W(f_1), \ldots, W(f_{k-1})\big)$ in $\mathfrak{D}_q$ and a state ω on $\mathfrak{A}_q$, then*

$$\mathsf{S}\big(\rho_\omega[\mathcal{X}]\big) \leq \log \#\big(\operatorname{Supp}(\mathcal{X})\big).$$

Proof: Consider the GNS representation $(\mathfrak{H}, \pi, \Omega)$ of ω. We compute $\mathsf{S}\big(\rho_\omega[\mathcal{X}]\big)$ in terms of the density matrix $|\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{X}}$ as in lemma 3.1. This density matrix is obviously supported by the subspace of $\mathfrak{H}$ generated by the vectors $\pi\big(W(\eta)\big)\,\Omega$ with η in $\operatorname{Supp}(\mathcal{X})$. As this space is at most $\#\big(\operatorname{Supp}(\mathcal{X})\big)$-dimensional, the lemma follows.

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Lemma 4.2 *Let $\mathfrak{A}$ be a C^*-algebra, $\mathcal{X} = (a_0, a_1, \dots, a_{k-1})$ and $\mathcal{Y} = (b_0, b_1, \dots, b_{k-1})$ two k-tuples of elements in $\mathfrak{A}$ and denote by $\Gamma_{\mathcal{X}}$ and $\Gamma_{\mathcal{Y}}$ the completely positive transformations of $\mathfrak{A}$ associated with $\mathcal{X}$ and $\mathcal{Y}$ as in (1), then*

$$\|\Gamma_{\mathcal{X}} - \Gamma_{\mathcal{Y}}\| \leq (A+B) \sum_{i=0}^{k-1} \|a_i - b_i\|$$

where $A = \max\{\|a_i\| \mid a_i \in \mathcal{X}\}$ and mutatis mutandis for B.

Proof: Choose an arbitrary element $x \in \mathfrak{A}$. We make the following estimate:

$$\begin{aligned}
\left\|\Gamma_{\mathcal{X}}(x) - \Gamma_{\mathcal{Y}}(x)\right\| &= \left\|\sum_{i=0}^{k-1} a_i^* x a_i - b_i^* x b_i\right\| \\
&\leq \left\|\sum_{i=0}^{k-1} a_i^* x (a_i - b_i)\right\| + \left\|\sum_{i=0}^{k-1} (a_i^* - b_i^*) x b_i\right\| \\
&\leq \sum_{i=0}^{k-1} (A+B) \|a_i - b_i\| \, \|x\|.
\end{aligned}$$

■

For $f \in \ell^1(\mathbf{Z}^2)$ we can estimate the C*-norm of $W(f)$ by

$$\|W(f)\| \leq \|f\|_1.$$

Combining this estimate with that of lemma 4.2 leads us for partitions

$$\mathcal{X} = \big(W(f_0), W(f_1), \dots, W(f_{k-1})\big) \qquad \text{and} \qquad \mathcal{Y} = \big(W(g_0), W(g_1), \dots, W(g_{k-1})\big)$$

in $\mathfrak{A}_q^\infty$ to

$$\begin{aligned}
\|\Gamma_{\mathcal{X}} - \Gamma_{\mathcal{Y}}\| &\leq (A+B) \sum_{i=0}^{k-1} \|W(f_i) - W(g_i)\| \\
&\leq \Big(\max_i(\|f_i\|_1) + \max_i(\|g_i\|_1)\Big) \sum_{i=0}^{k-1} \|f_i - g_i\|_1.
\end{aligned}$$

Proof of theorem 4.1: Fix a partition of unity $\mathcal{X} = \big(W(f_0), W(f_1), \dots, W(f_{k-1})\big)$ in $\mathfrak{A}^q_\infty$. After timestep n the partition $\mathcal{X}$ becomes

$$\mathcal{X}^n = \Big(\Theta_T^n\big(W(f_0)\big), \Theta_T^n\big(W(f_1)\big), \dots, \Theta_T^n\big(W(f_{k-1})\big)\Big)$$

where

$$\Theta_T^n\big(W(f_i)\big) = \sum_\eta f_i(\eta)\, W\big((T^{\,\mathrm{t}})^n(\eta)\big).$$

At each timestep n we construct a new partition $\mathcal{Y}^n$ supported in a finite volume $\Lambda^n \subset \mathbf{Z}^2$. For $n = 0$, we choose a disc $\Lambda_0 \subset \mathbf{Z}^2$:

$$\Lambda_0 := \{\eta \in \mathbf{Z}^2 \mid \|\eta\| \le r_0\}$$

with r_0 such that $|f_i(\eta)| \le r_0^{-4}$ for all $|\eta| \ge r_0$ and $i = 0, 1, \ldots, k-1$. Let $C = \max_i \|f_i\|_1$ and $\epsilon > 0$. Since

$$\sum_{\eta \in \Lambda_0^c} \frac{1}{\|\eta\|^4} \le 2\pi \int_{r_0}^\infty dr\, \frac{1}{r^3} = \frac{\pi}{r_0^2},$$

we may take r_0 such that $\sum_{\eta\in\Lambda_0^c} \|\eta\|^{-4} \le \epsilon/Ck$. We then define

$$\mathcal{Y}^0 := \big(W^{\Lambda_0}(f_0), W^{\Lambda_0}(f_1), \ldots, W^{\Lambda_0}(f_{k-1})\big)$$

with

$$W^\Lambda(f) := \sum_{\eta\in\Lambda} f(\eta)\, W(\eta).$$

Define for $n = 1, 2, \ldots$ volumes

$$\tilde{\Lambda}^n := \{\eta \in \mathbf{Z}^2 \mid \|\eta\| \le (n+1) r_0\}$$

and

$$\Lambda^n := \Theta_T^n\big(\tilde{\Lambda}^n\big).$$

Then, since

$$W(f_i) - W^{\Lambda_0}(f_i) = \sum_{\eta\in\Lambda_0^c} f_i(\eta)\, W(\eta)$$

we can apply lemma 1, which leads to the following estimate

$$\|\Gamma_{\mathcal{X}} - \Gamma_{\mathcal{Y}^0}\| \le C \sum_{i=0}^{k-1} \sum_{\eta\in\Lambda_0^c} |f_i(\eta)| < \epsilon.$$

In the same way

$$\begin{aligned}
\|\Gamma_{\mathcal{X}^n} - \Gamma_{\mathcal{Y}^n}\| &\le C \sum_{i=0}^{k-1} \Big\| \sum_{\eta\in\Lambda_n^c} f_i(\eta)\, W(\Theta_T^n(\eta)) \Big\| \\
&\le C \sum_{i=0}^{k-1} \sum_{\eta\in\Lambda_n^c} |f_i(\eta)| \\
&\le \frac{\epsilon}{(n+1)^2}.
\end{aligned}$$

Therefore

$$\sum_{\alpha=0}^{n} \|\Gamma_{\mathcal{X}^\alpha} - \Gamma_{\mathcal{Y}^\alpha}\| \leq \sum_{\alpha=0}^{n} \frac{\epsilon}{(\alpha+1)^2} \leq \frac{\pi^2}{6}\epsilon.$$

We now estimate the entropy $\mathsf{S}\big(\rho_\omega[\mathcal{Y}^n \circ \mathcal{Y}^{n-1} \circ \ldots \circ \mathcal{Y}^0]\big)$. The partitions $\mathcal{Y}^i$ are supported by finite volumes and hence we can use lemma 4.1. So we have to estimate the volume on which the partition $\mathcal{Y}^n \circ \mathcal{Y}^{n-1} \circ \ldots \circ \mathcal{Y}$ lives. Call the largest component of any vector η in Λ^0 in the e_+ direction η_+ and in the e_- direction η_-. Then the volume of the support of $\mathcal{Y}^n \circ \mathcal{Y}^{n-1} \circ \ldots \circ \mathcal{Y}$ is certainly not larger than

$$(\eta_+ + 2\lambda_+\eta_+ + \ldots + n\lambda_+^{n-1}\eta_+)(\eta_- + 2\lambda_-\eta_- + \ldots + n\lambda_-^{n-1}\eta_-).$$

Hence,

$$\begin{aligned}\frac{1}{n}\log \mathrm{Vol}\Big(\mathrm{Supp}(\mathcal{Y}^n \circ \mathcal{Y}^{n-1} \circ \ldots \circ \mathcal{Y})\Big) \leq \; & \frac{1}{n}\log(\eta_+ + 2\lambda_+\eta_+ + \ldots + n\lambda_+^{n-1}\eta_+) + \\ & \frac{1}{n}\log(\eta_- + 2\lambda_-\eta_- + \ldots + n\lambda_-^{n-1}\eta_-).\end{aligned}$$

Because $\sum_{i=1}^{n} i\lambda_-^i \leq (1-\lambda_-)^{-2}$, we do not have to take the second term into account. The first term behaves as

$$(\eta_+ + 2\lambda_+\eta_+ + \ldots + n\lambda_+^{n-1}\eta_+) = n^2 \mathrm{O}(\lambda_+^n)\,\eta_+.$$

This yields

$$\mathsf{S}\big(\rho_\omega[\mathcal{Y}^n \circ \mathcal{Y}^{n-1} \circ \ldots \circ \mathcal{Y}^0]\big) \leq 2\log n + n\log\lambda_+$$

and consequently:

$$\mathsf{h}_{(\Theta_T,\omega)}[\mathcal{Y}] \leq \log\lambda_+.$$

We can now compare by lemma 3.3 the dynamical entropy of $\mathcal{X}$ with

$$\limsup_{n\to\infty} \frac{1}{n}\mathsf{S}\big(\rho_\omega[\mathcal{Y}^n \circ \mathcal{Y}^{n-1} \circ \ldots \circ \mathcal{Y}^0]\big) \leq \log\lambda_+.$$

This yields:

$$|\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] - \limsup_{n\to\infty}\frac{1}{n}\mathsf{S}\big(\rho_\omega[\mathcal{Y}^n \circ \mathcal{Y}^{n-1} \circ \ldots \circ \mathcal{Y}^0]\big)| \leq 2k(\log 2 + \log k)\epsilon\frac{\pi^2}{6} - 2k\frac{\pi^2}{6}\epsilon\log 2k\frac{\pi^2}{6}\epsilon.$$

So that finally, after taking the limit $\epsilon \to 0$,

$$\mathsf{h}_{(\Theta,\mathcal{X})} \leq \log\lambda_+.$$

■

Proof of theorem 4.2 It is sufficient to consider as reference state ω a vector state in the GNS representation $(\mathfrak{H}, \pi, \Omega)$ of τ that is generated by a vector of the type $\pi(W(g))\,\Omega$ with g of finite support and $\tau(W(g)^*W(g)) = \|g\|_2^2 = 1$. A general normal state can be approximated by states of this kind. In [AFTA] it has been shown that there exists for any $\epsilon > 0$ a partition $\mathcal{X} = \{W(f_0), W(f_1), \ldots, W(f_{k-1})\}$ in $\mathfrak{D}_q$ such that

$$\mathsf{h}_{(\Theta_T,\tau)}[\mathcal{X}] \geq \log \lambda_+ - \epsilon.$$

Suppose that the points of the support of a function k are separated by an amount larger than twice the diameter of the support of g. We can then write that

$$\begin{aligned}
\langle \pi\bigl(W(g)\bigr)\,\Omega, \pi\bigl(W(k)\bigr)\,\bigl(W(g)\bigr)\,\Omega\rangle &= \tau\bigl(W(g)^*W(k)\,W(g)\bigr) \\
&= \tau\bigl(W(g)\,W(g)^*W(k)\bigr) \\
&= (g^\dagger *_q g)(0)\,k(0) \\
&= \tau\bigl(W(k)\bigr).
\end{aligned}$$

By applying a sufficiently high power n of Θ_T we can separate all the points in the supports of the f_j by an amount that is larger than twice the diameter of the support of $W(g)$. This leads to

$$\mathsf{h}_{(\Theta_T,\omega)}[\Theta_T^n(\mathcal{X})] = \mathsf{h}_{(\Theta_T,\tau)}[\Theta_T^n(\mathcal{X})] = \mathsf{h}_{(\Theta_T,\tau)}[\mathcal{X}] \geq \log \lambda_+ - \epsilon.$$

■

Remark It can be proven along similar lines that the dynamical entropy of non-expanding toral automorphisms Θ_T of $\mathfrak{A}_q$, i.e. automorphisms defined by a map T with $\mathrm{Tr}\,T = 2$, is zero if we restrict ourselves to partitions in $\mathfrak{A}^q_\infty$. It is also possible to treat the higher dimensional non-commutative tori using techniques similar to these in [AAFT].

5. Dynamical entropy of the non-commutative shift

5.1 The algebra of exponentially localized elements for spin chains

In this section, we introduce a subalgebra of the quasi-local observables of a quantum spin chain that consists of elements that are local up to terms that are exponentially small in the size of the region of localization. The quasi-local algebra $\mathfrak{A}^{\mathcal{L}}$ of a quantum spin chain with the $d \times d$ matrices $\mathcal{M}_d$ as single site algebra is obtained by completing the algebra $\mathfrak{A}_{\rm loc}$ of strictly local observables with respect to its natural operator-norm. $\mathfrak{A}_{\rm loc}$ is the union of the algebras $\mathfrak{A}_\Lambda$ over the finite subsets Λ of the chain $\mathbf{Z}$. To each site of the chain we attach a copy of $\mathcal{M}_d$ and $\mathfrak{A}_\Lambda$ is the tensor product of the copies attached to the sites of Λ. Algebras belonging to increasing sets are naturally related by identifying $x_1 \in \mathfrak{A}_{\Lambda_1}$ with $x_1 \otimes \big(\otimes_{j\in\Lambda_2\setminus\Lambda_1} \mathbb{1}\big)$ if $\Lambda_1 \subset \Lambda_2$. This identification preserves the operator-norm and induces a C^*-norm on $\mathfrak{A}_{\rm loc}$. If M is an arbitrary subset of $\mathbf{Z}$, $\mathfrak{A}_{\rm M}$ will denote the completion of the algebra generated by the $\mathfrak{A}_\Lambda$ with Λ running through the finite subsets of M. The algebra of quasi-local observables $\mathfrak{A}^{\mathcal{L}}$ corresponds to the choice $\mathrm{M} = \mathbf{Z}$. The support $\mathrm{Supp}(x)$ of $x \in \mathfrak{A}^{\mathcal{L}}$ is the smallest subset M of $\mathbf{Z}$ such that $x \in \mathfrak{A}_{\rm M}$.

An observable x with support in M depends only on the non-commuting variables at the sites $i \in \mathrm{M}$. This can be expressed as $x = u\,x\,u^*$ for any unitary u in $\mathfrak{A}_{\{j\}}$ with $j \in \mathrm{M}^c := \mathbf{Z} \setminus \mathrm{M}$. The converse also holds: any $x \in \mathfrak{A}^{\mathcal{L}}$ that commutes with all unitaries in $\mathfrak{A}_{\{j\}}$ for $j \in \mathrm{M}^c$ has support contained in M. The dependence of an observable x on the "quantum variable" at the site $j \in \mathbf{Z}$ can be measured in terms of the oscillation norm $|||\ \ |||_j$ at j:

$$|||x|||_j := \sup_{u\in\mathcal{U}_j} \|x - u\,x\,u^*\|.$$

In this formula $\mathcal{U}_j$ is the copy of the $d \times d$ unitary matrices at j. Clearly,

$$\begin{aligned}
|||x^*|||_j &= |||x|||_j \qquad \text{and} \\
|||x\,y|||_j &= \sup_{u\in\mathcal{U}_j} \|x\,y - u\,x\,y\,u^*\| \\
&\le \sup_{u\in\mathcal{U}_j} \Big(\|(x - u\,x\,u^*)y\| + \|u\,x\,u^*(y - u\,y\,u^*)\|\Big) \\
&\le \sup_{u\in\mathcal{U}_j} \|x - u\,x\,u^*\|\,\|y\| + \sup_{u\in\mathcal{U}_j} \|x\|\,\|y - u\,y\,u^*\| \\
&= |||x|||_j\,\|y\| + \|x\|\,|||y|||_j.
\end{aligned}$$

We will consider in this section observables that depend only weakly on quantum variables far away, more precisely, let

$$\mathfrak{A}_0 := \Big\{ x \in \mathfrak{A}^{\mathcal{L}} \ \Big|\ \lim_{|j|\to\infty} j^k\,|||x|||_j = 0 \quad \text{for all } k \in \mathbf{N} \Big\}.$$

$\mathfrak{A}_0$ is a $*$-subalgebra of $\mathfrak{A}^{\mathcal{L}}$ that is uniformly dense as it contains $\mathfrak{A}_{\rm loc}$. It is also invariant under the automorphism Θ of $\mathfrak{A}^{\mathcal{L}}$ induced by the shift on $\mathbf{Z}$.

Let τ_{M} be the normalized trace on $\mathfrak{A}_{\mathrm{M}}$. Taking the partial trace $\mathrm{id}_{\mathrm{M}} \otimes \tau_{\mathrm{M}^c}$ with respect to the observables in M^c is a conditional expectation onto $\mathfrak{A}_{\mathrm{M}}$ and

$$\left(\mathrm{id}_{\mathrm{M}_1} \otimes \tau_{\mathrm{M}_1^c}\right) \circ \left(\mathrm{id}_{\mathrm{M}_2} \otimes \tau_{\mathrm{M}_2^c}\right) = \mathrm{id}_{\mathrm{M}_1 \cup \mathrm{M}_2} \otimes \tau_{\mathrm{M}_1^c \cap \mathrm{M}_2^c}.$$

There is a simple estimate of the distance between an observable $x \in \mathfrak{A}_{\mathcal{L}}$ and its projection $\left(\mathrm{id}_{\mathrm{M}} \otimes \tau_{\mathrm{M}^c}\right)(x)$ on $\mathfrak{A}_{\mathrm{M}}$ in terms of oscillation norms. If μ is the Haar measure on $\mathcal{U}(d)$, then

$$\begin{aligned} \left\|x - \left(\mathrm{id}_{\{j\}} \otimes \tau_{\mathbf{Z} \setminus \{j\}}\right)\right\| &= \left\| \int_{\mathcal{U}_j} d\mu(u)\, (x - u\, x\, u^*) \right\| \\ &\leq \sup_{u \in \mathcal{U}_j} \|x - u\, x\, u^*\| \\ &= |||x|||_j. \end{aligned}$$

Telescoping this estimate leads to

$$\left\|x - \left(\mathrm{id}_{\mathrm{M}} \otimes \tau_{\mathrm{M}^c}\right)(x)\right\| \leq \sum_{j \in \mathrm{M}^c} |||x|||_j.$$

As a consequence, the elements in $\mathfrak{A}_0$ can be very well approximated by local elements in the following sense: for any $n \in \mathbf{N}$ there is sequence $\left(x_j(n)\right)_j$ in $\mathfrak{A}_{\mathrm{loc}}$ and a $j_0(n) \in \mathbf{N}$ such that

$$x_j(n) \in \mathfrak{A}_{\{-j,-j+1,\ldots,j\}} \qquad \text{and} \qquad \|x - x_j(n)\| \leq \frac{1}{j^n} \quad \text{for } j \geq j_0(n). \tag{2}$$

5.2 Dynamical entropy in the local algebra

The dynamical triple describing the shift on a quantum spin chain is $(\mathfrak{A}^{\mathcal{L}}, \Theta, \omega)$ with $\mathfrak{A}^{\mathcal{L}}$ the quasi-local algebra on $\mathbf{Z}$ with $\mathcal{M}_d$ as single site observables. We will in this subsection compute the dynamical entropy for partitions in local observables. For any finite $\Lambda \subset \mathbf{Z}$, the restriction of ω to $\mathfrak{A}_\Lambda$ is given by a density matrix ω^Λ on $(\mathbf{C}^d)^{\otimes\Lambda}$:

$$\omega(A) = \mathrm{Tr}_{(\mathbf{C}^d)^{\otimes\Lambda}}\, \omega^\Lambda\, A, \qquad A \in \mathfrak{A}_\Lambda.$$

The ω^Λ are called the reduced density matrices of ω. By $\sigma(\omega)$ we denote the mean von Neumann entropy of a shift-invariant state ω.

Theorem 5.1 *With the notations of above:*

$$\mathsf{h}_{(\mathfrak{A}_{\mathrm{loc}},\Theta,\omega)} = \sigma(\omega) + \log d.$$

Proof: Choose a partition $\mathcal{X} = (x_0, x_1, \ldots, x_{k-1})$ of size k in $\mathfrak{A}_{\mathrm{loc}}$. By translation invariance of the state ω we can without loss of generality assume that the x_i belong to $\mathfrak{A}_{\{0,1,\ldots,\ell-1\}}$ for some $\ell = 1, 2, \ldots$ Then, for $m > 0$, the elements of the partition $\Theta^{m-1}(\mathcal{X}) \circ \ldots \circ \Theta(\mathcal{X}) \circ \mathcal{X}$ belong to $\mathfrak{A}_{\{0,1,\ldots,m+\ell-1\}}$. Hence, using lemma 3.2, we obtain

$$\mathsf{S}\left(\rho_\omega^{\otimes\{0,1,\ldots,m+\ell-1\}}[\mathcal{X}]\right) \leq \mathsf{S}\left(\omega^{\otimes\{0,1,\ldots,m+\ell-1\}}\right) + \log d^{m+\ell-1}.$$

Dividing both sites by m and taking the limit $m \to \infty$, this inequality becomes

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] \leq \sigma(\omega) + \log d.$$

In order to prove that the upper bound is reached, consider the partition:

$$\mathcal{X} = \left(\frac{1}{\sqrt{d}}\, e_{ij} \;\middle|\; i, j = 1, 2, \ldots, d\right),$$

where the e_{ij} are matrix units for $\mathcal{M}_d$. An easy computation yields the following form of the density matrix $\rho_\omega^{\{0,1,\ldots,m-1\}}[\mathcal{X}]$ on $\mathcal{M}_{d^2}^{\otimes m}$

$$\rho_\omega^{\{0,1,\ldots,m-1\}}[\mathcal{X}] = \omega^{\{0,1,\ldots,m-1\}} \otimes \left(\frac{1}{d}\,\mathbb{I}_d\right)^{\otimes m}.$$

It is clear that for this partition $\mathcal{X}$:

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] = \sigma(\omega) + \log d.$$

$\blacksquare$

5.3 Dynamical entropy in the algebra of exponentially localized elements

We will now extend the result of theorem 5.1 to more general partitions, namely those that are generated by elements in $\mathfrak{A}_0$, that is to say by elements x with sufficiently weak dependence on sites far away:

$$\lim_{|j|\to\infty} j^k\, |||x|||_j = 0 \qquad \text{for all } k \in \mathbf{N}.$$

Theorem 5.2 *With the notation introduced in the previous paragraphs,*

$$\mathsf{h}_{(\mathfrak{A}_0,\Theta,\omega)} = \sigma(\omega) + \log d.$$

Proof: The inequality $\mathsf{h}_{(\mathfrak{A}_0,\Theta,\omega)} \geq \sigma(\omega) + \log d$ is obvious from $\mathfrak{A}_{\mathrm{loc}} \subset \mathfrak{A}_0$. It therefore remains to prove the upper bound $\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] \leq \sigma(\omega) + \log d$ for any partition $\mathcal{X} = (x_0, x_1, \ldots, x_{k-1})$ in $\mathfrak{A}_0$.

We will use the approximation property (2) for the case $n = 2$ i.e. for any $\epsilon > 0$ we can find a $j_0 \in \mathbf{N}$ such that for any $j \in \mathbf{N}$ there exist $x_i^{(j)} \in \mathfrak{A}_{\{-j-j_0,-j-j_0+1,\ldots,j+j_0\}}$ with

$$\|x_i - x_i^{(j)}\| \leq \frac{\epsilon}{(j+1)^2} \qquad \text{and} \qquad \|x_i^{(j)}\| \leq \|x_i\|, \qquad i = 1, 2, \ldots, k-1.$$

Choose a natural number N and a corresponding sequence of approximations $(\mathcal{Y}(n))_n$ to the partition $\mathcal{X}$:

$$\mathcal{Y}(n) = \big(x_0^{(j)}, x_1^{(j)}, \ldots, x_{k-1}^{(j)}\big) \qquad \text{for} \qquad jN \leq n < (j+1)N.$$

We can now, using lemma 4.2, estimate the norm difference between $\Gamma_{\mathcal{X}}$ and $\Gamma_{\mathcal{Y}(n)}$:

$$\big\|\Gamma_{\mathcal{X}} - \Gamma_{\mathcal{Y}(n)}\big\| \leq \frac{2\epsilon}{(j+1)^2} \qquad \text{for} \qquad jN \leq n < (j+1)N. \tag{3}$$

This estimate remains unchanged when we move the partitions $\mathcal{X}$ and $\mathcal{Y}(n)$ around by the dynamics. Using the same idea as in the proof of theorem 5.1, we count the size of the volume on which the partition

$$\Theta^{n-1}\big(\mathcal{Y}(n-1)\big) \circ \ldots \circ \Theta\big(\mathcal{Y}(1)\big) \circ \mathcal{Y}(0)$$

lives. By construction, it is clear that this size is bounded by $2j_0 + 1 + \frac{N+1}{N}n$. This yields by lemma (3.2) for $jN \leq n < (j+1)N$:

$$\mathsf{S}\Big(\rho_\omega\Big[\Theta^{n-1}\big(\mathcal{Y}(n-1)\big) \circ \ldots \circ \Theta\big(\mathcal{Y}(1)\big) \circ \mathcal{Y}(0)\Big]\Big) \leq \mathsf{S}(\omega^{\{-j_0,-j_0+1,\ldots,j_0+j\}}) + \log d^{\,2j_0+1+\frac{N+1}{N}n}.$$

If we divide both sides of the inequality by n and take the limit $n \to \infty$ we arrive at:

$$\limsup_{n\to\infty} \frac{1}{n}\, \mathsf{S}\Big(\rho_\omega\Big[\Theta^{n-1}\big(\mathcal{Y}(n-1)\big) \circ \ldots \circ \Theta\big(\mathcal{Y}(1)\big) \circ \mathcal{Y}(0)\Big]\Big) \leq \sigma(\omega) + \frac{N+1}{N}\log d.$$

By lemma 3.3 and the estimate (3) we then obtain

$$\Big|\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] - \limsup_{n\to\infty} \frac{1}{n}\, \mathsf{S}\Big(\rho_\omega\Big[\Theta^{n-1}\big(\mathcal{Y}(n-1)\big) \circ \ldots \circ \Theta\big(\mathcal{Y}(1)\big) \circ \mathcal{Y}(0)\Big]\Big)\Big|$$
$$\leq 2k(\log 2 + \log k)N\epsilon\frac{\pi^2}{6} + \eta(\frac{12k}{\pi^2}N\epsilon),$$

so that, after taking the limit $\epsilon \to 0$:

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] \leq \sigma(\omega) + \frac{N+1}{N}\log d.$$

But, since N was arbitrary this implies also that

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] \leq \sigma(\omega) + \log d.$$

■

6. Some miscellaneous results and remarks

The notion of dynamical entropy that we considered here is trivial for dynamical systems with discrete spectrum. Here, we think typically of systems consisting of a finite number of electrons confined to a compact region. The next result is a caricature of such a situation.

Proposition 6.1 *Let $(\mathfrak{A}, \Theta, \omega)$ be a quantum dynamical system with finite dimensional $\mathfrak{A}$, then*

$$\mathsf{h}_{(\mathfrak{A},\Theta,\omega)} = 0.$$

Proof: We can always assume that the algebra $\mathfrak{A}$ is a subalgebra of the $d \times d$ matrices $\mathcal{M}_d$. The state ω can be written in terms of a density matrix ρ. Then, using lemma 3.2, we have:

$$\mathsf{S}\big(\rho_\omega[\Theta^n(\mathcal{X}) \circ \Theta^{n-1}(\mathcal{X}) \circ \ldots \circ \mathcal{X}]\big) \leq \mathsf{S}(\rho) + \log d \leq 2 \log d.$$

Dividing both sides by n and taking the limit yields:

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] = 0.$$

Since, $\mathcal{X}$ was arbitrary $\mathsf{h}_{(\mathfrak{A},\Theta,\omega)} = 0$.

■

Hence, in our approach, finite quantum systems are too small to display a non-zero asymptotic entropy production. However, there is entropy production in finite systems as is pointed out in [AMM]. It saturates however after a finite time, typically proportional to $\log d$.

The following result shows that the entropy can jump to infinity if one allows a too large class of partitions. We show that in a properly infinite von Neumann algebra, which typically arise as a GNS-representation of a temperature state, even the identity automorphism produces an infinite entropy. This is a consequence of the existence of projections equivalent to the identity in these algebras, such that one can produce a trace of any dimension on the quantum spin chains. Physically, this means that the measurement associated to such a partition of unity disturbs the state of the system so drastically that it produces already entropy on its own.

In order to state the next proposition we consider a von Neumann dynamical system: $\mathfrak{M}$ is a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. In $\mathfrak{H}$ there is a distinguished normalized vector Ω that is cyclic for $\mathfrak{M}$ i.e. $\mathfrak{M}\,\Omega$ is dense in $\mathfrak{H}$. Ω defines a vector state ω on $\mathfrak{M}$ by putting $\omega(x) := \langle \Omega, x\,\Omega \rangle$. The dynamics is given in terms of a unitary operator U on $\mathfrak{H}$ that leaves Ω invariant and such that $\Theta(\mathfrak{M}) = \mathfrak{M}$ with $\Theta(x) := u\,x\,u^*$.

Proposition 6.2 *Let $(\mathfrak{M}, \Theta, \omega)$ be a von Neumann dynamical system where the algebra $\mathfrak{M}$ is taken to be a type I_∞ or a type III factor. Then,*

$$\mathsf{h}_{(\mathfrak{M},\Theta,\omega)} = \infty.$$

Proof: Fix $d \in \{2, 3, \ldots\}$. Since $\mathfrak{M}$ is a properly infinite factor, we can, by use of the halving lemma, find d orthogonal projections p_i in $\mathfrak{M}$ such that $\sum_{i=1}^d p_i = \mathbb{I}$ and each of the p_i equivalent to $\mathbb{I}$. Denote by v_i the isometries in $\mathfrak{M}$ satisfying $p_i = v_i v_i^*$ and $v_i^* v_i = \mathbb{I}$. Then, since

$$\sum_i p_i = \sum_i v_i v_i^* = \mathbb{I},$$

the Cuntz algebra O_d is contained in $\mathfrak{M}$. One can easily derive that for $i \neq j$,

$$(v_i^* v_j v_j^*)^* (v_i^* v_j v_j^*) = p_j^* p_i p_j = 0.$$

Therefore $v_i^* v_j v_j^* = 0$ and multiplying by v_j on the right, $v_i^* v_j = 0$. Consider now the following partition of unity in $\mathfrak{M}$:

$$\mathcal{X} := \frac{1}{\sqrt{d}}(v_1, v_2, \ldots, v_d).$$

For $m \in \{1, 2, \ldots\}$, let $\mathbf{i} = (i_0, i_1, \ldots, i_{m-1})$ and $\mathbf{j} = (j_0, j_1, \ldots, j_{m-1})$ be two m-tuples with values in $\{1, 2, \ldots, d\}$. We compute the density matrix:

$$\begin{aligned}\rho_\omega[\Theta^{m-1}(\mathcal{X}) \circ \ldots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]_{\mathbf{i},\mathbf{j}} &= \omega\big(v_{j_0}^* \cdots \Theta^{m-1}(v_{j_{m-1}}^*)\Theta^{m-1}(v_{i_{m-1}}) \cdots v_{i_0}\big) \\ &= \Big(\frac{1}{d}\Big)^m \delta_{\mathbf{i},\mathbf{j}}.\end{aligned}$$

Hence it is the tracial state on $\mathcal{M}_d^{\otimes \mathbf{N}}$. Then

$$\mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] \geq \log d.$$

Since d is arbitrary $\mathsf{h}_{(\mathfrak{M},\Theta,\omega)}$ can be made arbitrarily large.

$\blacksquare$

Finally, we would like to comment on the scaling relation which is highly desirable for dynamical entropy:

$$\mathsf{h}_{(\mathfrak{A}_0,\Theta^k,\omega)} = k\, \mathsf{h}_{(\mathfrak{A}_0,\Theta,\omega)}, \qquad k = 1, 2, \ldots$$

We cannot prove this relation for the time being, but we will at least mention a partial result.

Proposition 6.3 *Let $(\mathfrak{A}, \Theta, \omega)$ be a C^*-dynamical system, $\mathfrak{A}_0$ a Θ-invariant $*$-subalgebra of $\mathfrak{A}$ and $k \in \mathbf{N}_0$, then*

$$k\,\mathsf{h}_{(\mathfrak{A}_0,\Theta,\omega)} \leq \mathsf{h}_{(\mathfrak{A}_0,\Theta^k,\omega)}.$$

Proof: Consider a partition of unity $\mathcal{X}$ in $\mathfrak{A}_0$. Then, denoting $\hat{\mathcal{X}} = \Theta^{k-1}(\mathcal{X}) \circ \ldots \circ \Theta(\mathcal{X}) \circ \mathcal{X}$

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \mathsf{S}\big(\rho_\omega[\Theta^{k(n-1)}(\hat{\mathcal{X}}) \circ \ldots \circ \Theta^k(\mathcal{X}) \circ \hat{\mathcal{X}}]\big) &= \lim_{n\to\infty} \frac{k}{kn} \mathsf{S}\big(\rho[\Theta^{kn}(\mathcal{X}) \circ \ldots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]\big) \\
&= k \lim_{j\to\infty} \frac{1}{j} \mathsf{S}\big(\rho[\Theta^j \mathcal{X} \circ \ldots \circ \Theta(\mathcal{X}) \circ \mathcal{X}]\big) \\
&= k\,\mathsf{h}_{(\Theta,\omega)}(\mathcal{X}).
\end{aligned}$$

The last but one equality follows from the triangle inequality for the von Neumann entropy. Indeed, if j and n are natural numbers satisfying $(k-1)n \leq j < kn$ and if $\omega^{\{0,1,\ldots,j\}}$ denotes the reduced density matrix of a state ω on a spin half-chain at the sites $\{0, 1, \ldots, j\}$, then:

$$\big|\mathsf{S}\big(\omega^{\{0,1,\ldots,j\}}\big) - \mathsf{S}\big(\omega^{\{0,1,\ldots,(k-1)n\}}\big)\big| \leq \mathsf{S}\big(\omega^{\{(n-1)k,(n-1)k+1,\ldots,j\}}\big) \leq k\,\log d,$$

where d is the dimension of the single site spin. Thus,

$$k\,\mathsf{h}_{(\mathfrak{A}_0,\Theta,\omega)} = k\,\sup_{\mathcal{X}} \mathsf{h}_{(\Theta,\omega)}[\mathcal{X}] = \sup_{\mathcal{X}} \mathsf{h}_{(\Theta^k,\omega)}[\hat{\mathcal{X}}] \leq \sup_{\mathcal{X}} \mathsf{h}_{(\Theta^k,\omega)}[\mathcal{X}] = \mathsf{h}_{(\mathfrak{A}_0,\Theta^k,\omega)}.$$

■

The validity of the reverse inequality is an open problem. For the cat map, it is clear that the equality is satisfied. Indeed as Θ_T^k corresponds to the matrix T^k which has eigenvalues λ_+^k, λ_-^k, the dynamical entropy becomes $k \log \lambda_+$. For the shift, this can easily been proven using lemma 3.2. Also for the free shift, the equality holds. In fact we are not aware of any counterexample for this relation. Therefore, we conjecture that the additivity relation holds in general. For classical systems, the proof of the scaling behaviour of the dynamical entropy relies on the monotonicity of the entropy of a partition under refinements of the partition. It is well-known that classical monotonicity properties of the entropy generally don't extend to the non-commutative situation. The last proposition gives a partial result. The proof uses the notion of relative entropy. We introduce it here very briefly and refer the reader to any standard work on entropy for a more detailed discussion [OP]. The relative entropy $\mathsf{S}(\rho_1, \rho_2)$ of two density matrices ρ_1 and ρ_2 on a Hilbert space $\mathfrak{H}$ is given by:

$$\mathsf{S}(\rho_1, \rho_2) := \mathrm{Tr}\,\rho_1\,\big(\log \rho_1 - \log \rho_2\big).$$

Proposition 6.4 *Let $\mathfrak{A}$ be a C^*-algebra and $\mathcal{X}$, $\mathcal{Y}$ partitions of unity in $\mathfrak{A}$ with $\mathcal{X}$ bistochastic. Then, for an arbitrary state ω on $\mathfrak{A}$:*

$$\mathsf{S}(\rho_\omega[\mathcal{Y} \circ \mathcal{X}]) \geq \mathsf{S}(\rho_\omega[\mathcal{Y}]).$$

Proof: Let Ω denote the GNS vector representing the state ω in the GNS representation $(\mathfrak{H}, \Omega, \pi)$ of ω. By lemma 3.1 we know that

$$\mathsf{S}(\rho_\omega[\mathcal{Y} \circ \mathcal{X}]) = \mathsf{S}(|\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{Y}\circ\mathcal{X}}).$$

Let us denote by τ the normalized trace on the d-dimensional subspace of $\mathfrak{H}$ spanned by the vectors $\pi(\mathcal{Y})\,\Omega$ and $\pi(\mathcal{Y} \circ \mathcal{X})\,\Omega$. One easily verifies that $\mathsf{S}(\rho, \tau) = \log d - \mathsf{S}(\rho)$. Because $\Pi_{\mathcal{X}}$ is bistochastic: $\tau \circ \Pi_{\mathcal{X}} = \tau$. Uhlmann's monotonicity theorem for completely positive mappings [OP] gives then:

$$\begin{aligned}
\mathsf{S}\big(\rho_\omega[\mathcal{Y} \circ \mathcal{X}]\big) &= \mathsf{S}(|\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{Y}\circ\mathcal{X}}) \\
&= -\mathsf{S}(|\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{Y}\circ\mathcal{X}}, \tau) + \log d \\
&= -\mathsf{S}(|\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{Y}} \circ \Pi_{\mathcal{X}}, \tau) + \log d \\
&\geq -\mathsf{S}(|\Omega\rangle\langle\Omega| \circ \Pi_{\mathcal{Y}}, \tau) + \log d \\
&= \mathsf{S}(\rho_\omega[\mathcal{Y}]).
\end{aligned}$$

■

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