ACCUMULATION OF DISCRETE EIGENVALUES OF THE RADIAL DIRAC OPERATOR

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Abstract. For bounded potentials which behave like $-cx^{-\gamma}$ at infinity we investigate whether discrete eigenvalues of the radial Dirac operator H_κ accumulate at $+1$ or not. It is well known that $\gamma = 2$ is the critical exponent. We show that $c = 1/8 + \kappa(\kappa+1)/2$ is the critical coupling constant in the case $\gamma = 2$. Our approach is to transform the radial Dirac equation into a Sturm-Liouville equation nonlinear in the spectral parameter and to apply a new, general result on accumulation of eigenvalues of such equations.

1. Introduction

The Dirac operator on $\mathbb{R}^3$ with (bounded) potential behaving like $-c|\mathbf{x}|^{-\gamma}$ ($\gamma > 0$) at infinity has finitely many discrete eigenvalues if $\gamma > 2$ and infinitely many for $\gamma < 2$ [9]. In the case of a spherically symmetric potential this is still true for the Dirac operator restricted to a subspace of definite angular momentum and definite parity (radial Dirac operator) [9]. We rederive this result for the radial Dirac operator by a new method which allows us to determine the critical coupling constant in the case $\gamma = 2$. Our strategy is to transform the radial Dirac eigenvalue equations (see below) into a Sturm-Liouville equation nonlinear in the spectral parameter to which we then apply a new, general theorem (Theorem 2, Lutgen) on accumulation of eigenvalues of such equations. First and foremost our result is a nice application of this general theorem and a demonstration of its strength.

The eigenvalue equation for the Dirac operator with spherically symmetric potential V is equivalent to the system of first order differential equations

$$-F' + \frac{\kappa}{x}F + (1 + V - \lambda)G = 0 \tag{1}$$

$$G' + \frac{\kappa}{x}G + (-1 + V - \lambda)F = 0 \tag{2}$$

on $\mathbb{R}_+$, to be solved for $F, G \in L^2(\mathbb{R}_+)$, $\lambda \in \mathbb{R}$ and $\kappa \in \mathbb{Z}\backslash\{0\}$ with boundary conditions $F(0) = 0 = G(0)$. κ parameterizes both the total angular momentum and the parity. Let us assume the potential V is bounded, nonpositive, and $-V(x)x^\gamma \to c$ as $x = |\mathbf{x}|$ goes to infinity, the coupling constant c being positive. Under further assumptions on V' to be discussed below, we show that accumulation of eigenvalues at $+1$ ($= mc^2$) from below occurs whenever

$$\gamma < 2, \qquad \text{or} \qquad \gamma = 2 \quad \text{and} \quad c > \frac{1}{8} + \frac{\kappa(\kappa+1)}{2},$$

while for

$$\gamma > 2, \qquad \text{or} \qquad \gamma = 2 \quad \text{and} \quad c \leq \frac{1}{8} + \frac{\kappa(\kappa+1)}{2}$$

there is no accumulation. An analogous theorem (without conditions on V') with the same critical exponent and the same critical coupling constant holds true for the radial Schrödinger equation which emerges in the nonrelativistic (particle) limit

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of the Dirac equations (1) and (2). This should not come as a surprise, for if the potential is bounded, all but finitely many of the bound states of the Dirac operator are localized far away from zero where the potential is small by assumption; hence, the kinetic energy will also be small which means that relativity is of no importance for the question of whether eigenvalues accumulate at 1. This is true as long as relativistic corrections are not large enough to generate infinitely many bound states localized near the origin, a requirement which explains our condition on V' for finite x: at least for large values of $|\kappa|$ it says that the spin-orbit interaction, which is one of the leading relativistic corrections to the Schrödinger equation, is in some sense smaller than the kinetic energy due to the orbital angular momentum.

The general theorem from which we derive our result is a statement about accumulation of eigenvalues for a Sturm-Liouville eigenvalue problem nonlinear in the eigenvalue parameter. This is exactly the form of our problem once we have eliminated F by means of equation (2). Roughly speaking, the theorem says that the family of Sturm-Liouville equations we get has infinitely many L^2-solutions if and only if the equation for $\lambda = 1$ is oscillatory (i.e., every solution has infinitely many zeros). It thus generalizes folk wisdom from the theory of classical (i.e., linear in λ) Sturm-Liouville eigenvalue problems. To determine whether the equation for $\lambda = 1$ is oscillatory a standard result (Kneser's criterion) is applied.

Infinitude of the discrete spectrum for $\gamma < 2$ and finiteness for $\gamma > 2$ was proved by Kurbenin [9]. For the critical case $\gamma = 2$ and the special potential $V(x) = -\lambda(1 + x^2)^{-1}$ Klaus showed finiteness for $\lambda < 1/8$ and infinitude for $\lambda > 1/8$ [8]. This is the only previous result we are aware of which identifies a critical coupling constant. There is a variety of results in the case of finite discrete spectrum usually assuming $\gamma < 2$ or $V \in L^3 \cap L^{3/2}$, which excludes $\gamma = 2$ [13, 8, 1, 3]. Most interesting in the present context is one of Birman and Laptev, which compares the number of discrete eigenvalues of the Pauli and Dirac operators for large coupling constants [1]. It is found that the asymptotic numbers coincide, which nicely complements the picture given in the present paper (see [1, Theorem 4], the factor 2^{q+1} there is absent when the comparison operator in [1, eqn. 12] is chosen to describe the nonrelativistic particle limit).

This paper is organized as follows. In Section 2 we recall the angular momentum decomposition of the Dirac operator and in Section 3 we eliminate F from equation (2) to obtain a SL-equation equivalent to the system (1,2). The general theorem of Lutgen (Theorem 2) and our main result (Theorem 3) are contained in Section 4. In this section we also speculate about a more general relationship between the spectra of the radial Dirac and the Schrödinger operator. In the appendix we give a second proof of the accumulation result under different hypotheses.

2. The Radial Dirac Operator

This section serves to recall, without proofs, the angular momentum decomposition of the Dirac operator. More detailed rigorous expositions can be found in [14, 12, 16].

Consider a Dirac-electron subject to an external spherically symmetric potential V. Due to the $O(3)$-symmetry of this system, the total angular momentum and the parity of an electronic state are conserved quantities. This means that the Dirac operator H is reduced by the subspaces of definite angular momentum and definite parity. On such a subspace it is unitarily equivalent to a 2 by 2 matrix operator H_κ on $L^2(\mathbb{R}_+) \otimes \mathbb{C}^2$. In the following this is spelled out in more mathematical terms.

The three-dimensional Dirac operator with potential $V : \mathbb{R}^3 \to \mathbb{R}$ reads

$$H = -i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta + V \qquad \text{on } \mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$$

where $\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} = \sum_{i=1}^{3} \alpha_i \frac{\partial}{\partial x_i}$ and in the standard representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the Pauli matrices σ_i. In our units $\hbar = m = c = 1$. Assume for simplicity that V is bounded (and measurable). The Hamiltonian H is thus self-adjoint on $H^1(\mathbb{R}^3) \otimes \mathbb{C}^4$ [14].

The total angular momentum is represented in $\mathfrak{H}$ by the operator (actually the triple of operators) $\mathbf{J} = (J_1, J_2, J_3)$ which is the sum $\mathbf{J} = \mathbf{L} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{S}$ of the operators representing the orbital angular momentum $\mathbf{L} = -i\mathbf{x} \times \boldsymbol{\nabla}$ and the spin $\mathbf{S} = \frac{1}{2}\left(\begin{smallmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{smallmatrix}\right)$. The space reflection $\mathbf{x} \to -\mathbf{x}$ is represented by the parity operator P defined by $P\psi(\mathbf{x}) = \beta\psi(-\mathbf{x})$. Since $\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2$, J_3 and P are mutually commuting and have pure point spectrum there is a complete family of simultaneous eigenspaces $\mathfrak{H} = \bigoplus \mathfrak{H}_{j,m,\pi}$. The eigenvalues on $\mathfrak{H}_{j,m,\pi}$ are given by

$$\begin{aligned} &\mathbf{J}^2 = j(j+1) && j \in \{1/2, 3/2, 5/2, \dots\}, \\ &J_3 = m && m \in \{-j, \dots, j\}, \\ &P = \pi && \pi \in \{\pm 1\}. \end{aligned}$$

To achieve a economic notation we now introduce a new parameter $\kappa \in \mathbb{Z}\backslash\{0\}$ which replaces j and π. This is usually done in such a way that $j = |\kappa| - \frac{1}{2}$ and $\pi = \pm(-1)^\kappa$ if $\pm\kappa > 0$. With $\mathfrak{H}_{\kappa,m} = \mathfrak{H}_{j,m,\pi}$ one then has

$$\mathfrak{H} = \bigoplus_{\kappa \in \mathbb{Z}\backslash\{0\}} \bigoplus_{m=-(|\kappa|-1/2)}^{|\kappa|-1/2} \mathfrak{H}_{\kappa,m}$$

and $\mathfrak{H}_{\kappa,m}$ is of the form

$$\mathfrak{H}_{\kappa,m} = \left\{ \begin{pmatrix} \frac{iG}{x}\Omega_{\kappa,m} \\ \frac{F}{x}\Omega_{-\kappa,m} \end{pmatrix} \middle| \, G, F \in L^2(\mathbb{R}_+) \right\} \tag{3}$$

where $x = |\mathbf{x}|$ and $\Omega_{\kappa,m}$ are $\mathbb{C}^2$-valued functions on the unit sphere whose components are proportional to spherical harmonics $Y_{l,m}$, their index l being connected with j and κ by

$$\kappa = \pm\left(j + \frac{1}{2}\right) \quad \text{if} \quad l = j \pm \frac{1}{2}. \tag{4}$$

This relation is usually taken as the definition of κ (see [2]); it says that the sign of κ indicates whether spin and orbital angular momentum of the upper component are "parallel" or "anti-parallel".

Now let us assume that V is spherically symmetric. Then H commutes as well with $\mathbf{J}^2$, J_3 and P, and is thus reduced by the spaces $\mathfrak{H}_{\kappa,m}$. Its effect on the functions G and F in (3) is described by the radial Dirac operator

$$H_\kappa := \begin{pmatrix} 1 + V & -\frac{d}{dx} + \frac{\kappa}{x} \\ \frac{d}{dx} + \frac{\kappa}{x} & -1 + V \end{pmatrix}, \qquad D(H_\kappa) = H_0^1(\mathbb{R}_+) \otimes \mathbb{C}^2$$

in $L^2(\mathbb{R}_+) \otimes \mathbb{C}^2$, more precisely, the operator $H \restriction \mathfrak{H}_{\kappa,m}$ is unitarily equivalent to H_κ, the unitary map being $(G, F) \in L^2(\mathbb{R}_+, \mathbb{C}^2) \to (\frac{iG}{x}\Omega_{\kappa,m}, \frac{F}{x}\Omega_{-\kappa,m}) \in \mathfrak{H}_{\kappa,m}$. The domain of H_κ is the inverse image of $D(H) \cap \mathfrak{H}_{\kappa,m}$ under this map. It is equal to $H_0^1(\mathbb{R}_+) \otimes \mathbb{C}^2$ because $fx^{-1}Y_{l,m} \in H^1(\mathbb{R}^3)$ if and only if f belongs to $H_0^1(\mathbb{R}_+)$. The spectrum of H_κ is qualitatively the same as the spectrum of H: if $V \equiv 0$ then $\sigma(H_\kappa) = (-\infty, -1] \cup [1, \infty) = \sigma(H)$ and otherwise

$$\sigma_{ess}(H_\kappa) = (-\infty, -1] \cup [1, \infty) = \sigma_{ess}(H)$$

if at least $V(x) \to 0$ as $x \to \infty$ [16]. If in addition $V \leq 0$, then -1 is not an accumulation point of discrete eigenvalues [15, Thm. 10.37].

3. A Sturm-Liouville Equation equivalent to the Dirac Equation

The purpose of this section is to prove the equivalence of the radial Dirac equations $(H_\kappa - \lambda)\psi = 0$ to the λ-dependent Sturm-Liouville equation obtained by eliminating F. Formal equivalence is easy to see. The hard part is to show that the solutions belong to the prescribed spaces.

To begin with we recall that $f \in H_0^1(\mathbb{R}_+)$ if and only if f is absolutely continuous on $[0,\infty)$, $f, f' \in L^2(\mathbb{R}_+)$, and $f(0) = 0$ (we define absolute continuity locally). Suppose $\varepsilon := 2 - \sup_{x>0} V(x) > 0$, V is absolutely continuous and $\lambda \in (1-\varepsilon, 1)$. By the definition of H_κ, λ is an eigenvalue if and only if the equations (1,2) are satisfied for some $F, G \in H_0^1(\mathbb{R}_+)$, which may be assumed real-valued. By (2), G', and hence $(\lambda + 1 - V)^{-1}G'$, is absolutely continuous. Eliminating F with the help of (2) we see that G solves the boundary value problem

$$-(r(x,\lambda)G'(x))' + p(x,\lambda)G(x) = 0, \quad G(0) = 0, \tag{5}$$

where

$$r(x,\lambda) = (1 + \lambda - V(x))^{-1}$$

$$p(x,\lambda) = (1 + \lambda - V(x))^{-1}\frac{\kappa(\kappa+1)}{x^2} + (1 + V(x) - \lambda) - (1 + \lambda - V(x))^{-2}V'(x)\frac{\kappa}{x}.$$

Under some additional assumptions on V the converse is true as well:

Theorem 1. *Suppose $V \in L^\infty(\mathbb{R}^3;\mathbb{R})$ is absolutely continuous and $\varepsilon := 2 - \sup_{x>0} V(x)$ is positive. Furthermore, suppose $xV'(x) \to 0$ as $x \to 0$, $V'(x)/x \to 0$ as $x \to \infty$, $\liminf_{x\to\infty} V(x) \geq 0$ and $\kappa \neq -1$. Then $\lambda \in (1-\varepsilon, 1)$ is an eigenvalue of H_κ if and only if (5) has an L^2-solution G which belongs to $C^1(\mathbb{R}_+)$ and has absolutely continuous derivative. In this case (G, F) with*

$$F := (\lambda + 1 - V)^{-1}\left(G' + \frac{\kappa}{x}G\right)$$

is an eigenvector of H_κ belonging to λ.

Remark. Analogously, if $V(x) \geq -2 + \varepsilon$ and $\kappa \neq 1$, then $\lambda \in (-1, -1+\varepsilon)$ is an eigenvalue of H_κ if and only if there exists a solution $F \in L^2$ of the Sturm-Liouville equation obtained by eliminating G.

Proof. Suppose G is an L^2-solution of (5), $G \in C^1(\mathbb{R}_+)$ and G' is absolutely continuous. If $(\lambda + 1 - V)^{-1}\left(G' + \frac{\kappa}{x}G\right)$ is substituted for F in the Dirac equations (1,2), then equation (2) is trivially satisfied and (1) follows from the equation in (5). It remains to prove that G and F belong to $H_0^1(\mathbb{R}_+)$. To do this we show that

- (i) G' is square integrable at $+\infty$.
- (ii) There exists a $k > 1$ such that $|G(x)| \leq \text{const } x^{k+1/2}$ for small x.
- (iii) $G'(x)/x$ is square integrable at 0 and $G'(x) \to 0$ as $x \to 0$.

It then follows that $G', G/x \in L^2(\mathbb{R}_+)$ and hence that $F \in L^2(\mathbb{R}_+)$, that $F(0) = 0$, and that $F' \in L^2(\mathbb{R}_+)$ because $G'/x, G/x^2 \in L^2(\mathbb{R}_+)$ and F solves (1). Thus $F, G \in H_0^1(\mathbb{R}_+)$.

Henceforth the second argument of $r(x,\lambda)$ and $p(x,\lambda)$ will be fixed, and we therefore drop it. The conditions on V ensure that there exist constants $x_0, x_1, r_0, R \in \mathbb{R}_+$, depending on λ, such that

$$0 < r_0 \leq r(x) \leq R \qquad x \in \mathbb{R} \tag{6}$$

$$p(x) \geq 0 \qquad x \geq x_1 \tag{7}$$

$$p(x) \geq r(x)\frac{k^2}{x^2} \qquad x \leq x_0 \tag{8}$$

where $k^2 = \kappa(\kappa+1) - 1/2 > 1$. This is the only place in this proof where we need $\kappa \neq -1$. In the following the function $H(x) = rG'G(x)$ will be of importance. Its derivative is by (5)

$$H'(x) = pG^2(x) + r(G')^2(x). \tag{9}$$

Proof of (i). If $x \geq x_1$, then $H'(x) \geq 0$, i.e., $H(x)$ is monotonically increasing. There are thus two cases: Either $H(x_2) > 0$ for some $x_2 \geq x_1$ (case 1), or $H(x) \leq 0$ for all $x \geq x_1$ (case 2). In case 1

$$0 < H(x_2) \leq H(x) = \frac{1}{2} r (G^2)'(x) \leq \frac{1}{2} R (G^2)'(x), \qquad x \geq x_2$$

which implies

$$\frac{R}{2}[G^2(x) - G^2(x_2)] \geq (x - x_2) H(x_2) \to \infty \qquad (x \to \infty)$$

in contradiction to $G \in L^2(\mathbb{R}_+)$. Thus we must have case 2; hence,

$$r_0 \int_{x_1}^{x} (G')^2 dy \leq \int_{x_1}^{x} r (G')^2 dy \leq \int_{x_1}^{x} H'(y) dy \leq -H(x_1)$$

for all $x > x_1$. This proves (i).
Proof of (ii). By (8) and (9) we have for $x \leq x_0$

$$H'(x) \geq r \left(\frac{k}{x}\right)^2 G^2 + r(G')^2 \geq 2r\frac{k}{x}|GG'| = 2\frac{k}{x}|H(x)|. \tag{10}$$

This implies

$$\frac{d}{dx}\left(H(x) x^{\pm 2k}\right) \geq 0, \qquad x \leq x_0, \tag{11}$$

and $H'(x) \geq 0$ for $x \leq x_0$. There are again two possible cases: either $H(x_2) < 0$ for some $x_2 \in (0, x_0]$ (case 1), or $H(x) \geq 0$ for all $x \in (0, x_0]$ (case 2). In case 1

$$0 > H(x_2) \geq H(x) = \frac{1}{2} r (G^2)'(x) \geq \frac{1}{2} R (G^2)'(x) \qquad x \leq x_2$$

which, using $G(0) = 0$, leads to the contradiction

$$\frac{R}{2} G^2(x_2) = \frac{R}{2} \int_0^{x_2} (G^2)' dy \leq x_2 H(x_2) < 0.$$

Therefore, case 2 is realized and $H(0) = \lim_{x \to 0} H(x) \geq 0$. By (11), $H(x)x^{-2k} \leq H(x_0)x_0^{-2k} =: c_0$, that is

$$H(x) \leq c_0 x^{2k}, \qquad x \leq x_0, \tag{12}$$

and by (10)

$$\left|\frac{d}{dx} G^2(x)\right| = \frac{2}{r}|H(x)| \leq \frac{x}{kr} H'(x).$$

This with $G(0) = 0$ leads to

$$\begin{aligned} G^2(x) \leq \int_0^x \left|\frac{d}{dy} G^2(y)\right| dy &\leq \frac{1}{r_0} \int_0^x \frac{y}{k} H'(y) dy \\ &\leq \frac{x}{kr_0}[H(x) - H(0)] \leq \frac{c_0}{kr_0} x^{2k+1} \qquad x \leq x_0 \end{aligned}$$

which proves (ii).

Proof of (iii). Recall from above that $H(x) \geq 0$ for all $x \in (0, x_0]$. By (8) and (9), $rG'(x)^2/x^2 \leq H'(x)/x^2$ for $x \in (0, x_0]$. Now integrate by parts and use (12) to see that

$$\int_\varepsilon^{x_0} \frac{H'(y)}{y^2} dy \leq \frac{H(x_0)}{x_0^2} + 2c_0 \int_\varepsilon^{x_0} x^{2k-3} dx \leq \text{const}$$

where the constant is independent of $\varepsilon > 0$ because $2k - 3 > -1$. This shows that $G'(x)/x$ is square integrable near 0.

Finally, we prove that $G'(x) \to 0$ as $x \to 0$. For $x \leq x_0$ the function $G^2(x)$ is monotonically increasing because $G'(x)G(x) = H(x)/r(x) \geq 0$. We may assume $G(x) > 0$ for $x \in (0, x_0]$ (otherwise $-G(x) > 0$ near 0 or there is nothing to prove). Then $(rG')' = pG \geq 0$ for $x \leq x_0$, i.e., rG' is monotonically increasing near 0. Furthermore, $rG'(x) \geq 0$ near 0 because $G(x) > 0$ and $rG'G(x) = H(x) \geq 0$ near 0. Hence, the limit $c_1 = \lim_{x\to 0} rG'(x) \geq 0$ exists and $G'(x) \geq c_1/R$ near 0. If $c_1 > 0$ this is not compatible with $|G(x)| \leq \text{const}\ x^{k+1/2}$. Therefore, $c_1 = 0$ and $G'(x) \to 0$ as $x \to 0$. □

4. The Main Result

The main theorem, which is a special, simplified case of the results in [10, 11], treats accumulation/nonaccumulation of eigenvalues for the problem

$$-(r(x;\lambda)f'(x))' + p(x;\lambda)f(x) = 0 \quad x \in [a,\infty), \tag{13}$$

$$\alpha(\lambda)f(a) + \beta(\lambda)f'(a) = 0 \tag{14}$$

where the spectral parameter λ varies in an interval $\Lambda = (\mu, \nu]$ with ν finite (an eigenvalue is a value λ_0 such that for $\lambda = \lambda_0$ eqn. (13) has an L^2-solution satisfying (14)). The following conditions are assumed to hold:

Assumptions

i) $r, p : [a,\infty) \times \Lambda \to \mathbb{R}$ are continuous and r is positive.

ii) $\alpha, \beta : \Lambda \to \mathbb{R}$ are continuous, $\alpha(\lambda)^2 + \beta(\lambda)^2 \neq 0$ for all $\lambda \in \Lambda$, and β is either never zero or is identically zero.

iii) There exist functions $\chi : (\mu,\nu) \to [a,\infty)$ and $\eta : (\mu,\nu) \to (0,\infty)$, the first one being continuous, such that $p(x;\lambda) \geq \eta(\lambda)$ for all $(x;\lambda) \in [\chi(\lambda),\infty) \times (\mu,\nu)$.

Theorem 2. *(Main Theorem)*

i) *If equation (13) with $\lambda = \nu$ is oscillatory on $[a,\infty)$, then ν is an accumulation point of eigenvalues of (13, 14) from the left.*

ii) *If $r(a;\lambda)\alpha(\lambda)/\beta(\lambda)$ is increasing, $r(x;\lambda)$ is decreasing, and $p(x;\lambda)$ is strictly decreasing in λ for each x, and if equation (13) with $\lambda = \nu$ is nonoscillatory on $[a,\infty)$, then ν is not an accumulation point of eigenvalues of (13, 14) from the left.*

The question whether (13) is oscillatory for $\lambda = \nu$ can be settled by applying a generalization [10, 11] of Kneser's Criterion [4, Cor.XIII.7.37]. Specifically, if for some $\varrho \in \mathbb{R}$,

$$\limsup_{x\to\infty} \frac{r(x;\nu)}{x^\varrho} < \infty, \quad \limsup_{x\to\infty} \frac{p(x;\nu)}{x^{\varrho-2}} < -\frac{(1-\varrho)^2}{4} \limsup_{x\to\infty} \frac{r(x;\nu)}{x^\varrho},$$

then we have oscillation, whereas if

$$\liminf_{x\to\infty} \frac{r(x;\nu)}{x^\varrho} > 0, \quad \liminf_{x\to\infty} \frac{p(x;\nu)}{x^{\varrho-2}} > -\frac{(1-\varrho)^2}{4} \liminf_{x\to\infty} \frac{r(x;\nu)}{x^\varrho},$$

then we have nonoscillation.

Theorem 3. *(Main Result) Assume the hypotheses in Theorem 1, that V' is continuous, and that $V(x) \to 0$ as $x \to \infty$.*

i) *If $-\limsup_{x\to\infty} \{V(x)x^2 - \kappa V'(x)x/(2 - V(x))^2\} > 1/8 + \kappa(\kappa+1)/2$, then $+1$ is an accumulation point of eigenvalues of H_κ in the gap $(-1, 1)$.*

ii) *Suppose there is a constant $\delta < 1$ such that $\kappa V'(x)x < \delta\left\{\kappa(\kappa+1)+4x^2\right\}$ for all $x > 0$. If $-\liminf_{x\to\infty}\left\{V(x)x^2 - \kappa V'(x)x/(2-V(x))^2\right\} < 1/8+\kappa(\kappa+1)/2$, then $+1$ is not an accumulation point of eigenvalues of H_κ in the gap $(-1,1)$.*

That a condition similar to the one in part i) of the Theorem still implies accumulation when $\kappa = -1$ and $-2 < V(x) \leq 0$ is proved in the Appendix. The statement in part ii) also still holds for $\kappa = -1$ provided that in addition $V'(0) > 0$. This occurs because the "only if" part of Theorem 1 also holds for $\kappa = -1$, i.e., finiteness of the number of eigenvalues of (5) in the gap for $\kappa = -1$ implies the same for the Dirac problem (1,2). Further, the proof of Theorem 3 still works when $p(x;\lambda) \geq k/x$ for $x \in (0, x_0], \lambda \in (0,1)$ with some constants $k, x_0 > 0$ which is indeed the case in (5) when $\kappa = -1$ and $V'(0) > 0$. Applying these results under the additional assumptions that $-V(x)x^\gamma \to c$ and $V'(x)x \to 0$ as $x \to \infty$ for some positive constants c, γ proves the comments in the introduction for c not equal to the critical coupling constant (the critical case is treated in the following remarks).

Remarks i) It may happen that neither of the inequalities in Kneser's Criterion hold; nevertheless, it is still possible to obtain results in such "critical cases". For simplicity consider the critical case

$$0 < \lim_{x\to\infty} r(x;\nu) < \infty, \quad \lim_{x\to\infty} p(x;\nu)x^2 = -\frac{1}{4}\lim_{x\to\infty} r(x;\nu).$$

Employing a sort of "refined" Kneser's criterion in which we compare the function $p(x;\nu)$ with one of the form $x^{-2}\{-1/8+\text{const}\cdot(\log x)^{-2}\}$ near $+\infty$ rather than with $\text{const}\cdot x^{-2}$ (in fact, there is a whole sequence of such refinements [6, pp.325,362]) we find that now

$$\limsup_{x\to\infty} x^2(\log x)^2\left\{p(x;\nu)+\frac{1}{8x^2}\right\} < -\frac{1}{4}\lim_{x\to\infty} r(x;\nu)$$

is sufficient for oscillation of (13) with $\lambda = \nu$ and

$$\liminf_{x\to\infty} x^2(\log x)^2\left\{p(x;\nu)+\frac{1}{8x^2}\right\} > -\frac{1}{4}\lim_{x\to\infty} r(x;\nu)$$

is sufficient for nonoscillation. These comments can be applied to the critical case in the introduction: If $x(\log x)^2V'(x) \to 0$ and $(\log x)^2(c+V(x)x^2) \to 0$ as $x \to \infty$ with c equal to the critical coupling constant $1/8+\kappa(\kappa+1)/2$ (which is valid, for example, when $V(x) = -c/x^2$ near $+\infty$), then the limits on the left above both equal 0, and, consequently, $+1$ is not an accumulation point of eigenvalues.

ii) The procedure above can easily be applied to the radial Schrödinger equation

$$\left\{-\frac{1}{2}\frac{d^2}{dx^2}+\frac{\kappa(\kappa+1)}{2x^2}+V(x)-\lambda\right\}f = 0 \quad (x\in(0,\infty)), \quad f(0) = 0$$

as well. In particular, assuming V is continuous, bounded, and nonpositive on $(0,\infty)$, $-V(x)x^\gamma \to c$ as $x \to \infty$ for some constants $c, \gamma > 0$, and $\kappa(\kappa+1) > 0$, then 0 is an accumulation point of negative eigenvalues if $\gamma < 2$ or if $\gamma = 2$ and $c > 1/8+\kappa(\kappa+1)/2$, and 0 is not an accumulation point if $\gamma > 2$ or if $\gamma = 2$ and $c < 1/8+\kappa(\kappa+1)/2$. In the critical case $(c+V(x)x^2)(\log x)^2 \to 0$ as $x \to \infty$ where $c = 1/8+\kappa(\kappa+1)/2$ accumulation does not occur at 0. Here the monotonicity conditions in part ii) of the theorem are trivial, whereas for the radial Dirac equations we had to make additional assumptions on V' to guarantee that these hold.

iii) The radial Schrödinger equation above emerges as a nonrelativistic limit of the Dirac equations (1,2) (see [2, 7]). This explains the similarity of the results in Theorem 3 and remark ii) (see also the introduction). The condition on V' in Theorem 3 can be understood by considering the first relativistic corrections to the (radial) Schrödinger equation. It is due to the term $-(1+\lambda-V(x))^{-2}V'(x)\kappa/x$

in equation (5), which is close to $-V'(x)\kappa/(4x)$ for $\lambda = 1$ and $V(x) \simeq 0$. But this is exactly the term proportional to V' due to the Darwin term and the spin-orbit interaction, the spin-orbit interaction providing the larger part if $|\kappa| > 1$. Our condition on V' compares $-V'(x)\kappa/(4x)$ with $\kappa(\kappa+1)/(2x^2)$, i.e., the spin-orbit interaction with the energy due to the orbital angular momentum.

The similarity of the above results for the radial Dirac equations and the radial Schrödinger equation and the intuition discussed in the introduction concerning the reason for this similarity suggest the following conjecture:

Conjecture. *Suppose V satisfies the hypotheses in Theorem 1 and in ii) of Theorem 3, $V(x) \to 0$ and $xV'(x) \to 0$ as $x \to \infty$ (i.e., $V'(x)/x$ decays faster than x^{-2}). Then the eigenvalues of the system (1,2) accumulate at 1 from below if and only if the eigenvalues of the radial Schrödinger equation accumulate at 0 from below.*

The "if" part of this conjecture holds true whenever the hypotheses of Theorem 4 are satisfied. This follows from equation (17) in the proof of this theorem.

Sketch of the proof of the main theorem. See [10, 11] for the details. For each $\lambda \in \Lambda$ let y_∞ be the solution of (13) determined by the initial conditions $y_\infty(\chi(\lambda);\lambda) = 1 = r(\chi(\lambda);\lambda)y'_\infty(\chi(\lambda);\lambda)$. Then, as functions of x, y_∞ and ry'_∞ are increasing on $[\chi(\lambda),\infty)$, and a second solution is defined by

$$w(x;\lambda) := c_\infty(\lambda)y_\infty(x;\lambda)\int_x^\infty \frac{ds}{r(s;\lambda)y_\infty(s;\lambda)^2} \qquad \chi(\lambda) \le x < \infty,$$

where $c_\infty(\lambda)$ is a constant such that $w(\chi(\lambda);\lambda) = 1$. Extended as solutions on $[a,\infty)$, the $w(\cdot\,;\lambda)$ have the following properties: $w(\cdot\,;\lambda)$ is in $L^2[a,\infty)$ and is positive and decreasing on $[\chi(\lambda);\infty)$, w and $\partial w/\partial x$ are jointly continuous in x and λ, and, if $r(x;\lambda)$ is decreasing and $p(x;\lambda)$ is strictly decreasing in λ for fixed x, then $\lambda \mapsto r(a;\lambda)w'(a;\lambda)/w(a;\lambda)$ is strictly increasing on intervals where the denominator is nonzero. In view of assumption iii), (13) is in the limit point case at ∞ for each λ, i.e., there is, up to constant multiple, at most one L^2-solution; hence, λ is an eigenvalue if and only if $\alpha(\lambda)w(a;\lambda) + \beta(\lambda)w'(a;\lambda) = 0$. Extending the coefficients r, p and the solutions w to the whole real axis by setting $r(x;\lambda) := r(a;\lambda)$ and $p(x;\lambda) := p(a;\lambda) + (x-a)$ for $x \le a$, it follows that $w(\cdot\,;\lambda)$ has infinitely many zeros $\cdots < z_{n+1}(\lambda) < z_n(\lambda) < \cdots < z_1(\lambda) < \chi(\lambda)$ accumulating only at $-\infty$, and, with the help of the Implicit Function Theorem, that the z_n are continuous. Consider now a pair of successive crossings of zero curves with the line $x = a$, i.e., points λ_n and λ_{n+1} such that $z_n(\lambda_n) = a = z_{n+1}(\lambda_{n+1})$ and such that no zero curve crosses $x = a$ between λ_n and λ_{n+1}. Then there is at least one eigenvalue λ in $[\lambda_n, \lambda_{n+1})$: In case $\beta \equiv 0$, λ_n is obviously an eigenvalue. If β is never zero, then the right side of the equation

$$r(a;\lambda)\frac{w'(a;\lambda)}{w(a;\lambda)} = -r(a;\lambda)\frac{\alpha(\lambda)}{\beta(\lambda)} \tag{15}$$

is bounded on $(\lambda_n, \lambda_{n+1})$. Note that $w(a;\lambda)$ is nonzero on $(\lambda_n, \lambda_{n+1})$ but converges to zero at the endpoints and that $w'(a;\lambda)$ has opposite sign at the endpoints ($w'(\cdot\,;\lambda_{n+1})$ changes sign between the successive zeros $z_n(\lambda_{n+1})$ and $z_{n+1}(\lambda_{n+1}) = a$ of $w(\cdot\,;\lambda_{n+1})$, but $w'(z_n(\lambda);\lambda)$ is continuous in λ and nonzero; hence has constant sign). Thus, the range of the left side at (15) on $(\lambda_n, \lambda_{n+1})$ is $\mathbb{R}$ and, by continuity, there must be at least one solution. If (13) is oscillatory for $\lambda = \nu$, a continuity argument shows that the number of zeros $N(\lambda)$ of $w(\cdot\,;\lambda)$ on $[a,\infty)$ must go to ∞ as $\lambda \to \nu$; hence, there are infinitely many pairs of crossings as above and hence infinitely many eigenvalues (which accumulate at ν). Next assume the monotonicity conditions on r and p. Then, for each $\lambda \in \Lambda$, (13) is a Sturm majorant for the equation with $\lambda = \nu$. Thus, nonoscillation for $\lambda = \nu$ implies $N(\lambda)$ is bounded on Λ. Furthermore, it can also be shown that the zero curves are now increasing; hence, there can only be finitely many crossings with the line $x = a$. Since, as mentioned above, the left side of (15) is now strictly increasing and we assume the right side is decreasing, there can be at most one solution between crossings; hence, only finitely many eigenvalues. □

Proof of Theorem 3. Let r and p be as in (5), and choose constants $k, x_0, r_0 > 0$ such that $r(x;\lambda) \geq r_0$, $p(x;\lambda) \geq k/x^2$, and $\partial p/\partial\lambda < 0$ for all $(x;\lambda) \in (0, x_0] \times (0,1]$. Note that $\partial r/\partial\lambda < 0$ holds automatically for all x and λ. For each $\lambda \in (0,1]$ let $y_0(\cdot\,;\lambda)$ be the solution of the differential equation in (5) determined by the initial conditions $y_0(x_0;\lambda) = 1$, $r(x_0;\lambda)y_0'(x_0;\lambda) = -1$. As functions of x, y_0 is decreasing and ry_0' is increasing on $(0, x_0]$. For each $\lambda \in (0,1]$ define a new solution by

$$v(x;\lambda) := c_0(\lambda) y_0(x;\lambda) \int_0^x \frac{ds}{r(s;\lambda) y_0(s;\lambda)^2} \qquad 0 < x \leq x_0,$$

where $c_0(\lambda)$ is a constant such that $v(x_0;\lambda) = 1$. The following properties hold (see [10, 11] for the details): v is positive and increasing on $(0, x_0]$, $v(x;\lambda) \to 0$ $(x \to 0)$, v and $\partial v/\partial x$ are jointly continuous in x and λ, and $\lambda \mapsto r(x_0;\lambda)v'(x_0;\lambda)$ is decreasing on $(0,1]$. Since $\lim_{x\to 0} y_0(x;\lambda) > 0$, a solution G will satisfy the boundary condition in (5) if and only if it is a constant multiple of v, i.e., if and only if $v'(x_0;\lambda)G(x_0) - G'(x_0) = 0$. Thus, the eigenvalue problem (5) is equivalent to one of the type (13, 14) on $[x_0, \infty)$. The additional boundedness assumption in ii) is made to guarantee that $\partial p/\partial\lambda < 0$ for all $x \geq x_0$ and $\lambda \in (0,1)$. Simply calculating the limits in the comments before the Theorem with $\varrho = 0$ and $\nu = 1$ and applying the Main Theorem gives the result. □

5. Appendix

Here we give a second proof of the accumulation result which works for $\kappa = -1$ as well. It is based on the minimax principle of Siedentop and Griesemer [5] and the accumulation result for the radial Schrödinger equation explained in remark ii) after Theorem 3.

Theorem 4. *Suppose $V \in L^\infty(\mathbb{R}_+)$, $0 \geq V(x) > -2$ a.e. and $V(x) \to 0$ as $x \to \infty$. Then discrete eigenvalues of H_κ accumulate at $+1$ whenever*

$$-\limsup_{x\to\infty} x^2 V(x) > \frac{1}{8} + \frac{\kappa(\kappa+1)}{2}. \tag{16}$$

Proof. Notice that H_κ has the form of a perturbed supersymmetric Dirac operator. In fact $H_\kappa = Q + \beta + V$ where

$$Q = \begin{pmatrix} 0 & -\frac{d}{dx} + \frac{\kappa}{x} \\ \frac{d}{dx} + \frac{\kappa}{x} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the decomposition $L^2(\mathbb{R}_+) \otimes \mathbb{C}^2 = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$, and $Q\beta + \beta Q = 0$ on $D(Q)$. Hence by [5, Theorem 8]

$$\dim\ P_{(-1,1)}(\mathfrak{H}_\kappa)\mathfrak{H} \geq \dim\ P_{(-\infty,0)}(h) L^2(\mathbb{R}_+) \tag{17}$$

where h is the restriction of $Q^2/2 + V$ to the subspace of vectors with vanishing lower component. A straight forward computation shows that h is the radial Schrödinger operator, i.e., on $C_0^\infty(\mathbb{R}_+)$, $h = -1/2(d^2/dx^2) + \frac{\kappa(\kappa+1)}{2x^2} + V$. Moreover, as we shall prove below, any solution $f \in L^2(\mathbb{R}_+)$ of

$$-\frac{1}{2}f'' + \frac{\kappa(\kappa+1)}{2x^2}f + (V-\lambda)f = 0, \quad f(0) = 0, \tag{18}$$

f, f' being absolutely continuous on $[0,\infty)$, is an eigenvector of h with eigenvalue λ. But this equation has infinitely many L^2-solutions with eigenvalues λ accumulating at 0 from below by (16) and remark ii) after Theorem 3. Hence h has infinitely many negative eigenvalues. The theorem now follows from (17) and the fact that discrete eigenvalues of H_κ cannot accumulate at -1.

Clearly the operator Q is self-adjoint on $D(H_\kappa) = H_0^1(\mathbb{R}_+) \otimes \mathbb{C}^2$ which is thus the form domain of Q^2. Hence $H_0^1(\mathbb{R}_+)$ is the form domain of h and (by Hardy's inequality [15, Theorem 10.35]) $C_0(\mathbb{R}_+)$ is a form core of h. This and $h = h^*$ imply that $D(h) = \{f \in H_0^1(\mathbb{R}_+) | \ \exists f^* \in L^2 \text{ s.t. } \langle hg, f\rangle = \langle g, f^*\rangle \ \forall g \in C_0(\mathbb{R}_+)\}$. A solution f of (18) belongs to this set because it belongs to $H_0^1(\mathbb{R}_+)$ (see the proof of Theorem 1) and because

$$\langle hg, f\rangle = \langle g, \lambda f\rangle \quad \text{for all} \quad g \in C_0(\mathbb{R}_+) \tag{19}$$

(integrate twice by parts). But (19) shows that $hf = \lambda f$ which completes the proof. $\square$

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