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EFFECTIVE INTERACTIONS DUE TO QUANTUM FLUCTUATIONS

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Abstract. Quantum lattice systems are rigorously studied at low temperatures. When the Hamiltonian of the system consists of a potential (diagonal) term and a — small — off-diagonal matrix containing typically quantum effects, such as a hopping matrix, we show that the latter creates an effective interaction between the particles.

In the case that the potential matrix has infinitely many degenerate ground states, some of them may be stabilized by the effective potential. The low temperature phase diagram is thus a small deformation of the zero temperature phase diagram of the diagonal potential *and* the effective potential. As illustrations we discuss the asymmetric Hubbard model and the hard-core Bose-Hubbard model.

Keywords: effective potential, quantum instability, phase diagrams, quantum Pirogov-Sinai theory, asymmetric Hubbard model, Bose-Hubbard model

Contents

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1. Introduction

Physics of a large number of quantum particles at equilibrium is very interesting and difficult at the same time. Interesting, because it is treating such macroscopic phenomena as magnetization, crystallisation, superfluidity or superconductivity. And difficult, because their study has to combine Quantum Mechanics and Statistical Physics.

A natural approach is to decrease difficulties arising from this combination by starting from only one aspect. Thus one can use only Quantum Mechanics and treat the particles first as independent, trying next to add small interactions. In the present paper we are concerned with the other approach. Namely, to start with a model treated by Classical Statistical Physics, adding next a small quantum perturbation. Another simplification is to consider lattice systems (going back to a physical justification for the modeling process, we can invoke applications to condensed matter physics).

Quantum systems studied here have Hamiltonians consisting of two terms. The first term is a classical interaction between particles; formally, this operator is "function" of the position operators of the particles and it is diagonal with respect to the corresponding basis in occupation numbers. The second term is an off-diagonal operator that we suppose to be small with respect to the interaction. Typical example for this is a hopping matrix.

The aim of the paper is to show that a *new effective interaction* appears that is due to the combination of the potential and the kinetic term. An explicit formula is computed, and sufficient conditions are given in order that the low temperature behaviour is controlled by the sum of the original diagonal interaction and the effective potential. To be more precise, it is rigorously shown that the phase diagram of the original quantum model is only a small perturbation of the phase diagram of a classical lattice model with the effective interaction.

Thus, we will start by recalling some standard ideas of Classical Statistical Mechanics of lattice systems. The Peierls argument for proving the occurrence of a first order phase transition in the Ising model [Pei, Dob, Gri] marks the beginning of the perturbative studies of the low temperature regimes of classical statistical models. Partition functions and expectation values of observables may be expanded with respect to the excitations on top of the ground states, interpreting the excitations in geometric terms as *contours*. These ideas and methods are referred to as the Pirogov-Sinai theory; they were first introduced in [PS, Sin] and later further extended [Zah, BI, BS].

The intuitive picture is that a low temperature phase is essentially a ground state configuration with small excitations. A phase is stable whenever it is unprobable to install a large domain with another phase inside. For such an insertion one has to pay on its boundary, it is excited (two phases are separated by excitations), but, on the other side, one may gain on its volume if its *metastable free energy* (its ground energy minus the contribution of small thermal fluctuations) is smaller than the one of the external phase. It is important to take into account the fluctuations since they can play a role in determining which phase is dominant. A standard example here is the Blume-Capel model with an external field slightly favouring the "+1" phase; at low temperatures, the "0" phase may be still selected because it has more low energy excitations (theory of such dominant states chosen by thermal fluctuations may be found in [BS]).

The partition function of a quantum system $\operatorname{Tr} e^{-\beta H}$ may be expressed using Duhamel expansion (or Trotter formula), yielding a classical contour model in a space with one more (continuous) dimension. If the corresponding classical model (the diagonal part only) has stable low temperature phases, and if the off-diagonal terms of the Hamiltonian are small, the contours have low probability of occurrence and it is possible to extend the Peierls argument to quantum models [Gin]. More generally, one can formulate a "Quantum Pirogov-Sinai theory" [BKU, DFF1], in order to establish that (i) low temperature phases are very close to ground states of the diagonal interaction (more precisely: the density matrix $\frac{1}{Z} e^{-\beta H}$ is close to the projection operator $|g\rangle\langle g|$, where $|g\rangle$ is the ground state of the diagonal interaction only) and (ii) low temperature phase diagrams are small deformations of zero temperature phase diagrams of the interactions.

So far we have only discussed the case when the effect of the quantum perturbation is small, and the features of the phases are due to the classical interaction between the particles. It may happen, however, that the classical interaction alone is not sufficient to choose the low temperature behaviour. This is the case in the two models we introduce now and use later for illustration of our general approach.

- **The asymmetric Hubbard model.** It describes hopping spin $\frac{1}{2}$ particles on a lattice $\Lambda \subset \mathbb{Z}^\nu$. A basis of its Hilbert space is indexed by classical configurations $n \in \{0, \uparrow, \downarrow, 2\}^\Lambda$, and the Hamiltonian
$$H = -\sum_{\|x-y\|_2=1} \sum_{\sigma=\uparrow,\downarrow} t_\sigma c^\dagger_{x\sigma} c_{y\sigma} + U \sum_x n_{x\uparrow} n_{x\downarrow} - \mu \sum_x (n_{x\uparrow} + n_{x\downarrow}) \tag{1.1}$$
(the hopping parameter t_σ depends on the spin of the particle). In the atomic limit $t_\uparrow = t_\downarrow = 0$ the ground states are all the configurations with exactly one particle at each site. The degeneracy equals $2^{|\Lambda|}$, which means that it has nonvanishing residual entropy at zero temperature.
- **The hard-core Bose-Hubbard model.** We consider bosons moving on a lattice $\Lambda \subset \mathbb{Z}^2$. They interact through an infinite on-site repulsive potential (hard-core), nearest neighbour and next nearest neighbour repulsive potentials. A basis of its Hilbert space is the set of all configurations $n \in \{0, 1\}^\Lambda$, and its Hamiltonian:
$$H = -t \sum_{\|x-y\|_2=1} a^\dagger_x a_y + U_1 \sum_{\|x-y\|_2=1} n_x n_y + U_2 \sum_{\|x-y\|_2=\sqrt{2}} n_x n_y - \mu \sum_x n_x. \tag{1.2}$$
For $U_1 \geqslant 2U_2$, and if $0 < \mu < 8U_2$, the ground states of the potential part are those generated by $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, i.e. any configuration with each two lines empty, and the other anti-ferromagnetic, is a ground state (and similarly in the other direction). The degeneracy is of the order $2^{\frac{1}{2}|\Lambda|^{\frac{1}{2}}}$ (if Λ is a square), there is no residual entropy.

In these two situations, the smallest quantum fluctuations yield an effective interaction, and this interaction stabilizes phases displaying long-range order (there is neither superfluidity nor superconductivity).

In the case where classical and quantum particles are mixed in one model, like the Falicov-Kimball model, a method using Peierls argument was proposed by Kennedy and Lieb [KL]; it was extended in [LM] to situations that are not covered by the present paper, namely to cases of such mixed systems with continuous classical variables.

Results very similar to ours have already been obtained by Datta, Fernández, Fröhlich and Rey-Bellet [DFFR]. Their approach is different, however. Starting from a Hamiltonian $H(\lambda) = H^{(0)} + \lambda V$, $H^{(0)}$ being a diagonal operator with infinitely many ground states, and V the quantum perturbation, the idea is to choose an antisymmetric matrix $S = \lambda S^{(1)} + \lambda^2 S^{(2)}$ in such a way that the operator $H^{(2)}(\lambda) = \mathrm{e}^S H(\lambda) \mathrm{e}^{-S}$, expanded with the help of Lie-Schwinger series, turns out to be diagonal, up to terms of order λ^3 or higher. If the diagonal part of $H^{(2)}$ has a finite number of ground states and the excitations cost strictly positive energy, it can be shown that the ground states are stable. It is possible to include higher orders in this perturbation scheme (see [DFFR]).

In fact, our first intention was to study the stability of the results of [BS] with respect to a quantum perturbation, and we began the present study as a warm-up and the first simple step towards this goal. This simple step turned out however to be rather involved. Even though, at the end, the paper contains results similar to that of [DFFR], we think that the subject is important enough to justify an alternative approach, and that there are some advantages in an explicit formula for the effective potential and sufficient conditions for it to control the low temperature behaviour that may be useful in explicit applications.

The intuitive background of this paper owes much to the work of Bricmont and Slawny [BS] discussing the situation with infinite degeneracy of ground states, where only a finite number of ground states is dominating as a result of thermal fluctuations, and to the paper of Messager and Miracle-Solé [MM] which was useful to understand the structure of the quantum fluctuations. Having expanded the partition function Tr $\mathrm{e}^{-\beta H}$ using Duhamel formula and having defined *quantum contours* as excitations with respect to a well chosen classical configuration, we identify the smallest quantum contours (that we call *loops*). Given a set of big quantum contours, we can replace the sum over sets of loops by an *effective interaction* acting on the quantum configurations without loops. This effective interaction is long-range, but decays exponentially quickly with respect to the distance. This allows, for a class of models, to have an explicit control on the approximation given by effective interaction allowing to prove rigorous statements about the behaviour of original quantum model.

An important model that does not fall into the class of models we can treat, is the (symmetric) Hubbard model. Take $U = 1$ and $t_\uparrow = t_\downarrow = t$ in (1.1). Computing the effective potential stemming from one transition of a particle to a neighbouring site and back, we find an antiferromagnetic interaction of strength t^2. On the other hand, it is possible to make two transitions as a result of which the spins of nearest neighbours are interchanged,

$$|n_x, n_y\rangle = |\downarrow,\uparrow\rangle = -c^\dagger_{x\downarrow} c_{y\downarrow} c^\dagger_{y\uparrow} c_{x\uparrow} |\uparrow,\downarrow\rangle.$$

It turns out that this brings the factor t^2, which is of the same order as the strength of the effective interaction. In this case we cannot ensure the stability of the phases selected by the effective potential — we would need a stronger effective interaction. Otherwise the system jumps easily from a configuration with one particle per site to another such configuration, i.e. from a classical ground state to another classical ground state. We call *quantum instability* this property of the system. In the Hubbard model it is a manifestation of a continuous symmetry of the system, namely the rotation invariance.

In Section 2 the ideas discussed above are introduced with the precise definitions. The effective potential is written down in Sections 2.2 (a general formula) and 2.3 (a simpler formula in special cases). The results of the paper are summarized in Theorems 2.2 (a characterization of stable pure phases) and 2.3 (the structure of the phase diagram); experts will recognize standard formulations of Pirogov-Sinai theory. Taking into account that our aim is to describe in rigorous way the behaviour of a quantum system, some care must be given to the introduction of stable phases. We define them with the help of an external field perturbation of the state constructed with periodic boundary conditions. In Section 3 we apply the results to our two illustrative examples. The rest of the paper is devoted to the construction of a contour representation (Section 4), the proof of the exponential decay of the weights of the contours (Section 5), and, finally, the proofs of our claims with the help of contour expansions of the expectation values of local observables and the standard Pirogov-Sinai theory (Section 6).

Let us end this introduction by noting that given a model which enters our setting, it is not a straightforward task to apply our theorems. One still has to separate the correct leading orders that determine the behaviour of effective interaction. This situation has the utmost advantage that it should bring much more pleasure to users, since the most interesting part of the job remains to be done — to get intuition and to understand how the system behaves.

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2. Assumptions and statements

2.1. **Classical Hamiltonian with quantum perturbation.** Let $\mathbb{Z}^\nu$, $\nu \geq 2$, be the hypercubic lattice. We use $|x-y| := \|x-y\|_\infty$ to denote the distance between two sites $x, y \in \mathbb{Z}^\nu$. Ω is the finite state space of the system at site $x=0$, $|\Omega| = S < \infty$. Our standard setting will be to consider the system on a finite torus $\Lambda = (\mathbb{Z}/L\mathbb{Z})^\nu$ (i.e. a finite hypercube with periodic boundary conditions). With a slight abuse of notation we identify Λ with a subset of $\mathbb{Z}^\nu$ and always assume that it is sufficiently large (to surpass the range of considered finite range interactions). A *classical configuration* n_Λ (occasionally we suppress the index and denote it n) is an element of Ω^Λ. If $A \subset \Lambda$, the restriction of n_Λ to A is also denoted by n_A. $\mathcal{H}_\Lambda$ is the (finite-dimensional) Hilbert space spanned by the classical configurations, i.e. the set of vectors

$$|v\rangle = \sum_{n_\Lambda} a_{n_\Lambda} |n_\Lambda\rangle, \quad a_{n_\Lambda} \in \mathbb{C},$$

with the scalar product

$$\langle v|v'\rangle = \sum_{n_\Lambda} a^*_{n_\Lambda} a'_{n_\Lambda}.$$

Given two configurations $n_A \in \Omega^A$ and $n'_{A'} \in \Omega^{A'}$, with $A \cap A' = \emptyset$, it is convenient to define $n_A n'_{A'} \in \Omega^{A\cup A'}$ to be the configuration coinciding with n_A on A and with $n'_{A'}$ on A'.

The Hamiltonian is a sum of two terms, $\boldsymbol{H}_\Lambda^{\text{per}} = \boldsymbol{H}_\Lambda^{(0)\,\text{per}} + \boldsymbol{V}_\Lambda^{\text{per}}$. The former is the quantum equivalent of a classical interaction, the latter is the quantum perturbation.

Assumption 1. Classical Hamiltonian.
There exists a periodic interaction Φ (i.e. a collection of functions $\Phi_A : \Omega^A \to \mathbb{R}$) of finite range R_0 (i.e. $\Phi_A = 0$ whenever $\operatorname{diam} A > R_0$) and period ℓ_0 such that

$$\boldsymbol{H}_\Lambda^{(0)\,\text{per}} |n_\Lambda\rangle = H_\Lambda^{(0)\,\text{per}}(n_\Lambda)\,|n_\Lambda\rangle = \sum_{A\subset\Lambda} \Phi_A(n_A)\,|n_\Lambda\rangle;$$

for any torus $\Lambda \subset \mathbb{Z}^\nu$ of side L that is a multiple of ℓ_0 and any $n_\Lambda \in \Omega^\Lambda$.

When stressing the dependence on the interaction Φ, we will also use the notation $H_\Lambda^{(0)\,\text{per}}(n_\Lambda) = H_\Lambda^{\Phi\,\text{per}}(n_\Lambda)$.

Let us suppose that a fixed collection of reference configurations $G \subset \Omega^{\mathbb{Z}^\nu}$ is given[1]. For any $n \in \Omega^{\mathbb{Z}^\nu}$, $x \in \mathbb{Z}^\nu$, and any (finite range, but not necessarily translation invariant) interaction Ψ, we use $e_x^\Psi(n)$ to denote the local contribution to the "energy" Ψ of the configuration n at the site x, $e_x^\Psi(n) = \sum_{A\ni x} \frac{\Psi_A(n_A)}{|A|}$. Notice that $H_\Lambda^{\Psi\,\text{per}}(n_\Lambda) = \sum_{x\in\Lambda} e_x^\Psi(n)$ for every $n \in \Omega^\Lambda$ and Λ sufficiently large. Finally, let $U(x) = \{y \in \mathbb{Z}^\nu; |y-x| \leqslant R_0\}$, $\bar{A} = \cup_{x\in A} U(x)$ and $G_A = \{g_A; g \in G\}$, $A \subset \mathbb{Z}^\nu$.

We assume that the local energy gap of excitations is uniformly bounded from below, while the spread of local energies of reference states is not too big:

Assumption 2. Energy gap for classical excitations.
There exist strictly positive constants Δ_0 and δ_0 such that:

- *For any $x \in \mathbb{Z}^\nu$ and any $n_{U(x)} \notin G_{U(x)}$, one has the lower bound*

$$e_x^\Phi(n) - \max_{g\in G} e_x^\Phi(g) \geqslant \Delta_0, \tag{2.1}$$

[1] In some situations G is simply the set of all ground configurations of Φ. When discussing the full phase diagram, however, we will typically extend the interaction Φ to a class of interactions by adding certain "external fields" . The set G then will actually play the role of ground states of the interaction with a particular values of external fields (the point of maximal coexistence of ground state phase diagram).

- *and,*

$$\max_{g,g'\in G}\left|e_x^\Phi(g)-e_x^\Phi(g')\right|\leqslant\delta_0. \tag{2.2}$$

Furthermore, we assume the following extension property on the set of reference states G: if, for a connected $A\subset\mathbb{Z}^\nu$, a configuration n is such that $n_{U(x)}\in G_{U(x)}$ for any $x\in A$, then $n_{\bar A}\in G_{\bar A}$.

For later purpose, we note the following consequence of this assumption.

Property. *Let Φ satisfy Assumption 2, R be such that $R^\nu\leqslant\Delta_0/\delta_0$, and $A\subset\mathbb{Z}^\nu$ with* $\operatorname{diam}\bar A\leqslant R$. *Then any pair of configurations $g_{\bar A}\in G_{\bar A}$ and $n_{\bar A}\notin G_{\bar A}$, with $n_{\bar A\setminus A}=g_{\bar A\setminus A}$, satisfies the lower bound*

$$\sum_{A'\subset\bar A}\Big[\Phi_{A'}(n_{A'})-\Phi_{A'}(g_{A'})\Big]\geqslant R^{-\nu}\Delta_0. \tag{2.3}$$

Proof. We choose $g'\in G$ such that $g'_{\bar A}=g_{\bar A}$. Then we have

$$\sum_{A'\subset\bar A}\Big[\Phi_{A'}(n_{A'})-\Phi_{A'}(g_{A'})\Big]=\sum_{x\in\bar A}\Big(e_x^\Phi(n_Ag'_{A^c})-e_x^\Phi(g')\Big). \tag{2.4}$$

Since $n_{\bar A}\notin G_{\bar A}$, there exists at least one site $x\in A$ such that $n_{U(x)}\notin G_{U(x)}$. From the assumption, this implies that

$$\sum_{A'\subset\bar A}\Big[\Phi_{A'}(n_{A'})-\Phi_{A'}(g_{A'})\Big]\geqslant\Delta_0-\sum_{y\in\bar A,y\neq x}\delta_0.$$

Using $|\bar A|\leqslant R^\nu$, we obtain the property. □

The quantum perturbation $\boldsymbol{V}_\Lambda^{\mathrm{per}}$ is supposed to be a periodic quantum interaction. Namely, $\boldsymbol{V}_\Lambda^{\mathrm{per}}$ is a sum of local operators $\boldsymbol{V_A}$, $\boldsymbol{V}_\Lambda^{\mathrm{per}}=\sum_{\boldsymbol{A}}\boldsymbol{V_A}$, where $\boldsymbol{V_A}$ has support $\operatorname{supp}\boldsymbol{A}=A\subset\Lambda$ and $\boldsymbol{A}$ is, in general, a pair (A,α), where the index α specifies $\boldsymbol{V_A}$ from a possible finite set of operators with the same support. We found it useful to label quantum interactions $\boldsymbol{V_A}$ not only by the interaction domain A, but also, say, by quantum numbers of participating creation and annihilation operators. Thus, for example, the term $\boldsymbol{A}$ might, in the case of the Hubbard model, be a pair $(<x,y>,\uparrow)$ corresponding to the operator $\boldsymbol{V_A}=c^\dagger_{x,\uparrow}c_{y,\uparrow}$. We refer to $\boldsymbol{A}$ as a *quantum transition.*

Assumption 3. Quantum Perturbations.

The collection of operators $\boldsymbol{V_A}$ is supposed to be periodic[2]*, with period ℓ_0, with respect to the translations of* $\operatorname{supp}\boldsymbol{A}$. *The interactions $\boldsymbol{V_A}$ are assumed to satisfy the following condition, for fermions or bosons, respectively:*

- *(Fermions) $\boldsymbol{V_A}$ is a finite sum of even monomials in creation and annihilation operators of fermionic particles at a given site, i.e.*

$$\boldsymbol{V_A}=\sum_{\substack{(x_1,\sigma_1),\dots,(x_k,\sigma_k)\\(y_1,\sigma'_1)\dots,(y_\ell,\sigma'_\ell)}}c^\dagger_{x_1,\sigma_1}\cdots c^\dagger_{x_k,\sigma_k}c_{y_1,\sigma'_1}\cdots c_{y_\ell,\sigma'_\ell}$$

with $x_i,y_i\in A$ and σ_i, σ'_i are the internal degrees of freedom, such as spins; $k+\ell$ must be an even number. The creation and annihilation operators satisfy the anticommutation relations

$$\{c^\dagger_{x,\sigma},c^\dagger_{y,\sigma'}\}=0,\qquad\{c_{x,\sigma},c_{y,\sigma'}\}=0,\qquad\{c^\dagger_{x,\sigma},c_{y,\sigma'}\}=\delta_{x,y}\delta_{\sigma,\sigma'}.$$

[2]By taking the least common multiple, we can always suppose the same periodicity for Φ and $\boldsymbol{V}$. Moreover, whenever a torus Λ is considered, we suppose that its side is a multiple of ℓ_0.

- *(Spins or bosons) The matrix element*

$$\langle n_\Lambda | \boldsymbol{V_A} | n'_\Lambda \rangle$$

is zero whenever $n_{\Lambda\setminus A} \neq n'_{\Lambda\setminus A}$ *and otherwise it depends on* n_A *and* n'_A *only.*

In both cases $\boldsymbol{V}$ *is supposed to have an exponential decay with respect to its support: defining the norm of a (quantum) interaction* $\boldsymbol{V}$ *by*[3]

$$\|\boldsymbol{V}\| = \max_{\boldsymbol{A}, A\subset\mathbb{Z}^\nu} \Big[\max_{n_A, n'_A \in \Omega^A} |\langle n'_A | \boldsymbol{V_A} | n_A \rangle| \Big]^{1/|A|}, \tag{2.5}$$

we assume that $\|\boldsymbol{V}\| < \infty$.

When stating our theorems, we shall actually suppose $\|\boldsymbol{V}\|$ to be sufficiently small. Notice also that we do not assume that $\boldsymbol{V}$ is of finite range, the exponential decay suffices.

2.2. **The effective potential.** It is actually a cumbersome task to write down a compact formula for the effective potential in the general case. A lot of notation has to be introduced, and one pays for the generality by the fact that the resulting formulæ look rather obscure; nevertheless, the logic behind the following definitions and equations appears rather naturally along the steps in Section 4. In the next subsection we shall discuss a special case where the effective interaction is due to at most four transitions resulting in much simpler and straightforward formulæ. We would like to stress that for typical concrete models this is entirely sufficient. The reader might thus skip the present subsection on the first reading and consider only the simplified situation of the next subsection.

The real meaning of the next definitions (in particular, (2.7)) will appear more clearly only in Section 4, but, in general case, we cannot leave it aside. First of all, we assume that a list $\mathcal{S}$ of sequences of quantum transitions $\boldsymbol{A}$ is given to represent the leading quantum fluctuations. The particular choice of $\mathcal{S}$ depends on properties of the considered model. Often the obvious choice like "any sequence of transitions not surpassing a given order" is sufficient. In general case, certain conditions (specified later in Assumption 5) involving $\mathcal{S}$ are to be met. For any $g_A \in G_A$, the effective potential Ψ is defined to equal

$$\Psi_A(g_A) = -\sum_{n\geqslant 1} \frac{1}{n!} \sum_{k_1,\dots,k_n \geqslant 2} \sum_{\substack{(\boldsymbol{A}^1_1,\dots,\boldsymbol{A}^1_{k_1},\boldsymbol{A}^2_1,\dots,\boldsymbol{A}^n_{k_n})\in\mathcal{S} \\ \cup_{i,j}\bar{A}^i_j = A}}$$
$$\prod_{i=1}^n \Big\{ \sum_{n^{i,1}_A,\dots,n^{i,k_i-1}_A \notin G_A} \mathcal{I}(A^i_1,\dots,A^i_{k_i}; n^{i,1}_A g_{\Lambda\setminus A},\dots,n^{i,k_i-1}_A g_{\Lambda\setminus A}) \Big[\prod_{j=1}^{k_i} \langle n^{i,j-1}_A | \boldsymbol{V}_{\boldsymbol{A}^i_j} | n^{i,j}_A \rangle \Big]$$
$$\int_{-\infty<\tau^i_1<\dots<\tau^i_{k_i}<\infty} \mathrm{d}\tau^i_1 \dots \mathrm{d}\tau^i_{k_i} \Big[\prod_{j=1}^{k_i-1} \mathrm{e}^{-(\tau^i_{j+1}-\tau^i_j)\sum_{A'\subset A^i}[\Phi_{A'}(n^{i,j}_{A'})-\Phi_{A'}(g_{A'})]} \Big] \Big\}$$
$$\frac{\mathbb{I}\left[\min_{i,j}\tau^i_j < 0 \text{ and } \max_{i,j}\tau^i_j > 0\right]}{\max_{i,j}\tau^i_j - \min_{i,j}\tau^i_j} \varphi^{\mathrm{T}}(B_1,\dots B_n). \tag{2.6}$$

To begin to decode this formula, notice first that the second sum is over all sequences $(\boldsymbol{A}^1_1,\dots,\boldsymbol{A}^1_{k_1},\boldsymbol{A}^2_1,\dots,\boldsymbol{A}^n_{k_n})$ of transitions that are in the list $\mathcal{S}$ and are just covering the set A, $\cup_{i,j}\bar{A}^i_j = A$. The sum in the braces (for a given $i = 1,\dots,n$) is taken over collections of configurations $n^{i,1}_A,\dots,n^{i,k_i-1}_A \notin G_A$ with $n^{i,0}_A \equiv n^{i,k_i}_A \equiv g_A$, while the integral is taken over "times" attributed to transitions, with the energy term in the exponent taken over the set $A^i = \cup_{j=1}^{k_i} \bar{A}^i_j$, $\bar{A} = \cup_{x\in A} U(x)$.

[3]This is a norm, provided the multiplication of an interaction $\boldsymbol{V}$ by a scalar λ is defined to be $(\lambda\boldsymbol{V})_{\boldsymbol{A}} = \lambda^{|A|}\boldsymbol{V_A}$.

Finally, there are some restrictions on the sums and integrals encoded in functions $\frac{\mathbb{I}\left[\min_{i,j}\tau_j^i<0 \text{ and } \max_{i,j}\tau_j^i>0\right]}{\max_{i,j}\tau_j^i-\min_{i,j}\tau_j^i}$, $\varphi^{\mathrm{T}}(B_1,\dots B_n)$, and $\mathcal{I}(A_1^i,\dots,A_{k_i}^i;n_A^{i,1}g_{\Lambda\setminus A},\dots,n_A^{i,k_i-1}g_{\Lambda\setminus A})$. The easiest is the first one. One just assumes that the interval between the first and the last of concerned "times" contains the origin and the integrand is divided by the length of this interval. The function $\varphi^{\mathrm{T}}(B_1,\dots B_n)$ in terms of the sets $B_i=A^i\times[\tau_1^i,\tau_{k_i}^i]\subset\mathbb{Z}^\nu\times[-\infty,\infty]$, $i=1,\dots,n$, is the standard factor from the theory of cluster expansions defined as

$$\varphi^{\mathrm{T}}(B_1,\dots,B_n)=\begin{cases}1 & \text{if } n=1\\ \sum_{\mathcal{G}}\prod_{e(i,j)\in\mathcal{G}}\left(-\,\mathbb{I}\left[B_i\cup B_j \text{ is connected }\right]\right) & \text{if } n\geqslant 2\end{cases}$$

with the sum over all connected graphs $\mathcal{G}$ of n vertices. Connectedness of a set $B\subset\mathbb{Z}^\nu\times[-\infty,\infty]$ is defined by combining connection in continuous direction with connection in slices $\{x|(x,\tau)\in B\}\subset\mathbb{Z}^\nu$ through pairs of sites of distance one. The most difficult to define is the restriction given by the function $\mathcal{I}$ that characterizes whether the collection of transitions is connected, in some generalized sense, through the intertwining configurations. A consolation might be that in lowest orders it is always true. Namely, $\mathcal{I}(A_1,\dots,A_k;n_\Lambda^1,\dots,n_\Lambda^{k-1})=1$ whenever $k\leqslant 5$. To define it in a general case, consider $A_1,\dots,A_k\subset\mathbb{Z}^\nu$ and $n^1,\dots,n^{k-1}\in\Omega^{\mathbb{Z}^\nu}$. Taking $\bar{A}=\cup_{x\in A}U(x)$ and $E(n)=\{x\in\Lambda: n_{U(x)}\neq g_{U(x)} \text{ for any } g\in G\}$, we consider the set $\hat{B}^{(0)}\subset\mathbb{Z}^{\nu+1}$,

$$\hat{B}^{(0)}=\bigcup_{j=1}^{k}\left[\bar{A}_j\times\{2j-2\}\right]\cup\bigcup_{j=1}^{k-1}\left[E(n^j)\times\{2j-1\}\right].$$

Think of layers, one on top of another — configurations on odd levels interspersed with transitions on even levels. The set $\hat{B}^{(0)}$ decomposes into connected components, $\hat{B}^{(0)}=\cup_{\ell\geqslant 1}\hat{B}^{(0)}_\ell$. To any $\hat{B}^{(0)}_\ell$, define the box $\tilde{B}^{(0)}_\ell\subset\mathbb{Z}^{\nu+1}$ as the smallest rectangle containing $\hat{B}^{(0)}_\ell$. Then let $\hat{B}^{(1)}=\cup_{\ell\geqslant 1}\tilde{B}^{(0)}_\ell$; decompose into connected components $\hat{B}^{(1)}=\cup_{\ell\geqslant 1}\hat{B}^{(1)}_\ell$, and repeat the procedure until no change occurs any more, i.e. until $\hat{B}^{(m)}=\cup_{\ell\geqslant 1}\tilde{B}^{(m)}_\ell$. The function $\mathcal{I}$ characterizes whether this final set, the result of the above construction, is connected or not,

$$\mathcal{I}(A_1,\dots,A_k;n_\Lambda^1,\dots,n_\Lambda^{k-1})=\begin{cases}1 & \text{if } \hat{B}^{(m)} \text{ is connected}\\ 0 & \text{otherwise.}\end{cases}\tag{2.7}$$

2.3. **Quantum fluctuations with less than four transitions.** The equation (2.6) for the effective potential is hard to handle in general case. However, in many situations it is enough to consider only small sequences of less than four quantum transitions to define it. We rewrite in this section the explicit formulæ for the effective potential in such a case.

We assume thus that a list $\mathcal{S}$ of sequences of quantum transitions $\boldsymbol{A}$, containing at most 4 transitions, is given to represent the most important quantum fluctuations. Let us decompose $\mathcal{S}=\mathcal{S}^{(2)}\cup\mathcal{S}^{(3)}\cup\mathcal{S}^{(4)}$, with $\mathcal{S}^{(k)}$ denoting the list of sequences with exactly k transitions, and write

$$\Psi=\Psi^{(2)}+\Psi^{(3)}+\Psi^{(4)}.\tag{2.8}$$

Here $\Psi^{(k)}$ is the contribution to the effective potential due to the fluctuations from $\mathcal{S}^{(k)}$.

Let

$$\phi_A(n_A;g_A)=\sum_{A'\subset A}\Big[\Phi_{A'}(n_{A'})-\Phi_{A'}(g_{A'})\Big].$$

Then, for any connected $A \subset \mathbb{Z}^\nu$ and $g_A \in G_A$, we define

$$\Psi_A^{(2)}(g_A) = - \sum_{\substack{(\boldsymbol{A}_1,\boldsymbol{A}_2)\in\mathcal{S}^{(2)} \\ \bar{A}_1\cup\bar{A}_2=A}} \sum_{n_A\notin G_A} \frac{\langle g_A|\, \boldsymbol{V}_{\boldsymbol{A}_1}\, |n_A\rangle\langle n_A|\, \boldsymbol{V}_{\boldsymbol{A}_2}\, |g_A\rangle}{\phi_A(n_A;g_A)}, \tag{2.9}$$

$$\Psi_A^{(3)}(g_A) = - \sum_{\substack{(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_3)\in\mathcal{S}^{(3)} \\ \bar{A}_1\cup\bar{A}_2\cup\bar{A}_3=A}} \sum_{n_A,n'_A\notin G_A} \frac{\langle g_A|\, \boldsymbol{V}_{\boldsymbol{A}_1}\, |n_A\rangle\langle n_A|\, \boldsymbol{V}_{\boldsymbol{A}_2}\, |n'_A\rangle\langle n'_A|\, \boldsymbol{V}_{\boldsymbol{A}_3}\, |g_A\rangle}{\phi_A(n_A;g_A)\phi_A(n'_A;g_A)}. \tag{2.10}$$

The expression for $\Psi^{(4)}$ becomes more complicated (we shall see in Section 4 that clusters of excitations are actually occurring here),

$$\Psi_A^{(4)}(g_A) = - \sum_{\substack{(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{A}_4)\in\mathcal{S}^{(4)} \\ \bar{A}_1\cup\bar{A}_2\cup\bar{A}_3\cup\bar{A}_4=A}} \Bigg[\sum_{n_A,n'_A,n''_A\notin G_A} \frac{\langle g_A|\, \boldsymbol{V}_{\boldsymbol{A}_1}\, |n_A\rangle\langle n_A|\, \boldsymbol{V}_{\boldsymbol{A}_2}\, |n'_A\rangle\langle n'_A|\, \boldsymbol{V}_{\boldsymbol{A}_3}\, |n''_A\rangle\langle n''_A|\, \boldsymbol{V}_{\boldsymbol{A}_4}\, |g_A\rangle}{\phi_A(n_A;g_A)\phi_A(n'_A;g_A)\phi_A(n''_A;g_A)}$$
$$- \frac{1}{2} \sum_{n_A,n'_A\notin G_A} \frac{\langle g_A|\, \boldsymbol{V}_{\boldsymbol{A}_1}\, |n_A\rangle\langle n_A|\, \boldsymbol{V}_{\boldsymbol{A}_2}\, |g_A\rangle\langle g_A|\, \boldsymbol{V}_{\boldsymbol{A}_3}\, |n'_A\rangle\langle n'_A|\, \boldsymbol{V}_{\boldsymbol{A}_4}\, |g_A\rangle}{\phi_{A^1}(n_A;g_A)+\phi_{A^2}(n'_A;g_A)} \Big\{ \frac{1}{\phi_{A^1}(n_A;g_A)} + \frac{1}{\phi_{A^2}(n'_A;g_A)} \Big\}^2 \Bigg]. \tag{2.11}$$

Above we denoted $A^1 = \bar{A}_1 \cup \bar{A}_2$ and $A^2 = \bar{A}_3 \cup \bar{A}_4$. Property (2.3) implies that all the denominators are strictly positive.

These equations simplify further if $\boldsymbol{V}_{\boldsymbol{A}}$ is a monomial in creation and annihilation operators; indeed in the sums over intermediate configurations only one element has to be taken into account.

Notice, finally, that the diagonal terms in $\boldsymbol{V}$ are not playing any role in the previous definitions; we consider that they are small, since otherwise we would have included them into the diagonal potential.

2.4. **Stability of the dominant states.** The aim of rewriting a class of quantum transitions in terms of the effective potential was to get a control over stable low temperature phases. To this end, the three conditions, expressed first only vaguely and then in precise terms in the following Assumptions 4, 5, and 6, must be met. Namely, we suppose that

- the Hamiltonian corresponding to the sum $\Phi + \Psi$ of the classical (diagonal) and effective interactions has a finite number of ground configurations, and its excitations have strictly positive energy[4];
- the list $\mathcal{S}$ contains all the lowest quantum fluctuations;
- there is no "quantum instability"; the transition probability from a "ground state" g to another "ground state" g' is small compared to the energy cost of the excitations.

Each component of the effective interaction Ψ_A is a mapping $G_A \to \mathbb{R}$; let us first extend it to $\Omega^A \to \mathbb{R}$ by putting $\Psi_A(n_A) = 0$ if $n_A \notin G_A$. To give a precise meaning to the first condition, we suppose that a finite number of periodic reference configurations $D \subset G$ is given such that the interaction $\Phi + \Psi$ satisfies the Peierls condition with respect to D. We choose a formulation in which it is very easy to verify the condition and, in addition, it takes into account the fact that the configurations from D are not necessarily translation invariant. Namely, we will formulate the condition in terms of a potential Υ that is equivalent to $\Phi + \Psi$ and is chosen in a suitable way. Of course, in many particular cases this is not necessary and the condition as stated below is valid directly for $\Phi + \Psi$. However, in several important cases treated in Section 3, the interaction $\Phi + \Psi$ turns out not to be so called m-potential and the use of the equivalent

[4]Again, when exploring a region of phase diagram at once, we have a fixed finite set of reference configurations that, strictly speaking, turn out to be ground configurations of the corresponding Hamiltonian for a particular value of "external fields". See below for a more detailed formulation.

m-potential Υ not only simplifies the formulation of the Peierls condition, but also makes the task of its verification much easier.

We will consider the interactions φ and ϕ to be *equivalent*[5] if, for any finite torus Λ and any configuration $n \in \Omega^\Lambda$, one has

$$H_\Lambda^{\varphi\,\mathrm{per}}(n) = H_\Lambda^{\phi\,\mathrm{per}}(n).$$

Notice that the above amounts also to the equality

$$\sum_{x\in\Lambda} e_x^\varphi(n) = \sum_{x\in\Lambda} e_x^\phi(n).$$

Assumption 4. Peierls condition.

There exist a finite set of periodic configurations $D \subset G$ with the smallest common period L_0, a constant Δ such that $\Delta > \|\boldsymbol{V}\|^k$ for some finite constant k, and a periodic interaction $\Upsilon = \{\Upsilon_A\}$ (with period ℓ_0) that is equivalent to $\Phi + \Psi$ such that the following conditions are satisfied. The interaction Υ is of a finite range[6] *$R \in \mathbb{N}$ such that*

$$R^\nu \leqslant \Delta_0/\delta_0, \tag{2.12}$$

with the constants δ_0 and Δ_0 determined by the interaction Φ in Assumption 2. The value $e_x^\Upsilon(d)$ is supposed to be translation invariant with respect to x for any $d \in D$, and the interaction Υ satisfies the following conditions:

- *For any $x \in \Lambda$ and any n with $n_{U(x)} \notin G_{U(x)}$, one has*

$$e_x^\Upsilon(n) - \max_{g\in G} e_x^\Upsilon(g) \geqslant \tfrac{1}{2}\Delta_0.$$

- *For any $x \in \Lambda$ and any n with $n_{V(x)} \notin D_{V(x)}$, $V(x) = \{y \in \Lambda; |y-x| \le R\}$, one has*

$$e_x^\Upsilon(n) - \min_{d\in D} e_x^\Upsilon(d) \geqslant \Delta.$$

The following assumption is a condition demanding that the list $\mathcal{S}$ should contain all transitions that are relevant for the effective potential. We define

$$\boldsymbol{m}(\boldsymbol{V}_{\boldsymbol{A}_1}, \dots, \boldsymbol{V}_{\boldsymbol{A}_k}) = \max_{g\in G} \max_{n^1,\dots,n^{k-1}\notin G} |\langle g|\, \boldsymbol{V}_{\boldsymbol{A}_1} |n^1\rangle\langle n^1|\, \boldsymbol{V}_{\boldsymbol{A}_2} |n^2\rangle \dots \langle n^{k-1}|\, \boldsymbol{V}_{\boldsymbol{A}_k} |g\rangle|. \tag{2.13}$$

Assumption 5. Completeness of the set of quantum transitions.

There exists a function $b_1(\cdot)$ with $\lim_{\lambda\to 0} b_1(\lambda) = 0$ such that for any sequence $(\boldsymbol{A}_1, \dots, \boldsymbol{A}_m) \notin \mathcal{S}$ with connected $\cup_{i=1}^m \bar{A}_i$ one has

$$\boldsymbol{m}(\boldsymbol{V}_{\boldsymbol{A}_1}, \dots, \boldsymbol{V}_{\boldsymbol{A}_{k_1}}) \boldsymbol{m}(\boldsymbol{V}_{\boldsymbol{A}_{k_1+1}}, \dots, \boldsymbol{V}_{\boldsymbol{A}_{k_2}}) \dots \boldsymbol{m}(\boldsymbol{V}_{\boldsymbol{A}_{k_{n-1}+1}}, \dots, \boldsymbol{V}_{\boldsymbol{A}_m}) \leqslant b_1(\|\boldsymbol{V}\|)\Delta.$$

Finally, we have a condition assuring that there is *no quantum instability.*

Assumption 6. Absence of quantum instability.

There exists a function $b_2(\cdot)$ with $\lim_{\lambda\to 0} b_2(\lambda) = 0$ such that for any sequence $(\boldsymbol{A}_1, \dots, \boldsymbol{A}_m)$, and any $g, g' \in G$, $g \neq g'$, one has

$$\Big| \langle g|\, \boldsymbol{V}_{\boldsymbol{A}_1} \dots \boldsymbol{V}_{\boldsymbol{A}_m} |g'\rangle \Big| \leqslant b_2(\|\boldsymbol{V}\|)\Delta.$$

[5] Usual notion of (physically) equivalent interactions (see [Geo], [EFS]) is slightly weaker, but we will not need it here.

[6] We will suppose, taking larger R if necessary, that it is larger or equal to the range R_0 of Φ, as well as to the range of the effective interaction Ψ and to L_0.

2.5. **Characterization of stable phases.** Notice first that the specific energy per lattice site of the configuration $d \in D$, defined by

$$e(d) = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \sum_{A \subset \Lambda} [\Phi_A(d_A) + \Psi_A(d_A)], \tag{2.14}$$

is equal, according to Assumption 4, to $e_x^\Upsilon(d)$ (whose value does not depend on x).

Our first result concerns the existence of the thermodynamic limit for the state under periodic boundary conditions. Taking L_0 to be the smallest common period of periodic configurations from D, we always consider in the following the limit over tori $\Lambda \nearrow \mathbb{Z}^\nu$ whose sides are multiples of L_0 and ℓ_0.

Theorem 2.1. Thermodynamic limit.

Suppose that the Assumptions 1–6 are satisfied. There exist constants $\varepsilon_0 > 0$ and $\beta_0 = \beta_0(\Delta)$ (depending on ν, S, R, ℓ_0) such that the limit

$$\langle T \rangle_\beta^{\text{per}} = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \frac{\operatorname{Tr} T \, \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}}{\operatorname{Tr} \, \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}} \tag{2.15}$$

exists whenever $\|\boldsymbol{V}\| \leqslant \varepsilon_0$, $\beta \geqslant \beta_0$, and T is a local observable.

Notice the logic of constants in the theorem above (as well as in the remaining two theorems stated below). We first choose ε_0. Then, for any $\|\boldsymbol{V}\| \leqslant \varepsilon_0$ one can choose β_0 (depending on Δ that is determined in terms of $\boldsymbol{V}$ through the effective potential Ψ) such that the claim is valid for the given $\boldsymbol{V}$ and any $\beta \geqslant \beta_0(\Delta)$. With $\|\boldsymbol{V}\| \to 0$ we may have to go to lower temperatures (higher β) to keep the control. Of course, if Δ does not vanish with vanishing $\|\boldsymbol{V}\|$ (i.e. Assumption 4 is valid for Φ alone) as was the case in [BKU], one can choose the constant β_0 uniformly in $\|\boldsymbol{V}\|$.

If there are coexisting phases for a given temperature and Hamiltonian, the state $\langle\cdot\rangle_\beta^{\text{per}}$ will actually turn out to be a linear combination of several pure states. A standard way how to select such a pure state is to consider a thermodynamic limit with a suitably chosen fixed boundary condition. In many situations to which the present theory should apply, this approach is not easy to implement. The classical part of the Hamiltonian might actually consist only of on-site terms and to make the system "feel" the boundary, the truly quantum terms must be used. One possibility is, of course, to couple the system with the boundary with the help of the effective potential. The problem here is, however, that since we are interested in a genuine quantum model, we would have to introduce the effective potential directly in the finite volume quantum state. Expanding this state, in a similar manner as it will be done in the next section, we would actually obtain a new, boundary dependent effective potential. One can imagine that it would be possible to cancel the respective terms by assuming that the boundary potential satisfies certain "renormalizing self-consistency conditions". However, the details of such an approach remain to be clarified.

Here we have chosen another approach. Namely, we construct the pure states by limits of states $\langle\cdot\rangle_\beta^{\Phi^\alpha\,\text{per}}$, defined by (2.15) with $\boldsymbol{H}_\Lambda^{\text{per}} = \boldsymbol{H}_\Lambda^{\Phi^\alpha\,\text{per}} + \boldsymbol{V}_\Lambda^{\text{per}}$, where Φ^α is a perturbation of the interaction Φ suitably chosen in such a way that one approaches the coexistence point from the one phase region. Consider thus $\mathcal{F}_{R_0}$, the space of all periodic interactions of range R_0. We say that a state $\langle\cdot\rangle_\beta^{\phi\,\text{per}}$, $\phi \in \mathcal{F}_{R_0}$, is *unperturbable* if it is insensitive to small perturbations:

$$\langle T \rangle_\beta^{\phi,\,\text{per}} = \lim_{\alpha \to 0} \langle T \rangle_\beta^{(\phi + \alpha\psi)\,\text{per}} \tag{2.16}$$

for every $\psi \in \mathcal{F}_{R_0}$ and every local observable T. We define now a state $\langle\cdot\rangle_\beta^*$ to be a *pure stable state* (with classical potential Φ and quantum interaction $\boldsymbol{V}$) if there exists a function

$(0, \alpha_0) \ni \alpha \to \Phi^\alpha \in \mathcal{F}_{R_0}$ so that $\lim_{\alpha\to 0+} \Phi^\alpha = \Phi$, the states $\langle\cdot\rangle_\beta^{\Phi^\alpha \text{ per}}$ are unperturbable, and

$$\langle T\rangle_\beta^* = \lim_{\alpha\to 0+} \langle T\rangle_\beta^{\Phi^\alpha \text{ per}} \tag{2.17}$$

for every local observable T.

Theorem 2.2. Pure low temperature phases. *Under the Assumptions 1–6 and for any* $\eta > 0$*, there exist* $\varepsilon_0 > 0$ *and* $\beta_0 = \beta_0(\Delta)$ *(depending on* ν, S, R, ℓ_0*) such that if* $\|\boldsymbol{V}\| \leqslant \varepsilon_0$ *and* $\beta \geqslant \beta_0$*, there exists for every* $d \in D$ *a function* $f^\beta(d)$ *such that the set* $Q = \{d \in D;\, \operatorname{Re} f^\beta(d) = \min_{d'\in D} \operatorname{Re} f^\beta(d')\}$ *characterizes the set of pure stable phases. Namely, for any* $d \in Q$*:*

a) *The function* $f^\beta(d)$ *is equal to the free energy of the system, i.e.*

$$f^\beta(d) = -\frac{1}{\beta} \lim_{\Lambda\nearrow\mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \operatorname{Tr} \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}.$$

b) *There exists a pure stable state* $\langle\cdot\rangle_\beta^d$*. Moreover, it is close to the state* $|d_\Lambda\rangle$ *in the sense that for any bounded local observable* T *and any sufficiently large* Λ*, one has*

$$\Big| \langle T\rangle_\beta^d - \langle d_\Lambda|\, T\, |d_\Lambda\rangle \Big| \leqslant \eta |\operatorname{supp} T| \|T\|$$

where $\operatorname{supp} T$ *is the support of the operator* T*.*

c) *There is an exponential decay of correlations in the state* $\langle\cdot\rangle_\beta^d$*, i.e. there exists a constant* $\xi^d > 0$ *such that*

$$\Big| \langle TT'\rangle_\beta^d - \langle T\rangle_\beta^d \langle T'\rangle_\beta^d \Big| \leqslant |\operatorname{supp} T| |\operatorname{supp} T'| \|T\| \|T'\| \,\mathrm{e}^{-\operatorname{dist}(\operatorname{supp} T, \operatorname{supp} T')/\xi^d}$$

for any bounded local observables T *and* T'*.*

d) *The state* $\langle\cdot\rangle_\beta^{\text{per}}$ *is a linear combination of the states* $\langle\cdot\rangle_\beta^d$*,* $d \in Q$*, with equal weights,*

$$\langle T\rangle_\beta^{\text{per}} = \frac{1}{|Q|} \sum_{d\in Q} \langle T\rangle_\beta^d$$

for each local observable T*.*

2.6. **Phase diagram.** We now turn to the phase diagram at low temperatures. Let r be the number of dominant states, i.e. $r = |D|$. To be able to investigate the phase diagram, we suppose that $r-1$ suitable "external fields" are added to the Hamiltonian $\boldsymbol{H}_\Lambda^{\text{per}}$. Or, in other words, we suppose that classical potential Φ and quantum interaction $\boldsymbol{V}$ depend on a vector parameter $\mu = (\mu_1, \dots, \mu_{r-1}) \in \mathcal{U}$, where $\mathcal{U}$ is an open set of $\mathbb{R}^{r-1}$. The dependence should be such that the parameters μ *remove the degeneracy* on the set D of dominant states. One way how to formulate this condition is to assume a nonsingularity of the matrix of derivatives $\big(\frac{\partial e^\mu(d_j)}{\partial \mu_i}\big)$.

Assumption 7. *The potentials* Φ *and* $\boldsymbol{V}$ *are differentiable with respect to* μ *and there exist a constant* $M < \infty$ *such that*

$$\sup_{A\subset\mathbb{Z}^\nu, n_A\in\Omega^A} \Big| \frac{\partial}{\partial\mu_i} \Phi_A(n_A) \Big| < M$$

and

$$\|\boldsymbol{V}\| + \sum_{i=1}^{r-1} \Big\| \frac{\partial \boldsymbol{V}}{\partial \mu_i} \Big\| < M$$

for all $\mu \in \mathcal{U}$*.*

Further, there exists a point $\mu_0 \in \mathcal{U}$ *such that*

$$e^{\mu_0}(d) = e^{\mu_0}(d') \text{ for all } d, d' \in D,$$

and the inverse of the matrix of derivatives

$$\left(\frac{\partial}{\partial\mu_i}\big[e^{\mu}(d_j)-e^{\mu}(d_r)\big]\right)_{1\leqslant i,j\leqslant r-1}$$

has a uniform bound for all $\mu\in\mathcal{U}$.

Notice that if for some $d\in D$ one has $e^{\mu}(d)=e^{\mu}:=\min_{d'\in D}e^{\mu}(d')$, then, according to Peierls condition (Assumption 4), the configuration d is actually a ground state of Υ. Thus, the assumption above implies that the zero temperature phase diagram has a regular structure: there exists a point $\mu_0\in\mathcal{U}$ where all energies $e^{\mu_0}(d)$ are equal, $e^{\mu_0}(d)=e^{\mu_0}$, r lines ending in μ_0 with $r-1$ ground states, $\frac{1}{2}r(r-1)$ two-dimensional surfaces whose boundaries are the lines above with $r-2$ ground states, ..., r open $(r-1)$-dimensional domains with only one ground state. Denoting the $(r-|Q|)$-dimensional manifolds corresponding to the coexistence of a given set $Q\subset D$ of ground states by

$$\mathfrak{M}^*(Q)=\Big\{\mu\in\mathcal{U};\operatorname{Re}e^{\mu}(d)=\min_{d'\in D}\operatorname{Re}e^{\mu}(d')\text{ for all }d\in Q\Big\},\tag{2.18}$$

we can summarize the above structure by saying that the collection $\mathcal{P}^*=\{\mathfrak{M}^*(Q)\}_{Q\subset D}$ determines a *regular phase diagram.* Notice, in particular, that $\cup_{Q\subset D}\mathfrak{M}^*(Q)=\mathcal{U}$, $\mathfrak{M}^*(Q)\cap\mathfrak{M}^*(Q')=\emptyset$ whenever $Q\neq Q'$, while for the closures, $\overline{\mathfrak{M}}^*(Q)\cap\overline{\mathfrak{M}}^*(Q')=\overline{\mathfrak{M}}^*(Q\cup Q')$. Here we set $\mathfrak{M}(\emptyset)=\emptyset$.

The statement of the following theorem is that the similar collection $\mathcal{P}=\{\mathfrak{M}(Q)\}_{Q\subset D}$ of manifolds corresponding to existence of corresponding stable pure phases for the full model is also a regular phase diagram and differs only slightly from $\mathcal{P}^*$. To measure the distance of two manifolds $\mathfrak{M}$ and $\mathfrak{M}'$, we introduce the Hausdorff distance

$$\operatorname{dist}_{\mathrm{H}}(\mathfrak{M},\mathfrak{M}')=\max(\sup_{\mu\in\mathfrak{M}}\operatorname{dist}(\mu,\mathfrak{M}'),\ \sup_{\mu\in\mathfrak{M}'}\operatorname{dist}(\mu,\mathfrak{M})).$$

Theorem 2.3. Low temperature phase diagram

Under the Assumptions 1–7 there exist $\varepsilon_0>0$ *and* $\beta_0=\beta_0(\Delta)$ *such that if* $\|\boldsymbol{V}\|+\sum_{i=1}^{r-1}\|\frac{\partial}{\partial\mu_i}\boldsymbol{V}\|\leqslant\varepsilon_0$ *and* $\beta\geqslant\beta_0$, *there exists a collection of manifolds* $\mathcal{P}^{\beta}=\{\mathfrak{M}^{\beta}(Q)\}_{Q\subset D}$ *such that*

- (a) *The collection* $\mathcal{P}^{\beta}$ *determines a regular phase diagram;*
- (b) *If* $\mu\in\mathfrak{M}^{\beta}(Q)$, *the corresponding stable pure state* $\langle\cdot\rangle_{\beta}^{d}$ *exists for every* $d\in Q$ *and satisfies the properties b), c), and d), from Theorem 2.2;*
- (c) *The Hausdorff distance* $\operatorname{dist}_{\mathrm{H}}$ *between the manifolds of* $\mathcal{P}^{\beta}$ *and their correspondent in* $\mathcal{P}^*$ *is bounded,*

$$\operatorname{dist}_{\mathrm{H}}(\mathfrak{M}_{\lambda}^{\beta}(Q),\mathfrak{M}_{\lambda}^{*}(Q))\leqslant O(\mathrm{e}^{-\beta}+\|\boldsymbol{V}\|+\sum_{i=1}^{r-1}\Big\|\frac{\partial\boldsymbol{V}}{\partial\mu_i}\Big\|),$$

for all $Q\subset D$.

The proofs of these theorems are given in the rest of the paper. Expansions of the partition function and of expectation values of local observables are constructed, and interpreted as contours of a classical model in one additional dimension. Then we show that the assumptions for using the standard Pirogov-Sinai theory are fulfilled, and, with some special care to be taken due to our definition of stability, the validity of the three theorems follows.

3. Examples

3.1. The asymmetric Hubbard model. The state space is $\Omega = \{0, \uparrow, \downarrow, 2\}$ and the Hamiltonian is written in (1.1). Hence the classical interaction is

$$\Phi_{\{x\}}(n_x) = \begin{cases} 0 & \text{if } n_x = 0 \\ -\mu & \text{if } n_x = \uparrow \text{ or } n_x = \downarrow \\ U - 2\mu & \text{if } n_x = 2 \end{cases} \tag{3.1}$$

($R_0 = 0$). We choose the chemical potential such that $0 < \mu < U$. The set G is here the set of ground states of Φ, i.e.

$$G = \{n \in \Omega^{\mathbb{Z}^\nu} : n_x = \uparrow \text{ or } n_x = \downarrow \text{ for any } x \in \mathbb{Z}^\nu\}.$$

Assumption 2 holds with $\Delta_0 = \min(\mu, U - \mu)$ and $\delta_0 = 0$.

The quantum perturbation is defined to be

$$\boldsymbol{V}_{\boldsymbol{A}} = \begin{cases} t_\uparrow c^\dagger_{x\uparrow} c_{y\uparrow} & \text{if } \boldsymbol{A} = (<x,y>, \uparrow) \\ t_\downarrow c^\dagger_{x\downarrow} c_{y\downarrow} & \text{if } \boldsymbol{A} = (<x,y>, \downarrow) \end{cases} \tag{3.2}$$

and we always have $A = \{x, y\}$ for a pair of nearest neighbours $x, y \in \mathbb{Z}^\nu$. $\|\boldsymbol{V}\| = |t_\uparrow|^{\frac{1}{2}}$ (if $|t_\uparrow| \geqslant |t_\downarrow|$).

The sequence $\mathcal{S}$ of transitions that we consider is

$$\mathcal{S} = \{(\boldsymbol{A}, \boldsymbol{A}') : \boldsymbol{A} = (<x,y>, \uparrow) \text{ and } \boldsymbol{A}' = (<y,x>, \uparrow) \text{ for some } x, y \in \mathbb{Z}^\nu, \|x - y\|_2 = 1\}.$$

The effective potential is given by Equation (2.9). For any $x, y \in \mathbb{Z}^\nu$, nearest neighbours, any configuration n such that $|n\rangle = c^\dagger_{x\uparrow} c_{y\uparrow} |g\rangle$, $g \in G$, has an increase of energy of

$$\phi_{\{x,y\}}(n_{\{x,y\}}; g_{\{x,y\}}) = U.$$

Furthermore we have

$$\langle g_{\{x,y\}}| \, c^\dagger_{x\uparrow} c_{y\uparrow} c^\dagger_{y\uparrow} c_{x\uparrow} \, |g_{\{x,y\}}\rangle + \langle g_{\{x,y\}}| \, c^\dagger_{y\uparrow} c_{x\uparrow} c^\dagger_{x\uparrow} c_{y\uparrow} \, |g_{\{x,y\}}\rangle = \begin{cases} 1 & \text{if } g_{\{x,y\}} \in \{(\uparrow, \downarrow), (\downarrow, \uparrow)\} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Therefore

$$\Psi_{\{x,y\}}(g_{\{x,y\}}) = \begin{cases} -t_\uparrow^2/U & \text{if } g_{\{x,y\}} \in \{(\uparrow, \downarrow), (\downarrow, \uparrow)\} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $V(x) = \{y \in \mathbb{Z}^\nu : |y - x| \leqslant 1\}$. Taking $\Upsilon = \Phi + \Psi$, we have

$$e^\Upsilon_x(n_{V(x)}) = \Phi_{\{x\}}(n_x) + \tfrac{1}{2} \sum_{y : \|x-y\|_2 = 1} \Psi_{\{x,y\}}(n_{\{x,y\}}) \tag{3.5}$$

($e^\Upsilon_x(n_{V(x)})$ was initially a function $\Omega^{\mathbb{Z}^\nu} \to \mathbb{R}$, but is actually depends on $n_{V(x)}$ only).

The set D has two elements, namely the two chessboard configurations $d^{(1)}$ and $d^{(2)}$; if $(-1)^x := \prod_{i=1}^\nu (-1)^{x_i}$,

$$d^{(1)}_x = \begin{cases} \uparrow & \text{if } (-1)^x = 1 \\ \downarrow & \text{if } (-1)^x = -1 \end{cases} \qquad d^{(2)}_x = \begin{cases} \uparrow & \text{if } (-1)^x = -1 \\ \downarrow & \text{if } (-1)^x = 1. \end{cases}$$

The last inequality of Assumption 4 holds with $\Delta = \frac{1}{2} t_\uparrow^2 / U$.

The maximum of the expression in Assumption 5 is equal to $2U \max(t_\downarrow^2/t_\uparrow^2, t_\uparrow^2)$. If there exists $\varepsilon > 0$ such that $|t_\downarrow| \leqslant |t_\uparrow|^{1+\varepsilon}$, the bound of Assumption 5 can be chosen to be $b_1 = 2U|t_\uparrow|^{2\varepsilon}$. For Assumption 6 the expression has maximum equals to $2U|t_\downarrow|/|t_\uparrow|$ and we can take $b_2 = 2U|t_\uparrow|^\varepsilon$

(this Assumption is not true in the symmetric Hubbard model; the effective potential is not strong enough in order to forbid the model to jump from one g to another g').

Our results for the asymmetric Hubbard model can be stated in the following theorem (first obtained by [DFF2]).

Theorem 3.1 (Chessboard phases in asymmetric Hubbard model)**.** *Consider the lattice $\mathbb{Z}^\nu$, $\nu \geqslant 2$, and suppose $0 < \mu < U$ and $|t_\downarrow| \leqslant |t_\uparrow|^{1+\varepsilon}$ with $\varepsilon > 0$. Then for any $\delta > 0$, there exist $t_0 > 0$ and $\beta_0(t_\uparrow) < \infty$ ($\lim_{t_\uparrow \to 0} \beta_0(t_\uparrow) = \infty$) such that if $|t_\uparrow| \leqslant t_0$ and $\beta \geqslant \beta_0$,*

- *The free energy exists in the thermodynamic limit with periodic boundary conditions, as well as expectation values of observables.*
- *There are two pure periodic phases, $\langle\cdot\rangle_\beta^{(1)}$ and $\langle\cdot\rangle_\beta^{(2)}$, with exponential decay of correlations.*
- *One of these pure phases, $\langle\cdot\rangle_\beta^{(1)}$, is a small deformation of the chessboard state $|d^{(1)}\rangle$:*

$$\langle n_{x\uparrow}\rangle_\beta^{(1)} \begin{cases} \geqslant 1-\delta & \text{if } (-1)^x = 1 \\ \leqslant \delta & \text{if } (-1)^x = -1 \end{cases} \qquad \langle n_{x\downarrow}\rangle_\beta^{(1)} \begin{cases} \leqslant \delta & \text{if } (-1)^x = 1 \\ \geqslant 1-\delta & \text{if } (-1)^x = -1. \end{cases}$$

The other pure phase, $\langle\cdot\rangle_\beta^{(2)}$, is a small deformation of $|d^{(2)}\rangle$.

To construct the two pure phases, one way is to consider the Hamiltonian

$$\boldsymbol{H}_\Lambda^{\text{per}}(h) = \boldsymbol{H}_\Lambda^{\text{per}} - h \sum_{x\in\Lambda} (-1)^x (n_{x\uparrow} - n_{x\downarrow}).$$

Then

$$\langle\cdot\rangle_\beta^{(1)} = \lim_{h\to 0+} \langle\cdot\rangle_\beta^{\text{per}}(h)$$

and

$$\langle\cdot\rangle_\beta^{(2)} = \lim_{h\to 0-} \langle\cdot\rangle_\beta^{\text{per}}(h),$$

where $\langle\cdot\rangle_\beta^{\text{per}}(h)$ is defined by (2.15) with Hamiltonian $\boldsymbol{H}_\Lambda^{\text{per}}(h)$.

3.2. **The hard-core Bose-Hubbard model.** The state space is $\Omega = \{0,1\}$ and the Hamiltonian is written in (1.2). Let P a plaquette of four sites; the classical interaction is

$$\Phi_P(n_P) = \tfrac{1}{2}U_1 \sum_{\substack{x,y\in P \\ \|x-y\|_2=1}} n_x n_y + U_2 \sum_{\substack{x,y\in P \\ \|x-y\|_2=\sqrt{2}}} n_x n_y - \tfrac{1}{4}\mu \sum_{x\in P} n_x, \tag{3.6}$$

and $\Phi_A = 0$ if A is not a plaquette. Here $R_0 = 1$. Remark that we have

$$\Phi_P(n_P) = (\tfrac{1}{4}U_1 - \tfrac{1}{2}U_2) \sum_{\substack{x,y\in P \\ \|x-y\|_2=1}} (n_x + n_y - \tfrac{1}{2})^2 + U_2 \Bigl(\sum_{x\in P} n_x - \tfrac{1}{2} - \frac{\mu}{8U_2}\Bigr)^2 + C \tag{3.7}$$

with $C = -\frac{1}{2}U_1 + \frac{3}{4}U_2 - \frac{1}{8}\mu - \frac{1}{64}\mu^2/U_2$. $\Phi_P(n_P)$ is minimum if $n_P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, or any configuration obtained from $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ by rotation. Hence we define

$$G = \Bigl\{ n \in \{0,1\}^{\mathbb{Z}^2} : n_P \in \bigl\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \bigr\} \text{ for any plaquette } P \Bigr\}$$

(G is here the set of ground states of the interaction Φ). Since $\Phi_P(n_P) - \Phi_P(g_P) \geqslant \frac{1}{4}\min(\mu, 8U_2 - \mu)$, for any $n_P \notin G_P$, $g_P \in G_P$, Assumption 2 holds with $\Delta_0 = \frac{1}{16}\min(\mu, 8U_2 - \mu)$, provided $0 < \mu < 8U_2$. $\delta_0 = 0$.

We take as sequence of transitions for the smallest quantum fluctuations

$$\mathcal{S} = \{(\boldsymbol{A}, \boldsymbol{A}') : \boldsymbol{A} = <x,y> \text{ and } \boldsymbol{A}' = <y,x> \text{ for some } x,y \in \mathbb{Z}^2, \|x-y\|_2 = 1\}.$$

The effective potential follows from (2.9). Let $\mathcal{P}_{xy} = \cup_{P \cap \{x,y\} \neq \emptyset} P$ and more generally we denote by $\mathcal{P}$ any 3×4 or 4×3 rectangle. Up to rotations, we have to take into account five configurations, namely

$$\underset{g_{\mathcal{P}}^{(A)}}{\begin{array}{ccc}0&1&0\\0&0&0\\0&1&0\\0&0&0\end{array}} \qquad \underset{g_{\mathcal{P}}^{(B)}}{\begin{array}{ccc}0&1&0\\0&0&0\\1&0&1\\0&0&0\end{array}} \qquad \underset{g_{\mathcal{P}}^{(C)}}{\begin{array}{ccc}1&0&1\\0&0&0\\0&1&0\\0&0&0\end{array}} \qquad \underset{g_{\mathcal{P}}^{(D)}}{\begin{array}{ccc}1&0&1\\0&0&0\\1&0&1\\0&0&0\end{array}} \qquad \underset{g_{\mathcal{P}}^{(E)}}{\begin{array}{ccc}1&0&0\\0&0&1\\1&0&0\\0&0&1\end{array}}$$

We find $\Psi_{\mathcal{P}}(g_{\mathcal{P}}^{(A)}) = -t^2/2U_1$, $\Psi_{\mathcal{P}}(g_{\mathcal{P}}^{(C)}) = -t^2/4U_2$, and $\Psi_{\mathcal{P}}(g_{\mathcal{P}}^{(B)}) = \Psi_{\mathcal{P}}(g_{\mathcal{P}}^{(D)}) = \Psi_{\mathcal{P}}(g_{\mathcal{P}}^{(E)}) = 0$.

The equivalent potential Υ should have a range bigger or equal to 3; in this case, the corresponding energies $e_x^{\Upsilon}(n)$ would depend on the configuration on the square 7×7 centered at x. We could proceed in this way, but actually it simplifies a lot to do the following. Looking in the derivation of the contour model (Section 4.1), we see that beside of Assumption 4, the only property that e_x^{Υ} has to satisfy is that

$$\sum_{A \subset \Lambda^{\text{per}}} \Big[\Phi_A(n_A) + \Psi_A(n_A) \Big] = \sum_{x \in \Lambda^{\text{per}}} e_x^{\Upsilon}(n)$$

[see equation (4.29)]. Therefore we do not define Υ, but we do define

$$e_x^{\Upsilon}(n_{V(x)}) = \tfrac{1}{4} \sum_{P \ni x} \Phi_P(n_P) + \tfrac{1}{4} \sum_{y, \|y-x\|_2 = 1} \Psi_{\mathcal{P}_{xy}}(n_{\mathcal{P}_{xy}}) + \tfrac{1}{4} \sum_{\mathcal{P} \subset V(x)} \Psi_{\mathcal{P}}(n_{\mathcal{P}}), \tag{3.8}$$

where $V(x)$ is the square 5×5 centerd at x.

The configurations $g_{V(x)} \in G_{V(x)}$ are (up to rotations and reflections)

$$\underset{g_{V(x)}^{(a)}}{\begin{array}{ccccc}1&0&1&0&1\\0&0&0&0&0\\1&0&1&0&1\\0&0&0&0&0\\1&0&1&0&1\end{array}} \quad \underset{g_{V(x)}^{(b)}}{\begin{array}{ccccc}1&0&1&0&1\\0&0&0&0&0\\1&0&1&0&1\\0&0&0&0&0\\0&1&0&1&0\end{array}} \quad \underset{g_{V(x)}^{(c)}}{\begin{array}{ccccc}1&0&1&0&1\\0&0&0&0&0\\0&1&0&1&0\\0&0&0&0&0\\0&1&0&1&0\end{array}} \quad \underset{g_{V(x)}^{(d)}}{\begin{array}{ccccc}1&0&1&0&1\\0&0&0&0&0\\0&1&0&1&0\\0&0&0&0&0\\1&0&1&0&1\end{array}} \quad \underset{g_{V(x)}^{(e)}}{\begin{array}{ccccc}0&1&0&1&0\\0&0&0&0&0\\1&0&1&0&1\\0&0&0&0&0\\0&1&0&1&0\end{array}} \quad \underset{g_{V(x)}^{(f)}}{\begin{array}{ccccc}0&0&0&0&0\\1&0&1&0&1\\0&0&0&0&0\\1&0&1&0&1\\0&0&0&0&0\end{array}} \quad \underset{g_{V(x)}^{(g)}}{\begin{array}{ccccc}0&0&0&0&0\\1&0&1&0&1\\0&0&0&0&0\\0&1&0&1&0\\0&0&0&0&0\end{array}} \quad \underset{g_{V(x)}^{(h)}}{\begin{array}{ccccc}0&0&0&0&0\\0&1&0&1&0\\0&0&0&0&0\\0&1&0&1&0\\0&0&0&0&0\end{array}}$$

We find that $e_x^{\Upsilon}(g_{V(x)}^{(d)}) = e_x^{\Upsilon}(g_{V(x)}^{(e)}) = e_x^{\Upsilon}(g_{V(x)}^{(g)}) = -t^2/2U_1 - t^2/4U_2$, otherwise $e_x^{\Upsilon}(g_{V(x)}) \geqslant -3t^2/4U_1 - t^2/8U_2$ (if $U_1 \geqslant 2U_2$). This allows to take $\Delta = t^2(\frac{1}{8U_2} - \frac{1}{4U_1})$ in Assumption 4. The set of dominant states D consists in all the configurations generated by $g_{V(x)}^{(d)}$ and $g_{V(x)}^{(f)}$. $|D| = 8$.

The maximum of the expression in Assumption 5 is $b_1 = t^2(\frac{1}{8U_2} - \frac{1}{4U_1})^{-1}$. In Assumption 6 $b_2 = 0$, because $g \neq g'$ means that g and g' must differ on a whole row, and the matrix element is zero for any finite m.

These eight dominant states bring eight pure periodic phases, $\langle \cdot \rangle_\beta^{(1)}, \dots, \langle \cdot \rangle_\beta^{(8)}$; each one can be constructed by adding a suitable field in the Hamiltonian (e.g. the projector onto the dominant state).

Theorem 3.2 (Hard-core Bose-Hubbard model). *Consider the hard-core Bose-Hubbard model on the lattice $\mathbb{Z}^2$, and suppose $U_1 > 2U_2$ and $0 < \mu < 8U_2$. There exist $t_0 > 0$ and $\beta_0(t) < \infty$ ($\lim_{t \to 0} \beta_0(t) = \infty$) such that if $t \leqslant t_0$ and $\beta \geqslant \beta_0$,*

- *the free energy exists in the thermodynamic limit with periodic boundary conditions, as well as expectation values of observables,*
- *there are 8 pure periodic phases with exponential decay of correlations.*

Each of these eight phases is a perturbation of a dominant state d, and the expectation value of any operator is close to its value in the state d, see Theorem 2.2 for more precise statement.

4. Contour representation of a quantum model

Our Hamiltonian has periodicity $\ell_0 < \infty$. Without loss of generality, however, one can consider only translation invariant Hamiltonians, applying the standard trick. Namely, if Ω is the single site phase space, we let $\Omega' = \Omega^{\{1,\dots,\ell_0\}^\nu}$; $S' = |\Omega'| = S^{\ell_0^\nu}$. Then we consider the torus $\Lambda' \subset \mathbb{Z}^\nu$, $\ell_0^\nu|\Lambda'| = |\Lambda|$, each point of which is representing a block of sites in Λ of size ℓ_0^ν, and identify

$$\Omega'^{\Lambda'} \simeq \Omega^{\Lambda}.$$

Constructing $\mathcal{H}'$ as the Hilbert space spanned by the elements of $\Omega'^{\Lambda'}$, it is clear that $\mathcal{H}'$ is isomorphic to $\mathcal{H}$. The new translation invariant interactions Φ' and $\boldsymbol{V}'$ are defined by resumming, for each $A \subset \Lambda'$, the corresponding contributions with supports in the union of corresponding blocks. Notice the change in range of interactions. Namely, it decreased to $\lceil R/\ell_0 \rceil$ (the lowest integer bigger or equal to R/ℓ_0).

¿From now on, keeping the original notation $\mathcal{H}$, S, ... , we suppose that the Hamiltonian is translation invariant.

The partition function of a quantum model is a trace over a Hilbert space. But expanding $\mathrm{e}^{-\beta \boldsymbol{H}}$ with the help of Duhamel formula we can reformulate it in terms of the partition function of a classical model in a space with one additional dimension (the extra dimension being continuous). In this section we present such an expansion, leading to a contour representation, of the partition function $Z_\Lambda^{\text{per}} := \operatorname{Tr} \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}$ in a finite torus Λ.

Expansion with the help of Duhamel formula yields

$$\mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}} = \sum_{m \geqslant 0} \sum_{\substack{\boldsymbol{A}_1,\dots,\boldsymbol{A}_m \\ \bar{A}_i \subset \Lambda}} \int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m$$
$$\mathrm{e}^{-\tau_1 \boldsymbol{H}_\Lambda^{(0)\,\text{per}}} \boldsymbol{V}_{\boldsymbol{A}_1} \mathrm{e}^{-(\tau_2-\tau_1)\boldsymbol{H}_\Lambda^{(0)\,\text{per}}} \boldsymbol{V}_{\boldsymbol{A}_2} \dots \boldsymbol{V}_{\boldsymbol{A}_m} \mathrm{e}^{-(\beta-\tau_m)\boldsymbol{H}_\Lambda^{(0)\,\text{per}}}. \tag{4.1}$$

Inserting the expansion of unity $\mathbb{1}_{\mathcal{H}_\Lambda} = \sum_{n_\Lambda} |n_\Lambda\rangle\langle n_\Lambda|$ to the right of operators $\boldsymbol{V}_{\boldsymbol{A}_j}$, we obtain

$$Z_\Lambda^{\text{per}} = \sum_{m \geqslant 0} \sum_{n_\Lambda^1,\dots n_\Lambda^m} \sum_{\substack{\boldsymbol{A}_1,\dots,\boldsymbol{A}_m \\ \bar{A}_i \subset \Lambda}} \int_{0<\tau_1<\dots<\tau_m<\beta} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m$$
$$\mathrm{e}^{-\tau_1 H_\Lambda^{(0)\,\text{per}}(n_\Lambda^1)} \langle n_\Lambda^1| \boldsymbol{V}_{\boldsymbol{A}_1} |n_\Lambda^2\rangle \mathrm{e}^{-(\tau_2-\tau_1) H_\Lambda^{(0)\,\text{per}}(n_\Lambda^2)} \dots \langle n_\Lambda^m| \boldsymbol{V}_{\boldsymbol{A}_m} |n_\Lambda^1\rangle \mathrm{e}^{-(\beta-\tau_m) H_\Lambda^{(0)\,\text{per}}(n_\Lambda^1)}. \tag{4.2}$$

This expansion can be interpreted as a classical partition function on the $(\nu+1)$-dimensional space $\Lambda \times [0,\beta]$. Namely, calling the additional dimension "time direction", the partition function Z_Λ^{per} is a (continuous) sum over all space-time configurations $\boldsymbol{n}_\Lambda = \boldsymbol{n}_\Lambda(\tau)$, $\tau \in [0,\beta]$, and all possible transitions at times corresponding to discontinuities of $\boldsymbol{n}_\Lambda(\tau)$. Notice that $\boldsymbol{n}_\Lambda(\tau)$ is periodic in the time direction. Thus, actually, we obtain a classical partition function on the $d+1$-dimensional torus $\mathbb{T}_\Lambda = \Lambda \times [0,\beta]_{\text{per}}$ with a circle $[0,\beta]_{\text{per}}$ in time direction (for simplicity we omit in $\mathbb{T}_\Lambda$ a reference to β). Introducing the *quantum configuration* $\boldsymbol{\omega}_{\mathbb{T}_\Lambda}$ consisting of the space-time configuration $\boldsymbol{n}_\Lambda(\tau)$ and the transitions $(\boldsymbol{A}_i, \tau_i)$ at corresponding times, we can rewrite (4.2) in a compact form

$$Z_\Lambda^{\text{per}} = \int \mathrm{d}\boldsymbol{\omega}_{\mathbb{T}_\Lambda} \rho^{\text{per}}(\boldsymbol{\omega}_{\mathbb{T}_\Lambda}) \tag{4.3}$$

with $\rho^{\text{per}}(\boldsymbol{\omega}_{\mathbb{T}_\Lambda})$ standing for the second line of (4.2).

Now, we are going to specify excitations within a spacetime configuration $\boldsymbol{n}$ and identify classes of small excitations — *the loops*[7] — and large ones — *the quantum contours.*

A configuration $n \in \Omega^{\mathbb{Z}^\nu}$ is said to be in the state $g \in G$ at site x whenever $n_{U(x)} = g_{U(x)}$ (notice that, in general, g is not unique). If there is no such $g \in G$, the configuration n is said to be *classically excited* at x. We use $E(n)$ to denote the set of all classically excited sites of $n \in \Omega^{\mathbb{Z}^\nu}$. For any $\Lambda \subset \mathbb{Z}^\nu$, let us consider the set $\mathcal{Q}_\Lambda^{\rm per}$ of quantum configurations on the torus $\mathbb{T}_\Lambda$. Whenever $\boldsymbol{\omega} \in \mathcal{Q}_\Lambda^d$, its *boundary* $\boldsymbol{B}^{(0)}(\boldsymbol{\omega}) \subset \mathbb{T}_\Lambda$ is defined as the union

$$\boldsymbol{B}^{(0)}(\boldsymbol{\omega}) = (\cup_{\tau\in[0,\beta]}(E(\boldsymbol{n}(\tau)) \times \tau)) \cup (\cup_{i=1}^m(\bar{A}_i \times \tau_i)). \tag{4.4}$$

The sets $\bar{A}_i \times \tau_i \subset \mathbb{T}_\Lambda$ represent the effect of the operator $\boldsymbol{V}$ and for this reason are called *quantum transitions.* It is worth to notice that the set $\boldsymbol{B}^{(0)}(\boldsymbol{\omega})$ is closed.

Next step is to identify the smallest quantum excitations — those consisting of a sequence of transitions from the list $\mathcal{S}$. First, let us use $\mathcal{B}^{(0)}(\boldsymbol{\omega})$ to denote the set of connected components of $\boldsymbol{B}^{(0)}(\boldsymbol{\omega})$ (so that $\boldsymbol{B}^{(0)}(\boldsymbol{\omega}) = \cup_{B\in\mathcal{B}^{(0)}(\boldsymbol{\omega})}B$). To any $B \in \mathcal{B}^{(0)}(\boldsymbol{\omega})$ that is not wrapped around the cylinder (i.e., for which there exists time $\tau_B \in [0,\beta]_{\rm per}$ with $B \cap (\mathbb{Z}^\nu \times \tau_B) = \emptyset$) we assign its sequence of transitions, $S(B,\boldsymbol{\omega})$, ordered according to their times (starting from τ_B to β and proceeding from 0 to τ_B) as well as the smallest box $\tilde{B}$ containing B. Here, a box is any subset of $\mathbb{T}_{\mathbb{Z}^\nu}$ of the form $A \times [\tau_1,\tau_2]$ with connected $A \subset \mathbb{Z}^\nu$ and $[\tau_1,\tau_2] \subset [0,\beta]_{\rm per}$ (if $\tau_1 > \tau_2$, we interpret the segment $[\tau_1,\tau_2]$ as that interval in $[0,\beta]_{\rm per}$ (with endpoints τ_1 and τ_2) that contains the point $0 \equiv \beta$).

We would like to declare the excitations with $S(B,\boldsymbol{\omega}) \in \mathcal{S}$ to be small. However, we need to be sure that there are no other excitations in their close neighbourhood. If this were the case, we would "glue" the neighbouring excitations together. This motivates the following iterative procedure.

Given $\boldsymbol{\omega}$, let us first consider the set $\mathcal{B}_0^{(0)}(\boldsymbol{\omega})$ of those components $B \in \mathcal{B}^{(0)}(\boldsymbol{\omega})$ that are not wrapped around the cylinder and for which $S(B,\boldsymbol{\omega}) \in \bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ is the set of all subsequences of sequences from $\mathcal{S}$. Next, we define the first extension of the boundary,

$$\boldsymbol{B}^{(1)}(\boldsymbol{\omega}) = (\cup_{B\in\mathcal{B}^{(0)}(\boldsymbol{\omega})\setminus\mathcal{B}_0^{(0)}(\boldsymbol{\omega})}B) \cup (\cup_{B\in\mathcal{B}_0^{(0)}(\boldsymbol{\omega})}\tilde{B}).$$

Using $\mathcal{B}^{(1)}(\boldsymbol{\omega})$ to denote the set of connected components of $\boldsymbol{B}^{(1)}(\boldsymbol{\omega})$ and $\mathcal{B}_0^{(1)}(\boldsymbol{\omega}) \subset \mathcal{B}^{(1)}(\boldsymbol{\omega})$ the set of those components B in $\mathcal{B}^{(1)}(\boldsymbol{\omega})$ that are not wrapped around the cylinder and for which[8] $S(B,\boldsymbol{\omega}) \in \bar{\mathcal{S}}$, we define

$$\boldsymbol{B}^{(2)}(\boldsymbol{\omega}) = (\cup_{B\in\mathcal{B}^{(1)}(\boldsymbol{\omega})\setminus\mathcal{B}_0^{(1)}(\boldsymbol{\omega})}B) \cup (\cup_{B\in\mathcal{B}_0^{(1)}(\boldsymbol{\omega})}\tilde{B}).$$

Iterating this procedure, it is clear that after finite number of steps we obtain the final extension of the boundary,

$$\boldsymbol{B}(\boldsymbol{\omega}) = (\cup_{B\in\mathcal{B}^{(k)}(\boldsymbol{\omega})\setminus\mathcal{B}_0^{(k)}(\boldsymbol{\omega})}B) \cup (\cup_{B\in\mathcal{B}_0^{(k)}(\boldsymbol{\omega})}B).$$

Here, every $B \in \mathcal{B}_0^{(k)}(\boldsymbol{\omega})$ is actually a box of the form $A\times[\tau_1,\tau_2]$ (that is not wrapped around the cylinder) and $S(B,\boldsymbol{\omega}) \in \bar{\mathcal{S}}$. Let us denote $\mathcal{B}(\boldsymbol{\omega}) \equiv \mathcal{B}_0^{(k)}(\boldsymbol{\omega})$ and consider the set $\mathcal{B}_0(\boldsymbol{\omega}) \subset \mathcal{B}(\boldsymbol{\omega})$ of all those sets $B \in \mathcal{B}_0^{(k)}(\boldsymbol{\omega})$ for which actually $S(B,\boldsymbol{\omega}) \in \mathcal{S}$ and, moreover, $n_A(\tau_1 - 0) = n_A(\tau_2 + 0)$. Finally, let $\mathcal{B}_l(\boldsymbol{\omega}) = \mathcal{B}(\boldsymbol{\omega}) \setminus \mathcal{B}_0(\boldsymbol{\omega})$ — it represents the set of all excitations of $\boldsymbol{\omega}$ that are not loops. Taking, for any closed $B \subset \mathbb{T}_\Lambda$, the restriction $\boldsymbol{n}_B$ of a space-time

[7]Even though the present framework is more general, the name comes from thinking about simplest excitations in Hubbard type models. Namely, a jump of an electron to a neighbouring site and returning afterwards to its original position.

[8]A set $B \in \mathcal{B}_0^{(1)}(\boldsymbol{\omega})$ may actually contain several original components from $\mathcal{B}_0^{(0)}(\boldsymbol{\omega})$. We take for $S(B,\boldsymbol{\omega})$ the sequence of all transitions in all those components.

configuration $\boldsymbol{n}$ to be defined by $(\boldsymbol{n}_B)_x(\tau) = \boldsymbol{n}_x(\tau)$ for any $x \times \tau \in B$, we introduce the useful notion of the restriction $\boldsymbol{\omega}_B$ of a quantum configuration $\boldsymbol{\omega}$ to B as to consist of $\boldsymbol{n}_B$ and those quantum transitions from $\boldsymbol{\omega}$ that are contained in B, $A \times \tau \subset B$ (we suppose here that $\boldsymbol{\omega}$ and B are such that no transition intersects both B and its complement; we do not define $\boldsymbol{\omega}_B$ in this case).

Now the loops and and the quantum contours can be defined. First, the *loops* of a quantum configuration $\boldsymbol{\omega}$ are the triplets $\xi \equiv (B, \boldsymbol{\omega}_B, g_A^{\xi})$; $B \equiv A \times [\tau_1, \tau_2] \in \mathcal{B}_0(\boldsymbol{\omega})$ is the *support* of the loop ξ and $g_A^{\xi} = n_A(\tau_1 - 0) = n_A(\tau_2 + 0)$, a restriction of a configuration $g \in G$. (While the configuration g is not unique, its restriction to A is determined by the loop ξ in a unique way.) We say that ξ is *immersed* in g. Given a quantum configuration $\boldsymbol{\omega}$, we obtain a new configuration $\check{\boldsymbol{\omega}}$ by erasing all loops $(B, \boldsymbol{\omega}_B, g_A^{\xi})$, i.e. for each ξ we remove all the transitions in its support B and change the space-time configuration on B into $g \in G$ into which ξ is immersed. Let us remark that $\mathcal{B}(\check{\boldsymbol{\omega}}) = \mathcal{B}_l(\boldsymbol{\omega})$. Notice that, since we started our construction from (4.4), we have automatically $\operatorname{diam} A \geq 2R_0$ for a support $A \times [\tau_1, \tau_2]$ of any loop ξ.

Quantum contours of a configuration $\boldsymbol{\omega}$ will be constructed by extending pairs $(B, \boldsymbol{\omega}_B)$ with $B \in \mathcal{B}_l(\boldsymbol{\omega})$ by including also the regions of nondominating states from G. Namely, summing over loops we will see that "loop free energy" favours the regions with dominating configurations from $D \subset G$. However, to recognize the influence of loops, we have to look on regions of size comparable to the size of loops. This motivates the following definitions with $V(x) = \{y \in \mathbb{Z}^{\nu}, |x - y| < R\}$ being an extension of original neighbourhood $U(x)$. Thus, we enlarge the set $E(n)$ of classically excited sites to $\tilde{E}(n)$, with

$$\tilde{E}(n) = \{x \in \mathbb{Z}^{\nu} : n_{V(x)} \neq g_{V(x)} \text{ for any } g \in G\}$$

and we introduce the set $F(n)$ of *softly excited sites* by

$$F(n) = \{x \in \mathbb{Z}^{\nu} \setminus \tilde{E}(n) : n_{V(x)} \neq d_{V(x)} \text{ for any } d \in D\}.$$

Then, for a quantum configuration such that $\boldsymbol{\omega} = \check{\boldsymbol{\omega}}$, we define the new extended boundary

$$\boldsymbol{B}_{\mathrm{e}}(\check{\boldsymbol{\omega}}) = \bigcup_{\tau \in [0,\beta]_{\mathrm{per}}} \Big([\tilde{E}(\boldsymbol{n}(\tau)) \cup F(\boldsymbol{n}(\tau))] \times \tau \Big) \bigcup \bigcup_{i=1}^{m} \Big([\underset{x \in A_i}{\cup} V(x)] \times \tau_i \Big),$$

and if $\boldsymbol{\omega} \neq \check{\boldsymbol{\omega}}$, we set $\boldsymbol{B}_{\mathrm{e}}(\boldsymbol{\omega}) = \boldsymbol{B}_{\mathrm{e}}(\check{\boldsymbol{\omega}})$. Notice that $\boldsymbol{B}(\check{\boldsymbol{\omega}}) \subset \boldsymbol{B}_{\mathrm{e}}(\boldsymbol{\omega})$, since the first set is the union of classical excitations, quantum transitions and boxes; obviously the classical excitations and the quantum transitions also belong to $\boldsymbol{B}_{\mathrm{e}}(\boldsymbol{\omega})$, and the boxes being such that their diameter is smaller than R and they contain $U(x)$-excited sites at each time, they are $V(x)$-excited. Decomposing $\boldsymbol{B}_{\mathrm{e}}(\boldsymbol{\omega})$ into connected components, we get our quantum contours, namely $\gamma = (B, \boldsymbol{\omega}_B)$. Notice that the configuration $\boldsymbol{\omega}_B$ contains actually also the information determining which dominant ground state lies outside B. We call the set B the *support* of γ, $B = \operatorname{supp} \gamma$, and introduce also its "truly excited part", the *core*, $\operatorname{core} \gamma \subset \operatorname{supp} \gamma$, by taking $\operatorname{core} \gamma = \operatorname{supp} \gamma \bigcap \Big(\cup_{\tau \in [0,\beta]_{\mathrm{per}}} (\tilde{E}(\boldsymbol{n}(\tau)) \times \tau) \cup \cup_{i=1}^{m} \Big([\cup_{x \in A_i} V(x)] \times \tau_i \Big) \Big)$. Finally, notice that if the contour is not wrapped around the torus in its spatial direction, there exists a space-time configuration $\boldsymbol{\omega}^{\gamma}$ and we have $B = \boldsymbol{B}_{\mathrm{e}}(\boldsymbol{\omega}^{\gamma})$.

A set of quantum contours $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is called admissible if there exists a quantum configuration $\boldsymbol{\omega}^{\Gamma} \in \mathcal{Q}_{\Lambda}^{\mathrm{per}}$ which has Γ as set of quantum contours. Clearly, if it exists, it is unique under assumption that it contains no loops ($\boldsymbol{\omega}^{\Gamma} = \check{\boldsymbol{\omega}}^{\Gamma}$). We use $\mathcal{D}_{\Lambda}^{\mathrm{per}}$ to denote the set of all collections Γ of admissible quantum contours.

Given $\Gamma \in \mathcal{D}_{\Lambda}^{\mathrm{per}}$, a set of loops $\Xi = \{\xi_1, \ldots, \xi_{\ell}\}$ is said admissible and compatible with Γ if there exists $\boldsymbol{\omega}^{\Gamma \cup \Xi}$ which has Ξ as set of loops and Γ as set of quantum contours (it is also unique whenever it exists). More explicitly,

- two loops $\xi = (B, \boldsymbol{\omega}_B, g_A^{\xi})$ and $\xi' = (B', \boldsymbol{\omega}'_{B'}, g_{A'}^{\xi'})$ are compatible, $\xi \sim \xi'$, iff $B \cup B'$ is not connected;
- using $\operatorname{core}\Gamma = \cup_{\gamma\in\Gamma}\operatorname{core}\gamma$, a loop $\xi = (B, \boldsymbol{\omega}_B, g_A^{\xi})$, with $B = A \times [\tau_1, \tau_2]$, is compatible with Γ, $\xi \sim \Gamma$, iff

$$B \cup \operatorname{core}\Gamma \text{ is not connected,} \tag{4.5}$$

$$g_A^{\xi} = \boldsymbol{n}_A^{\Gamma}(\tau_1 - 0) = \boldsymbol{n}_A^{\Gamma}(\tau_2 + 0); \tag{4.6}$$

- a collection of loops $\Xi = \{\xi_1, \dots, \xi_\ell\}$ is admissible and compatible with Γ iff any two loops from Ξ are compatible and each loop from Ξ is compatible with Γ.

We use $\mathcal{D}_\Lambda^{\mathrm{loop}}(\Gamma)$ to denote the set of all admissible collections Ξ that are compatible with Γ.

The conditions of admissibility and compatibility above can be, for any given set of transitions $\{\boldsymbol{A}_1, \dots, \boldsymbol{A}_m\}$, formulated as a finite number of restrictions on corresponding transition times $\{\tau_1, \dots, \tau_m\}$. Given the restrictions on admissibility of $\Gamma \in \mathcal{D}_\Lambda^{\mathrm{per}}$, the restrictions on Ξ to belong to $\mathcal{D}_\Lambda^{\mathrm{loop}}(\Gamma)$ factorize. As a result, the partition function Z_Λ^{per} in (4.3) can be rewritten in terms of integrations over $\mathcal{D}_\Lambda^{\mathrm{per}}$ and $\mathcal{D}_\Lambda^{\mathrm{loop}}(\Gamma)$ [the summation over Γ and Ξ accompanied with the integration, *a priori* over the interval $[0, \beta]$, over times τ_i of corresponding transitions, subjected to above formulated restrictions, c.f. (4.2)]. Furthermore the contribution of $\Gamma \cup \Xi$ factorizes as a contribution of Γ times a product of terms for $\xi \in \Xi$ [BKU, DFF1][9], we get

$$\begin{aligned} Z_\Lambda^{\mathrm{per}} &= \int_{\mathcal{D}_\Lambda^{\mathrm{per}}} \mathrm{d}\Gamma \int_{\mathcal{D}_\Lambda^{\mathrm{loop}}(\Gamma)} \mathrm{d}\Xi\, \rho^{\mathrm{per}}(\boldsymbol{\omega}^{\Gamma\cup\Xi}) \\ &= \int_{\mathcal{D}_\Lambda^{\mathrm{per}}} \mathrm{d}\Gamma \rho^{\mathrm{per}}(\boldsymbol{\omega}^{\Gamma}) \int_{\mathcal{D}_\Lambda^{\mathrm{loop}}(\Gamma)} \mathrm{d}\Xi \prod_{\xi\in\Xi} z(\xi). \end{aligned} \tag{4.7}$$

Here, using $\{(\boldsymbol{A}_i, \tau_i), i = 1, \dots, m\}$ to denote the quantum transitions of $\Gamma \cup \Xi$, we put

$$\rho^{\mathrm{per}}(\boldsymbol{\omega}^{\Gamma\cup\Xi}) = \prod_{i=1}^{m} \langle \boldsymbol{n}_{A_i}^{\Gamma\cup\Xi}(\tau_i - 0)|\, \boldsymbol{V}_{\boldsymbol{A}_i}\, |\boldsymbol{n}_{A_i}^{\Gamma\cup\Xi}(\tau_i + 0)\rangle \exp\{-\int_{\mathbb{T}_\Lambda} \mathrm{d}(A,\tau)\Phi_A(\boldsymbol{n}_A^{\Gamma\cup\Xi}(\tau))\}, \tag{4.8}$$

where $\int_B \mathrm{d}(A,\tau)$ is the shorthand for $\int_0^\beta \mathrm{d}\tau \sum_{A: A\times\tau\subset B}$ (used here for $B = \mathbb{T}_\Lambda$). Similarly for $\rho^{\mathrm{per}}(\boldsymbol{\omega}^{\Gamma})$. Further, the weight of a loop $\xi = (B^\xi, \boldsymbol{\omega}_{B^\xi}, g_A^{\xi})$ with the set of quantum transitions $\{(\boldsymbol{A}_i, \tau_i), i = 1, \dots, \ell\}$ and $\boldsymbol{n}^\xi$ the space-time configuration corresponding to $\boldsymbol{\omega}_{B^\xi}$, is

$$\begin{aligned} z(\xi) = \exp\Bigl\{-\int_{B^\xi} \mathrm{d}(A,\tau)[\Phi_A(\boldsymbol{n}_A^\xi(\tau)) - \Phi_A(g_A^\xi)]\Bigr\} \langle g_{A_1}^\xi|\, \boldsymbol{V}_{\boldsymbol{A}_1}\, |\boldsymbol{n}_{A_1}^\xi(\tau_1 + 0)\rangle \times \\ \times \langle \boldsymbol{n}_{A_2}^\xi(\tau_2 - 0)|\, \boldsymbol{V}_{\boldsymbol{A}_2}\, |\boldsymbol{n}_{A_2}^\xi(\tau_2 + 0)\rangle \dots \langle \boldsymbol{n}_{A_\ell}^\xi(\tau_\ell - 0)|\, \boldsymbol{V}_{\boldsymbol{A}_\ell}\, |g_{A_\ell}^\xi\rangle. \end{aligned} \tag{4.9}$$

Given $\Gamma \in \mathcal{D}_\Lambda^{\mathrm{per}}$, the second integral in (4.7) is over the collections of the loops that interact only through a condition of non-intersection. This is the usual framework for applying the cluster expansion of polymers. The only technical difficulty is that the set of our loops is uncountable (the loops depend on continuous transition times), and thus we cannot simply quote the existing literature. Nevertheless, the needed extension is rather straightforward and often implicitly used.

[9] For spin or boson systems factorization is true simply because any two operators with disjoint supports commute. In the case of fermion systems there is an additional sign due to anticommutation relations between creation and annihilation operators, and factorization is no more obvious. That it indeed factorizes was nicely proved in Section 4.2 of [DFF1].

Given a collection $\boldsymbol{C} = (\xi_1, \ldots, \xi_n)$ of loops, we define the truncated function

$$\Phi^{\mathrm{T}}(\boldsymbol{C}) = \frac{1}{n!}\varphi^{\mathrm{T}}(\boldsymbol{C}) \prod_{\xi \in \boldsymbol{C}} z(\xi), \tag{4.10}$$

with

$$\varphi^{\mathrm{T}}(\boldsymbol{C}) = \varphi^{\mathrm{T}}(\xi_1, \ldots, \xi_n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{\mathcal{G}} \prod_{e(i,j) \in \mathcal{G}} \big(\mathbb{I}\left[\xi_i \sim \xi_j\right] - 1 \big) & \text{if } n \geqslant 2, \end{cases}$$

where the sum is over all connected graphs $\mathcal{G}$ of n vertices. Notice that $\Phi^{\mathrm{T}}(\boldsymbol{C}) = 0$ whenever $\boldsymbol{C}$ is not a cluster, i.e. if the union of the supports of its loops is not connected. We use $\mathcal{L}_\Lambda$ and $\mathcal{C}_\Lambda$ to denote the set of all loops and clusters, respectively, and use $\int_{\mathcal{C}_\Lambda} \mathrm{d}\boldsymbol{C}$ as a shorthand for $\sum_{n \geqslant 1} \int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_1 \cdots \int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_n$, in obvious meaning. Whenever $\Gamma \in \mathcal{D}_\Lambda^{\mathrm{per}}$ is fixed, we use $\mathcal{L}_\Lambda(\Gamma)$ to denote the set of all loops compatible with Γ and write $\boldsymbol{C} \in \mathcal{C}_\Lambda(\Gamma)$ whenever the cluster $\boldsymbol{C}$ contains only loops from $\mathcal{L}_\Lambda(\Gamma)$. Again, $\int_{\mathcal{C}_\Lambda(\Gamma)} \mathrm{d}\boldsymbol{C}$ is a shorthand for $\sum_{n \geqslant 1} \int_{\mathcal{L}_\Lambda(\Gamma)} \mathrm{d}\xi_1 \cdots \int_{\mathcal{L}_\Lambda(\Gamma)} \mathrm{d}\xi_n$. Finally, we also need similar integrals conditioned by the time of the first transition encountered in the loop ξ or the cluster $\boldsymbol{C}$. Namely, using C to denote the support of $\boldsymbol{C}$, i.e. the union of the supports of the loops of $\boldsymbol{C}$, and $I_C = \{\tau_1(\boldsymbol{C}), \tau_2(\boldsymbol{C})\}$ to denote its vertical projection[10], $I_C = \{\tau \in [0, \beta]_{\mathrm{per}}; \mathbb{Z}^\nu \times \tau \cap C \neq \emptyset\}$, we use $\mathcal{C}_\Lambda^{(x,\tau)}$ for the set of all clusters $\boldsymbol{C} \in \mathcal{C}_\Lambda$ with the first transition time $\tau_1(\boldsymbol{C}) = \tau$, for which their first loop ξ_1 with support $B_1 = A_1 \times [\tau_1(\boldsymbol{C}), \tau_2]$, contains the site x, $A_1 \ni x$. Then $\int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi$ and $\int_{\mathcal{C}_\Lambda^{(x,\tau)}} \mathrm{d}\boldsymbol{C}$ are shorthands for the corresponding integrals with first transition time fixed — formally one replaces $\int \mathrm{d}\xi_1$ by $\int \mathbb{I}\left[A_1 \ni x\right] \delta(\tau_1(\xi_1) - \tau) \mathrm{d}\xi_1$. With this notation we can formulate the cluster expansion lemma.

Lemma 4.1 (Cluster expansion). *For any $c \in \mathbb{R}$, $\alpha_1 < (2R_0)^{-\nu}$, $\alpha_2 < R^{-2\nu}\Delta_0$ and $\delta > 0$, there exists $\varepsilon_0 > 0$ such that whenever $\|\boldsymbol{V}\| \leqslant \varepsilon_0$ and $\Gamma \in \mathcal{D}_\Lambda^{\mathrm{per}}$, we have the loop cluster expansion,*

$$\int_{\mathcal{D}_\Lambda^{\mathrm{loop}}(\Gamma)} \mathrm{d}\Xi \prod z(\xi) = \exp\Big\{ \int_{\mathcal{C}_\Lambda(\Gamma)} \mathrm{d}\boldsymbol{C} \Phi^{\mathrm{T}}(\boldsymbol{C}) \Big\}. \tag{4.11}$$

Moreover, the weights of the clusters are exponentially decaying (uniformly in Λ and β):

$$\int_{\mathcal{C}_\Lambda} \mathrm{d}\boldsymbol{C}\, \mathbb{I}\left[C \ni (x,\tau)\right] |\Phi^{\mathrm{T}}(\boldsymbol{C})| \prod_{\xi \in \boldsymbol{C}} \mathrm{e}^{(c - \alpha_1 \log \|\boldsymbol{V}\|)|A| + \alpha_2 |B|} \leqslant \delta \tag{4.12}$$

and

$$\int_{\mathcal{C}_\Lambda^{(x,\tau)}} \mathrm{d}\boldsymbol{C} |\Phi^{\mathrm{T}}(\boldsymbol{C})| \prod_{\xi \in \boldsymbol{C}} \mathrm{e}^{(c - \alpha_1 \log \|\boldsymbol{V}\|)|A| + \alpha_2 |B|} \leqslant \delta \tag{4.13}$$

for every $(x,\tau) \in \mathbb{T}_\Lambda$.

Proof. One can follow any standard reference concerning cluster expansions for continuum systems, for example [Bry]. We are using here [Pfi] whose formulation is closer to our purpose. Assuming that inequality (4.13) holds true, we have a finite bound

$$\sum_{n \geqslant 1} \frac{1}{n!} \int_{\mathcal{L}_\Lambda(\Gamma)^n} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n |\varphi^{\mathrm{T}}(\xi_1, \ldots, \xi_n)| \prod_{i=1}^n |z(\xi_i)| \leqslant \delta\beta|\Lambda|. \tag{4.14}$$

Lemma 4.1 then follows from Lemma 3.1 of [Pfi]. Let us turn to the proof of the two inequalities. Let

$$f(\xi) = |z(\xi)|\, \mathrm{e}^{(c - \alpha_1 \log \|\boldsymbol{V}\|)|A| + \alpha_2 |B|}.$$

[10] Again, if $\tau_1 > \tau_2$, the segment $[\tau_1, \tau_2] \subset [0, \beta]_{\mathrm{per}}$ contains the point $0 \equiv \beta$.

Skipping the conditions $\xi_j \sim \Gamma$, we define

$$I_n = n\Big[\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_1\, \mathbb{I}\big[B_1 \ni (x,\tau)\big] + \int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi_1\Big] \int_{\mathcal{L}_\Lambda^{n-1}} \mathrm{d}\xi_2 \dots \mathrm{d}\xi_n |\varphi^{\mathrm{T}}(\xi_1,\dots,\xi_n)| \prod_{i=1}^n f(\xi_i) \tag{4.15}$$

(it does not depend on $(x,\tau) \in \mathbb{T}_\Lambda$). The lemma will be completed once we shall have established that $I_n \leqslant n!(\frac{1}{2}\delta)^n$ (assuming that $\delta \leqslant 1$; otherwise, we show that $I_n \leqslant n!/2^n$). ¿From Lemma 3.4 of [Pfi], we get

$$|\varphi^{\mathrm{T}}(\xi_1,\dots,\xi_n)| \leqslant \sum_{\mathcal{T} \text{ tree on } n \text{ vertices}} \prod_{e(i,j)\in\mathcal{T}} \mathbb{I}\big[B_i \cup B_j \text{ connected}\big]. \tag{4.16}$$

Denoting $d_1,\dots,d_n$ the incidence numbers of vertices $1,\dots,n$, we first proceed with the integration on the loops $j \neq 1$ for which $d_j = 1$; in the tree $\mathcal{T}$, such j shares an edge only with one vertex i. The incompatibility between ξ_i and ξ_j, with $\xi = (B_i, \boldsymbol{\omega}_{B_i}^{(i)}, g_{A_i}^{\xi_i})$, $B_i = A_i \times [\tau_1^{(i)}, \tau_2^{(i)}]$, and similarly for ξ_j, means that either $B_j \cup [A_i \times \tau_1^{(i)}]$ is connected, or $[A_j \times \tau_1^{(j)}] \cup B_i$ is connected. Hence, the bound for the integral over the ξ_j that are incompatible with ξ_i is

$$\begin{aligned}\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_j\, \mathbb{I}\big[B_j \cap B_i \text{ connected}\big] f(\xi_j) &\leqslant 2\nu|A_i| \int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_j\, \mathbb{I}\big[B_j \ni (x,\tau)\big] f(\xi_j) + 2\nu|B_i| \int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi_j f(\xi_j) \\ &\leqslant 2\nu\Big(|A_i| + \alpha|B_i|\Big)\Big(\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_j\, \mathbb{I}\big[B_j \ni (x,\tau)\big] f(\xi_j) + \frac{1}{\alpha}\int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi_j f(\xi_j)\Big).\end{aligned} \tag{4.17}$$

(The constant α has been introduced in order to match with the conditions of the next lemma). Then

$$\begin{aligned}I_n \leqslant n(2\nu)^{n-1} \sum_{\mathcal{T} \text{ tree of } n \text{ vertices}} \Big[\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_1\, \mathbb{I}\big[B_1 \ni (x,\tau)\big] + \int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi_1\Big] f(\xi_1)\Big(|A_1| + \alpha|B_1|\Big)^{d_1} \\ \prod_{j=2}^n \Big[\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi_j\, \mathbb{I}\big[B_j \ni (x,\tau)\big] f(\xi_j)\Big(|A_j| + \alpha|B_j|\Big)^{d_j-1} + \frac{1}{\alpha}\int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi_j f(\xi_j)\Big(|A_j| + \alpha|B_j|\Big)^{d_j-1}\Big].\end{aligned} \tag{4.18}$$

Now summing over all trees, knowing that the number of trees with n vertices and incidence numbers $d_1,\dots,d_n$ is equal to

$$\frac{(n-2)!}{(d_1-1)!\dots(d_n-1)!} \leqslant \frac{(n-1)!}{d_1!(d_2-1)!\dots(d_n-1)!},$$

we find a bound

$$I_n \leqslant n!(2\nu)^{n-1}(1+\alpha)\Big[\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi\, \mathbb{I}\big[B \ni (x,\tau)\big] f(\xi)\, \mathrm{e}^{|A|+\alpha|B|} + \frac{1}{\alpha}\int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi f(\xi)\, \mathrm{e}^{|A|+\alpha|B|}\Big]^n. \tag{4.19}$$

We conclude by using the following lemma which implies that the quantity between the brackets is small.

□

Lemma 4.2. *Let $\alpha_1 < (2R_0)^{-\nu}$ and $\alpha_2 < R^{-2\nu}\Delta_0$. For any $c \in \mathbb{R}$ and $\delta > 0$, there exists $\varepsilon_0 > 0$ such that whenever $\|\boldsymbol{V}\| \leqslant \varepsilon_0$ the following inequality holds true,*

$$\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi\, \mathbb{I}\big[B \ni (x,\tau)\big] |z(\xi)|\, \mathrm{e}^{(c-\alpha_1 \log\|\boldsymbol{V}\|)|A| + \alpha_2|B|} + \int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi |z(\xi)|\, \mathrm{e}^{(c-\alpha_1 \log\|\boldsymbol{V}\|)|A| + \alpha_2|B|} \leqslant \delta,$$

where (x,τ) is any space-time site of $\mathbb{T}_\Lambda$.

Proof. Let us first consider the integral over ξ such that its box contains a given space-time site. We denote by ℓ_1 the number of quantum transitions of ξ at times bigger than τ, and ℓ_2 the number of the other quantum transitions. The integral over ξ can be done by summing over $(\ell_1+\ell_2)$ quantum transitions $\boldsymbol{A}_1^1,\dots,\boldsymbol{A}_{\ell_1}^1,\boldsymbol{A}_1^2,\dots,\boldsymbol{A}_{\ell_2}^2$, by summing over $(\ell_1+\ell_2)$ configurations $n_{A_j^i}^{i,j}$, and by integrating over times $\tau_1^1 < \dots < \tau_{\ell_1}^1$, $\tau_1^2 < \dots < \tau_{\ell_2}^2$. Let us do the change of variables $\tilde\tau_1^1 = \tau_1^1 - \tau$, $\tilde\tau_2^1 = \tau_2^1 - \tau_1^1$, ..., $\tilde\tau_{\ell_1}^1 = \tau_{\ell_1}^1 - \tau_{\ell_1-1}^1$, and $\tilde\tau_1^2 = \tau - \tau_1^2$, ..., $\tilde\tau_{\ell_2}^2 = \tau_{\ell_2-1}^2 - \tau_{\ell_2}^2$. Then we can write the following upper bound

$$\begin{aligned}\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi\, \mathbb{I}\big[B \ni (x,\tau)\big] |z(\xi)|\, \mathrm{e}^{(c-\alpha_1 \log\|\boldsymbol{V}\|)|A| + \alpha_2|B|} \leqslant \sum_{\ell_1,\ell_2 \geqslant 1} \sum_{\substack{\boldsymbol{A}_1^1,\dots\boldsymbol{A}_{\ell_2}^2 \\ \cup_{i,j}\bar A_j^i = A \ni x \\ A \text{ connected}}} \sum_{n_{A_1^1}^{1,1},\dots,n_{A_{\ell_2}^2}^{2,\ell_2} \notin G_A} \int_0^\infty \mathrm{d}\tilde\tau_1^1 \dots \mathrm{d}\tilde\tau_{\ell_2}^2 \\ \prod_{i=1,2}\prod_{j=1}^{\ell_i} |\langle n_A^{i,j}|\, \boldsymbol{V}_{\boldsymbol{A}_j^i}\, |n_A^{i,j+1}\rangle|\, \mathrm{e}^{(c-\alpha_1 \log\|\boldsymbol{V}\|)|\bar A_j^i|}\, \mathrm{e}^{-\tilde\tau_j^i \sum_{A'\subset A}[\Phi_{A'}(n_{A'}^{i,j}) - \Phi_{A'}(g_{A'})]}\, \mathrm{e}^{\tilde\tau_j^i R^\nu \alpha_2}\end{aligned} \tag{4.20}$$

where $g_A \in G_A$ is the configuration in which the loop ξ is immersed (if the construction does not lead to a possible loop, we find a bound by picking any $g_A \in G_A$). Remark that we neglected a constraint on the sum over configurations, namely $n_A^{1,1} = n_A^{2,1}$. It is useful to note that the sums over ℓ_1, ℓ_2 and over the quantum transitions are finite, otherwise they cannot constitute a loop.

Using the definition (2.5) of the norm of a quantum interaction, we have

$$|\langle n'_A |\, \boldsymbol{V_A}\, | n_A\rangle| \leqslant \|\boldsymbol{V}\|^{|A|}.$$

Furthermore

$$\sum_{A' \subset A} [\Phi_{A'}(n_{A'}^{i,j}) - \Phi_{A'}(g_{A'})] \geqslant R^{-\nu} \Delta_0$$

as claimed in Property (2.3). Hence we have, since the number of configurations on A is bounded with $S^{|A|}$,

$$\int_{\mathcal{L}_\Lambda} \mathrm{d}\xi\, \mathbb{I}\big[B \ni (x,\tau)\big] |z(\xi)|\, \mathrm{e}^{(c - \alpha_1 \log \|\boldsymbol{V}\|)|A| + \alpha_2 |B|} \leqslant \sum_{\ell_1, \ell_2 \geqslant 1} \sum_{\substack{\boldsymbol{A}_1^1, \dots \boldsymbol{A}_{\ell_2}^2 \\ \cup_{i,j} \bar{A}_j^i = A \ni x \\ A \text{ connected}}} \prod_{i=1,2} \prod_{j=1}^{\ell_i} \frac{\big[\|\boldsymbol{V}\|^{1-\alpha_1 (2R_0)^\nu} S \mathrm{e}^{c(2R_0)^\nu}\big]^{|A_j^i|}}{R^{-\nu}\Delta_0 - R^\nu \alpha_2}. \tag{4.21}$$

This is a small quantity since the sums are finite, by taking $\|\boldsymbol{V}\|$ small enough. Now we turn to the second term, namely

$$\int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi |z(\xi)|\, \mathrm{e}^{(c-\alpha_1 \log \|\boldsymbol{V}\|)|A| + \alpha_2 |B|}.$$

The proof is similar; we first sum over the number of transitions ℓ, then over ℓ transitions $\boldsymbol{A}_1, \dots \boldsymbol{A}_\ell$ with $A = \cup_i \bar{A}_i \ni x$, A connected. Then we choose $\ell - 1$ intermediate configurations. Finally, we integrate over $\ell - 1$ time intervals. The resulting equation looks very close to (4.20) and is small for the same reasons.

□

Now, we single out the class of *small clusters*. Namely, a cluster is small if the sequence of its quantum transitions belongs to the list $\mathcal{S}$. To be more precise, we have to specify the order of transitions: considering a cluster $\boldsymbol{C} \equiv (\xi_1, \dots, \xi_k)$ and using $S(\xi^{(\ell)})$, $\ell = 1, \dots, k$, to denote the sequence of quantum transitions of the loop $\xi^{(\ell)} = (B^{(\ell)}, \boldsymbol{\omega}_{B^{(\ell)}}, g_A^{\xi^{(\ell)}})$, $S(\xi^{(\ell)}) \equiv S(B^{(\ell)}, \boldsymbol{\omega}_{B^{(\ell)}})$, we take the sequence $S(\boldsymbol{C})$ obtained by combining the sequences $S(\xi^{(1)}), \dots, S(\xi^{(k)})$ in this order. A cluster $\boldsymbol{C}$ is said to be *small* if $S(\boldsymbol{C}) \in \mathcal{S}$, it is *large* otherwise. We use $\mathcal{C}_\Lambda^{\text{small}}$ to denote the set of all small clusters on the torus $\mathbb{T}_\Lambda$.

The local contribution to the energy at time τ, when the system is in a state $\boldsymbol{n}_A(\tau)$, is $\Phi_A(\boldsymbol{n}_A(\tau))$. Similarly, we will introduce the local contribution of loops (and small clusters of loops) in the expansion of the partition function — the effective potential $\Psi_A^\beta(\boldsymbol{n}_A(\tau))$. The latter is a local quantity in the sense that it depends on $\boldsymbol{n}$ only on the set A at time τ. An explicit expression of $\Psi_A^\beta(g_A)$ with $g \in G$ is, in terms of small clusters,

$$\Psi_A^\beta(g_A) \;:=\; -\int_{\mathcal{C}_\Lambda^{\text{small}}} \mathrm{d}\boldsymbol{C} \frac{\Phi^{\mathrm{T}}(\boldsymbol{C})}{|I_C|}\, \mathbb{I}\big[\boldsymbol{C} \sim g_A, A_C = A, I_C \ni 0\big]. \tag{4.22}$$

Here, again, C is the support of $\boldsymbol{C}$, A_C its horizontal projection onto $\mathbb{Z}^\nu$, $A_C = \{x \in \mathbb{Z}^\nu; x \times [0,\beta]_{\text{per}} \cap C \neq \emptyset\}$, and I_C its vertical projection, $|A_C|$ and $|I_C|$ their corresponding areas, and the condition $\boldsymbol{C} \sim g_A$ means that each loop of $\boldsymbol{C}$ is immersed in the ground state g. Notice that "horizontal extension" of any small cluster is at most R: if $\boldsymbol{C}$ is a small cluster, $\operatorname{diam}(A_C) \leqslant R$. The definitions of Section 2.2 are now clear, once we identify the effective potential Ψ defined in (2.6) as the limit $\beta \to \infty$ of (4.22). Namely,

$$\Psi = \lim_{\beta \to \infty} \Psi^\beta.$$

Our assumptions in Section 2.4 concern the limit $\beta \to \infty$ of the effective potential, but at non zero temperature we have to work with Ψ^β. To trace down the difference, we introduce $\psi^\beta = \Psi^\beta - \Psi$. Notice that (4.22) implies $\Psi_A^\beta(n_A) = 0$ whenever $n_A \notin G_A$ or $\operatorname{diam} A < 2R_0$.

Recalling that if $C \subset \mathbb{T}_\Lambda$, $\tilde{C}$ is the smallest box containing C, we introduce, for any cluster $\boldsymbol{C} \in \mathcal{C}_\Lambda^{\mathrm{small}}$, the function

$$\Phi^{\mathrm{T}}(\boldsymbol{C};\Gamma) = \frac{\Phi^{\mathrm{T}}(\boldsymbol{C})}{|I_C|}\int_0^\beta \mathrm{d}\tau\, \mathbb{I}\left[I_C \ni \tau\right]\Big(\mathbb{I}\left[\boldsymbol{C}\sim\Gamma\right] - \mathbb{I}\left[\boldsymbol{n}^\Gamma_{A_C}(\tau)\in G_{A_C}, \boldsymbol{C}\sim \boldsymbol{n}^\Gamma_{A_C}(\tau)\right]\Big). \quad (4.23)$$

Here, the first indicator function in the parenthesis singles out the clusters whose each loop is compatible with Γ, while the second indicator concerns the clusters for which $\boldsymbol{n}^\Gamma_{A_C}(\tau)\in G_{A_C}$ and each their loop is immersed in the configuration $\boldsymbol{n}^\Gamma_A(\tau)$ (extended as a constant to all the time interval I_C). Observing that $\Phi^{\mathrm{T}}(\boldsymbol{C};\Gamma)=0$ whenever $\tilde{C}\cap \operatorname{core}\Gamma=\emptyset$, we split the integral over small clusters into its bulk part expressed in terms of the effective potential and boundary terms "decorating" the quantum contours from Γ.

Lemma 4.3. *For any fixed $\Gamma\in\mathcal{D}_\Lambda$, one has*

$$\int_{\mathcal{C}_\Lambda^{\mathrm{small}}(\Gamma)} \mathrm{d}\boldsymbol{C}\,\Phi^{\mathrm{T}}(\boldsymbol{C}) = -\int_{\mathbb{T}_\Lambda}\mathrm{d}(A,\tau)\Psi_A(\boldsymbol{n}^\Gamma_A(\tau)) - \int_{\mathbb{T}_\Lambda}\mathrm{d}(A,\tau)\psi^\beta_A(\boldsymbol{n}^\Gamma_A(\tau)) + \int_{\mathcal{C}_\Lambda^{\mathrm{small}}}\mathrm{d}\boldsymbol{C}\,\Phi^{\mathrm{T}}(\boldsymbol{C};\Gamma).$$

The term $\Phi^{\mathrm{T}}(\boldsymbol{C};\Gamma)$ vanishes whenever $\tilde{C}\cap\operatorname{core}\Gamma=\emptyset$.

Proof. To get the equality of integrals, it is enough to rewrite

$$\int_{\mathcal{C}_\Lambda^{\mathrm{small}}(\Gamma)} \mathrm{d}\boldsymbol{C}\,\Phi^{\mathrm{T}}(\boldsymbol{C}) = \int_{\mathcal{C}_\Lambda^{\mathrm{small}}} \mathrm{d}\boldsymbol{C}\,\Phi^{\mathrm{T}}(\boldsymbol{C})\,\mathbb{I}\left[\boldsymbol{C}\sim\Gamma\right] \quad (4.24)$$

and

$$-\int_{\mathbb{T}_\Lambda}\mathrm{d}(A,\tau)\Psi^\beta_A(\boldsymbol{n}^\Gamma_A(\tau)) = \int_{\mathcal{C}_\Lambda^{\mathrm{small}}}\mathrm{d}\boldsymbol{C}\,\Phi^{\mathrm{T}}(\boldsymbol{C})\int_0^\beta \mathrm{d}\tau\,\mathbb{I}\left[\boldsymbol{n}^\Gamma_{A_C}(\tau)\in G_{A_C}, \boldsymbol{C}\sim\boldsymbol{n}^\Gamma_{A_C}(\tau)\right]. \quad (4.25)$$

Moreover, whenever $\tilde{C}\cap\operatorname{core}\Gamma=\emptyset$, the configuration $\boldsymbol{n}^\Gamma_{A_C}(\tau)$ belongs to G_{A_C}, and it is constant, for all $\tau\in I_C$. Under these circumstances, the condition $\boldsymbol{C}\sim\Gamma$ is equivalent to $\boldsymbol{C}\sim\boldsymbol{n}^\Gamma_{A_C}(\tau)$ and the right hand side of (4.23) vanishes. □

Whenever $\Gamma\in\mathcal{D}_\Lambda$ is fixed, let $W_d(\Gamma)\subset\mathbb{T}_\Lambda$ be the set of space-time sites in the state d, i.e.

$$W_d(\Gamma) = \{(x,\tau)\in\mathbb{T}_\Lambda : \boldsymbol{n}^\Gamma_{V(x)}(\tau) = d_{V(x)}\}.$$

Notice that

$$\mathbb{T}_\Lambda = \operatorname{supp}\Gamma \cup \bigcup_{d\in D} W_d(\Gamma); \quad W_d(\Gamma)\cap W_{d'}(\Gamma)=\emptyset \text{ if } d\neq d',$$

and the set $\operatorname{supp}\Gamma\cap W_d(\Gamma)$ is of measure zero (with respect to the measure $\mathrm{d}(x,\tau)$ on $\mathbb{T}_\Lambda$). Let us recall that the equivalent potential Υ satisfies the equality $\sum_{A\subset\Lambda}\Upsilon_A(n_A) = \sum_{A\subset\Lambda}(\Phi_A(n_A)+\Psi_A(n_A))$ for any configuration n on the torus Λ and that we defined $e(d)=\sum_{A\ni 0}\frac{\Upsilon_A(d_A)}{|A|}$ for every $d\in D$.

Lemma 4.4. *The partition function* (4.7) *can be rewritten as*

$$Z^{\mathrm{per}}_\Lambda = \int_{\mathcal{D}^{\mathrm{per}}_\Lambda}\mathrm{d}\Gamma \prod_{d\in D}\mathrm{e}^{-|W_d(\Gamma)|e(d)}\prod_{\gamma\in\Gamma} z(\gamma)\,\mathrm{e}^{\mathcal{R}(\Gamma)}\,.$$

Here the weight $z(\gamma)$ of a quantum contour $\gamma=(B,\boldsymbol{\omega}_B)$ with the sequence of transitions $(\boldsymbol{A}_1,\dots,\boldsymbol{A}_m)$ at times $(\tau_1,\dots,\tau_m)$ is

$$z(\gamma) = \prod_{i=1}^m \langle \boldsymbol{n}^\gamma_{A_i}(\tau_i-0)|\,\boldsymbol{V}_{\boldsymbol{A}_i}\,|\boldsymbol{n}^\gamma_{A_i}(\tau_i+0)\rangle \exp\Big\{-\int_B \mathrm{d}(x,\tau) e^\Upsilon_x(\boldsymbol{n}^\gamma(\tau))\Big\}. \quad (4.26)$$

The rest $\mathcal{R}(\Gamma)$ is given by

$$\mathcal{R}(\Gamma) = \int_{\mathcal{C}_\Lambda(\Gamma)\setminus\mathcal{C}_\Lambda^{\mathrm{small}}(\Gamma)}\mathrm{d}\boldsymbol{C}\,\Phi^{\mathrm{T}}(\boldsymbol{C}) - \int_{\mathbb{T}_\Lambda}\mathrm{d}(A,\tau)\psi^\beta_A(\boldsymbol{n}^\Gamma_A(\tau)) + \int_{\mathcal{C}_\Lambda^{\mathrm{small}}}\mathrm{d}\boldsymbol{C}\,\Phi^{\mathrm{T}}(\boldsymbol{C};\Gamma). \quad (4.27)$$

Proof. Using the Lemmas 4.1 and 4.3 to substitute in (4.7) the contribution of loops by the action of the effective potential, we get

$$Z_\Lambda^{\text{per}} = \int_{\mathcal{D}_\Lambda^{\text{per}}} \mathrm{d}\Gamma \Big\{ \prod_{i=1}^m \langle n_{A_i}^\Gamma(\tau_i - 0)| \boldsymbol{V}_{\boldsymbol{A}_i} |n_{A_i}^\Gamma(\tau_i + 0)\rangle \Big\} \exp\Big\{ -\int_{\mathbb{T}_\Lambda} \mathrm{d}(A,\tau)(\Phi_A(\boldsymbol{n}_A^\Gamma(\tau)) + \Psi_A(\boldsymbol{n}_A^\Gamma(\tau))) \Big\} \,\mathrm{e}^{\mathcal{R}(\Gamma)} . \tag{4.28}$$

Replacing $\Phi + \Psi$ by the physically equivalent potential Υ, we get

$$Z_\Lambda^{\text{per}} = \int_{\mathcal{D}_\Lambda^{\text{per}}} \mathrm{d}\Gamma \Big\{ \prod_{i=1}^m \langle n_{A_i}^\Gamma(\tau_i - 0)| \boldsymbol{V}_{\boldsymbol{A}_i} |n_{A_i}^\Gamma(\tau_i + 0)\rangle \Big\} \exp\Big\{ -\int_{\operatorname{supp}\Gamma} \mathrm{d}(x,\tau) e_x^\Upsilon(\boldsymbol{n}^\gamma(\tau)) \Big\} \prod_{d\in D} \mathrm{e}^{-e(d)|W_d(\Gamma)|} \,\mathrm{e}^{\mathcal{R}(\Gamma)} . \tag{4.29}$$

We get our lemma by observing that the product over quantum transitions and the first exponential factorize with respect to the quantum contours, as it was the case for the loops (for fermions the sign is arising because of anticommutation relations also factorize; we again refer to [DFF1] for the proof).

□

Our goal is to obtain a classical lattice system in $\nu + 1$ dimensions. Thus we introduce a discretization of the continuous time direction, by choosing suitable parameters $\tilde\beta > 0$ and $N \in \mathbb{N}$ with $\beta = N\frac{\tilde\beta}{\Delta}$.[11] Setting $\mathbb{L}_\Lambda$ to be the $(\nu+1)$-dimensional discrete torus $\mathbb{L}_\Lambda = \Lambda \times \{0, 1, \dots, N-1\}^{\text{per}}$ — let us recall that Λ has periodic boundary conditions in all spatial directions — and using $C(x,t) \subset \mathbb{R}^{\nu+1}$ to denote, for any $(x,t) \in \mathbb{L}_\Lambda$, the cell centered in $(x, \frac{\tilde\beta}{\Delta} t)$ with vertical length $\tilde\beta/\Delta$, we have $\mathbb{T}_\Lambda = \cup_{(x,t)\in\mathbb{L}_\Lambda} C(x,t)$.

For any $M \subset \mathbb{L}_\Lambda$, we set $C(M)$ to be the union of all cells centered at sites of M, $C(M) = \cup_{(x,t)\in M} C(x,t) \subset \mathbb{T}_\Lambda$. Conversely, if $B \subset \mathbb{T}_\Lambda$, we take $M(B) \subset \mathbb{L}_\Lambda$ to be the smallest set such that $C(M(B)) \supset B$. Given a connected[12] set $M \subset \mathbb{L}_\Lambda$ and a collection of quantum contours $\Gamma \in \mathcal{D}_\Lambda^{\text{per}}$, we define

$$\begin{aligned} \varphi(M;\Gamma) = &\int_{\mathcal{C}_\Lambda(\Gamma)\setminus\mathcal{C}_\Lambda^{\text{small}}(\Gamma)} \mathrm{d}\boldsymbol{C}\, \mathbb{I}\left[M(C) = M\right] \Phi^{\mathrm{T}}(\boldsymbol{C}) + \\ &+ \int_{\mathcal{C}_\Lambda^{\text{small}}} \mathrm{d}\boldsymbol{C}\, \mathbb{I}\left[M(C) = M, C \not\subset C(\operatorname{supp}\Gamma)\right] \Phi^{\mathrm{T}}(\boldsymbol{C};\Gamma) - \int_{M(A\times\tau)=M} \mathrm{d}(A,\tau) \psi_A^\beta(\boldsymbol{n}_A^\Gamma(\tau)) \end{aligned} \tag{4.30}$$

and

$$\tilde{\mathcal{R}}(\Gamma) = \int_{\mathcal{C}_\Lambda^{\text{small}}} \mathrm{d}\boldsymbol{C}\, \mathbb{I}\left[C \subset C(\operatorname{supp}\Gamma)\right] \Phi^{\mathrm{T}}(\boldsymbol{C};\Gamma). \tag{4.31}$$

We have separated the contributions of the small clusters inside $C(\operatorname{supp}\Gamma)$, because they are not necessarily a small quantity, and it is impossible to expand them. On the contrary, $\varphi(M;\Gamma)$ is small, and hence it is natural to write

$$\mathrm{e}^{\mathcal{R}(\Gamma)} = \mathrm{e}^{\tilde{\mathcal{R}}(\Gamma)} \sum_{\mathcal{M}} \prod_{M\in\mathcal{M}} \left(\mathrm{e}^{\varphi(M;\Gamma)} - 1 \right), \tag{4.32}$$

with the sum running over all collections $\mathcal{M}$ of connected subsets of $\mathbb{L}_\Lambda$.

Let $\operatorname{supp}\mathcal{M} = \cup_{M\in\mathcal{M}} M$. Given a set of quantum contours $\Gamma \in \mathcal{D}_\Lambda^{\text{per}}$ and a collection $\mathcal{M}$, we introduce contours on $\mathbb{L}_\Lambda$ by decomposing the set $M(\operatorname{supp}\Gamma) \cup \operatorname{supp}\mathcal{M}$ into connected components [notice that if $(x,t) \notin M(\operatorname{supp}\Gamma) \cup \operatorname{supp}\mathcal{M}$, then $C(x,t) \subset \cup_{d\in D} W_d(\Gamma)$]. Namely, a *contour* Y is a pair $(\operatorname{supp} Y, \alpha_Y)$ where $\operatorname{supp} Y \subset \mathbb{L}_\Lambda$ is a (non-empty) connected subset of $\mathbb{L}_\Lambda$, and α_Y is a labeling of connected components F of $\partial C(\operatorname{supp} Y)$, $\alpha_Y(F) = 1, \dots, r$. We write $|Y|$ for the length (area) of the contour Y, i.e. the number of sites in $\operatorname{supp} Y$. A set of contours $\mathcal{Y} = \{Y_1, \dots, Y_k\}$ is *admissible* if the contours are mutually disjoint and if the

[11] Remark the difference from [BKU]; here the vertical length of a unit cell $\tilde\beta/\Delta$ depends on $\|\boldsymbol{V}\|$, since so does the quantum Peierls constant Δ.

[12] Connectedness in $\mathbb{L}_\Lambda$ is meant in standard way via nearest neighbours.

labeling is constant on the boundary of each connected component of $\mathbb{T}_\Lambda \setminus \cup_{Y\in\mathcal{Y}} C(\operatorname{supp} Y)$. Finally, given an admissible set of contours $\mathcal{Y}$, we define $\mathcal{W}_d(\mathcal{Y})$ to be the union of all connected components M of $\mathbb{L}_\Lambda \setminus \cup_{Y\in\mathcal{Y}} \operatorname{supp} Y$ such that $C(M)$ has label d on its boundary.

Consider now any configuration $\boldsymbol{\omega} \in \mathcal{Q}_\Lambda^{\rm per}$ yielding, together with a collection $\mathcal{M}$, a fixed set of contours $\mathcal{Y}$. Summing over all such configurations $\boldsymbol{\omega}$ and collections $\mathcal{M}$, we get the weight to be attributed to the set $\mathcal{Y}$. Let $\Gamma^{\boldsymbol{\omega}}$ be the collection of quantum contours corresponding to $\boldsymbol{\omega}$, $\cup_{Y\in\mathcal{Y}} \operatorname{supp} Y = M(\operatorname{supp}\Gamma^{\boldsymbol{\omega}}) \cup \operatorname{supp}\mathcal{M}$. Given that the configurations $\boldsymbol{\omega}$ are necessarily constant with no transition on $\mathbb{T}_\Lambda \setminus C(M(\operatorname{supp}\Gamma^{\boldsymbol{\omega}}) \cup \operatorname{supp}\mathcal{M})$, we easily see that the weight factor splits into product of weight factors of single contours $Y \in \mathcal{Y}$. Namely, for the weight $\mathfrak{z}$ of a contour Y we get the expression

$$\mathfrak{z}(Y) = \int_{\mathcal{D}_\Lambda^{\rm per}(Y)} \mathrm{d}\Gamma \prod_{\gamma\in\Gamma} z(\gamma) \prod_{d\in D} \mathrm{e}^{-e(d)|W_d(\Gamma)\cap C(\operatorname{supp} Y)|}\, \mathrm{e}^{\tilde{\mathcal{R}}(\Gamma)}$$
$$\sum_{\mathcal{M}} \mathbb{I}\bigl[M(\operatorname{supp}\Gamma)\cup\operatorname{supp}\mathcal{M} = \operatorname{supp} Y\bigr] \prod_{M\in\mathcal{M}} \Bigl(\mathrm{e}^{\varphi(M;\Gamma)} - 1\Bigr), \qquad (4.33)$$

where $\mathcal{D}_\Lambda^{\rm per}(Y)$ is the set of collections Γ of quantum contours compatible with Y, $\Gamma \in \mathcal{D}_\Lambda^{\rm per}(Y)$ if $\operatorname{supp}\Gamma \subset \operatorname{supp} Y$ and the labels on the boundary of $\operatorname{supp}\Gamma$ match with labels of Y. Thus, we can finally rewrite the partition function in a form that agrees with the standard Pirogov-Sinai setting, namely

$$Z_\Lambda^{\rm per} = \sum_{\mathcal{Y}} \prod_{d\in D} \mathrm{e}^{-\frac{\tilde\beta}{\Delta} e(d)|\mathcal{W}_d(\mathcal{Y})|} \prod_{Y\in\mathcal{Y}} \mathfrak{z}(Y), \qquad (4.34)$$

with the sum being over all admissible sets of contours on $\mathbb{L}_\Lambda$.

In the next section we will evaluate the decay rate of contours weights in a preparation to apply, in Section 6, the Pirogov-Sinai theory to prove Theorems 2.1, 2.2, and 2.3.

5. Exponential decay of the weight of the contours

In this section we show that the weight $\mathfrak{z}$ has exponential decay with respect to the length of the contours. We begin by a lemma proving that the contribution of $\mathcal{M}$ is small, that we shall use in Lemma 5.2 below for the bound of $\mathfrak{z}$.

Lemma 5.1. *Under the Assumptions 1–6, for any $c < \infty$ there exist constants β_0, $\tilde\beta_0 < \infty$, and $\varepsilon_0 > 0$ such that for any $\beta \geqslant \beta_0$, $\tilde\beta_0 \leqslant \tilde\beta < 2\tilde\beta_0$, and $\|\boldsymbol{V}\| \leqslant \varepsilon_0$, one has*

$$\sum_{M\ni(x,t)} \bigl|\mathrm{e}^{\varphi(M;\Gamma)} - 1\bigr|\, \mathrm{e}^{c|M|} \leqslant 1$$

for any contour Y and any set of quantum contours $\Gamma \in \mathcal{D}_\Lambda^{\rm per}(Y)$.

Proof. We show that

$$\sum_{M\ni(x,t)} \bigl|\varphi(M;\Gamma)\bigr|\, \mathrm{e}^{c|M|} \leqslant 1.$$

This implies that $|\varphi(M;\Gamma)| \leqslant 1$ and consequently Lemma 5.1 holds — with a slightly smaller constant c.

Let us consider separately, in (4.30), the three terms on the right hand side: *(a)* the integral over big clusters, *(b)* the integral over small clusters, and *(c)* the expression involving ψ^β.

(a) Big clusters. Our aim is to estimate

$$J = \sum_{M\ni(x,t)} \mathrm{e}^{c|M|} \int_{\mathcal{C}_\Lambda(\Gamma)\setminus\mathcal{C}_\Lambda^{\rm small}(\Gamma)} \mathrm{d}\boldsymbol{C}\, \mathbb{I}\bigl[M(C) = M\bigr]\bigl|\Phi^{\rm T}(\boldsymbol{C})\bigr|.$$

Since $M(C) = M$ and $M \ni (x,t)$, the cell $C(x,t)$ either intersects a quantum transition of $\boldsymbol{C}$, or it is contained in a box B belonging to a loop of $\boldsymbol{C}$ (both possibilities may occur at the same time). In the first case we start the integral over clusters by choosing the time for the first quantum transition, which yields a factor $\tilde\beta/\Delta$. In the second case we simply integrate over all loops containing the given site. In the same time, given a cluster $\boldsymbol{C}=(\xi_1,\dots,\xi_n)$, $\xi_i=(B_i,\boldsymbol{\omega}^{(i)}_{B_i},g^{\xi_i}_{A_i})$ and $B_i=A_i\times[\tau_1^{(i)},\tau_2^{(i)}]$, the condition $M(C)=M$ implies that

$$\sum_{i=1}^{n}\Bigl\{|A_i|+\frac{\Delta}{\tilde\beta}|B_i|\Bigr\}\geqslant|M|. \tag{5.1}$$

Using it to bound $|M|$, we get the estimate

$$J\leqslant\frac{\tilde\beta}{\Delta}\int_{\mathcal{C}_\Lambda^{(x,\tau)}\setminus\mathcal{C}_\Lambda^{\rm small}}\mathrm{d}\boldsymbol{C}|\Phi^{\rm T}(\boldsymbol{C})|\prod_{\xi\in\boldsymbol{C}}\mathrm{e}^{c|A|+c\frac{\Delta}{\tilde\beta}|B|}+\int_{\mathcal{C}_\Lambda\setminus\mathcal{C}_\Lambda^{\rm small}}\mathrm{d}\boldsymbol{C}\,\mathbb{I}\bigl[C\ni(x,\tau)\bigr]|\Phi^{\rm T}(\boldsymbol{C})|\prod_{\xi\in\boldsymbol{C}}\mathrm{e}^{c|A|+c\frac{\Delta}{\tilde\beta}|B|}. \tag{5.2}$$

Taking, in Lemma 4.1, the constant c as above as well as $\alpha_1=\frac12(2R_0)^{-\nu}$, $\alpha_2=c\Delta/\tilde\beta$, $\delta=1$, and choosing the corresponding $\varepsilon_0(c,\alpha_1,\alpha_2,\delta)$, we can bound the second term of (5.2), for any $\|\boldsymbol{V}\|\leqslant\varepsilon_0$, with the help of (4.12) once $\tilde\beta$ is chosen large enough to satisfy

$$\frac{\tilde\beta}{\Delta}>\frac{c}{\Delta_0}R^{2\nu}. \tag{5.3}$$

To estimate the first term of (5.2), we first consider the contribution of those clusters for which

$$\frac{\tilde\beta}{\Delta}\leqslant\prod_{\xi\in\boldsymbol{C}}\|\boldsymbol{V}\|^{\frac12(2R_0)^{-\nu}|A|}.$$

Applying it together with (5.3) we can directly use the bound (4.13).

Thus it remains to estimate the contribution of those terms for which

$$\frac{2}{(2R_0)^{\nu}}\sum_{\xi\in\boldsymbol{C}}|A|<\frac{\log(\Delta/\tilde\beta)}{\log\|\boldsymbol{V}\|}. \tag{5.4}$$

Let us first fix $\tilde\beta$ and $\varepsilon_0\leqslant\varepsilon_0(c,\alpha_1,\alpha_2,\delta)$ with the constants c, α_1, α_2, and δ as above, so that

$$\frac{\tilde\beta}{\varepsilon_0}>\frac{c}{\Delta_0}R^{2\nu} \tag{5.5}$$

and, in the same time,

$$\tilde\beta\leqslant\varepsilon_0^{k-2k'(2R_0)^{-\nu}} \tag{5.6}$$

for a suitable large k' (we also assume that $\varepsilon_0\leqslant1$). Here k is the constant that appears in Assumption 4, $\Delta(\|\boldsymbol{V}\|)\geqslant\|\boldsymbol{V}\|^k$. Observing further that $\Delta(\|\boldsymbol{V}\|)$ can be taken to increase with $\|\boldsymbol{V}\|$ (one can always consider a weaker lower bound Δ when taking smaller $\|\boldsymbol{V}\|$), we conclude that (5.3), as well as the condition

$$\tfrac12(2R_0)^{\nu}\frac{\log(\Delta/\tilde\beta)}{\log\|\boldsymbol{V}\|}\leqslant k',$$

are satisfied for every $\|\boldsymbol{V}\|\leqslant\varepsilon_0$. Thus, it suffices to find an upper bound to

$$J'=\frac{\tilde\beta}{\Delta}\int_{\mathcal{C}_\Lambda^{(x,\tau)}\setminus\mathcal{C}_\Lambda^{\rm small}}\mathrm{d}\boldsymbol{C}|\Phi^{\rm T}(\boldsymbol{C})|\,\mathbb{I}\bigl[\sum_{\xi\in\boldsymbol{C}}|A|<k'\bigr]. \tag{5.7}$$

The main problem in estimating this term stems from the factor $1/\Delta$ that may be large if $\|\boldsymbol{V}\|$ is small. Thus, to have a bound valid for all small $\|\boldsymbol{V}\|$, some terms, coming from the integral, that would suppress this factor must be displayed.

The condition $\sum_{\xi\in\boldsymbol{C}}|A|<k'$ will be used several times by applying its obvious consequences: (i) the number of loops in $\boldsymbol{C}$ is smaller than k', (ii) the number of transitions for each loop is smaller than k', (iii) each transition $\boldsymbol{A}$ is such that $|A|<k'$, and (iv) the distance between each transition and x is smaller than k'.

Furthermore, we use Assumption 5 to bound the contribution of the transitions of $\boldsymbol{C}$; recalling the definition (4.9) of the weight of ξ, we have, for any large $\boldsymbol{C}$,

$$\begin{aligned}\prod_{\xi\in\boldsymbol{C}}|z(\xi)|&\leqslant b_1(\|\boldsymbol{V}\|)\Delta\prod_{\xi\in\boldsymbol{C}}\exp\Bigl\{-\int_B\mathrm{d}(A,\tau)[\Phi_A(n_A^\xi(\tau))-\Phi_A(g_A^\xi(\tau))]\Bigr\}\\&\leqslant b_1(\|\boldsymbol{V}\|)\Delta\prod_{\xi\in\boldsymbol{C}}\mathrm{e}^{-R^{-2\nu}\Delta_0|B|}.\end{aligned} \tag{5.8}$$

In the last inequality we used Assumption 2 in the form of the bound (2.3) as well as the lower bound $|\tau_2-\tau_1|=\frac{|B|}{|A|}\geqslant\frac{|B|}{R^\nu}$ for the support $B=A\times[\tau_1,\tau_2]$ of the loop ξ.

For any $\xi \in \boldsymbol{C} = (\xi_1, \dots \xi_n)$, let τ be the time at which the first transition in C occurs (we assume that it happens for the "first" loop ξ_1) and τ^ξ be such that $\tau + \tau^\xi$ is the time at which the first transition in ξ occurs ($\tau^{\xi_1} = 0$). Referring to the condition (i) on the number of loops in $\boldsymbol{C}$, we get the inequality

$$\sum_{\xi \neq \xi_1} |\tau^\xi| \leqslant k' \sum_{\xi} |B|,$$

and thus also

$$1 \leqslant \prod_{\xi} \mathrm{e}^{-\frac{\Delta_0}{2k' R^{2\nu}} |\tau^\xi|} \prod_{\xi} \mathrm{e}^{\frac{1}{2} R^{-2\nu} \Delta_0 |B|}.$$

Integrating now over the time of the first transition for each $\xi \in \boldsymbol{C}$, $\xi \neq \xi_1$, and taking into account that $|\varphi^{\mathrm{T}}(\xi_1, \dots, \xi_n)| \leqslant n^{n-2}$, we get

$$J' \leqslant \tilde{\beta} b_1(\|\boldsymbol{V}\|) \sum_{n=1}^{k'} \frac{n^{n-2}}{(n-1)!} \Big(\frac{2k' R^{2\nu}}{\Delta_0}\Big)^{n-1} \Big\{ \int_{\mathcal{L}_\Lambda^{(x,\tau)}} \mathrm{d}\xi \, \mathrm{e}^{-\frac{1}{2} R^{-2\nu} \Delta_0 |B|} \, \mathbb{I}\left[\xi : k'\right] \Big\}^n. \tag{5.9}$$

Here the constraint $\mathbb{I}\left[\xi_i : k'\right]$ means that the loop ξ_i satisfies the conditions (ii)–(iv) above. We have then a finite number of finite terms, the contribution of which is bounded by a fixed number $K < \infty$ (depending on ε_0, $\tilde{\beta}$, and k'). Thus $J' \leqslant \tilde{\beta} b_1(\|\boldsymbol{V}\|) K$ which can be made small by taking $\|\boldsymbol{V}\|$ sufficiently small.

(b) Small clusters. Let us first notice that $|\Phi^{\mathrm{T}}(\boldsymbol{C}; \Gamma)| \leqslant |\Phi^{\mathrm{T}}(\boldsymbol{C})|$, and since $M(C) = M$, inequality (5.1) is valid. Moreover C must contain at least one of the two boundary points $(y, t\frac{\tilde{\beta}}{\Delta} \pm \frac{\tilde{\beta}}{2\Delta})$ of some cell $C(y,t)$ for which $\operatorname{dist}(x,y) \leqslant R$. Indeed, given that $\boldsymbol{C}$ is small and in the same time $\tilde{C} \cap \operatorname{core} \Gamma \neq \emptyset$ (c.f. Lemma 4.3), this is the only way to satisfy also $C \not\subset C(\operatorname{supp} \Gamma)$ [c.f. (4.30)]. Thus it suffices to use again (4.12) and (5.3) to estimate

$$(2R)^\nu \int_{\mathcal{C}_\Lambda^{\text{small}}} \mathrm{d}\boldsymbol{C} \, \mathbb{I}\left[C \ni (x, \tau)\right] |\Phi^{\mathrm{T}}(\boldsymbol{C})| \prod_{\xi \in \boldsymbol{C}} \mathrm{e}^{c|A| + c\frac{\Delta}{\beta}|B|}.$$

(c) Bound for ψ^β. Finally, we estimate the expression involving ψ^β. We first observe that

$$\mathrm{e}^{\alpha\beta} \, |\psi_A^\beta(g_A)| \leqslant 1 \tag{5.10}$$

for any $A \subset \mathbb{Z}^\nu$ and with $\alpha = \frac{1}{2} R^{-2\nu} \Delta_0$, Indeed,

$$\begin{aligned} \mathrm{e}^{\alpha\beta} \, |\psi_A^\beta(g_A)| &= \mathrm{e}^{\alpha\beta} \, |\Psi_A^\beta(g_A) - \Psi_A(g_A)| = \\ &= \mathrm{e}^{\alpha\beta} \Big| - \int_{\mathcal{C}_\Lambda^{\text{small}}} \mathrm{d}\boldsymbol{C} \, \mathbb{I}\left[\boldsymbol{C} \sim g_A, A_C = A, I_C \ni 0, C \subset \Lambda \times [0,\beta]_{\text{per}}, |I_C| = \beta\right] \frac{\Phi^{\mathrm{T}}(\boldsymbol{C})}{|I_C|} + \\ &\qquad + \int_{\mathcal{C}_\Lambda^{\text{small}}} \mathrm{d}\boldsymbol{C} \, \mathbb{I}\left[\boldsymbol{C} \sim g_A, A_C = A, I_C \ni 0, C \subset \Lambda \times [-\infty, \infty], |I_C| \geqslant \beta\right] \frac{\Phi^{\mathrm{T}}(\boldsymbol{C})}{|I_C|} \Big|. \end{aligned} \tag{5.11}$$

The first integral above corresponds to clusters wrapped around the torus in vertical direction, while the second one assumes integration over all clusters in $\Lambda \times [-\infty, \infty]$. For any $\boldsymbol{C}$ above it is $|I_C| \geqslant \beta$ and thus

$$\mathrm{e}^{\alpha\beta} \leqslant \prod_{\xi \in \boldsymbol{C}} \mathrm{e}^{\alpha|B|}.$$

Observing now that every cluster in both integrals necessarily contains in its support at least one of the points $(x, 0)$, $x \in A$, and using the fact that $\operatorname{diam} A \leqslant R$, we can bound the first integral by

$$\frac{R^\nu}{\beta} \int_{\mathcal{C}_\Lambda^{\text{small}}} \mathrm{d}\boldsymbol{C} \, \mathbb{I}\left[C \ni (x, 0)\right] |\Phi^{\mathrm{T}}(\boldsymbol{C})| \prod_{\xi \in \boldsymbol{C}} \mathrm{e}^{\alpha|B|},$$

which can be directly evaluated by (4.12). The same bound can be actually used also for the second integral, once we realize that the estimate (4.12) is uniform in β.

Using now the fact that $\psi_A^\beta = 0$ if $\operatorname{diam} A \geqslant R$, the condition $M(A \times \{\tau\}) = M$ implies that M has less than R^ν sites, hence $\mathrm{e}^{c|M|} \leqslant \mathrm{e}^{cR^\nu}$. Furthermore, referring to (5.10), we have

$$\int_{\mathbb{T}_\Lambda} \mathrm{d}(A, \tau) |\psi_A^\beta(\cdot)| \, \mathbb{I}\left[M(A \times \{\tau\}) = M\right] \mathrm{e}^{c|M|} \leqslant \frac{\tilde{\beta}}{\Delta} \mathrm{e}^{-\frac{1}{2} R^{-2\nu} \Delta_0 \beta + cR^\nu}, \tag{5.12}$$

which can be made small for β sufficiently large and concludes thus the proof of the lemma.

□

Using Lemma 5.1 and introducing $e_0 = \min_{d \in D} e(d)$, we can estimate the weight $\mathfrak{z}$ of the contours in the discrete space of cells.

Lemma 5.2. *Under the Assumptions 1–6, for any $c<\infty$, there exist $\beta_0,\tilde\beta_0<\infty$ and $\varepsilon_0>0$ such that for any $\beta\geqslant\beta_0$, $\tilde\beta_0\leqslant\tilde\beta<2\tilde\beta_0$, and $\|\boldsymbol{V}\|\leqslant\varepsilon_0$, one has*

$$|\mathfrak{z}(Y)|\leqslant \mathrm{e}^{-\frac{\tilde\beta}{\Delta}e_0|Y|}\,\mathrm{e}^{-c|Y|}$$

for any contour Y.

Proof. For a given Γ (such that $M(\operatorname{supp}\Gamma)\subset\operatorname{supp}Y$) with transitions $\{\boldsymbol{A}_1,\dots,\boldsymbol{A}_m\}$ at times $\{\tau_1,\dots,\tau_m\}$, we define $A(\Gamma)=\cup_{i=1}^m\cup_{x\in A_i}[V(x)\times\tau_i]$, $\mathcal{A}=M(A(\Gamma))$, and $\mathcal{E}\subset\operatorname{supp}Y\setminus\mathcal{A}$ to be the set of sites (x,t) such that $\boldsymbol{n}^\Gamma_{V(x)}(\tau)\notin D_{V(x)}$ for some $(x,\tau)\in C(x,t)$. The latter can be split into two disjoint subsets, $\mathcal{E}=\mathcal{E}^{\rm core}\cup\mathcal{E}^{\rm soft}$, with $(x,t)\in\mathcal{E}^{\rm core}$ whenever $\boldsymbol{n}^\Gamma_{V(x)}(\tau)\notin G_{V(x)}$ for some $(x,\tau)\in C(x,t)$. The condition $M(\operatorname{supp}\Gamma)\cup\operatorname{supp}\mathcal{M}=\operatorname{supp}Y$ in (4.33) implies the inequality

$$\mathrm{e}^{c|Y|}\leqslant \mathrm{e}^{c(2R)^\nu|A(\Gamma)|}\,\mathrm{e}^{c|\mathcal{E}|}\prod_{M\in\mathcal{M}}\mathrm{e}^{c|M|}.$$

¿From definitions (4.33) of $\mathfrak{z}(Y)$ and (4.26) of $z(\gamma)$, and using Assumption 4, we have

$$\mathrm{e}^{c|Y|}|\mathfrak{z}(Y)|\leqslant\sum_{\mathcal{A}\subset\operatorname{supp}Y}\mathrm{e}^{-\frac{\tilde\beta}{\Delta}e_0|\operatorname{supp}Y\setminus\mathcal{A}|}\sum_{\mathcal{E}\subset\operatorname{supp}Y\setminus\mathcal{A}}\ \sum_{\mathcal{E}^{\rm core}\subset\mathcal{E}}\mathrm{e}^{-(\tilde\beta-c)|\mathcal{E}\setminus\mathcal{E}^{\rm core}|}\,\mathrm{e}^{-(\frac{\tilde\beta}{\Delta}\frac{\Delta_0}{2}(2R)^{-\nu}-c)|\mathcal{E}^{\rm core}|}\times$$
$$\times\int_{\mathcal{D}_\Lambda^{\rm per}}\mathrm{d}\Gamma\,\mathbb{I}\big[M(A(\Gamma))=\mathcal{A},M(\operatorname{core}\Gamma)=\mathcal{E}^{\rm core}\big]\prod_{i=1}^m\big|\langle\boldsymbol{n}^\Gamma_{A_i}(\tau_i-0)|\,\boldsymbol{V}_{\boldsymbol{A}_i}\,|\boldsymbol{n}^\Gamma_{A_i}(\tau_i+0)\rangle\big|\,\mathrm{e}^{c(2R)^\nu|A_i|}\times$$
$$\times\exp\Big\{-\int_{C(\mathcal{A})}\mathrm{d}(x,\tau)e^\Upsilon_x(\boldsymbol{n}^\Gamma(\tau))\Big\}\,\mathrm{e}^{|\tilde{\mathcal{R}}(\Gamma)|}\sum_{\mathcal{M},\operatorname{supp}\mathcal{M}\subset\operatorname{supp}Y}\ \prod_{M\in\mathcal{M}}\big|\mathrm{e}^{\varphi(M;\Gamma)}-1\big|\,\mathrm{e}^{c|M|}.\tag{5.13}$$

All elements in $\mathcal{M}$ are different, because it is so in the expansion (4.32). Therefore we have

$$\sum_{\mathcal{M},\operatorname{supp}\mathcal{M}\subset\operatorname{supp}Y}\ \prod_{M\in\mathcal{M}}\big|\mathrm{e}^{\varphi(M;\Gamma)}-1\big|\,\mathrm{e}^{c|M|}\leqslant\sum_{n\geqslant0}\frac1{n!}\Big[\sum_{M\subset\operatorname{supp}Y}\big|\mathrm{e}^{\varphi(M;\Gamma)}-1\big|\,\mathrm{e}^{c|M|}\Big]^n$$
$$\leqslant\sum_{n\geqslant0}\frac1{n!}\Big[|Y|\sum_{M\ni(x,t)}\big|\mathrm{e}^{\varphi(M;\Gamma)}-1\big|\,\mathrm{e}^{c|M|}\Big]^n\tag{5.14}$$

and using Lemma 5.1 this may be bounded by $\mathrm{e}^{|Y|}$.

In (4.31) clusters are small, and they must contain a space-time site (x,τ) such that there exists x' with $(x',\tau)\in\operatorname{core}\Gamma$ and $\operatorname{dist}(x,x')<R$. So we have the bound

$$|\tilde{\mathcal{R}}(\Gamma)|\leqslant(2R)^\nu|\operatorname{core}\Gamma|\int_{\mathcal{C}_\Lambda^{\rm small}}\mathrm{d}\boldsymbol{C}\,\mathbb{I}\big[C\ni(x,\tau)\big]\big|\Phi^{\rm T}(\boldsymbol{C})\big|,$$

since $|\Phi^{\rm T}(\boldsymbol{C};\Gamma)|\leqslant|\Phi^{\rm T}(\boldsymbol{C})|$. Taking now, in Lemma 4.1, the constants $c=\alpha_1=\alpha_2=0$ and $\delta=\frac{\Delta_0}{4(2R)^{2\nu}}$, and choosing the corresponding ε_0, we apply (4.12) to get, for any $\|\boldsymbol{V}\|\leqslant\varepsilon_0$, the bound

$$|\tilde{\mathcal{R}}(\Gamma)|\leqslant\frac{\Delta_0}{4}(2R)^{-\nu}|\operatorname{core}\Gamma|\leqslant\frac{\tilde\beta}{\Delta}\frac{\Delta_0}{4}(2R)^{-\nu}|\mathcal{E}^{\rm core}|.$$

Assuming $\tilde\beta\geqslant c$ and $\frac{\tilde\beta}{\Delta}\frac{\Delta_0}{4}\geqslant(2R)^\nu c$ [c.f. (5.3)], we bound $\mathrm{e}^{-(\tilde\beta-c)|\mathcal{E}\setminus\mathcal{E}^{\rm core}|}\,\mathrm{e}^{-(\frac{\tilde\beta}{\Delta}\frac{\Delta_0}{4}(2R)^{-\nu}-c)|\mathcal{E}^{\rm core}|}\leqslant1$. Inserting these estimates into (5.13), we get

$$\mathrm{e}^{c|Y|}|\mathfrak{z}(Y)|\leqslant\mathrm{e}^{-\frac{\tilde\beta}{\Delta}e_0|Y|}\,\mathrm{e}^{|Y|}\sum_{\mathcal{A}\subset\operatorname{supp}Y}3^{|\operatorname{supp}Y\setminus\mathcal{A}|}\int_{\mathcal{D}_\Lambda^{\rm per}}\mathrm{d}\Gamma\,\mathbb{I}\big[M(A(\Gamma))=\mathcal{A}\big]\prod_{i=1}^m\big|\langle\boldsymbol{n}^\Gamma_{A_i}(\tau_i-0)|\,\boldsymbol{V}_{\boldsymbol{A}_i}\,|\boldsymbol{n}^\Gamma_{A_i}(\tau_i+0)\rangle\big|\times$$
$$\times\,\mathrm{e}^{c(2R)^\nu|A_i|}\exp\Big\{-\int_{C(\mathcal{A})}\mathrm{d}(x,\tau)\big[e^\Upsilon_x(\boldsymbol{n}^\Gamma(\tau))-e_0\big]\Big\}.\tag{5.15}$$

To estimate the above expression, we will split the "transition part" of the considered quantum contours into connected components, to be called *fragments*, and deal with them separately. Even though the weight of a quantum contour cannot be partitioned into the corresponding fragments, we will get an upper bound combined from fragment bounds. Consider thus the set

$$\hat A(\Gamma)=\operatorname{core}\Gamma\cap C(A(\Gamma))$$

and the fragments $\zeta_i=(B_i,\boldsymbol{\omega}_{B_i})$ on the components B_i of $\hat A(\Gamma)$, $\hat A(\Gamma)=\cup_{i=1}^nB_i$, $\boldsymbol{\omega}_{B_i}$ is the restriction of $\boldsymbol{\omega}^\Gamma$ onto B_i.

¿From Assumption 4, we have

$$\int_{C(\mathcal{A})} \mathrm{d}(x,\tau)\Big[e_x^{\Upsilon}(\boldsymbol{n}^{\Gamma}(\tau)) - e_0\Big] \geqslant \tfrac{1}{2}(2R)^{-\nu}\Delta_0 \sum_{i=1}^{n} |B_i|.$$

Let us introduce a bound for the contribution of a fragment ζ with transitions $\boldsymbol{A}_j, j = 1,\dots,k$,

$$\hat{z}(\zeta) = \mathrm{e}^{-\frac{1}{2}(2R)^{-\nu}\Delta_0 |B|} \prod_{j=1}^{k} |\langle \boldsymbol{n}^{\zeta}_{A_j}(\tau_1 - 0)|\, \boldsymbol{V}_{\boldsymbol{A}_j}\, |\boldsymbol{n}^{\zeta}_{A_j}(\tau_1 + 0)\rangle|\, \mathrm{e}^{c(2R)^{\nu}|A_j|}.$$

Then, integrating over the set $\mathcal{F}_{C(\mathcal{A})}$ of all fragments in $C(\mathcal{A})$, we get

$$\mathrm{e}^{c|Y|}\, |\mathfrak{z}(Y)| \leqslant \mathrm{e}^{-\frac{\tilde{\beta}}{\Delta}e_0|Y|}\, \mathrm{e}^{|Y|} \sum_{\mathcal{A}\subset \operatorname{supp} Y} 3^{|\operatorname{supp} Y\setminus \mathcal{A}|} \sum_{n\geqslant 0} \frac{1}{n!}\Big(\int_{\mathcal{F}_{C(\mathcal{A})}} \mathrm{d}\zeta\, \hat{z}(\zeta)\Big)^{n}. \tag{5.16}$$

Anticipating the bound $\int_{\mathcal{F}_{C(\mathcal{A})}} \mathrm{d}\zeta\, \hat{z}(\zeta) \leqslant |\mathcal{A}|$, we immediately get the claim,

$$\mathrm{e}^{c|Y|}\, |\mathfrak{z}(Y)| \leqslant \mathrm{e}^{-\frac{\tilde{\beta}}{\Delta}e_0|Y|}\, \mathrm{e}^{3|Y|},$$

with a slight change of constant $c \to c - 3$.

A bound on the integral of fragments. Let us first consider *short* fragments $\zeta = (B, \boldsymbol{\omega}_B)$ satisfying the condition

$$\frac{1}{2}\sum_{j=1}^{k} |A_j| \leqslant \frac{\log(\Delta/\tilde{\beta})}{\log \|\boldsymbol{V}\|}. \tag{5.17}$$

The integral over the time of occurrence of the first transition yields the factor $\tilde{\beta}/\Delta$. Notice that ζ is not a loop. This follows from the construction of quantum contours and the fact that B is a connected component of $\hat{A}(\Gamma)$, where every transition is taken together with its R-neighbourhood. Thus, either its sequence of transitions does not belong to $\mathcal{S}$, or the starting configuration does not coincide with the ending configuration. In the first case we use Assumption 5, in the second case Assumption 6, and since (5.17) means that the sum over transitions is bounded, we can write

$$\int_{\mathcal{F}^{\text{short}}_{C(\mathcal{A})}} \mathrm{d}\zeta\, \hat{z}(\zeta) \leqslant \tfrac{1}{2}|\mathcal{A}|. \tag{5.18}$$

Finally, we estimate the integral over ζ's that are not short. We have

$$\int_{\mathcal{F}_{C(\mathcal{A})}\setminus \mathcal{F}^{\text{short}}_{C(\mathcal{A})}} \mathrm{d}\zeta\, \hat{z}(\zeta) \leqslant |\mathcal{A}| \frac{\tilde{\beta}}{\Delta} \int_{\mathcal{F}^{(x,\tau)}_{C(\mathcal{A})}\setminus \mathcal{F}^{\text{short}}_{C(\mathcal{A})}} \mathrm{d}\zeta\, \hat{z}(\zeta). \tag{5.19}$$

Here $\mathcal{F}^{(x,\tau)}_{C(\mathcal{A})}$ is the set of all fragments ζ whose first quantum transition (A_1, τ_1) is such that $x \in A_1$ and $\tau = \tau_1$. Whenever ζ is not short, we have

$$1 \leqslant \frac{\Delta}{\tilde{\beta}} \prod_{j=1}^{k} \|\boldsymbol{V}\|^{-\frac{1}{2}|A_j|}.$$

Thus, defining

$$\hat{z}'(\zeta) = \mathrm{e}^{-\frac{1}{2}(2R)^{-\nu}\Delta_0|B|} \prod_{j=1}^{k} \Big[\|\boldsymbol{V}\|^{\frac{1}{2}}\, \mathrm{e}^{c(2R)^{\nu}+1}\Big]^{|A_j|}, \tag{5.20}$$

we find the bound

$$|\mathcal{A}| \int_{\mathcal{F}(x,\tau)} \mathrm{d}\zeta\, \hat{z}'(\zeta).$$

Here, slightly overestimating, we take for $\mathcal{F}(x,\tau)$ the set of all fragments containing a quantum transition (A,τ) with $x \in A$.

The support B of a fragment $\zeta = (B, \boldsymbol{\omega}_B) \in \mathcal{F}(x,\tau)$, is a finite union of vertical segments (i.e. sets of the form $\{y\} \times [\tau_1, \tau_2] \subset \mathbb{T}_\Lambda$) and k horizontal quantum transitions $A_1, \dots, A_k$.

We will finish the proof by proving by induction the bound

$$\int_{\mathcal{F}(x,\tau;k)} \mathrm{d}\zeta\, \hat{z}'(\zeta) \leq 1 \tag{5.21}$$

with $\mathcal{F}(x,\tau;k)$ denoting the set of fragments from $\mathcal{F}(x,\tau)$ with at most k quantum transitions.

Consider thus a fragment ζ with k horizontal quantum transitions connected by vertical segments. Let (A,τ) be the transition containing the point (x,τ) and let $(A_1, \tau+\tau_1), \dots, (A_\ell, \tau+\tau_\ell)$ be the transitions that are connected by (one or several) vertical segments of the respective lengths $|\tau_1|, \dots, |\tau_\ell|$ with the transition (A,τ).

If we remove all those segments, the fragment ζ will split into the "naked" transition (A,τ) and additional $\bar\ell \le \ell$ fragments $\zeta_1,\dots,\zeta_{\bar\ell}$, such that each fragment ζ_j, $j=1,\dots,\bar\ell$, belongs to $\mathcal{F}(y_j,\tau+\tau_j;k-1)$ with $y_j\in A$. Taking into account that the number of configurations (determining the possible vertical segments attached to A) above and below A is bounded by $S^{2|A|}$ and that the number of possibilities to choose the points y_j is bounded by $|A|^{\bar\ell}$, we get

$$\int_{\mathcal{F}(x,\tau;k)} \mathrm{d}\zeta\, \hat z'(\zeta) \leqslant$$

$$\leqslant \sum_{A,\operatorname{dist}(A,x)<R} \bigl[\|\boldsymbol{V}\|^{\frac12}\,\mathrm{e}^{c(2R)^\nu+1}\,S^2\bigr]^{|A|} \sum_{\bar\ell=1}^{\infty} \frac{|A|^{\bar\ell}}{\bar\ell!} \int \mathrm{d}\tau_1 \dots \int \mathrm{d}\tau_{\bar\ell}\, \mathrm{e}^{-\frac12 (2R)^{-\nu}\Delta_0(\tau_1+\dots+\tau_{\bar\ell})} \prod_{j=1}^{\bar\ell} \int_{\mathcal{F}(y_j,\tau+\tau_j;k-1)} \mathrm{d}\zeta\, \hat z'(\zeta_j)$$

$$\leqslant \sum_{A,\operatorname{dist}(A,x)<R} \bigl[\|\boldsymbol{V}\|^{\frac12} S^2\,\mathrm{e}^{c(2R)^\nu+2}\bigr]^{|A|}\,\mathrm{e}^{2(2R)^\nu/\Delta_0} \leqslant 1 \qquad (5.22)$$

once $\|\boldsymbol{V}\|$ is sufficiently small. □

In the application of Pirogov-Sinai theory we shall also need a bound on derivatives of the weight of contours.

Lemma 5.3. *Under the Assumptions 1–7, for any $c<\infty$, there exist constants $\alpha,\beta_0,\tilde\beta_0<\infty$ and $\varepsilon_0>0$ such that if $\beta\geqslant\beta_0$, $\tilde\beta_0\leqslant\tilde\beta<2\tilde\beta_0$, and $\|\boldsymbol{V}\|+\sum_{i=1}^{r-1}\|\frac{\partial}{\partial\mu_i}\boldsymbol{V}\|\leqslant\varepsilon_0$, one has*

$$\bigl|\frac{\partial}{\partial\mu_i}\mathfrak{z}(Y)\bigr| \leqslant \alpha\tilde\beta|Y|\,\mathrm{e}^{-\frac{\tilde\beta}{\Delta}e_0^\mu|Y|}\ \mathrm{e}^{-c|Y|}$$

for any contour Y.

Proof. ¿From the definition (4.33) of $\mathfrak{z}$, one has

$$\bigl|\frac{\partial}{\partial\mu_i}\mathfrak{z}(Y)\bigr| \leqslant |\mathfrak{z}(Y)|\Bigl\{\sum_{\gamma\in\Gamma}\bigl|\frac{\partial}{\partial\mu_i}z(\Gamma)\bigr| + \sum_{d\in D}\bigl|W_d\cap C(\operatorname{supp}Y)\bigr|\bigl|\frac{\partial}{\partial\mu_i}e^\mu(d)\bigr| + \bigl|\frac{\partial}{\partial\mu_i}\tilde{\mathcal{R}}(\Gamma)\bigr|\Bigr\}$$

$$+\int_{\mathcal{D}_\Lambda^{\mathrm{per}}(Y)} \mathrm{d}\Gamma \prod_{\gamma\in\Gamma}|z(\gamma)| \prod_{d\in D} \mathrm{e}^{-e^\mu(d)|W_d\cap C(\operatorname{supp}Y)|}\ \mathrm{e}^{|\tilde{\mathcal{R}}(\Gamma)|}$$

$$\sum_{\mathcal{M}} \mathbb{I}\bigl[\mathcal{M}(\operatorname{supp}\Gamma)\cup\operatorname{supp}\mathcal{M}=\operatorname{supp}Y\bigr] \sum_{M\in\mathcal{M}} \Bigl|\mathrm{e}^{\varphi(M;\Gamma)}\frac{\partial}{\partial\mu_i}\varphi(M;\Gamma)\Bigr| \prod_{M'\in\mathcal{M},M'\neq M} \bigl|\mathrm{e}^{\varphi(M';\Gamma)}-1\bigr|. \qquad (5.23)$$

The bound for $|\frac{\partial}{\partial\mu_i}z(\Gamma)|$ is standard, see [BKU], and $|\frac{\partial}{\partial\mu_i}e^\mu(d)|$ is assumed to be bounded in Assumption 7. For the other terms we have to control clusters of loops. Since we have exponential decay for $z(\xi)$ with any strength (by taking β large and $\|\boldsymbol{V}\|$ small), we have the same for $\frac{\partial}{\partial\mu_i}z(\xi)$ (by taking β larger and $\|\boldsymbol{V}\|$ smaller). The integrals over $\boldsymbol{C}$ can be estimated as before, the only effect of the derivative being an extra factor n (when the clusters have n loops). □

6. Expectation values of local observables and construction of pure states

So far we have obtained an expression (4.34) for the partition function Z_Λ^{per} of the quantum model on torus Λ in terms of that of a classical lattice contour model with the weights of the contours showing an exponential decay with respect to their length. Using the same weights $\mathfrak{z}(Y)$, we can also introduce the partition functions $Z^d_{\Lambda(L)}$ with the torus Λ replaced by a hypercube $\Lambda(L)$ and with fixed boundary conditions d. Namely, we take simply the sum only over those collections $\mathcal{Y}$ of contours whose external contours are labeled by d and are not close to the boundary[13] . Notice, however, that here we are defining $Z^d_{\Lambda(L)}$ directly in terms of the classical contour model, without ensuring existence of corresponding partition function directly for the original model. We will use these partition functions only as a tool for proving our Theorems that are stated directly in terms of quantum models.

[13]In the terminology of Pirogov-Sinai theory we rather mean *diluted partition functions* — see the more precise definition below.

To be more precise, we can extend the definition even more and consider, instead of the torus Λ, any finite set $V \subset \mathbb{L} = \mathbb{Z}^\nu \times \{0,1,\dots,N-1\}^{\text{per}}$. There is a class of contours that can be viewed as having their support contained in $V \subset \mathbb{L}$. For any such contour Y we introduce its interior $\operatorname{Int} Y$ as the union of all finite components of $\mathbb{L} \setminus \operatorname{supp} Y$ and $\operatorname{Int}_d Y$ as the union of all components of $\operatorname{Int} Y$ whose boundary is labeled by d. Recalling that we assumed $\nu \geqslant 2$, we note that the set $\mathbb{L} \setminus (\operatorname{supp} Y \cup \operatorname{Int} Y)$ is a connected set, implying that the label $\alpha_Y(\cdot)$ is constant on the boundary of the set $V(Y) = \operatorname{supp} Y \cup \operatorname{Int} Y$. We say that Y is a d-contour, if $\alpha_Y = d$ on this boundary. Two contours Y and Y' are called *mutually external* if $V(Y) \cap V(Y') = \emptyset$. Given an admissible set $\mathcal{Y}$ of contours, we say that $Y \in \mathcal{Y}$ is an *external contour* in $\mathcal{Y}$, if $\operatorname{supp} Y \cap V(Y') = \emptyset$ for all $Y' \in \mathcal{Y}$, $Y' \neq Y$. The sets $\mathcal{Y}$ contributing to Z_V^d are such that all their external contours are d-contours and $\operatorname{dist}(Y, \partial V) > 1$ for every $Y \in \mathcal{Y}$.

In this way we found ourselves exactly in the setting of standard Pirogov-Sinai theory, or rather, the reformulation for "thin slab" (cylinder $\mathbb{L}$ of fixed temporal size N) as presented in Sections 5–7 and Appendix of [BKU]. In particular, for sufficiently large β and sufficiently small $\|\boldsymbol{V}\| + \sum_{i=1}^{r-1} \|\frac{\partial}{\partial \mu_i} \boldsymbol{V}\|$, there exist functions $f^{\beta,\mu}(d)$, metastable free energies, such that the condition $\operatorname{Re} f^{\beta,\mu}(d) = f_0$, with $f_0 \equiv f_0^{\beta,\mu}$ defined by $f_0 = \min_{d' \in D} \operatorname{Re} f^{\beta,\mu}(d')$, characterizes the existence of pure stable phase d. Namely, as will be shown next, a pure stable phase $\langle \cdot \rangle_\beta^d$ exists and is close to the pure ground state $|d\rangle$.

There is one subtlety in the definition of $f^{\beta,\mu}(d)$. Namely, after choosing a suitable $\tilde{\beta}_0$, given β, there exist several pairs $(\tilde{\beta}, N)$ such that $\tilde{\beta} \in (\tilde{\beta}_0, 2\tilde{\beta}_0)$ and $N\tilde{\beta} = \beta$. To be specific, we may agree to choose among them that one with maximal N. The function $f^{\beta,\mu}(d)$ is then uniquely defined for each $\beta > \tilde{\beta}_0$. Notice, however, that while increasing β, we pass, at the particular value $\beta_N = N\tilde{\beta}_0$, from discretization of temporal size N to $N+1$. As a result, the function $f^{\beta,\mu}(d)$ might be discontinuous at β_N with $\beta = \infty$ being an accumulation point of such discontinuities. Nevertheless, these discontinuities are harmless. They can appear only when $\operatorname{Re} f^{\beta,\mu}(d) > f_0$ and do not change anything in the following argument.

Before we come to the construction of pure stable phases, notice that the first claim of Theorem 2.2 (equality of f_0 with the limiting free energy) is now a direct consequence of the bound

$$\left| Z_\Lambda^{\text{per}} - |Q| e^{-\tilde{\beta} f_0 N L^\nu} \right| \leqslant e^{-\tilde{\beta} f_0 N L^\nu} O(e^{-\text{const } c L}) \tag{6.1}$$

(c.f. [BKU], (7.14)). Here $Q = \{d; \operatorname{Re} f^{\beta,\mu}(d) = f_0\}$.

The expectation value of a local observable T is defined as

$$\langle T \rangle_\Lambda^{\text{per}} = \frac{\operatorname{Tr} T \, \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}}{\operatorname{Tr} \, \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}}. \tag{6.2}$$

In Section 4 we have obtained a contour expression for $Z_\Lambda^{\text{per}} = \operatorname{Tr} \, \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}$. We retrace here the same steps for $Z_\Lambda^{\text{per}}(T) := \operatorname{Tr} T \, \mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}$. Duhamel expansion (4.1) for $Z_\Lambda^{\text{per}}(T)$ leads to an equation analogous to (4.2),

$$Z_\Lambda^{\text{per}}(T) = \sum_{m \geqslant 0} \sum_{n_\Lambda^0, \dots n_\Lambda^m} \sum_{\substack{\boldsymbol{A}_1, \dots, \boldsymbol{A}_m \\ \bar{A}_i \subset \Lambda}} \int_{0 < \tau_1 < \dots < \tau_m < \beta} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m \langle n_\Lambda^0 | \, T \, | n_\Lambda^1 \rangle$$
$$\mathrm{e}^{-\tau_1 \tilde{H}_\Lambda^{(0)\,\text{per}}(n_\Lambda^1)} \langle n_\Lambda^1 | \, \boldsymbol{V}_{\boldsymbol{A}_1} \, | n_\Lambda^2 \rangle \, \mathrm{e}^{-(\tau_2 - \tau_1) \tilde{H}_\Lambda^{(0)\,\text{per}}(n_\Lambda^2)} \dots \langle n_\Lambda^m | \, \boldsymbol{V}_{\boldsymbol{A}_m} \, | n_\Lambda^0 \rangle \, \mathrm{e}^{-(\beta - \tau_m) \tilde{H}_\Lambda^{(0)\,\text{per}}(n_\Lambda^0)}. \tag{6.3}$$

Configurations n_Λ^0 and n_Λ^1 match on $\Lambda \setminus \operatorname{supp} T$ ($\operatorname{supp} T \subset \Lambda$ is a finite set due to the locality of T), but may differ on $\operatorname{supp} T$ if T is an operator with non zero off-diagonal terms. Let $\mathcal{Q}_\Lambda^{\text{per}}(T)$ be the set of quantum configurations with $\boldsymbol{n}_\Lambda(\tau)$ that is constant except possibly at

$\cup_{i=1}^{m}(A_i \times \tau_i) \cup (\operatorname{supp} T \times 0)$. Then

$$Z_\Lambda^{\rm per}(T) = \int_{\mathcal{Q}_\Lambda^{\rm per}(T)} \mathrm{d}\boldsymbol{\omega}_{\mathbb{T}_\Lambda} \langle n_\Lambda^0 |\, T \,| n_\Lambda^1 \rangle \rho^{\rm per}(\boldsymbol{\omega}_{\mathbb{T}_\Lambda}). \tag{6.4}$$

We identify loops with the same iteration scheme as in Section 4, starting with the set $\boldsymbol{B}^{(0)}(\boldsymbol{\omega}) \cup (\operatorname{supp} T \times 0)$ instead of $\boldsymbol{B}^{(0)}(\boldsymbol{\omega})$ only. This leads to the set $\boldsymbol{B}^T(\boldsymbol{\omega})$. Removing the loops, we define $\boldsymbol{B}_{\rm e}^T(\boldsymbol{\omega})$, whose connected components form quantum contours. There is one special quantum contour, namely that which contains $\operatorname{supp} T \times 0$. Let us denote it by γ^T and define its weight [see (4.26)]

$$z^T(\gamma^T) = \langle \boldsymbol{n}_{\operatorname{supp} T}^{\gamma^T}(-0) |\, T \,| \boldsymbol{n}_{\operatorname{supp} T}^{\gamma^T}(+0) \rangle \prod_{i=1}^{m} \langle \boldsymbol{n}_{A_i}^{\gamma^T}(\tau_i - 0) |\, \boldsymbol{V}_{\boldsymbol{A}_i} \,| \boldsymbol{n}_{A_i}^{\gamma^T}(\tau_i + 0) \rangle$$
$$\exp\Big\{ - \int_B \mathrm{d}(x,\tau) e_x^{\Upsilon}(\boldsymbol{n}^{\gamma^T}(\tau)) \Big\}. \tag{6.5}$$

Let $\Gamma^T = \{\gamma^T, \gamma_1, \ldots, \gamma_k\}$ be an admissible set of quantum contours, defining a quantum configuration $\boldsymbol{\omega}^{\Gamma^T} \in \mathcal{Q}_\Lambda^{\rm per}(T)$. Then we have an expression similar to that of Lemma 4.4,

$$Z_\Lambda^{\rm per}(T) = \int_{\mathcal{D}_\Lambda^{\rm per}(T)} \mathrm{d}\Gamma^T \prod_{d \in D} \mathrm{e}^{-|W_d(\Gamma^T)| e(d)}\, z^T(\gamma^T) \prod_{\gamma \in \Gamma^T \setminus \{\gamma^T\}} z(\gamma)\, \mathrm{e}^{\mathcal{R}(\Gamma^T)}, \tag{6.6}$$

with $\mathcal{R}(\Gamma^T)$ as in (4.27) with Γ replaced by Γ^T.

Next step is to discretize the lattice, to expand $\mathrm{e}^{\mathcal{R}(\Gamma^T)}$, and if Y^T is the contour that contains $\operatorname{supp} T \times 0 \subset \mathbb{L}_\Lambda$, to define $\mathfrak{z}^T(Y^T)$ [see (4.33)]:

$$\mathfrak{z}^T(Y^T) = \int_{\mathcal{D}_\Lambda^{\rm per}(Y^T)} \mathrm{d}\Gamma^T z^T(\gamma^T) \prod_{\gamma \in \Gamma^T \setminus \{\gamma^T\}} z(\gamma) \prod_{d \in D} \mathrm{e}^{-e(d)|W_d(\Gamma^T) \cap C(\operatorname{supp} Y^T)|}\, \mathrm{e}^{\tilde{\mathcal{R}}(\Gamma^T)}$$
$$\sum_{\mathcal{M}} \mathbb{I}\left[M(\operatorname{supp} \Gamma^T \cup \operatorname{supp} \mathcal{M} = \operatorname{supp} Y^T \right] \prod_{M \in \mathcal{M}} \left(\mathrm{e}^{\varphi(M;\Gamma^T)} - 1 \right). \tag{6.7}$$

We also need a bound for $\mathfrak{z}^T(Y^T)$. It is clear that the situation is the same as for Lemmas 5.1 and 5.2, except for a factor $\langle \boldsymbol{n}_{\operatorname{supp} T}^{\gamma^T}(-0) |\, T \,| \boldsymbol{n}_{\operatorname{supp} T}^{\gamma^T}(+0) \rangle$ that is bounded by $\|T\|$. We can thus summarize:

Lemma 6.1. *Under the Assumptions 1–6, for any $c < \infty$, there exist $\beta_0, \tilde{\beta}_0 < \infty$, and $\varepsilon_0 > 0$ such that if $\beta \geqslant \beta_0$, $\tilde{\beta}_0 \leqslant \tilde{\beta} < 2\tilde{\beta}_0$ and $\|\boldsymbol{V}\| \leqslant \varepsilon_0$, we have*

$$Z_\Lambda^{\rm per}(T) = \sum_{\mathcal{Y}^T = \{Y^T, Y_1, \ldots, Y_k\}} \prod_{d \in D} \mathrm{e}^{-\frac{\tilde{\beta}}{\Delta} e(d) |\mathcal{W}_d(\mathcal{Y}^T)|}\, \mathfrak{z}^T(Y^T) \prod_{Y \in \mathcal{Y}^T \setminus \{Y^T\}} \mathfrak{z}(Y), \tag{6.8}$$

for every local observable T, with

$$|\mathfrak{z}^T(Y^T)| \leqslant \|T\|\, \mathrm{e}^{c|\operatorname{supp} T|}\, \mathrm{e}^{-\frac{\tilde{\beta}}{\Delta} e_0 |Y^T|}\, \mathrm{e}^{-c|Y^T|}$$

for any contour Y^T.

In a similar manner as at the beginning of this section, we can introduce $Z_V^d(T)$ for any $V \subset \mathbb{L}$ by restricting ourselves in the sum (6.8) to the collections $\mathcal{Y}^T$ whose all external contours are d-contours and $\operatorname{dist}(Y, \partial V) > 1$ for every $Y \in \mathcal{Y}^T$. Thus we can *define* the expectation value

$$\langle T \rangle_V^d = \frac{Z_V^d(T)}{Z_V^d} \tag{6.9}$$

for any $V \subset \mathbb{L}$ and, in particular, the expectation $\langle T \rangle_{\Lambda(L)}^d$ for a hypercube $\Lambda(L)$.

Again, this is exactly the setting discussed in detail in [BKU]. We can use directly the corresponding results (c.f. [BKU], Lemma 6.1) to prove first that the limiting state $\langle\cdot\rangle_\beta^d$ exists. Further, retracing the proof of Theorem 2.2 in [BKU] we prove that the limit

$$\langle T\rangle_\beta^{\text{per}} = \lim_{\Lambda\nearrow\mathbb{Z}^\nu} \frac{\operatorname{Tr} T\,\mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}}{\operatorname{Tr}\,\mathrm{e}^{-\beta \boldsymbol{H}_\Lambda^{\text{per}}}} \tag{6.10}$$

exists for every local T (proving thus Theorem 2.1). Moreover,

$$\langle T\rangle_\beta^{\text{per}} = \frac{1}{Q}\sum_{d\in Q}\langle T\rangle_\beta^d, \tag{6.11}$$

where, again, Q denotes the set of stable phases, $Q=\{d;\operatorname{Re} f^{\beta,\mu}(d)=f_0\}$. Thus we proved the claim d) of Theorem 2.2.

Also the assertion c) follows in standard manner from contour representation employing directly the exponential decay of contour activities and corresponding cluster expansion (c.f. [BKU], (2.27)).

Before passing to the proof of b), we shall verify that $\langle\cdot\rangle_\beta^d$ is actually a pure stable state according to our definition, i.e. a limit of unperturbable states[14]. To this end, let us first discuss how metastable free energies $f^{\beta,\mu}(d)$ change with μ. The standard construction yields $f^{\beta,\mu}(d)$ in the form of a sum $e^\mu(d)+s^{\beta,\mu}(d)$, where $s^{\beta,\mu}(d)$ is the free energy of "truncated" contour model $K'_d(Y)$ (see [BKU], (5.13) and (5.6)) constructed from labeled contour model (4.34), which is under control by cluster expansions. As a result, we have bounds of the form $O\big(e^{-\beta}+\|\boldsymbol{V}\|+\sum_{i=1}^{r-1}\big\|\frac{\partial \boldsymbol{V}}{\partial\mu_i}\big\|\big)$ on $|s^{\beta,\mu}(d)|$ as well as on the derivatives with respect to μ. Hence, in view of Assumption 7, the leading behaviour is yielded by $e^\mu(d)$.

Starting thus from a given potential Φ^μ with $Q^\mu=\{d\in D;\operatorname{Re} f^{\beta,\mu}(d)=f_0^\mu\}$, one can easily add to Φ^μ a suitable "external field" that favours a chosen $d\in Q^\mu$. For example, one can take

$$\Phi_A^{\mu,\alpha}(n)=\Phi_A^\mu(n)+\alpha\delta_A^d(n)$$

with δ_A^d defined by taking $\delta_A^d(n)=0$ for $n_A=d_A$ and $\delta_A^d(n)=1$ otherwise[15]. Now, since $\frac{\partial e^{\mu,\alpha}(d)}{\partial\alpha}$ is bounded from below by a positive constant (while $\frac{\partial e^{\mu,\alpha}(d')}{\partial\alpha}=0$ for $d'\neq d$), for any $\alpha>0$ the only stable phase is d, $\operatorname{Re} f^{\beta,\mu,\alpha}(d)=f_0^{\beta,\mu,\alpha}\equiv\min_{d'\in D}\operatorname{Re} f^{\beta,\mu,\alpha}(d')$, and, in the same time, $\operatorname{Re} f^{\beta,\mu,\alpha}(d')>f_0^{\beta,\mu,\alpha}$ for $d'\neq d$. Thus, $Q^{\mu,\alpha}=\{d\}$ and $\langle\cdot\rangle_{\beta,\mu,\alpha}^d=\langle\cdot\rangle_{\beta,\mu,\alpha}^{\text{per}}$. This state is unperturbable — when adding any small perturbation, metastable free energies will change only a little and that one corresponding to the state d will still be the only one attaining the minimum. The fact that in the limit of vanishing perturbation we recover $\langle\cdot\rangle_{\beta,\mu,\alpha}^d$, as well as the fact that

$$\lim_{\alpha\to0+}\langle\cdot\rangle_{\beta,\mu,\alpha}^{\text{per}}\equiv\lim_{\alpha\to0+}\langle\cdot\rangle_{\beta,\mu,\alpha}^d=\langle\cdot\rangle_{\beta,\mu}^d,$$

follows by inspecting the contour representations of the corresponding expectations and observing that it can be expressed in terms of converging cluster expansions whose terms depend smoothly on α as well as on the additional perturbation.

To prove, finally, the claim b) of Theorem 2.2, it suffices to show that it is valid for $\langle\cdot\rangle_{\beta,\mu,\alpha}^{\text{per}}=\langle\cdot\rangle_{\beta,\mu,\alpha}^d$ for every $\alpha>0$. Abbreviating $\langle\cdot\rangle_{\beta,\mu,\alpha}^{\text{per}}=\langle\cdot\rangle^{\text{per}}$ and $\boldsymbol{H}_\Lambda^{\mu,\alpha,\text{per}}=\boldsymbol{H}_\Lambda^{\text{per}}$, we first notice that the expectation value of the projector onto the configuration d on $\operatorname{supp}T$, $P_{\operatorname{supp}T}^d:=|d_{\operatorname{supp}T}\rangle\langle d_{\operatorname{supp}T}|$, is close to 1, since its complement $\langle(\mathbb{1}-P_{\operatorname{supp}T}^d)\rangle^{\text{per}}=\langle(\mathbb{1}-P_{\operatorname{supp}T}^d)\rangle^d$ is related to the presence of a contour intersecting or surrounding $\operatorname{supp}T$ (loops intersecting

[14] Recall that, up to now, the state $\langle\cdot\rangle_\beta^d$ is defined only in terms of the contour representation (see (6.9), (6.8), and (4.34)), and the only proven connection with a state of original quantum model is the equality (6.11).

[15] Actually, we can restrict δ_A^d only to a particular type of sets A — for example all hypercubes of side R.

$\operatorname{supp} T \times \{0\}$ are considered here as part of quantum contours), whose weight is small. More precisely, for any $\delta > 0$ we have

$$\langle (\mathbb{1} - P^d_{\operatorname{supp} T}) \rangle^{\text{per}} \leqslant \delta |\operatorname{supp} T|,$$

whenever $\|\boldsymbol{V}\|$ is small enough and β large enough. Furthermore,

$$\langle T \rangle^{\text{per}}_\Lambda = \frac{1}{Z^{\text{per}}_\Lambda} \Big[\operatorname{Tr} \big(P^d_{\operatorname{supp} T} T P^d_{\operatorname{supp} T} \, \mathrm{e}^{-\beta \boldsymbol{H}^{\text{per}}_\Lambda} \big) + \\ + \operatorname{Tr} \big((\mathbb{1} - P^d_{\operatorname{supp} T}) T P^d_{\operatorname{supp} T} \, \mathrm{e}^{-\beta \boldsymbol{H}^{\text{per}}_\Lambda} \big) + \operatorname{Tr} \big(T (\mathbb{1} - P^d_{\operatorname{supp} T}) \, \mathrm{e}^{-\beta \boldsymbol{H}^{\text{per}}_\Lambda} \big) \Big] \tag{6.12}$$

and

$$\operatorname{Tr} \big(P^d_{\operatorname{supp} T} T P^d_{\operatorname{supp} T} \, \mathrm{e}^{-\beta \boldsymbol{H}^{\text{per}}_\Lambda} \big) = \langle d_\Lambda | \, T \, | d_\Lambda \rangle \operatorname{Tr} \big(P^d_{\operatorname{supp} T} \, \mathrm{e}^{-\beta \boldsymbol{H}^{\text{per}}_\Lambda} \big) \\ = \langle d_\Lambda | \, T \, | d_\Lambda \rangle \big[\operatorname{Tr} \big(\mathrm{e}^{-\beta \boldsymbol{H}^{\text{per}}_\Lambda} \big) - \operatorname{Tr} \big((\mathbb{1} - P^d_{\operatorname{supp} T}) \, \mathrm{e}^{-\beta \boldsymbol{H}^{\text{per}}_\Lambda} \big) \big], \tag{6.13}$$

so that we have

$$\big| \langle T \rangle^{\text{per}}_\Lambda - \langle d_\Lambda | \, T \, | d_\Lambda \rangle \big| \leqslant \\ \leqslant \big| \langle d_\Lambda | \, T \, | d_\Lambda \rangle \big| \langle (\mathbb{1} - P^d_{\operatorname{supp} T}) \rangle^{\text{per}}_\Lambda + \big| \langle (\mathbb{1} - P^d_{\operatorname{supp} T}) T P^d_{\operatorname{supp} T} \rangle^{\text{per}}_\Lambda \big| + \big| \langle T (\mathbb{1} - P^d_{\operatorname{supp} T}) \rangle^{\text{per}}_\Lambda \big|. \tag{6.14}$$

The mapping $(T, T') \mapsto \langle T^\dagger T' \rangle^{\text{per}}_\Lambda$, with any two local operators T, T', is a scalar product; therefore the Schwarz inequality yields

$$\big| \langle T \rangle^{\text{per}}_\Lambda - \langle d_\Lambda | \, T \, | d_\Lambda \rangle \big| \leqslant \big| \langle d_\Lambda | \, T \, | d_\Lambda \rangle \big| \langle (\mathbb{1} - P^d_{\operatorname{supp} T}) \rangle^{\text{per}}_\Lambda \\ + \Big(\langle (\mathbb{1} - P^d_{\operatorname{supp} T}) \rangle^{\text{per}}_\Lambda \Big)^{\frac{1}{2}} \Big(\big[\langle P^d_{\operatorname{supp} T} T^\dagger T P^d_{\operatorname{supp} T} \rangle^{\text{per}}_\Lambda \big]^{\frac{1}{2}} + \big[\langle T^\dagger T \rangle^{\text{per}}_\Lambda \big]^{\frac{1}{2}} \Big) \\ \leqslant \|T\| \Big[\langle (\mathbb{1} - P^d_{\operatorname{supp} T}) \rangle^{\text{per}}_\Lambda + 2 \big(\langle (\mathbb{1} - P^d_{\operatorname{supp} T}) \rangle^{\text{per}}_\Lambda \big)^{1/2} \Big] \leqslant \|T\| |\operatorname{supp} T| (\delta + 2\delta^{\frac{1}{2}}). \tag{6.15}$$

The proof of the remaining Theorem 2.3 is a standard application of the implicit function theorem. Thus, for example, the point $\bar{\mu}_0$ of maximal coexistence, $\operatorname{Re} f^{\beta, \bar{\mu}_0}(d) = \operatorname{Re} f^{\beta, \bar{\mu}_0}(d')$ for every pair $d, d' \in D$, can be viewed as the solution of the vector equation $f(\bar{\mu}_0) = 0$, with $f(\mu) = (\operatorname{Re} f^{\beta, \mu}(d_i) - \operatorname{Re} f^{\beta, \mu}(d_r))_{i=1}^{r-1}$. Now, $f = e + s$, $e(\mu) = (e^\mu(d_i) - e^\mu(d_r))_{i=1}^{r-1}$, $s(\mu) = (\operatorname{Re} s^{\beta, \mu}(d_i) - \operatorname{Re} s^{\beta, \mu}(d_r))_{i=1}^{r-1}$, with $\|s\|$ as well as $\big\| \frac{\partial s}{\partial \mu} \big\|$ bounded by a small constant once $\|\boldsymbol{V}\| + \sum_{i=1}^{r-1} \Big\| \frac{\partial \boldsymbol{V}}{\partial \mu_i} \Big\|$ is sufficiently small β is sufficiently large. The existence of a unique solution $\bar{\mu}_0 \in \mathcal{U}$ then follows once we notice the existence of the solution $\mu_0 \in \mathcal{U}$ of the equation $e(\mu_0) = 0$ (equivalent with $e^{\mu_0}(d) = e^{\mu_0}(d')$, $d, d' \in D$) and the fact that the mapping

$$\mathcal{T} : \mu \to A^{-1} \big(\frac{\partial e}{\partial \mu} \big|_{\mu = \mu_0} (\mu - \mu_0) - f(\mu) \big)$$

with A^{-1} the matrix inverse to $\big(\frac{\partial e}{\partial \mu} \big)$, is a contraction. To this end it is enough just to recall Assumption 7 and the bounds on $s^{\beta, \mu}(d)$, $d \in D$, and its derivatives.

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