On Formal Quasi-Periodic Solutions of the Schrödinger Equation for a Two-Level System with a Hamiltonian Depending Quasi-Periodically on Time

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Abstract: *We consider the Schrödinger equation for a class of two-level atoms in a quasi-periodic external field for large coupling, i.e., for which the energy difference 2ϵ between the unperturbed levels is sufficiently small. We show that this equation has a solution in terms of a formal power series in ϵ, with coefficients which are quasi-periodical functions of the time, in analogy to the Lindstedt-Poincaré series in classical mechanics.*

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1 Introduction and Summary

The problem of existence of quasi-periodic solutions of the Schrödinger equation with quasi-periodic coefficients is a classical problem in analysis, with recent applications to the theory of "quantum chaos" [1] (see also [2] for a review with recent results). In the scalar almost-periodic case, the solution is provided by a theorem of H. Bohr [3] (generalizing previous results by P. Bohl for the quasi-periodic case [4]). We now pass from the scalar case to the simplest non-commutative setting, i.e., the Schrödinger equation for two-level systems (e.g. atoms) in a quasi-periodic (external) field, with Hamiltonian

$$H(t) \ = \ \epsilon\sigma_3 - f(t)\sigma_1, \tag{1.1}$$

where σ_1, σ_2 and σ_3 denote the Pauli matrices, which satisfy the SU(2)-commutation relations $[\sigma_1,\ \sigma_2] = 2i\sigma_3$ plus cyclic permutations.

In (1.1), $f(t)$ is a quasi-periodic function of time t

$$f(t) \ = \ \sum_{\underset{\sim}{m}\in\mathbb{Z}^B} f_{\underset{\sim}{m}}\, e^{i\underset{\sim}{m}\cdot\underset{\sim}{\omega}_f t} \tag{1.2}$$

and ϵ is a "small" parameter (half of the energy difference between the unperturbed levels), which characterizes the large coupling domain [5]. Above $\underset{\sim}{\omega}_f$ is a vector of frequencies $\underset{\sim}{\omega}_f = (\omega_f^1, \ldots, \omega_f^B)$ for some $B \geq 1$, where we will assume $\omega_f^j > 0$ for all $1 \leq j \leq B$. Moreover, $\underset{\sim}{m} \cdot \underset{\sim}{\omega}_f$ denotes $m_1\omega_f^1 + \cdots + m_B\omega_f^B$.

We have chosen the regime where ϵ is small because the result which is equivalent [1] to the existence of quasi-periodic solutions in this case – the pure point spectrum of the so-called generalized quasi-energy operator [6] – has not yet been proven in full generality [5].

The analogous problem in classical mechanics has been very nicely sketched in [7] – albeit just as a pedagogical introduction to perturbative renormalization in quantum field theory. The main theorem of [7] – existence of formal <u>quasi-periodic</u> solutions of a class of quasi-periodic Hamiltonian systems as a series in ϵ – the Lindstedt-Poincaré series – has a counterpart – for (1.1) – in Theorem 2.2 of the present paper. In classical mechanics there is, however, a method, due to Eliasson [8], to isolate the divergent contributions to the Lindstedt-Poincaré series (see [7] for a simple but illuminating discussion on these terms) and introduce further renormalization counter-terms ensuring convergence. See [9] for a realization of this idea in a special model and [10] for a review of the models and method.

The question now poses itself whether methods similar to those discussed above might yield the existence of a quasi-periodic solution to the Schrödinger equation

$$i\frac{\partial\psi(t)}{\partial t} \ = \ H(t)\psi(t) \tag{1.3}$$

with $H(t)$ given by (1.1). The natural guide is the (convergent) Dyson perturbation series, which we describe now. By a rotation of $\pi/2$ around the 2-axis, we may replace $H(t)$ by

$$H_1(t) \ = \ e^{-i\pi\sigma_2/4}H(t)e^{i\pi\sigma_2/4} \ = \ \epsilon\sigma_1 + f(t)\sigma_3 \tag{1.4}$$

and the Schrödinger equation becomes

$$i\frac{\partial\Phi(t)}{\partial t} \ = \ H_1(t)\Phi(t), \tag{1.5}$$

with

$$\Phi(t) \ := \ e^{-i\pi\sigma_2/4}\psi(t). \tag{1.6}$$

Defining now

$$\tilde{\Phi}(t) \ \equiv \ \exp\left(i\alpha(t)\sigma_3\right)\Phi(t), \tag{1.7}$$

where

$$\alpha(t) \equiv \int_0^t d\tau \ f(\tau) \tag{1.8}$$

the Schrödinger equation (1.5) becomes

$$i\frac{\partial\tilde{\Phi}(t)}{\partial t} \ = \ \epsilon\widetilde{H}(t)\tilde{\Phi}(t), \tag{1.9}$$

where

$$\widetilde{H}(t) \equiv \begin{pmatrix} 0 & e^{2i\alpha(t)} \\ e^{-2i\alpha(t)} & 0 \end{pmatrix}. \tag{1.10}$$

The Dyson series solution to (1.10) is

$$\tilde{\Phi}(t) \ = \ \tilde{\Phi}(0) + \sum_{n=1}^{\infty}\frac{(-i\epsilon)^n}{n!}\int_0^t d\tau_1\cdots\int_0^{\tau_{n-1}} d\tau_n \ \widetilde{H}(\tau_1)\cdots\widetilde{H}(\tau_n)\ \tilde{\Phi}(0). \tag{1.11}$$

Let, now, f be such that

$$q(t) \ := \ e^{i\alpha(t)} \ = \ \exp\left(i\int_0^t \ f(\tau)\ d\tau\right) \tag{1.12}$$

is quasi-periodic (see Appendix B), and write

$$q(t)^2 \ = \ \sum_{\underline{n}\in\mathbb{Z}^A} g_{\underline{n}}e^{i\underline{n}\cdot\underline{\omega}t}, \tag{1.13}$$

where $\underline{\omega} \equiv (\omega_1, \ \ldots, \ \omega_A)$ is a frequency vector satisfying some Diophantine conditions to be stated later on, and assume that (for the purpose of this introduction)

$$g_{\underline{0}} \in \mathbb{R}. \tag{1.14}$$

Relation (1.14) is seen to hold explicitly, for instance, if $f(t) = f(\omega_1 t, \ldots, \omega_B t) = \sum_{i=1}^{B}\cos(\omega_i t)$. Then, explicit summation of all orders in (1.11) of the $\underline{n} = \underline{0}$ term in (1.10), which is, by (1.13) and (1.14) of the form $g_{\underline{0}}\sigma_1$, we find

$$\tilde{\Phi}(t) = \left[\cos(tg_{\underline{0}}) - i\sin(tg_{\underline{0}})\sigma_1\right]\tilde{\Phi}(0), \tag{1.15}$$

which is quasi-periodic. This illustrates that non-uniform convergence of a sequence of non-quasi-periodic functions may yield a quasi-periodic function and, in fact, this does occur in the general solution (1.11). Coupled with the fact that the general n-th term in (1.11) grows as t^n, it is obvious that (1.11) is grossly unsuitable to show quasi-periodicity of the wave function. We therefore try to subtract (1.15) explicitly from the solution of (1.9) by writing

$$\epsilon\widetilde{H}(t) \ = \ M_\epsilon + \widetilde{H}_1(t), \tag{1.16}$$

where

$$M_\epsilon \;=\; \epsilon \begin{pmatrix} 0 & g_{\underline{0}} \\ \overline{g_{\underline{0}}} & 0 \end{pmatrix} = \epsilon g_{\underline{0}} \sigma_1. \tag{1.17}$$

under assumption (1.14). In correspondence to (1.16) we set

$$\widetilde{\Phi}_1(t) \;=\; e^{iM_\epsilon t} \widetilde{\Phi}(t). \tag{1.18}$$

The equation for $\widetilde{\Phi}_1$ is

$$i \frac{\partial \widetilde{\Phi}_1(t)}{\partial t} \;=\; \widetilde{H}^{(I)}_{1,\,\epsilon}(t) \widetilde{\Phi}_1(t), \tag{1.19}$$

where

$$\widetilde{H}^{(I)}_{1,\,\epsilon}(t) \;\equiv\; e^{iM_\epsilon t} \widetilde{H}_1(t) e^{-iM_\epsilon t}. \tag{1.20}$$

The Dyson series (1.11) for (1.20) has the first order term

$$-i\epsilon \int_0^t d\tau \; \widetilde{H}^{(I)}_{1,\,\epsilon}(\tau) \widetilde{\Phi}_1(0) =$$

$$-i\epsilon \int_0^t d\tau \begin{pmatrix} i\sin(\epsilon g_{\underline{0}}\tau)\cos(\epsilon g_{\underline{0}}\tau)(\overline{\lambda(\tau)} - \lambda(\tau)) & \sin^2(\epsilon g_{\underline{0}}\tau)\overline{\lambda(\tau)} + \cos^2(\epsilon g_{\underline{0}}\tau)\lambda(\tau) \\ \sin^2(\epsilon g_{\underline{0}}\tau)\lambda(\tau) + \cos^2(\epsilon g_{\underline{0}}\tau)\overline{\lambda(\tau)} & i\sin(\epsilon g_{\underline{0}}\tau)\cos(\epsilon g_{\underline{0}}\tau)(\lambda(\tau) - \overline{\lambda(\tau)}) \end{pmatrix} \widetilde{\Phi}_1(0), \tag{1.21}$$

where $\lambda(t) := q(t)^2 - g_{\underline{0}}$.

Note, however, that, due to the ϵ-dependence of M_ϵ in (1.17), this Dyson series is <u>not</u> a power series in ϵ: there has been a nonlinear in ϵ, non-perturbative "renormalization" embodied in (1.20). Looking at the first matrix element in (1.21) (the discussion of the others is similar), we find

$$\begin{aligned} \epsilon \int_0^t \sin(\epsilon g_{\underline{0}}\tau)\cos(\epsilon g_{\underline{0}}\tau)\left(q(\tau)^{-2} - q(\tau)^2\right) d\tau \;&=\; \frac{i\epsilon}{2} \int_0^t \sin(2\epsilon g_{\underline{0}}\tau) \sum_{\underline{n}\neq\underline{0}} (\overline{g_{-\underline{n}}} - g_{\underline{n}}) e^{i\underline{n}\cdot\underline{\omega}\tau} \; d\tau \\ &=\; -\frac{i\epsilon}{4} \sum_{\underline{n}\neq\underline{0}} (\overline{g_{-\underline{n}}} - g_{\underline{n}}) \left[\frac{e^{i(\underline{n}\cdot\underline{\omega}+2\epsilon g_{\underline{0}})t}}{\underline{n}\cdot\underline{\omega} + 2\epsilon g_{\underline{0}}} - \frac{e^{i(\underline{n}\cdot\underline{\omega}-2\epsilon g_{\underline{0}})t}}{\underline{n}\cdot\underline{\omega} - 2\epsilon g_{\underline{0}}} \right]. \end{aligned} \tag{1.22}$$

In addition to the small denominators of the form $\underline{n}\cdot\underline{\omega} \pm 2\epsilon g_{\underline{0}}$ as found above, there occurs in (1.21) also small denominators of the form $\underline{n}\cdot\underline{\omega}$ and it is necessary to require, for instance

$$\left|\underline{n}\cdot\underline{\omega} \pm 2\epsilon g_{\underline{0}}\right| \;\geq\; \gamma_{\underline{n}} = \gamma(1 + |\underline{n}|^2)^{-\alpha}, \tag{1.23}$$

$$|\underline{n}\cdot\underline{\omega}| \;\geq\; \gamma'_{\underline{n}} = \gamma'(1 + |\underline{n}|^2)^{-\alpha'} \tag{1.24}$$

in order to ensure convergence in (1.22) with $|g_{\underline{n}}| \leq c\exp(-d|\underline{n}|)$, with $d > 0$, the exponential decay being a consequence of analyticity of f in a strip, (1.8) and (1.13). Above, $|\underline{n}|$ is the l^1 norm of the vector $\underline{n}$.

Assume now, for simplicity, that $A = 2$ in (1.13), and set $\alpha = \omega_1/\omega_2 \in (0,\,1)$. Suppose, now, that

$$g_{\underline{0}} \;\neq\; 0 \tag{1.25}$$

and consider (without loss, because ϵ is supposed "small") ϵ such that $\beta \equiv 2\epsilon g_{\underline{0}} \in (0,\ 1)$. It is easy to show using Cassels theorem [11] that, if $\alpha,\ \alpha' > 1$ in (1.23)-(1.24), the set of $(\alpha,\ \beta)$ in the open unit square such that (1.23)-(1.24) are satisfied for all $\gamma,\ \gamma' > 0$ is of full Lebesgue measure. The intersection of this set with the ϵ-axis is thus, however, not an open set and, therefore, no convergence may be expected for ϵ in any (small) open set around the origin. Hence, any power series in ϵ cannot be expected to be analytic, or to converge uniformly in some open set containing the origin. Other strong indications of this fact (not a proof!) will appear later in the context of (pure) power series in ϵ (see the heuristic discussion in Section 7).

Our discussion above relied on (1.25) (otherwise ϵ does not appear in (1.23)). It is interesting to remark that (1.25) holds whenever $\underset{\sim}{f_0} = 0$, (see Appendix C), which was the case left open in [5].

Is it possible to go further and analyze the whole Dyson series for (1.20)? The problem is that higher order terms require further subtractions and we were not able to show that they can be "renormalized" as in (1.20) (with a different M_ϵ, of course). It seems that the natural way to obtain a formal (asymptotic) series in ϵ with quasi-periodic coefficients is to write an Ansatz for (1.1) in exponential form and then to "renormalize" the exponential in an inductive way. This is the purpose of the present paper.

As we will see in Section 2.1 and in Appendix A, the solutions of the Schrödinger equation (1.5) can be studied in terms of the solutions of a complex and quasi-periodic version of Hill's equation:

$$\phi''(t) + \left(\epsilon^2 + if'(t) + f(t)^2\right)\phi(t) = 0. \tag{1.26}$$

We attempt to solve this last equation using the Ansatz $\phi(t) = \exp\left(-i\int_0^t (g(\tau) + f(\tau))d\tau\right)$, where g satisfies a generalized Riccati equation, and we try to find solutions for g in terms of a (formal) power expansion in ϵ like $g(t) = q(t)\sum_{n=1}^{\infty} \epsilon^n\, c_n(t)$.

As discussed above, perturbative solutions of quasi-periodically time-dependent systems are usually plagued by small denominators and by the presence of the so-called "secular terms", i.e. polynomials in t, that spoil the analysis of convergence of the series and the proofs of quasi-periodicity of the perturbative terms. We discovered a particular way to eliminate completely the secular terms from the perturbative expansion of g and we were able, under some special assumptions, to show that the coefficients $c_n(t)$ are all quasi-periodic functions.

We prove explicitly the convergence of our perturbative solution in the somewhat trivial case where $f(t)$ is a non-zero constant function. Unfortunately, however, no conclusion could be drawn about the convergence of the perturbative expansion for g in the general case. We conjecture, however, that our expansion is uniformly convergent at least in the situation where $f(t) - \underset{\sim}{f_0}$ is uniformly small.

Remarks on the notation. In this paper $\mathbb{R}_+$ will denote the open interval $(0,\ \infty)$. Given the Fourier representation (1.2) of the quasi-periodic function f, we will denote by $\underline{\omega}$ the vector of frequencies defined by

$$\underline{\omega} \ := \begin{cases} \underset{\sim}{\omega}_f \in \mathbb{R}^B, & \text{if } f_{\underset{\sim}{0}} = 0 \\ \\ (\underset{\sim}{\omega}_f,\ f_{\underset{\sim}{0}}) \in \mathbb{R}^{B+1}, & \text{if } f_{\underset{\sim}{0}} \neq 0, \end{cases} . \tag{1.27}$$

Since we are assuming that $\underset{\sim}{\omega}_f \in \mathbb{R}^B_+$, the definition above says that all components of $\underline{\omega}$ are always non-zero. Moreover, we will denote

$$A \ := \begin{cases} B, & \text{if } f_{\underset{\sim}{0}} = 0 \\ \\ B+1, & \text{if } f_{\underset{\sim}{0}} \neq 0 \end{cases} . \tag{1.28}$$

We will denote vectors in $\mathbb{Z}^B$ (or $\mathbb{R}^B$) by $\underset{\sim}{v}$ and vectors in $\mathbb{Z}^A$ (or $\mathbb{R}^A$) by $\underline{v}$. The symbol $|\underline{n}|$ will denote the $l^1(\mathbb{Z}^A)$ norm of a vector $\underline{n} = (n_1, \ldots, n_A) \in \mathbb{Z}^A$: $|\underline{n}| := |n_1| + \cdots + |n_A|$.

We will use the symbol $\mathbb{1}$ for the identity matrix. $\mathrm{Mat}(n,\ \mathbb{C})$ is the set of all $n \times n$ matrices with complex entries.

For an almost-periodic (in particular, quasi-periodic) function h we denote by $M(h)$ the *mean value* of h, defined as

$$M(h) \ := \ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(t)\, dt. \tag{1.29}$$

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2 The Main Results

In this section we will state our two main results. The first one describes the solution of the Schrödinger equation (1.5) in terms of a particular solution of a generalized Riccati equation. The reason for considering the Schrödinger equation in the form (1.5) is that, as we will see below and in Appendix A, we will be able to decouple the two components of $\Phi(t)$ by increasing the order of the equation.

Theorem 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and $\epsilon \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{C}$, $g \in C^1(\mathbb{R})$ be a particular solution of the generalized Riccati equation*

$$G' - iG^2 - 2ifG + i\epsilon^2 = 0. \tag{2.1}$$

Then, the function $\Phi : \mathbb{R} \to \mathbb{C}^2$ given by

$$\Phi(t) \;=\; \begin{pmatrix} \phi_+(t) \\ \phi_-(t) \end{pmatrix} \;=\; U(t)\Phi(0), \tag{2.2}$$

where

$$U(t) \;:=\; \begin{pmatrix} R(t)\left(1 + ig(0)S(t)\right) & -i\epsilon R(t)S(t) \\ \\ -i\epsilon\overline{R(t)}\;\overline{S(t)} & \overline{R(t)}\left(1 - i\,\overline{g(0)}\;\overline{S(t)}\right) \end{pmatrix}, \tag{2.3}$$

with

$$R(t) \;:=\; \exp\left(-i\int_0^t (f(t') + g(t'))\,dt'\right) \tag{2.4}$$

and

$$S(t) \;:=\; \int_0^t R(t')^{-2}\,dt' \tag{2.5}$$

is a solution of the Schrödinger equation (1.5) with initial value $\Phi(0) \;=\; \begin{pmatrix} \phi_+(0) \\ \phi_-(0) \end{pmatrix} \in \mathbb{C}^2$. $\square$

In Section 2.1 we will present a proof of Theorem 2.1. The proof and the solution will be further discussed in Appendix A.

Theorem 2.1 reduces the problem of solving (1.5) to the not necessarily easier question of finding solutions for (2.1). Somewhat surprisingly, however, some interesting results could be proven about the nature of some particular solutions of (2.1) for the case where f is a quasi-periodic function subjected to some additional restrictions. These results are described in Theorem 2.2.

Theorem 2.2 *Let f be quasi-periodic with*

$$f(t) \;=\; \sum_{\underset{\sim}{n} \in \mathbb{Z}^B} f_{\underset{\sim}{n}}\, e^{i\underset{\sim}{\omega}_f \cdot \underset{\sim}{n} t},$$

and such that the sum above contains only a finite number of terms. Assume that the vector $\underline{\omega}$ (defined in (1.27)) satisfies Diophantine conditions, i.e., assume the existence of constants $\Delta > 0$ and $\sigma > 0$ such that, for all $\underline{n} \in \mathbb{Z}^A$, $\underline{n} \neq \underline{0}$,

$$|\underline{n} \cdot \underline{\omega}| \geq \Delta^{-1}|\underline{n}|^{-\sigma}.$$

I. *Assume that f satisfies the condition $M(q^2) \neq 0$. Then, there exists a formal power series*

$$g(t) \;=\; q(t) \sum_{n=1}^{\infty} c_n(t)\epsilon^n,$$

representing a particular solution of the generalized Riccati equation (2.1) such that all coefficients c_n can be chosen to be quasi-periodic and can be represented as

$$c_n(t) \;=\; \sum_{\underline{m} \in \mathbb{Z}^A} C^{(n)}_{\underline{m}}\, e^{i\underline{m}\cdot\underline{\omega}t},$$

where, for the Fourier coefficients $C^{(n)}_{\underline{m}}$, we have

$$|C^{(n)}_{\underline{m}}| \;\leq\; \mathcal{K}_n e^{-\chi_0 |\underline{m}|},$$

where $\chi_0 > 0$ is a constant and $\mathcal{K}_n \geq 0$.

II. *Assume that f satisfies the conditions $M(q^2) = 0$ and $M(\mathcal{Q}_1) \neq 0$, where*

$$\mathcal{Q}_1(t) \;:=\; q(t)^2 \int_0^t q^{-2}(t')dt'.$$

Then, there exists a formal power series

$$g(t) \;=\; q(t) \sum_{n=1}^{\infty} e_n(t)\epsilon^{2n},$$

representing a particular solution of the generalized Riccati equation (2.1) such that all coefficients e_n can be chosen to be quasi-periodic and can be represented as

$$e_n(t) \;=\; \sum_{\underline{m} \in \mathbb{Z}^A} E^{(n)}_{\underline{m}}\, e^{i\underline{m}\cdot\underline{\omega}t},$$

where, for the Fourier coefficients $E^{(n)}_{\underline{m}}$, we have

$$|E^{(n)}_{\underline{m}}| \;\leq\; \mathcal{L}_n e^{-\chi_0 |\underline{m}|},$$

where $\chi_0 > 0$ is a constant and $\mathcal{L}_n \geq 0$. □

Sections 3 and 5 are dedicated to the proof of part I of this theorem and Sections 4 and 6 to the proof of part II. The proof of part I consists in establishing inductively the bound

$$|C^{(n)}_{\underline{m}}| \;\leq\; \mathcal{K}_n e^{-(\chi - \delta_n)|\underline{m}|}, \tag{2.6}$$

for all $n \in \mathbb{N}$, with positive $\mathcal{K}_n$ and χ and with an increasing positive sequence δ_n converging to some $\delta_\infty < \chi$. We set $\chi_0 := \chi - \delta_\infty$. Quasi-periodicity of the functions c_n follows from the bound (2.6). The proof of part II is analogous.

Unfortunately, as discussed at the end of Section 5, the behaviour for large n of the constants $\mathcal{K}_n$ and $\mathcal{L}_n$ is apparently too bad to guarantee absolute convergence of the formal power series above. Uniform convergence of that power series in ϵ would imply quasi-periodicity of g.

The hypothesis that the Fourier series of f contains only a finite number of terms is not really crucial and may be eliminated with more work. Actually this hypothesis is used only in Appendices C and D. This hypothesis is anyway closer to the physical reality, since f represents an external interaction coupled to the system.

2.1 Proof of Theorem 2.1

Let $\Phi(t) = V(t)\Phi(0)$ be the solution of the Schrödinger equation (1.5) with initial value $\Phi(0)$, where $V : \mathbb{R} \to \mathrm{Mat}(2, \mathbb{C})$. Then V is the solution of

$$iV'(t) = H_1(t)V(t) \tag{2.7}$$

with $V(0) = \mathbb{1}$. Hence, V also satisfies $iV'(0) = H_1(0)$ and

$$iV''(t) = D(t)V(t), \tag{2.8}$$

where $D(t) := H_1'(t) - iH_1(t)^2$. An explicit computation shows that $D(t)$ is the diagonal matrix

$$D(t) = \begin{pmatrix} f'(t) - i(\epsilon^2 + f(t)^2) & 0 \\ 0 & -f'(t) - i(\epsilon^2 + f(t)^2) \end{pmatrix}. \tag{2.9}$$

We will show that the matrix $U(t)$ defined in (2.3) satisfies a) $U(0) = \mathbb{1}$, b) $iU'(0) = H_1(0)$ and c) $iU''(t) = D(t)U(t)$. By the uniqueness of solutions of systems of linear second order differential equations with continuous coefficients we conclude that $U(t) = V(t)$.

The proof of a) follows from the facts that $R(0) = 1$ and $S(0) = 0$. To show b) we note that, since $R'(t) = -i(f(t) + g(t))R(t)$ and $S'(t) = R(t)^{-2}$, an explicit computation gives

$$iU'(t) = \begin{pmatrix} f(t) + g(t) & 0 \\ 0 & -\left(f(t) + \overline{g(t)}\right) \end{pmatrix} U(t) + \begin{pmatrix} R(t)^{-1} & 0 \\ 0 & \overline{R(t)^{-1}} \end{pmatrix} \begin{pmatrix} -g(0) & \epsilon \\ \epsilon & \overline{g(0)} \end{pmatrix}. \tag{2.10}$$

Since $R(0) = 1$ and $U(0) = \mathbb{1}$, it follows from (2.10) that

$$iU'(0) = \begin{pmatrix} f(0) & \epsilon \\ \epsilon & -f(0) \end{pmatrix} = H_1(0), \tag{2.11}$$

proving b). From (2.10) we have $iU''(t) = E(t)U(t)$ where

$$E(t) := \begin{pmatrix} f'(t) + g'(t) - i(f(t) + g(t))^2 & 0 \\ 0 & -f'(t) - \overline{g'(t)} - i\left(f(t) + \overline{g(t)}\right)^2 \end{pmatrix}. \tag{2.12}$$

Now, under the assumption that g satisfies (2.1), we easily check that $E(t) = D(t)$, completing the proof of Theorem 2.1. ■

The solution (2.3) has not been guessed out of nothing. In Appendix A we present a more constructive proof of Theorem 2.1. There we also discuss some questions related to the solution (2.3), as its relation to the *general* solution of the generalized Riccati equation.

Our starting point in Appendix A will be the fact that, translated back to the components $\phi_\pm(t)$ of the wave function, equation $iU''(t) = D(t)U(t)$ becomes a complex and quasi-periodic version of Hill's equation: $\phi_\pm''(t) + (\pm if'(t) + \epsilon^2 + f(t)^2)\,\phi_\pm(t) = 0$.

3 The Solution g and its Dependence on ϵ

Now we will start our analysis of the formal particular solution of the generalized Riccati equation (2.1) and the proof of Theorem 2.2.

The solution presented in (2.3) is still incomplete, since no particular solution g of the generalized Riccati equation (2.1) has been presented. First we note that for the case $\epsilon \equiv 0$, the solution for $U(t)$ is well known and is a diagonal matrix, whose diagonal elements are $\exp\left(\mp i\int_0^t f(\tau)d\tau\right)$. Comparing with (2.3), we see that for the case $\epsilon \equiv 0$ we should consider the particular solution $g(t) \equiv 0$. For more about this, see the discussion of Appendix A. A natural proposal would be to express g as a formal power expansion on ϵ which vanishes at $\epsilon = 0$. For convenience, we write this expansion as

$$g(t) \;=\; q(t)\sum_{n=1}^{\infty} c_n(t)\epsilon^n, \tag{3.1}$$

where $q(t)$ was defined in (1.12). This would give the desired solution, provided the infinite sum converges for all values of ϵ and t contained in some open intervals. Inserting (3.1) into (2.1) leads to

$$\sum_{n=1}^{\infty}\left((qc_n)' - i\sum_{p=1}^{n-1} q^2 c_p c_{n-p} - 2ifqc_n\right)\epsilon^n + i\epsilon^2 = 0. \tag{3.2}$$

Under the assumptions above, we conclude

$$(qc_1)' - 2ifqc_1 = 0, \tag{3.3}$$

$$(qc_2)' - iq^2c_1^2 - 2ifqc_2 + i = 0, \tag{3.4}$$

$$(qc_n)' - i\sum_{p=1}^{n-1} q^2 c_p c_{n-p} - 2ifqc_n = 0, \quad n \geq 3. \tag{3.5}$$

The solutions of (3.3)-(3.4) are

$$c_1(t) \;=\; \alpha_1\, q(t), \tag{3.6}$$

$$c_2(t) \;=\; q(t)\left[i\int_0^t \left(\alpha_1^2 q(t')^2 - q(t')^{-2}\right)dt' + \alpha_2\right], \tag{3.7}$$

$$c_n(t) \;=\; q(t)\left[i\left(\sum_{p=1}^{n-1}\int_0^t c_p(t')c_{n-p}(t')\,dt'\right) + \alpha_n\right], \quad \text{for } n \geq 3, \tag{3.8}$$

where the α_n's above, $n = 1, 2, \ldots$, are arbitrary integration constants.

Since the differential equation (2.1) involves ϵ^2, the reader may wonder why we try in (3.1) to find solutions given in terms of power series on ϵ and not on $\lambda \equiv \epsilon^2$. One should note that, in the case $f \equiv 0$, a particular solution of (2.1) is given simply by $g(t) = \epsilon$, which leads to solutions of the form $\mathcal{A}\cos(\epsilon t) + \mathcal{B}\sin(\epsilon t)$ for $\phi_\pm$, as expected. Therefore, solutions like (3.1) must also be also considered, in principle. It is interesting to note, however, that we will meet in Section 4 a situation where the solution g involves a power expansion on λ.

In the next two subsections we analyze two different choices of the constants α_n.

3.1 First Choice of the Constants α_n

In order to illustrate some potential problems, we choose first $\alpha_1 = 1$ and $\alpha_n = 0$, $\forall n \geq 2$. We get

$$
\begin{aligned}
c_1(t) &= q(t), && (3.9)\\
c_2(t) &= iq(t) \int_0^t \left(q(t')^2 - q(t')^{-2} \right) dt', && (3.10)\\
c_n(t) &= iq(t) \sum_{p=1}^{n-1} \int_0^t c_p(t') c_{n-p}(t')\, dt', \quad \text{for } n \geq 3. && (3.11)
\end{aligned}
$$

In this case we can easily find a condition guaranteeing the convergence of the formal power expansion (3.1). We have

$$
|c_1(t)| = 1, \tag{3.12}
$$

$$
|c_2(t)| \leq \int_0^{|t|} 2dt' = 2|t|. \tag{3.13}
$$

Admitting that $|c_k(t)| \leq (2|t|)^{k-1}$ for all k with $1 \leq k < n$, we have

$$
\begin{aligned}
|c_n(t)| &\leq \sum_{p=1}^{n-1} \int_0^{|t|} (2|t'|)^{p-1} (2|t'|)^{n-p-1}\, dt' = 2^{n-2} |t|^{n-1}\\
&\leq (2|t|)^{n-1}. && (3.14)
\end{aligned}
$$

This proved by induction that

$$
|c_n(t)| \leq (2|t|)^{n-1}, \tag{3.15}
$$

for all $n \geq 1$. Hence, $|g(t)| \leq \sum_{n=1}^{\infty} (2|t|)^{n-1} |\epsilon|^n$, which converges, provided $2|t||\epsilon| < 1$.

3.2 Second Choice of the Constants α_n. Elimination of Polynomial Terms on t

Let us return to (3.6)-(3.8). We will show that there is a choice of the α_n's for which all functions c_n are quasi-periodic functions of t. If this holds, there is a chance that g be also quasi-periodic, provided the sum (3.1) converges uniformly on $\mathbb{R}$.

To fix some ideas, let us start recalling some well-known facts about almost-periodic functions. For an almost-periodic function h, let us denote by $\sigma(h)$ the spectrum of h (see, e.g. [12]). The function h has the Fourier decomposition $h(t) = \sum_{\omega_i \in \sigma(h)} h_i e^{i\omega_i t}$ that converges uniformly on $\mathbb{R}$ (see, e.g. [12] for a precise statement). According to a celebrated theorem by H. Bohr (see, e.g. [12]), a necessary and sufficient condition for $H(t) := \int_0^t h(t')dt'$ being also almost-periodic is $H \in L^\infty(\mathbb{R})$.

For this, it is sufficient that $M(h) = 0$ and $\sum_{\substack{\omega_i \in \sigma(h) \\ \omega_i \neq 0}} \frac{|h_i|}{|\omega_i|} < \infty$. Bohr's theorem generalizes a previous result by P. Bohl on the quasi-periodic case [4].

A quasi-periodic function h is an almost-periodic function with the following property. There exists $N \in \mathbb{N}$, $N > 0$, and a N-tuple $\underline{\omega} = (\omega_1, \ldots, \omega_N) \in (\mathbb{R}_+)^N$, such that $\sigma(h) \subset \{\underline{n} \cdot \underline{\omega}, \; \underline{n} \in \mathbb{Z}^N\}$. For a given quasi-periodic function h we will denote the associated N-tuple $\underline{\omega}$ by $\beta(h)$.

For the analysis below we will admit that f is such that $M(q^2) \neq 0$. A sufficient condition for this will be discussed in Appendix C. The case $M(q^2) = 0$ will be considered in Section 4.

Let us start with c_2. In order to have c_2 quasi-periodic one needs at least to guarantee that the integrand in (3.7) contains no constant term, i.e.,

$$M\left(\alpha_1^2 q^2 - q^{-2}\right) = 0, \tag{3.16}$$

that means,

$$\alpha_1^2 = \frac{M(q^{-2})}{M(q^2)} = \frac{\overline{M(q^2)}}{M(q^2)}. \tag{3.17}$$

More generally, in order to have c_n quasi-periodic, $n \geq 3$, one needs to guarantee at least that the integrand in (3.8) contains no constant term, i.e.,

$$\sum_{p=1}^{n-1} M\left(c_p c_{n-p}\right) = 0. \tag{3.18}$$

This means

$$2M(c_1 c_2) = 2\alpha_1 M(q c_2) = 0 \tag{3.19}$$

for $n = 3$ and

$$2\alpha_1 M(q c_{n-1}) = -\sum_{p=2}^{n-2} M\left(c_p c_{n-p}\right). \tag{3.20}$$

for $n \geq 4$. Defining for $n \geq 2$

$$d_n(t) \; := \; q(t)(c_n(t) - \alpha_n q(t)) \; = \; \begin{cases} iq(t)^2 \displaystyle\int_0^t \left(\alpha_1^2 q(t')^2 - q(t')^{-2}\right) dt', & n = 2, \\[2ex] iq(t)^2 \displaystyle\sum_{p=1}^{n-1} \int_0^t c_p(t') c_{n-p}(t')\, dt', & n \geq 3, \end{cases} \tag{3.21}$$

we get from (3.19)

$$\alpha_2 = -\frac{M(d_2)}{M(q^2)}, \tag{3.22}$$

which fixes α_2. For $n \geq 4$ we have

$$M(q c_{n-1}) = M(d_{n-1}) + \alpha_{n-1} M(q^2). \tag{3.23}$$

Condition (3.20), combined with (3.23), says that

$$\alpha_{n-1} = -\frac{1}{M(q^2)} \left(M(d_{n-1}) + \frac{1}{2\alpha_1} \sum_{p=2}^{n-2} M\left(c_p c_{n-p}\right) \right). \tag{3.24}$$

for $n \geq 4$. Now, note that d_{n-1} depends only on $\{c_1, \ldots, c_{n-2}\}$ and, therefore, relation (3.24) fixes α_{n-1} for given $\{\alpha_1, \ldots, \alpha_{n-2}\}$. This, together with (3.17) and (3.22), fixes recursively all constants α_* and guarantees (3.16) and (3.18) for all $n \in \mathbb{N}$.

Fixing the constants α_n in the way described above is an important step towards the proof of quasi-periodicity of the functions c_n and, eventually, of g. With this choice for the α_n's no polynomial terms on t will appear after performing the integrations found in (3.6)-(3.8). As a consequence, one should expect to get here better estimates for the behaviour on t of the functions c_n than that found in (3.15) for our first choice of the α_n's. As a matter of fact we will be able to prove that all functions c_n are quasi-periodic by analysing recursively their Fourier coefficients. This will be performed in Section 5. Now we will consider the case $M(q^2) = 0$.

4 A Solution g in the Case $M(q^2) = 0$

In the case where $M(q^2) = 0$ there are two equivalent procedures to start with. In the first we adopt the Ansatz (3.1) and choose $\alpha_1 = 0$, what implies $c_n \equiv 0$ for all odd n. In the second, which we follow here, we adopt directly the Ansatz

$$g(t) \;=\; q(t) \sum_{n=1}^{\infty} e_n(t)\lambda^n, \tag{4.1}$$

with $\lambda := \epsilon^2$. Inserting this into (2.1) leads to

$$\sum_{n=1}^{\infty} \left((qe_n)' - i\sum_{p=1}^{n-1} q^2 e_p e_{n-p} - 2ifqe_n \right) \lambda^n + i\lambda = 0, \tag{4.2}$$

or

$$(qe_1)' - 2ifqe_1 + i = 0, \tag{4.3}$$

$$(qe_n)' - i\sum_{p=1}^{n-1} q^2 e_p e_{n-p} - 2ifqe_n = 0, \quad n \geq 2. \tag{4.4}$$

The solutions are

$$e_1(t) \;=\; q(t)\left(-i\int_0^t q^{-2}(t')\,dt' + \beta_1\right), \tag{4.5}$$

$$e_n(t) \;=\; q(t)\left[i\left(\sum_{p=1}^{n-1}\int_0^t e_p(t')e_{n-p}(t')\,dt'\right) + \beta_n\right], \quad \text{for } n \geq 2, \tag{4.6}$$

where the β_n's above, $n = 1, 2, \ldots$, are arbitrary integration constants.

Since now $M(q^2) = \overline{M(q^{-2})} = 0$, e_1 is bounded and, therefore, quasi-periodic. For e_2 we have

$$e_2(t) = q(t)\left[i\int_0^t e_1(t')^2\, dt' + \beta_2\right]. \tag{4.7}$$

Using the same strategy used in the case $M(q^2) \neq 0$ above, we require $M(e_1^2) = 0$. Since

$$e_1(t)^2 = q(t)^2 \left(- \left(\int_0^t q^{-2}(t')dt' \right)^2 - 2i\beta_1 \left(\int_0^t q^{-2}(t')dt' \right) + \beta_1^2 \right), \tag{4.8}$$

we must have

$$-2i\beta_1 M(\mathcal{Q}_1) = M(\mathcal{Q}_2), \tag{4.9}$$

with

$$\mathcal{Q}_1(t) \ := \ q(t)^2 \int_0^t q^{-2}(t')dt' \quad \text{and} \tag{4.10}$$

$$\mathcal{Q}_2(t) \ := \ q(t)^2 \left(\int_0^t q^{-2}(t')dt' \right)^2 . \tag{4.11}$$

Hence we can choose

$$\beta_1 = \frac{i}{2} \frac{M(\mathcal{Q}_2)}{M(\mathcal{Q}_1)}, \tag{4.12}$$

provided $M(\mathcal{Q}_1) \neq 0$, i.e.

$$M(\mathcal{Q}_1) \ = \ i \sum_{\substack{\underline{n} \in \mathbb{Z}^A \\ \underline{n} \neq \underline{0}}} \frac{|Q^{(2)}_{\underline{n}}|^2}{\underline{n} \cdot \underline{\omega}} \neq 0, \tag{4.13}$$

what we will assume here.

For all $n \geq 3$ we require $\sum_{p=1}^{n-1} M(e_p e_{n-p}) = 0$, that means,

$$2M(e_1 e_{n-1}) + \sum_{p=2}^{n-2} M(e_p e_{n-p}) = 0, \tag{4.14}$$

with the convention that $\sum_{p=2}^{n-2} M(e_p e_{n-p}) = 0$ for $n = 3$. Define,

$$j_n(t) \ := \ q(t)(e_n(t) - \beta_n q(t)) = iq(t)^2 \sum_{p=1}^{n-1} \int_0^t e_p(t') e_{n-p}(t') \, dt', \tag{4.15}$$

for $n \geq 2$. We get for $n \geq 3$

$$\begin{aligned} M(e_1 e_{n-1}) \ &= \ M(q^{-1} e_1 j_{n-1}) + \beta_{n-1} M(e_1 q) \\ &= \ M(q^{-1} e_1 j_{n-1}) + \beta_{n-1} \left(-iM(\mathcal{Q}_1) + \beta_1 M(q^2) \right) \\ &= \ M(q^{-1} e_1 j_{n-1}) - i\beta_{n-1} M(\mathcal{Q}_1). \end{aligned} \tag{4.16}$$

Hence, condition (4.14) implies

$$\beta_{n-1} \ = \ -i \left[\frac{M(q^{-1} e_1 j_{n-1}) + \frac{1}{2} \sum_{p=2}^{n-2} M(e_p e_{n-p})}{M(\mathcal{Q}_1)} \right], \qquad n \geq 3. \tag{4.17}$$

With the choices (4.12) and (4.17) we guarantee that no polynomial terms on t will appear after performing the integrations in (4.5) and (4.6). In Section 6 we will establish that the functions e_n are all quasi-periodic.

4.1 The Power Series Solutions for the Case $f(t)$ = Constant

Before we finish this section let us consider one particular but interesting situation.

For the case $f(t) = f_0 \equiv \underset{\sim}{f_0}$, constant, the Riccati equation (2.1) has $g = -f_0 \pm \sqrt{f_0^2 + \epsilon^2}$ as particular solutions and for both $U(t)$ becomes

$$U(t) \;=\; \frac{1}{\omega_0}\begin{pmatrix} \omega_0\cos(\omega_0 t) - if_0\sin(\omega_0 t) & -\epsilon\sin(\omega_0 t) \\ -\epsilon\sin(\omega_0 t) & \omega_0\cos(\omega_0 t) + if_0\sin(\omega_0 t) \end{pmatrix}, \tag{4.18}$$

as expected, with $\omega_0 := \sqrt{f_0^2 + \epsilon^2}$.

It is important to take notice of the fact that, in the situation where $f(t) = \underset{\sim}{f_0} \equiv f_0$, constant, with $f_0 \neq 0$, we have precisely the conditions described above, namely, we have $\widetilde{M}(q^2) = 0$ and $M(\mathcal{Q}_1) = i(2f_0)^{-1} \neq 0$. We should therefore expect that the power series (4.1) reproduces the solution $g = -f_0 + \sqrt{f_0^2 + \epsilon^2}$ (we take $f_0 > 0$ without loss).

Let us prove that this is indeed the case, at least for $|\epsilon| < |f_0|$. We will show by induction, using (4.6) and (4.17), that

$$e_m(t) \;=\; l_m\, q(t)^{-1} \tag{4.19}$$

and

$$\beta_m \;=\; l_m, \tag{4.20}$$

for all $m \in \mathbb{N}$, where

$$l_m \;:=\; (-1)^{m-1}\,\frac{(2m-2)!}{m!(m-1)!}\,\frac{1}{(2f_0)^{2m-1}}. \tag{4.21}$$

An explicit computation using (4.12) and (4.5) shows that $e_1(t) = (2f_0)^{-1}q(t)^{-1} = l_1 q(t)^{-1}$ and that $\beta_1 = (2f_0)^{-1} = l_1$. Let us assume (4.19) and (4.20) valid for $1 \le m \le n-1$. By (4.17) we have

$$\beta_n \;=\; -(2f_0)\left[M(q^{-1}e_1 j_n) + \frac{1}{2}\sum_{p=2}^{n-1} M(e_p e_{n+1-p})\right], \tag{4.22}$$

but, by (4.19), $M(e_p e_{n+1-p}) = l_p l_{n+1-p}\overline{M(q^2)} = 0$ and, hence,

$$\begin{aligned}
\beta_n \;&=\; -(2f_0)M(q^{-1}e_1 j_n) = -iM\left(\sum_{p=1}^{n-1}\int_0^t e_p(t')e_{n-p}(t')\,dt'\right) \\
&=\; -\frac{1}{2f_0}\sum_{p=1}^{n-1} l_p\, l_{n-p} \\
&=\; \frac{(-1)^{n-1}}{(2f_0)^{2n-1}}\sum_{p=1}^{n-1}\frac{(2p-2)!\,(2n-2p-2)!}{p!(p-1)!(n-p)!(n-p-1)!}.
\end{aligned} \tag{4.23}$$

Using the identity

$$\sum_{p=1}^{n-1} \frac{(2p-2)!\,(2n-2p-2)!}{p!(p-1)!(n-p)!(n-p-1)!} \;=\; \frac{(2n-2)!}{n!(n-1)!}, \tag{4.24}$$

the proof of (4.20) for $m = n$ is complete and from (4.20) we have

$$\begin{aligned} e_n(t) &= q(t)\left[i\int_0^t q(t')^{-2}dt' \left(\sum_{p=1}^{n-1} l_p\, l_{n-p}\right) + l_n\right] \\ &= l_n\, q(t)^{-1}, \end{aligned} \tag{4.25}$$

using (4.6) and (4.24) again. This proves (4.19) and (4.20) by induction for all $n \in \mathbb{N}$.

Relation (4.24) can be obtained from the more general identity

$$\sum_{k=0}^{n} \binom{tk+r}{k}\binom{tn-tk+s}{n-k} \frac{r}{tk+r}\;\frac{s}{tn-tk+s} \;=\; \binom{tn+r+s}{n}\frac{r+s}{tn+r+s} \tag{4.26}$$

by taking $t = 2$, $r = s = -1$ and by moving the terms with $k = 0$ and $k = n$ to the right hand side of (4.26). Relation (4.26) is one of the so-called "convolution identities" for binomial coefficients and is valid for $n \in \mathbb{N}$, $t,\ r,\ s \in \mathbb{R}$. Its proof is indicated in [13], Chapter 5.

With (4.19) one can easily check that the series (4.1) becomes exactly the Taylor series in ϵ (about $\epsilon = 0$) of the function $-f_0 + \sqrt{f_0^2 + \epsilon^2}$, which is a particular solution of (2.1) for the case $f(t) = f_0$. In this case the convergence of (4.1) is clearly restricted to $|\epsilon| < |f_0|$.

In the general case we learn from this example that one should not expect convergence of (4.1) for all values of ϵ. Further, we see that, in the situation where $M(f) \neq 0$ and f has small oscillations around its mean value, i.e., in the situation where $f(t) - M(f)$ is small (in some sense to be precised), the series (4.1) should be expected to converge (at least for small values of ϵ). Unfortunately our analysis below was insufficient to provide a proof of this conjecture.

5 Quasi-Periodicity of the Functions c_n

In this section we will prove the quasi-periodicity of the functions c_n for our second choice of the constants α_n and for the case $M(q^2) \neq 0$.

Let us describe briefly the general strategy we will follow and the problems we will face. The strategy and the problems will be the same in the case $M(q^2) = 0$, treated in Section 6. We first express formally the recursive relations (3.6)-(3.8) for the functions c_n (with the choices (3.17), (3.22) and (3.24)) in terms of its Fourier coefficients $C^{(n)}_{\underline{m}}$, $\underline{m} \in \mathbb{Z}^A$. The recursive relations for the coefficients $C^{(n)}_{\underline{m}}$ involve convolutions (a consequence, lately, of the quadratic character of the generalized Riccati equation) and, inevitably, small denominators.

Assuming an exponential decay of the form $|Q_{\underline{m}}| \leq \mathcal{Q}\, e^{-\chi|\underline{m}|}$ for the Fourier coefficients $Q_{\underline{m}}$ of the function q, a fact that will be proven in Appendix D, we prove inductively bounds of the form $|C^{(n)}_{\underline{m}}| \leq \mathcal{K}_n e^{-(\chi-\delta_n)|\underline{m}|}$ for the Fourier coefficients of the functions c_n. This exponential decay justifies the formal expression of the functions c_n in terms of the coefficients $C^{(n)}_{\underline{m}}$ and the corresponding recursion relations and is enough to establish the quasi-periodicity of all the functions c_n by induction.

Due to the convolutions and to the small denominators appearing in the recursive relations for the coefficients $C^{(n)}_{\underline{m}}$, the sequence δ_n is positive and increasing. It is possible to keep the sequence δ_n bounded and smaller than χ, but this ruins the behaviour in n of the constants $\mathcal{K}_n$ and obstructs the control of the absolute convergence of the formal power series (3.1).

5.1 The Fourier Decomposition of the Functions c_n

To prepare the proof that the functions c_n are all quasi-periodic, we will first prove that c_1 is quasi-periodic. To prove that q is quasi-periodic we note that

$$q(t) \ := \ e^{if_{\underset{\sim}{0}}t} \exp\left(i\int_0^t \left(f(t') - f_{\underset{\sim}{0}}\right) dt'\right) . \tag{5.1}$$

Assuming now that

$$2 \sum_{\substack{\underset{\sim}{n}\in\mathbb{Z}^B \\ \underset{\sim}{n}\neq\underset{\sim}{0}}} \frac{|f_{\underset{\sim}{n}}|}{|\underset{\sim}{n}\cdot\underset{\sim}{\omega}_f|} < \infty \tag{5.2}$$

one has

$$F(t) := \int_0^t \left(f(t') - f_{\underset{\sim}{0}}\right) dt' \in L^\infty(\mathbb{R}) \tag{5.3}$$

and we conclude, by Bohr's theorem, that F is almost-periodic and, by the uniform convergence of the Fourier series (see, e.g. [12]), that

$$F(t) = -i \sum_{\substack{\underset{\sim}{n}\in\mathbb{Z}^B \\ \underset{\sim}{n}\neq\underset{\sim}{0}}} \frac{f_{\underset{\sim}{n}}}{\underset{\sim}{n}\cdot\underset{\sim}{\omega}_f} \left(e^{i\underset{\sim}{n}\cdot\underset{\sim}{\omega}_f t} - 1\right) , \tag{5.4}$$

what shows that F is quasi-periodic. A standard argument (see Appendix B) shows that $\exp(iF(t))$ is also quasi-periodic with $\beta(\exp(iF(t))) = \beta(F) = \beta(f)$. Therefore, we have established that q is quasi-periodic with $\sigma(q) = \sigma(\exp(iF)) + f_{\underset{\sim}{0}}$. We write

$$q(t) \ = \ \sum_{\underline{n}\in\mathbb{Z}^A} Q_{\underline{n}}\, e^{i\underline{n}\cdot\underline{\omega}t}, \tag{5.5}$$

where $\underline{\omega}$ and A are defined in (1.27) and (1.28), respectively. See Appendix C for the relation between the Fourier coefficients of q and those of f.

For $n \geq 1$, let us write

$$c_n(t) \ = \ \sum_{\underline{m}\in\mathbb{Z}^A} C^{(n)}_{\underline{m}}\, e^{i\underline{m}\cdot\underline{\omega}t} . \tag{5.6}$$

In the next we will be concerned with the recursion relations for the Fourier coefficients $C^{(n)}_{\underline{m}}$. For $C^{(2)}_{\underline{m}}$ we have from (3.7)

$$C^{(2)}_{\underline{m}} \ = \ Q_{\underline{m}} \left[\alpha_2 - \sum_{\substack{\underline{n}\in\mathbb{Z}^A \\ \underline{n}\neq\underline{0}}} \frac{\alpha_1^2 Q^{(2)}_{\underline{n}} - \overline{Q^{(2)}_{-\underline{n}}}}{\underline{n}\cdot\underline{\omega}}\right] + \sum_{\substack{\underline{n}\in\mathbb{Z}^A \\ \underline{n}\neq\underline{0}}} \frac{Q_{\underline{m}-\underline{n}}}{\underline{n}\cdot\underline{\omega}} \left(\alpha_1^2 Q^{(2)}_{\underline{n}} - \overline{Q^{(2)}_{-\underline{n}}}\right) . \tag{5.7}$$

Above, $Q^{(2)}_{\underline{n}}$ are the Fourier coefficients of q^2, namely

$$q(t)^2 \;=\; \sum_{\underline{m}\in\mathbb{Z}^A} Q^{(2)}_{\underline{m}} e^{i\underline{m}\cdot\underline{\omega}t} \;, \tag{5.8}$$

with

$$Q^{(2)}_{\underline{m}} \;=\; \sum_{\underline{n}\in\mathbb{Z}^A} Q_{\underline{n}} Q_{\underline{m}-\underline{n}} \;. \tag{5.9}$$

For $n \geq 3$ we have from (3.8)

$$\begin{aligned} C^{(n)}_{\underline{m}} \;=\; & \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} Q_{\underline{m}-\underline{n}_1-\underline{n}_2} \left(\sum_{p=1}^{n-1} C^{(p)}_{\underline{n}_1} C^{(n-p)}_{\underline{n}_2}\right) \frac{1}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} \;+ \\ & Q_{\underline{m}} \left[\alpha_n - \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \left(\sum_{p=1}^{n-1} C^{(p)}_{\underline{n}_1} C^{(n-p)}_{\underline{n}_2}\right) \frac{1}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}}\right]. \end{aligned} \tag{5.10}$$

For the constants α_n we have, according to (3.17), (3.22) and (3.24), the following expressions

$$\alpha_1^2 \;=\; Q^{(2)}_{\underline{0}} \Big/ \overline{Q^{(2)}_{\underline{0}}} \;, \tag{5.11}$$

$$\alpha_2 \;=\; -\frac{1}{M(q^2)} \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{\left(\alpha_1^2 Q^{(2)}_{\underline{n}} - \overline{Q^{(2)}_{-\underline{n}}}\right)}{\underline{n}\cdot\underline{\omega}} \left(Q^{(2)}_{-\underline{n}} - Q^{(2)}_{\underline{0}}\right), \tag{5.12}$$

$$\begin{aligned} \alpha_n \;=\; & -\frac{1}{M(q^2)} \left\{ \sum_{p=1}^{n-1} \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \frac{C^{(p)}_{\underline{n}_1} C^{(n-p)}_{\underline{n}_2}}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} \left(Q^{(2)}_{-(\underline{n}_1+\underline{n}_2)} - Q^{(2)}_{\underline{0}}\right) \right. \\ & \left. + \frac{1}{2\alpha_1} \sum_{p=2}^{n-1} \sum_{\underline{n}\in\mathbb{Z}^A} C^{(p)}_{\underline{n}} C^{(n+1-p)}_{-\underline{n}} \right\}, \qquad n \geq 3. \end{aligned} \tag{5.13}$$

Inserting these expressions into (5.7) and (5.10) we get

$$C^{(1)}_{\underline{m}} \;=\; \alpha_1 Q_{\underline{m}}, \tag{5.14}$$

$$C^{(2)}_{\underline{m}} \;=\; \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{\left(\alpha_1^2 Q^{(2)}_{\underline{n}} - \overline{Q^{(2)}_{-\underline{n}}}\right)}{\underline{n}\cdot\underline{\omega}} \left[Q_{\underline{m}-\underline{n}} - \frac{Q_{\underline{m}} Q^{(2)}_{-\underline{n}}}{Q^{(2)}_{\underline{0}}}\right], \tag{5.15}$$

$$C^{(n)}_{\underline{m}} \;=\; \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \frac{1}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} \left(\sum_{p=1}^{n-1} C^{(p)}_{\underline{n}_1} C^{(n-p)}_{\underline{n}_2}\right) \left[Q_{\underline{m}-(\underline{n}_1+\underline{n}_2)} - \frac{Q_{\underline{m}} Q^{(2)}_{-\underline{n}_1-\underline{n}_2}}{Q^{(2)}_{\underline{0}}}\right]$$

$$- \frac{Q_{\underline{m}}}{2\alpha_1 Q^{(2)}_{\underline{0}}} \sum_{\underline{n}\in\mathbb{Z}^A} \sum_{p=2}^{n-1} C^{(p)}_{\underline{n}} C^{(n+1-p)}_{-\underline{n}}, \qquad n \geq 3. \tag{5.16}$$

Above we used $M(q^2) = Q^{(2)}_{\underline{0}}$.

5.2 An Upper Bound for $|Q^{(2)}_{\underline{m}}|$

At this moment we have to introduce some restrictions, which ultimately reflect on restrictions on the function f. The first concerns the frequencies $\underline{\omega}$. We assume a so called Diophantine condition, namely, we assume the existence of constants $\Delta > 0$ and $\sigma > 0$ such that, for all $\underline{n} \in \mathbb{Z}^A$, $\underline{n} \neq \underline{0}$,

$$|\underline{n} \cdot \underline{\omega}| \geq \Delta^{-1} |\underline{n}|^{-\sigma}. \tag{5.17}$$

The second restriction concerns the decay properties of the Fourier coefficients $Q_{\underline{m}}$ for $|\underline{m}| \to \infty$. We assume that there are constants $\mathcal{Q} > 0$ and $\chi > 0$ such that, for all $\underline{m} \in \mathbb{Z}^A$,

$$|Q_{\underline{m}}| \ \leq \ \mathcal{Q} e^{-\chi |\underline{m}|}. \tag{5.18}$$

This will be proven in Appendix D for the case where f has a Fourier decomposition given by a finite sum. As a consequence of (5.18), we have

$$|Q^{(2)}_{\underline{m}}| \ \leq \ \mathcal{Q}^2 \, \Xi_0(\chi, \ \underline{m}) \tag{5.19}$$

where, for $\alpha > 0$,

$$\Xi_0(\alpha, \ \underline{m}) \ := \ \sum_{\underline{n}\in\mathbb{Z}^A} e^{-\alpha(|\underline{n}| + |\underline{m} - \underline{n}|)}. \tag{5.20}$$

One has

$$\Xi_0(\alpha, \ \underline{m}) = \prod_{a=1}^{A} \sum_{n\in\mathbb{Z}} e^{-\alpha(|n| + |m_a - n|)}. \tag{5.21}$$

A simple computation shows that

$$\sum_{n\in\mathbb{Z}} e^{-\alpha(|n| + |m_a - n|)} = e^{-\alpha |m_a|} \left(|m_a| + (\tanh(\alpha))^{-1} \right). \tag{5.22}$$

Hence

$$\Xi_0(\alpha, \ \underline{m}) \leq \left(|\underline{m}| + (\tanh(\alpha))^{-1} \right)^A e^{-\alpha |\underline{m}|}. \tag{5.23}$$

Using now the inequality

$$|x|^\sigma \leq \left(\frac{\sigma}{e\delta} \right)^\sigma e^{\delta |x|}, \tag{5.24}$$

valid for all $x \in \mathbb{C}$ and all $\delta > 0$, $\sigma > 0$, we arrive at

$$\Xi_0(\alpha, \ \underline{m}) \leq \left[\left(\frac{A}{e\delta} \right)^A e^{\delta / \tanh(\alpha)} \right] e^{-(\alpha - \delta)|\underline{m}|}. \tag{5.25}$$

Defining

$$\mathcal{Q}^{(2)}(\delta) := \mathcal{Q}^2 \left(\frac{A}{e\delta} \right)^A e^{\delta / \tanh(\chi)} \tag{5.26}$$

for $\delta > 0$, we conclude that, for some δ_2 positive and small enough,

$$|Q^{(2)}_{\underline{m}}| \ \leq \ \mathcal{Q}^{(2)}(\delta_2/2) e^{-(\chi - \delta_2/2)|\underline{m}|}. \tag{5.27}$$

Below we will frequently denote $\mathcal{Q}^{(2)} \equiv \mathcal{Q}^{(2)}(\delta_2/2)$.

5.3 An Upper Bound for $|C^{(2)}_{\underline{m}}|$

Since c_2 is expressed as the quasi-periodic function q times the integral of a quasi-periodic function, one has only to show, for proving that c_2 is quasi-periodic, that

$$i\int_0^t (\alpha_1^2 q(t')^2 - q(t')^{-2})\, dt' \quad \in L^\infty(\mathbb{R}). \tag{5.28}$$

But, by the hypotheses above, and using the fact that $|\alpha_1| = 1$, we have

$$\left| i\int_0^t (\alpha_1^2 q(t')^2 - q(t')^{-2})\, dt' \right| \leq 2 \sum_{\substack{\underline{n}\in\mathbb{Z}^A \\ \underline{n}\neq\underline{0}}} \frac{|Q^{(2)}_{\underline{n}}|}{|\underline{n}\cdot\underline{\omega}|} \leq 2\Delta\mathcal{Q}^{(2)} \sum_{\underline{n}\in\mathbb{Z}^A} |\underline{n}|^\sigma e^{-(\chi-\delta_2/2)|\underline{n}|}. \tag{5.29}$$

Above the term corresponding to $\underline{n} = \underline{0}$ has been eliminated by the choice of α_1. Using (5.24) we get

$$2 \sum_{\substack{\underline{n}\in\mathbb{Z}^A \\ \underline{n}\neq\underline{0}}} \frac{|Q^{(2)}_{\underline{n}}|}{|\underline{n}\cdot\underline{\omega}|} \leq 2\Delta\mathcal{Q}^{(2)} \left(\frac{2\sigma}{e\delta_2}\right)^\sigma \sum_{\underline{n}\in\mathbb{Z}^A} e^{-(\chi-\delta_2)|\underline{n}|} \leq \infty \tag{5.30}$$

by choosing $\delta_2 < \chi$. This concludes the proof of (5.28) and shows that c_2 is quasi-periodic.

Relation (5.15) gives us the following upper bound for $|C^{(2)}_{\underline{m}}|$:

$$\begin{aligned} |C^{(2)}_{\underline{m}}| &\leq 2\Delta\mathcal{Q}^{(2)}\mathcal{Q} \sum_{\underline{n}\in\mathbb{Z}^A} |\underline{n}|^\sigma e^{-(\chi-\delta_2/2)|\underline{n}|} \left[e^{-\chi|\underline{m}-\underline{n}|} + \frac{\mathcal{Q}^{(2)}}{|Q^{(2)}_{\underline{0}}|} e^{-\chi|\underline{m}|-(\chi-\delta_2/2)|\underline{n}|} \right] \\ &\leq 2\Delta\mathcal{Q}^{(2)}\mathcal{Q} \left(\frac{\sigma}{e\delta}\right)^\sigma \left(L_1(\underline{m}) + e^{-\chi|\underline{m}|} \frac{\mathcal{Q}^{(2)}}{|Q^{(2)}_{\underline{0}}|} L_2(2\chi - \delta_2 - \delta), \right), \end{aligned} \tag{5.31}$$

where we used again (5.24) and the Diophantine condition (5.17). Above

$$L_1(\underline{m}) = \Xi(\chi - \delta_2/2 - \delta,\ \chi,\ \underline{m}),$$

where, for $\alpha,\ \beta > 0$,

$$\Xi(\alpha,\ \beta,\ \underline{m}) := \sum_{\underline{n}\in\mathbb{Z}^A} e^{-\alpha|\underline{n}|-\beta|\underline{m}-\underline{n}|} \qquad \textit{and} \qquad L_2(\alpha) := \sum_{\underline{n}\in\mathbb{Z}^A} e^{-\alpha|\underline{n}|}.$$

Note that $\Xi(\alpha,\ \beta,\ \underline{m}) = \Xi(\beta,\ \alpha,\ \underline{m})$ and that

$$\Xi(\alpha,\ \beta,\ \underline{m}) \leq \Xi_0(\min\{\alpha, \beta, \},\ \underline{m}). \tag{5.32}$$

To proceed we have to find upper bounds for L_1 and L_2. We have for $L_2(\alpha)$ simply

$$L_2(\alpha) = \left(1 + 2\sum_{a=1}^\infty e^{-\alpha a}\right)^A = \left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\right)^A \tag{5.33}$$

for $\alpha > 0$. For L_1 we have, using (5.32),

$$L_1(\underline{m}) \leq \Xi_0(\chi - \delta_2/2 - \delta,\ \underline{m}) \leq e^{-(\chi-\delta_2/2-2\delta)|\underline{m}|} e^{\delta/\tanh(\chi-\delta_2/2-\delta)} \left(\frac{A}{e\delta}\right)^A \tag{5.34}$$

by (5.25), for any $\delta > 0$, small enough. Choosing $\delta = \delta_2/4$ we get

$$L_1(\underline{m}) \ \leq \ e^{-(\chi-\delta_2)|\underline{m}|}\, e^{\delta_2/(4\tanh(\chi-3\delta_2/4))} \left(\frac{4A}{e\delta_2}\right)^A . \tag{5.35}$$

Putting the estimates above together we have for all $\delta_2 > 0$ sufficiently small

$$|C^{(2)}_{\underline{m}}| \ \leq \ \mathcal{K}_2\, e^{-(\chi-\delta_2)|\underline{m}|}, \tag{5.36}$$

where,

$$\mathcal{K}_2 \ := \ 2\Delta\mathcal{Q}^{(2)}\mathcal{Q}\left(\frac{4\sigma}{e\delta_2}\right)^\sigma \left(e^{\delta_2/\tanh(\chi-3\delta_2/4)}\left(\frac{4A}{e\delta_2}\right)^A + \frac{\mathcal{Q}^{(2)}}{|\mathcal{Q}^{(2)}_{\underline{0}}|}L_2(2\chi-5\delta_2/4)\right). \tag{5.37}$$

Note that $\mathcal{K}_2 \sim \delta_2^{-(2A+\sigma)}$ for $\delta_2 \to 0$.

5.4 Recursive Upper Bounds for $|C^{(n)}_{\underline{m}}|$ with $n \geq 3$

In order to show that all c_n's, $n \geq 3$, are quasi-periodic we note that the kind of reasoning used above can be applied inductively and indicates to be sufficient to show that $c_n \in L^\infty(\mathbb{R})$. For this it is enough to establish that

$$\sum_{\underline{m}\in\mathbb{Z}^A} |C^{(n)}_{\underline{m}}| \leq \infty. \tag{5.38}$$

Using the Diophantine condition (5.17) together with (5.18) and (5.27) we get from (5.16)

$$\begin{aligned} |C^{(n)}_{\underline{m}}| \ \leq \ & \Delta\mathcal{Q}\sum_{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A} |\underline{n}_1+\underline{n}_2|^\sigma \left(\sum_{p=1}^{n-1}|C^{(p)}_{\underline{n}_1}|\,|C^{(n-p)}_{\underline{n}_2}|\right)\left[e^{-\chi|\underline{m}-\underline{n}_1-\underline{n}_2|} + \frac{\mathcal{Q}^{(2)}}{|Q^{(2)}_{\underline{0}}|}e^{-\chi|\underline{m}|-(\chi-\delta_2)|\underline{n}_1+\underline{n}_2|}\right] \\ & + e^{-\chi|\underline{m}|}\frac{\mathcal{Q}}{2|Q^{(2)}_{\underline{0}}|}\sum_{\underline{n}\in\mathbb{Z}^A}\sum_{p=2}^{n-1}|C^{(p)}_{\underline{n}}|\,|C^{(n+1-p)}_{-\underline{n}}|, \qquad\qquad n\geq 3. \end{aligned} \tag{5.39}$$

The strategy we will follow is to use (5.39) for recursively proving a bound on the absolute value of the Fourier coefficients $C^{(n)}_{\underline{m}}$, from which (5.38) should follow. We will assume that for $1 \leq a \leq n-1$ there are positive constants $\mathcal{K}_a$ and δ_a, $\delta_a < \chi$, such that, for all $\underline{m} \in \mathbb{Z}^A$,

$$|C^{(a)}_{\underline{m}}| \ \leq \ \mathcal{K}_a\, e^{-(\chi-\delta_a)|\underline{m}|}. \tag{5.40}$$

We have already seen that we can adopt $\mathcal{K}_1 := \mathcal{Q}$, $\delta_1 := 0$ and $\mathcal{K}_2$ as in (5.37) with δ_2 arbitrary and small.

Inserting (5.40) into (5.39) and using once more the inequality (5.24), for some $\delta > 0$ small enough to be chosen conveniently, gives

$$\begin{aligned} |C^{(n)}_{\underline{m}}| \ \leq \ & \Delta\mathcal{Q}\left(\frac{\sigma}{e\delta}\right)^\sigma \sum_{p=1}^{n-1}\mathcal{K}_p\mathcal{K}_{n-p}\left(L_3(p,\ \underline{m}) + e^{-\chi|\underline{m}|}\frac{\mathcal{Q}^{(2)}}{|Q^{(2)}_{\underline{0}}|}\ L_4(p)\right) \\ & + e^{-\chi|\underline{m}|}\frac{\mathcal{Q}}{2|Q^{(2)}_{\underline{0}}|}\sum_{p=2}^{n-1}\ \mathcal{K}_p\mathcal{K}_{n+1-p}\ L_5(p), \end{aligned} \tag{5.41}$$

where

$$L_3(p,\ \underline{m}) \ := \ L_3(\chi-\delta_p,\ \chi-\delta_{n-p},\ \delta,\ \chi,\ \underline{m}), \tag{5.42}$$

$$L_4(p) \ := \ L_4(\chi-\delta_p,\ \chi-\delta_{n-p},\ \chi-\delta_2-\delta), \tag{5.43}$$

$$L_5(p) \ := \ L_2(2\chi--\delta_p-\delta_{n+1-p}), \tag{5.44}$$

with $1 \le p \le n-1$ for L_3 and for L_4 and $2 \le p \le n-1$ for L_5, where

$$L_3(\alpha,\ \beta,\ \mu,\ \nu,\ \underline{m}) \ := \ \sum_{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A} \exp\left(-\alpha|\underline{n}_1|-\beta|\underline{n}_2|+\mu|\underline{n}_1+\underline{n}_2|-\nu|\underline{m}-\underline{n}_1-\underline{n}_2|\right), \tag{5.45}$$

$$L_4(\alpha,\ \beta,\ \gamma) \ := \ \sum_{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A} \exp\left(-\alpha|\underline{n}_1|-\beta|\underline{n}_2|-\gamma|\underline{n}_1+\underline{n}_2|\right), \tag{5.46}$$

with $\alpha,\ \beta,\ \gamma,\ \mu,\ \nu > 0$ and μ small enough.

We have to find convenient upper bounds for the sums above. For $L_5(p)$ we have

$$L_5(p) = L_2(2\chi-\delta_p-\delta_{n+1-p}) \le L_2\left(2\chi-2\delta[n]\right), \tag{5.47}$$

where

$$\delta[n] \ := \ \max_{p'<n}\delta_{p'}.$$

For $L_4(\alpha,\ \beta,\ \gamma)$ we have

$$L_4(\alpha,\ \beta,\ \gamma) \le L_4(\alpha,\ \beta,\ 0) = L_2(\alpha)L_2(\beta). \tag{5.48}$$

Hence,

$$L_4(p) \le L_2(\chi-\delta_p)L_2(\chi-\delta_{n-p}) \le L_2\left(\chi-\delta[n]\right)^2. \tag{5.49}$$

Let us now consider $L_3(p,\ \underline{m})$. For later use let us try to find bounds for $L_3(\alpha,\ \beta,\ \mu,\ \nu,\ \underline{m})$. We have, for any $\delta > 0$ sufficiently small,

$$\begin{aligned}
L_3(\alpha,\ \beta,\ \mu,\ \nu,\ \underline{m}) \ &\le \ \sum_{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A} \exp\left(-(\alpha-\mu)|\underline{n}_1|-(\beta-\mu)|\underline{n}_2|-\nu|\underline{m}-\underline{n}_1-\underline{n}_2|\right)\\
&= \ \sum_{\underline{n}_1\in\mathbb{Z}^A} \exp\left(-(\alpha-\mu)|\underline{n}_1|\right) \ \ \Xi(\beta-\mu,\ \nu,\ \underline{m}-\underline{n}_1)\\
&\le \ \sum_{\underline{n}_1\in\mathbb{Z}^A} \exp\left(-(\alpha-\mu)|\underline{n}_1|\right) \ \ \Xi_0(\kappa,\ \underline{m}-\underline{n}_1)\\
&\le \ \left(\frac{A}{e\delta}\right)^A \exp\left(\frac{\delta}{\tanh(\kappa)}\right) \ \ \Xi(\alpha-\mu,\ \kappa-\delta,\ \underline{m})\\
&\le \ \left(\frac{A}{e\delta}\right)^A \exp\left(\frac{\delta}{\tanh(\kappa)}\right) \ \ \Xi_0(\zeta,\ \underline{m})\\
&\le \ \mathcal{S}\, e^{-(\zeta-\delta)|\underline{m}|},
\end{aligned} \tag{5.50}$$

with

$$\mathcal{S} \ := \ \left(\frac{A}{e\delta}\right)^{2A} \exp\left(\frac{\delta}{\tanh(\kappa)} + \frac{\delta}{\tanh(\zeta)}\right),$$
$$\kappa \ := \ \min\{\beta - \mu,\ \nu\}$$

and

$$\zeta \ := \ \min\{\alpha - \mu,\ \kappa - \delta\} = \min\{\alpha - \mu,\ \beta - \mu - \delta,\ \nu - \delta\}.$$

Above, we made use of inequality (5.25) in the third and in the last inequalities, as well as of (5.32). After the corresponding replacements we conclude that

$$L_3(p,\ \underline{m}) \ \leq \ \left[\left(\frac{A}{e\delta}\right)^{2A} \exp\left(\frac{2\delta}{\tanh(\chi - \delta[n] - 2\delta)}\right)\right] \exp\left(-\left(\chi - \delta[n] - 3\delta\right)|\underline{m}|\right). \tag{5.51}$$

Note that the bounds (5.47), (5.49) and (5.51) do not depend on p, but depend on n.

We choose the sequence δ_a, $a \geq 1$ as a strictly increasing bounded sequence converging to some $\delta_\infty < \chi$. This makes $\delta[n] = \delta_{n-1}$ We also choose δ such that $\delta_n = \delta_{n-1} + 3\delta$. These choices give

$$L_3(p,\ \underline{m}) \ \leq \ \left[\left(\frac{3A}{e(\delta_n - \delta_{n-1})}\right)^{2A} \exp\left(\frac{2\delta_\infty}{3\tanh(\chi - \delta_\infty)}\right)\right] \exp\left(-\left(\chi - \delta_n\right)|\underline{m}|\right), \tag{5.52}$$

$$L_4(p) \ \leq \ L_2(\chi - \delta_\infty)^2, \tag{5.53}$$

$$L_5(p) \ \leq \ L_2(2\chi - 2\delta_\infty), \tag{5.54}$$

with $1 \leq p \leq n-1$ for L_3 and for L_4 and $2 \leq p \leq n-1$ for L_5.

Returning to (5.41), we get

$$\begin{aligned} |C^{(n)}_{\underline{m}}| \ &\leq \ \Delta\mathcal{Q}\left(\frac{3\sigma}{e(\delta_n - \delta_{n-1})}\right)^{\sigma} D_n \left(e^{-(\chi-\delta_n)|\underline{m}|} + e^{-\chi|\underline{m}|}\right) \sum_{p=1}^{n-1} \mathcal{K}_p \mathcal{K}_{n-p} \\ &+ e^{-\chi|\underline{m}|} \frac{\mathcal{Q}}{2|Q^{(2)}_{\underline{0}}|} L_2(2\chi - 2\delta_\infty) \sum_{p=2}^{n-1} \mathcal{K}_p \mathcal{K}_{n+1-p}, \end{aligned} \tag{5.55}$$

with

$$D_n \ := \ \max\left\{\left(\frac{3A}{e(\delta_n - \delta_{n-1})}\right)^{2A} \exp\left(\frac{2\delta_\infty}{3\tanh(\chi - \delta_\infty)}\right), \quad L_2(\chi - \delta_\infty)^2 \frac{\mathcal{Q}^{(2)}}{|Q^{(2)}_{\underline{0}}|}\right\}. \tag{5.56}$$

We finally got the bound

$$|C^{(n)}_{\underline{m}}| \leq \mathcal{K}_n e^{-(\chi-\delta_n)|\underline{m}|}, \tag{5.57}$$

with

$$\mathcal{K}_n \ := \ 2\Delta\mathcal{Q}\left(\frac{3\sigma}{e(\delta_n - \delta_{n-1})}\right)^{\sigma} D_n \sum_{p=1}^{n-1} \mathcal{K}_p \mathcal{K}_{n-p} + \frac{\mathcal{Q}}{2|Q^{(2)}_{\underline{0}}|} L_2(2\chi - 2\delta_\infty) \sum_{p=2}^{n-1} \mathcal{K}_p \mathcal{K}_{n+1-p}. \tag{5.58}$$

By induction, this establishes the bound

$$|C_{\underline{m}}^{(n)}| \leq \mathcal{K}_n e^{-(\chi-\delta_\infty)|\underline{m}|} \tag{5.59}$$

for all $n \in \mathbb{N}$ and this proves (5.38) for all $n \in \mathbb{N}$. This, in turn, finally establishes that all functions c_n are quasi-periodic on $\mathbb{R}$ since it implies that $\sum_{\underline{m}\in\mathbb{Z}^A} |C_{\underline{m}}^{(n)}| < \infty$.

5.5 The Convergence of the Power Expansion on ϵ

We have now to deal with the question of the absolute convergence of the formal power series in (3.1). Since $|c_n(t)| \leq \mathcal{K}_n \sum_{\underline{m}\in\mathbb{Z}^A} e^{-(\chi-\delta_\infty)|\underline{m}|}$ we have to analyze the convergence of the formal infinite sum

$$\sum_{n=1}^{\infty} \mathcal{K}_n |\epsilon|^n. \tag{5.60}$$

Since the increasing sequence δ_n, $n \in \mathbb{N}$, introduced in the definition of $\mathcal{K}_n$, has to converge to some $\delta_\infty < \chi$, the difference $\delta_n - \delta_{n-1}$ occurring in (5.58) has to decrease to zero for $n \to \infty$. Hence, the leading behaviour for n large of the right hand side of (5.58) is governed by

$$\mathcal{K}_n \approx \frac{C}{(\delta_n - \delta_{n-1})^{\sigma+2A}} \sum_{p=1}^{n-1} \mathcal{K}_p \mathcal{K}_{n-p}, \tag{5.61}$$

for some positive constant C. Even choosing a slowly converging sequence δ_n, for instance such that $\delta_n - \delta_{n-1} \simeq n^{-(1+\epsilon_0)}$, with some small $\epsilon_0 > 0$, we would probably find a large-n behaviour for $\mathcal{K}_n$ like $\mathcal{K}_n \approx (n!)^{(1+\epsilon_0)(\sigma+2A)} c^n$ for some constant $c > 1$.

Unfortunately we conclude that the upper bounds on the $\mathcal{K}_n$ obtainable from the recursive definition (5.58) are too bad, for whatever choice of the converging sequence δ_n, to guarantee the convergence of the sum in (5.60) for any finite ϵ and, hence, analyticity on ϵ cannot be established through the above analysis. Actually, we will discuss heuristically in Section 7 that, perhaps, analyticity around $\epsilon = 0$ should not be generally expected to hold in the system we considered. On the other hand, *uniform* convergence of (3.1) for all $t \in \mathbb{R}$ for a special set (eventually dense, but with empty interior) of values of ϵ is not to be excluded, but to prove this conjecture would require a more refined analysis than that presented here.

6 Quasi-Periodicity of the Functions e_n

In this section we will prove the quasi-periodicity of the functions e_n.

6.1 The Fourier Decomposition of the Functions c_n

For all $n \geq 1$ we write,

$$e_n(t) \;=\; \sum_{\underline{m}\in\mathbb{Z}^A} E_{\underline{m}}^{(n)}\, e^{i\underline{m}\cdot\underline{\omega}t}. \tag{6.1}$$

We will study the Fourier coefficients $E^{(n)}_{\underline{m}}$. We have from (4.5)

$$E^{(1)}_{\underline{m}} = \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{Q_{\underline{m}-\underline{n}}\overline{Q^{(2)}_{-\underline{n}}}}{\underline{n}\cdot\underline{\omega}} + Q_{\underline{m}}\left(\beta_1 - \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{\overline{Q^{(2)}_{-\underline{n}}}}{\underline{n}\cdot\underline{\omega}}\right), \tag{6.2}$$

and, for $n\geq 2$, we have from (4.6)

$$\begin{aligned} E^{(n)}_{\underline{m}} &= \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} Q_{\underline{m}-\underline{n}_1-\underline{n}_2}\left(\sum_{p=1}^{n-1} E^{(p)}_{\underline{n}_1}E^{(n-p)}_{\underline{n}_2}\right)\frac{1}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} + \\ & Q_{\underline{m}}\left[\beta_n - \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}}\left(\sum_{p=1}^{n-1} E^{(p)}_{\underline{n}_1}E^{(n-p)}_{\underline{n}_2}\right)\frac{1}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}}\right], \qquad n\geq 2. \end{aligned} \tag{6.3}$$

For the constants β_* we have (see (4.12) and (4.17))

$$\beta_1 = \frac{1}{2iM(\mathcal{Q}_1)}\sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1\neq\underline{0},\ \underline{n}_2\neq\underline{0}}} \frac{Q^{(2)}_{\underline{n}_1+\underline{n}_2}\,\overline{Q^{(2)}_{\underline{n}_1}}\;\overline{Q^{(2)}_{\underline{n}_2}}}{(\underline{n}_1\cdot\underline{\omega})(\underline{n}_2\cdot\underline{\omega})} + \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}}\frac{\overline{Q^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}}, \tag{6.4}$$

$$\begin{aligned} \beta_n &= \frac{1}{iM(\mathcal{Q}_1)}\left[\sum_{p=1}^{n-1}\sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}}\left(Q^{(2)}_{-\underline{n}_1-\underline{n}_2}\mathcal{R} - \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}}\frac{Q^{(2)}_{\underline{n}+\underline{n}_1+\underline{n}_2}\overline{Q^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}}\right)\frac{E^{(p)}_{\underline{n}_1}E^{(n-p)}_{\underline{n}_2}}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}}\right. \\ &\left. +\frac{1}{2}\sum_{p=2}^{n-1}\sum_{\underline{n}\in\mathbb{Z}^A} E^{(p)}_{\underline{n}}E^{(n+1-p)}_{-\underline{n}}\right] + \sum_{p=1}^{n-1}\sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}}\frac{E^{(p)}_{\underline{n}_1}E^{(n-p)}_{\underline{n}_2}}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}}, \qquad \text{for } n\geq 2, \end{aligned} \tag{6.5}$$

where

$$\mathcal{R} := \beta_1 + \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}}\frac{\overline{Q^{(2)}_{-\underline{n}}}}{\underline{n}\cdot\underline{\omega}} = \frac{1}{2iM(\mathcal{Q}_1)}\sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1\neq\underline{0},\ \underline{n}_2\neq\underline{0}}}\frac{Q^{(2)}_{\underline{n}_1+\underline{n}_2}\,\overline{Q^{(2)}_{\underline{n}_1}}\;\overline{Q^{(2)}_{\underline{n}_2}}}{(\underline{n}_1\cdot\underline{\omega})(\underline{n}_2\cdot\underline{\omega})}. \tag{6.6}$$

Finally, for the Fourier coefficients we get,

$$\begin{aligned} E^{(1)}_{\underline{m}} &= \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}}\frac{Q_{\underline{m}+\underline{n}}\overline{Q^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}} + \frac{Q_{\underline{m}}}{2iM(\mathcal{Q}_1)}\sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1\neq\underline{0},\ \underline{n}_2\neq\underline{0}}}\frac{Q^{(2)}_{\underline{n}_1+\underline{n}_2}\,\overline{Q^{(2)}_{\underline{n}_1}}\;\overline{Q^{(2)}_{\underline{n}_2}}}{(\underline{n}_1\cdot\underline{\omega})(\underline{n}_2\cdot\underline{\omega})} \\ &= \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}}\left(Q_{\underline{m}+\underline{n}} + \frac{Q_{\underline{m}}}{2iM(\mathcal{Q}_1)}\sum_{\substack{\underline{n}'\in\mathbb{Z}^A\\ \underline{n}'\neq\underline{0}}}\frac{Q^{(2)}_{\underline{n}+\underline{n}'}\overline{Q^{(2)}_{\underline{n}'}}}{\underline{n}'\cdot\underline{\omega}}\right)\frac{\overline{Q^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}}, \end{aligned} \tag{6.7}$$

$$
\begin{aligned}
E^{(n)}_{\underline{m}} &= \sum_{p=1}^{n-1} \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \left[Q_{\underline{m}-\underline{n}_1-\underline{n}_2} + \frac{Q_{\underline{m}}}{iM(\mathcal{Q}_1)} \left(Q^{(2)}_{-\underline{n}_1-\underline{n}_2}\mathcal{R} - \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{Q^{(2)}_{\underline{n}+\underline{n}_1+\underline{n}_2}\overline{Q^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}} \right)\right] \frac{E^{(p)}_{\underline{n}_1}E^{(n-p)}_{\underline{n}_2}}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} \\
&\quad + \frac{Q_{\underline{m}}}{2iM(\mathcal{Q}_1)} \sum_{p=2}^{n-1} \sum_{\underline{n}\in\mathbb{Z}^A} E^{(p)}_{\underline{n}} E^{(n+1-p)}_{-\underline{n}}, \qquad \text{for } n\geq 2. \qquad (6.8)
\end{aligned}
$$

In the next two subsections we will follow the same steps of the case $M(q^2)\neq 0$ and we will try to find convenient upper bounds for the Fourier coefficients given recursively in (6.7) and (6.8).

6.2 An Upper Bound for $|E^{(1)}_{\underline{m}}|$

We will make use again of (5.18) and of (5.27) in the form $|Q^{(2)}_{\underline{m}}| \leq \mathcal{Q}^{(2)}(\delta)e^{-(\chi-\delta)|\underline{m}|}$, for $\delta>0$ arbitrary but conveniently small. From (6.7) we have, using the same inequalities as in the previous case,

$$
\begin{aligned}
|E^{(1)}_{\underline{m}}| &\leq \mathcal{Q}\mathcal{Q}^{(2)}(\delta) \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \left(e^{-\chi|\underline{m}+\underline{n}|} + e^{-\chi|\underline{m}|} \frac{\mathcal{Q}^{(2)}(\delta)^2}{2|M(\mathcal{Q}_1)|} \sum_{\substack{\underline{n}'\in\mathbb{Z}^A\\ \underline{n}'\neq\underline{0}}} \frac{e^{-(\chi-\delta)(|\underline{n}+\underline{n}'|+|\underline{n}'|)}}{|\underline{n}'\cdot\underline{\omega}|} \right) \frac{e^{-(\chi-\delta)|\underline{n}|}}{|\underline{n}\cdot\underline{\omega}|} \\
&\leq W_1 \sum_{\underline{n}\in\mathbb{Z}^A} \left(e^{-\chi|\underline{m}+\underline{n}|} + e^{-\chi|\underline{m}|} W_2 \sum_{\underline{n}'\in\mathbb{Z}^A} e^{-(\chi-\delta)|\underline{n}+\underline{n}'|-(\chi-2\delta)|\underline{n}'|)} \right) e^{-(\chi-2\delta)|\underline{n}|}, \qquad (6.9)
\end{aligned}
$$

with $W_1 := \Delta\mathcal{Q}\mathcal{Q}^{(2)}(\delta)\left(\frac{\sigma}{e\delta}\right)^{\sigma}$ and $W_2 := \Delta\frac{\mathcal{Q}^{(2)}(\delta)^2}{2|M(\mathcal{Q}_1)|}\left(\frac{\sigma}{e\delta}\right)^{\sigma}$. Using the previously defined function Ξ_0, the last inequality in (6.9) leads to

$$
|E^{(1)}_{\underline{m}}| \leq W_1\Xi_0(\chi-2\delta,\ \underline{m}) + e^{-\chi|\underline{m}|}W_2 \sum_{\underline{n}\in\mathbb{Z}^A} \Xi_0(\chi-2\delta,\ \underline{n})e^{-(\chi-2\delta)|\underline{n}|} \qquad (6.10)
$$

and using (5.25) we get

$$
|E^{(1)}_{\underline{m}}| \leq \mathcal{L}_1 e^{-(\chi-3\delta)|\underline{m}|} \qquad (6.11)
$$

for some constant $\mathcal{L}_1$ (depending on χ, δ, σ etc.). Of course, one sees here that, choosing $0<\delta<\chi/3$, we have proven that e_1 is a quasi-periodic function on $\mathbb{R}$.

6.3 Recursive Upper Bounds for $|E^{(n)}_{\underline{m}}|$ with $n\geq 2$

We start now from relation (6.8) and try to find recursive bounds for $|E^{(n)}_{m}|$, with $n\geq 2$, as in the case of the $|C^{(n)}_{m}|$'s. For $1\leq p\leq n-1$ we assume the bound

$$
|E^{(p)}_{m}| \leq \mathcal{L}_p\, e^{-(\chi-\delta_p)|\underline{m}|} \qquad (6.12)
$$

for $\mathcal{L}_p \geq 0$ and $\delta_p > 0$, small enough. This assumption has been proven above for the case $p = 1$ (see (6.11)). From (6.8) we have

$$
\begin{aligned}
|E^{(n)}_{\underline{m}}| \;\; \leq \;\; & \mathcal{Q} \sum_{p=1}^{n-1} \mathcal{L}_p \mathcal{L}_{n-p} \Bigg\{ \sum_{\substack{\underline{n}_1,\, \underline{n}_2 \in \mathbb{Z}^A \\ \underline{n}_1 + \underline{n}_2 \neq \underline{0}}} \Bigg[e^{-\chi |\underline{m} - \underline{n}_1 - \underline{n}_2|} + \frac{e^{-\chi |\underline{m}|} \mathcal{Q}^{(2)}(\delta)}{|M(\mathcal{Q}_1)|} \Bigg(|\mathcal{R}| e^{-(\chi - \delta)|\underline{n}_1 + \underline{n}_2|} + \\
& \mathcal{Q}^{(2)}(\delta) \sum_{\substack{\underline{n} \in \mathbb{Z}^A \\ \underline{n} \neq \underline{0}}} \frac{e^{-(\chi - \delta)(|\underline{n} + \underline{n}_1 + \underline{n}_2| + |\underline{n}|)}}{|\underline{n} \cdot \underline{\omega}|} \Bigg) \Bigg] \frac{e^{-(\chi - \delta_p)|\underline{n}_1| - (\chi - \delta_{n-p})|\underline{n}_2|}}{|(\underline{n}_1 + \underline{n}_2) \cdot \underline{\omega}|} \Bigg\} + \\
& e^{-\chi |\underline{m}|} \frac{\mathcal{Q}}{2|M(\mathcal{Q}_1)|} \sum_{p=2}^{n-1} \mathcal{L}_p \mathcal{L}_{n+1-p} \Bigg\{ \sum_{\underline{n} \in \mathbb{Z}^A} e^{-(2\chi - \delta_p - \delta_{n+1-p})|\underline{n}|} \Bigg\},
\end{aligned}
$$

for $n \geq 2$. Using once more (5.17) we have

$$
\begin{aligned}
|E^{(n)}_{\underline{m}}| \;\; \leq \;\; & \mathcal{Q} \Delta \left(\frac{\sigma}{e\delta} \right)^{\sigma} \sum_{p=1}^{n-1} \mathcal{L}_p \mathcal{L}_{n-p} \Bigg\{ L_3(p,\ \underline{m}) + e^{-\chi |\underline{m}|} \frac{\mathcal{Q}^{(2)}(\delta)}{|M(\mathcal{Q}_1)|} \Bigg[|\mathcal{R}|\ L_4(\chi - \delta_p,\ \chi - \delta_{n-p},\ \chi - 2\delta) + \\
& \Delta \left(\frac{\sigma}{e\delta} \right)^{\sigma} \Bigg(\sum_{\underline{n} \in \mathbb{Z}^A} e^{-(\chi - 2\delta)|\underline{n}|} L_3(\chi - \delta_p,\ \chi - \delta_{n-p},\ 0,\ \chi - \delta,\ \underline{n}) \Bigg) \Bigg] \Bigg\} + \\
& e^{-\chi |\underline{m}|} \frac{\mathcal{Q}}{2|M(\mathcal{Q}_1)|} \sum_{p=2}^{n-1} \ \mathcal{L}_p \, \mathcal{L}_{n+1-p} \ L_2(2\chi - \delta_p - \delta_{n+1-p}).
\end{aligned}
$$

Using

$$L_3(\chi - \delta_p,\ \chi - \delta_{n-p},\ 0,\ \chi - \delta,\ \underline{n}) \leq L_2(\chi - \delta[n])^2\ L_2(\chi - \delta)$$

and

$$L_4(\chi - \delta_p,\ \chi - \delta_{n-p},\ \chi - 2\delta) \leq L_2(\chi - \delta[n])^2$$

(see (5.48) and (5.51)), we get

$$|E^{(n)}_{\underline{m}}| \;\leq\; \mathcal{L}_n \, e^{-(\chi - \delta[n] - 3\delta)|\underline{m}|}, \tag{6.13}$$

where

$$
\begin{aligned}
\mathcal{L}_n \;\; := \;\; & \mathcal{Q} \Delta \left(\frac{\sigma}{e\delta} \right)^{\sigma} \left(\sum_{p=1}^{n-1} \mathcal{L}_p \mathcal{L}_{n-p} \right) \Bigg\{ \left(\frac{A}{e\delta} \right)^{2A} \exp \left(\frac{2\delta}{\tanh(\chi - \delta[n] - 2\delta)} \right) + \\
& \frac{\mathcal{Q}^{(2)}(\delta)}{|M(\mathcal{Q}_1)|} \left(|\mathcal{R}| L_2(\chi - \delta[n])^2 + \Delta \left(\frac{\sigma}{e\delta} \right)^{\sigma} L_2(\chi - \delta[n])^2 L_2(\chi - \delta)^2 \right) \Bigg\} + \\
& \frac{\mathcal{Q}}{2|M(\mathcal{Q}_1)|} L_2(2\chi - 2\delta[n]) \left(\sum_{p=2}^{n-1} \ \mathcal{L}_p \, \mathcal{L}_{n+1-p} \right).
\end{aligned} \tag{6.14}
$$

Choosing δ_n as an increasing sequence (converging to $\delta_\infty < \chi$) one has $\delta[n] = \delta_{n-1}$ and, with the choice $\delta = (\delta_n - \delta_{n-1})/3$, we get from (6.13)

$$|E^{(n)}_{\underline{m}}| \;\leq\; \mathcal{L}_n \, e^{-(\chi - \delta_n)|\underline{m}|}, \tag{6.15}$$

thus proving inductively the bound (6.12) for $\mathcal{L}_n$ given by the recurrence (6.14). As in the case of the functions c_n considered previously, this establishes that the functions e_n are, for all $n \geq 1$, quasi-periodic on $\mathbb{R}$ since it implies that $\sum_{\underline{m} \in \mathbb{Z}^A} |E_{\underline{m}}^{(n)}| < \infty$.

Note that from (6.14) the leading behaviour of $\mathcal{L}_n$ for large n is expected to be like

$$\mathcal{L}_n \approx \frac{C}{(\delta_n - \delta_{n-1})^{\sigma+2A}} \sum_{p=1}^{n-1} \mathcal{L}_p \mathcal{L}_{n-p}, \tag{6.16}$$

as in the case of $\mathcal{K}_n$ above. As in that case, we draw the same negative conclusion about the possibility of proving absolute convergence of the infinite sum in (4.1) with the methods used in this paper.

7 Final Discussion

To finish these notes let us now briefly discuss in a non-rigorous way some problems involving our analysis and our results.

Our original intention was to prove quasi-periodicity of the solutions of the generalized Riccati equation (2.1) by analysing its power series expansion in ϵ. A step in this direction was the method of elimination of the secular terms, another was the proof of quasi-periodicity of the coefficients c_n and e_n. However, the series expansions for g in ϵ could not be proven to be uniformly convergent for all $t \in \mathbb{R}$ and ϵ in some open set containing the point $\epsilon = 0$.

The question is, when should we expect that g is quasi-periodic? As discussed in Appendix A, the function g appeared in the Ansatz $\phi(t) = \exp\left(-i \int_0^t (f(t') + g(t'))dt'\right)$ for one of the particular solutions of one of the components of the wave function Φ, or more precisely, for a particular solution of the quasi-periodic Hill's equation (1.26). It is clear that g could not be expected to be quasi-periodic if $\phi(t)$ becomes zero at some finite t, because quasi-periodic functions are bounded. A theorem by H. Bohr [14] asserts, modulo technicalities, that if $\phi(t)$ is bounded away from zero then g should be quasi-periodic. Hence, one of the questions would be to know whether there are particular solutions ϕ of the quasi-periodic Hill's equation (1.26) such that $|\phi(t)| > \delta$ for all $t \in \mathbb{R}$ and some $\delta > 0$.

Its is hard to prove a priori this condition. However, in the situation where $f(t) = f_0$ is constant, we know that the particular solutions of (1.26) are of the form $e^{\pm i\omega_0 t}$ with $\omega_0 = \sqrt{\epsilon^2 + f_0^2}$. Each of these particular solutions do not vanish for $t \in \mathbb{R}$. In this case we have been able to show that the series expansion for g is convergent for $|\epsilon| < |f_0|$. This leads to the conjecture that g should be quasi-periodic at least for small perturbations of the case $f(t) =$ constant, for instance if $f(t) - M(f)$ is uniformly small.

We have not been able to prove this conjecture but there is another problem which has to be faced for a better understanding of the whole picture. The unitarity condition $U(t)U(t)^* = \mathbb{1}$ reads

$$|R(t)|^2 \left(1 - 2\text{Im}\,(g(0)S(t)) + \left(|g(0)|^2 + \epsilon^2\right) |S(t)|^2\right) = 1. \tag{7.1}$$

Looking at (7.1) we see that some additional constraints have to be fulfilled for a physically reasonable solution.

First we note that, if we assume that a quasi-periodic solution g for the generalized Riccati equation (2.1) was found, then we must have $M(g) \in \mathbb{R}$ for real ϵ. For, if $\mathrm{Im}(M(g)) > 0$, $R(t)$ diverges exponentially for $t \to +\infty$ while $S(t)$ stays bounded, violating (7.1) for large t. If, on the other hand, $\mathrm{Im}(M(g)) < 0$, $R(t)$ will decay like $\exp(-|\mathrm{Im}(M(g))|t)$ for $t \to +\infty$ while $S(t)$ will behave as $\exp(+2|\mathrm{Im}(M(g))|t)$. Hence, the product $|R(t)|^2|S(t)|^2$ will diverge for $t \to +\infty$ making (7.1) impossible again. It is difficult to verify the condition $M(g) \in \mathbb{R}$ for real ϵ directly from our perturbative expansions for g but, from the unitarity of the time evolution, this must be true.

This, however, is not the whole story, since an additional problem involving the unitarity condition (7.1) exists. Under the assumption that g is quasi-periodic with $M(g) \in \mathbb{R}$ for real ϵ it could still happen that $M(R^{-2}) \neq 0$. In this case the function $S(t)$ will develop a term like $M(R^{-2})t$, which is linearly increasing in t, violating again (7.1) for large t. Hence, if we assume unitarity an additional mechanism must impose the condition $M(R^{-2}) = 0$. A non-rigorous argument indicates, however, that this is a tricky condition. If we assume that our perturbative expansions for g converge, the function $R(t)^{-2}$ must be, presumably, of the form

$$R(t)^{-2} = e^{2i(M(f)+M(g))t} \sum_{\underline{m} \in \mathbb{Z}^A} h_{\underline{m}}(\epsilon) e^{-i\underline{m}\cdot\underline{\omega}t}.$$

Hence, except for some miraculous cases, we can only guarantee that $M(R^{-2}) = 0$ if the equation $2M(g) = \underline{m} \cdot \underline{\omega} - 2M(f)$ has only solutions for $\underline{m} \in \mathbb{Z}^A$ if $h_{\underline{m}}(\epsilon) = 0$ for the corresponding values of $\underline{m}$.

Now, if the perturbative expansions we attempted converge absolutely and uniformly on $t \in \mathbb{R}$, then $M(g)$ is analytic on ϵ on some open neighbourhood of $\epsilon = 0$ and vanishes for $\epsilon = 0$, mapping thus a real open neighbourhood of $\epsilon = 0$ onto another real open neighbourhood of $\epsilon = 0$. Moreover, the set $\{\underline{m} \cdot \underline{\omega} - 2M(f),\ \underline{m} \in \mathbb{Z}^A\}$ is dense on $\mathbb{R}$ due to the Diophantine conditions imposed on the frequencies. Thus, there must be a countable and dense set $\mathcal{E}_0$ of values of ϵ on a small enough real open neighbourhood of $\epsilon = 0$ for which $2M(g) = \underline{m} \cdot \underline{\omega} - 2M(f)$ has a solution for some $\underline{m} \in \mathbb{Z}^A$. For these values of ϵ and $\underline{m}$ we must have $h_{\underline{m}}(\epsilon) = 0$, according to the argument above.

Such a condition is not impossible, but it shows how subtle is the mechanism which imposes unitarity of the time evolution for a quasi-periodic function g.

Another possibility would be that for ϵ in the set $\mathcal{E}_0$ the function g fails to be almost-periodic, due to some sort of resonance, and the linearly increasing term $M(R^{-2})t$ appearing in (7.1) would make no sense. This could explain the apparent impossibility of proving uniform convergence of our perturbative expansions for g.

It would be very interesting to investigate which, if any, of the mechanisms above is actually in action in the system considered. This should involve a deeper analysis than that attempted here.

Appendices

A The Wave Function, Hill's Equation and a Generalized Riccati Equation

Let $\Phi : \mathbb{R} \to \mathbb{C}^2$, $\mathbb{R} \ni t \mapsto \begin{pmatrix} \phi_+(t) \\ \phi_-(t) \end{pmatrix}$, with $\phi_\pm \in C^2(\mathbb{R})$, be a solution of the Schrödinger equation (1.5) with Hamiltonian H_1 above:

$$i\Phi'(t) \; = \; (\epsilon\sigma_1 + f(t)\sigma_3)\Phi(t), \tag{A.1}$$

for $f : \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and $\epsilon \in \mathbb{R}$, with initial data $\begin{pmatrix} \phi_+(0) \\ \phi_-(0) \end{pmatrix} \in \mathbb{C}^2$. Above σ_i are the Pauli matrices in their usual representations.

Taking the time derivative of both sides of (A.1) we get

$$\begin{aligned} i\Phi''(t) \; &= \; f'(t)\sigma_3\Phi(t) - i(\epsilon\sigma_1 + f(t)\sigma_3)^2\Phi(t) \\ &= \; f'(t)\sigma_3\Phi(t) - i\left[(\epsilon^2 + f(t)^2)\mathbb{1} + \epsilon f(t)(\sigma_1\sigma_3 + \sigma_3\sigma_1)\right]\Phi(t). \end{aligned} \tag{A.2}$$

Hence,

$$i\Phi''(t) = \left(f'(t)\sigma_3 - i(\epsilon^2 + f(t)^2)\mathbb{1}\right)\Phi(t), \tag{A.3}$$

which is a diagonal equation for $\Phi(t)$. For the components of Φ, (A.3) means

$$\phi_+''(t) \; + \; \left(+if'(t) + \epsilon^2 + f(t)^2\right)\phi_+(t) = 0, \tag{A.4}$$

$$\phi_-''(t) \; + \; \left(-if'(t) + \epsilon^2 + f(t)^2\right)\phi_-(t) = 0. \tag{A.5}$$

The solutions of (A.1) can be recovered from the solutions of the differential equations (A.4) and (A.5) with initial data $(\phi_+(0),\ \phi_+'(0))$ and $(\phi_-(0),\ \phi_-'(0))$, respectively, by imposing

$$i\Phi'(0) \; = \; (\epsilon\sigma_1 + f(0)\sigma_3)\Phi(0) \tag{A.6}$$

as a restriction to the initial data, i.e., by imposing

$$i\phi_+'(0) \; = \; f(0)\phi_+(0) + \epsilon\phi_-(0), \tag{A.7}$$

$$i\phi_-'(0) \; = \; -f(0)\phi_-(0) + \epsilon\phi_+(0). \tag{A.8}$$

Since (A.5) is obtained from (A.4) by the change $f \to -f$, we will study only the equation

$$\phi''(t) + \left(if'(t) + \epsilon^2 + f(t)^2\right)\phi(t) = 0, \tag{A.9}$$

with initial data $(\phi(0),\ \phi'(0))$.

In order to motivate the Ansatz we are going to use, let us consider first a particular way of solving (A.9) for the case $\epsilon \equiv 0$.

A.1 The Case $\epsilon = 0$

In this case equation (A.9) reduces to

$$\phi''(t) + \left(if'(t) + f(t)^2\right)\phi(t) = 0. \tag{A.10}$$

A natural Ansatz is given by solutions of the form

$$\phi(t) = \phi_0 \exp\left(-i\int_0^t (G(t') + f(t'))\,dt'\right),$$

where ϕ_0 is an arbitrary constant. This leads to the following equation for G:

$$G' - iG^2 - 2ifG = 0. \tag{A.11}$$

This is an equation of the Bernoulli type, and has one obvious solution: $g \equiv 0$. Another solution can be found defining $v = 1/G$, which, in face of (A.11), leads to the linear equation

$$v' + 2ifv + i = 0, \tag{A.12}$$

whose general solution is

$$v(t) \;=\; \frac{1}{p_0(t)}\left(a_0 - i\int_0^t p_0(t')dt'\right), \tag{A.13}$$

where a_0 is an arbitrary constant and

$$p_0(t) \;:=\; \exp\left(2i\int_0^t f(t')dt'\right). \tag{A.14}$$

This leads to the following general expression for the solution of (A.11):

$$G(t) = \frac{p_0(t)}{a_0 - i\displaystyle\int_0^t p_0(t')dt'}. \tag{A.15}$$

Since $G \equiv 0$ is also a solution of (A.11), we allow formally $a_0 \in \mathbb{C} \cup \{+\infty\}$.

Hence, the general solution of (A.10) is

$$\phi(t) = \phi_0 \exp\left(-i\int_0^t f(t')\,dt'\right)\exp\left(-i\int_0^t \frac{p_0(t')}{a_0 - i\displaystyle\int_0^{t'} p_0(t'')dt''}\,dt'\right), \tag{A.16}$$

for arbitrary constants ϕ_0 and a_0. Since

$$\frac{p_0(t)}{a_0 - i\displaystyle\int_0^t p_0(t')dt'} = i\left(\ln\left(a_0 - i\int_0^t p_0(t')dt'\right)\right)'$$

we get, finally, the general solution of (A.10),

$$\phi(t) = \exp\left(-i\int_0^t f(t')\,dt'\right)\left[\phi_1 + \phi_2\int_0^t p_0(t')dt'\right], \tag{A.17}$$

with $\phi_1 = \phi_0 a_0^2$ and $\phi_2 = -i\phi_0 a_0$, or in terms of the initial data,

$$\begin{aligned} \phi_1 &= \phi(0), \\ \phi_2 &= if(0)\phi(0) + \phi'(0). \end{aligned} \tag{A.18}$$

After imposing (A.7) and (A.8) (with $\epsilon = 0$, of course) we get, as expected, the solutions

$$\phi_+(t) = \phi_+(0) \exp\left(-i\int_0^t f(t')\, dt'\right), \tag{A.19}$$

$$\phi_-(t) = \phi_-(0) \exp\left(+i\int_0^t f(t')\, dt'\right). \tag{A.20}$$

A.2 The Case $\epsilon \neq 0$

For $\epsilon \neq 0$ we will follow steps analogous to the previous case, but the situation here is more complicated. For solving (A.9) we start with the same Ansatz, namely,

$$\phi(t) = \phi_0 \exp\left(-i\int_0^t (G(t') + f(t'))\, dt'\right).$$

This leads to equation (2.1) for G.

Equation (2.1) is a differential equation of the generalized Riccati type. According to the theory of the generalized Riccati equations, given a particular solution g of (2.1), a general solution is of the form $G = g + u$, where u satisfies the Bernoulli equation

$$u' - 2i(f+g)u - iu^2 = 0. \tag{A.21}$$

This, in turn, can be transformed into a linear equation by defining $v := 1/u$, which gives for v

$$v' + 2i(f+g)v + i = 0. \tag{A.22}$$

The general solution of (A.22) is

$$v(t) = \frac{1}{p(t)}\left(a_0 - i\int_0^t p(t')dt'\right), \tag{A.23}$$

where a_0 is an arbitrary constant and

$$p(t) := \exp\left(2i\int_0^t (f(t') + g(t'))dt'\right). \tag{A.24}$$

Hence, the general solution of (2.1) is

$$G(t) = g(t) + \frac{p(t)}{a_0 - i\int_0^t p(t')dt'}. \tag{A.25}$$

Since $u \equiv 0$ is, in principle, also a solution of (A.21), we allow formally $a_0 \in \mathbb{C} \cup \{+\infty\}$.

For (A.9) we get, in complete analogy with the $\epsilon = 0$ case, the general solution

$$\phi(t) = \exp\left(-i\int_0^t (f(t') + g(t'))\,dt'\right)\left[\phi_1 + \phi_2\int_0^t p(t')dt'\right], \tag{A.26}$$

with ϕ_1, $\phi_2 \in \mathbb{C}$, arbitrary constants. In terms of the initial data one easily checks that

$$\begin{aligned} \phi_1 &= \phi(0), \\ \phi_2 &= i(f(0) + g(0))\phi(0) + \phi'(0). \end{aligned} \tag{A.27}$$

Expression (A.26) indicates the solution for ϕ_+. The solution for ϕ_- can be obtained, as we pointed before, by interchanging $f \to -f$. Since g depends on f, we must first discover what happens to g by changing the sign of f. The generalized Riccati equation (2.1) indicates that, if g is a solution for f, then $-\overline{g}$ is a solution for $-f$. Hence, using the constraints (A.7)-(A.8), we get finally equations (2.2) and (2.3) which express $\phi_\pm$ in terms of g and of the initial conditions.

In the general case it is interesting to note that the whole dependence on ϵ in (A.21) is hidden in the still unknown function g. For $\epsilon = 0$, equation (2.1) reduces to (A.11). Comparing (A.21) with (A.11), (A.14) with (A.24) and (A.15) with (A.25) we conclude that we should have $g \equiv 0$ for $\epsilon = 0$.

In the case where $f(t) = f_0$ is a constant, the solution $g = -f_0 + \mathrm{sign}(f_0)\sqrt{f_0^2 + \epsilon^2}$ for $f_0 \neq 0$ satisfies this condition. For $f_0 = 0$ we have $g = \pm\epsilon$.

B The Quasi-Periodicity of $q(t)$ and $\exp(iF(t))$

It is already clear that $\exp(iF(t))$ is an almost-periodic function since F (defined in (5.3)) is quasi-periodic and since the set of all almost-periodic functions on $\mathbb{R}$ is a closed sub-algebra of $L^\infty(\mathbb{R})$ (see, e.g. [12]).

Let P_n be the Taylor polynomial of degree n of the exponential function: $P_n(x) = \sum_{a=0}^{n} \frac{x^a}{a!}$. We know that

$$|\exp(x) - P_n(x)| \;\leq\; \left(1 - \frac{|x|}{n+2}\right)^{-1} \frac{|x|^{n+1}}{(n+1)!}, \tag{B.1}$$

for $n + 2 > |x|$, $x \in \mathbb{C}$. Let F_n, $n \in \mathbb{N}$, be given by

$$F_n(t) \;:=\; -i \sum_{\substack{\underset{\sim}{n} \in \mathbb{Z}^B,\ \underset{\sim}{n} \neq \underset{\sim}{0} \\ |\underset{\sim}{n}|_\infty \leq n}} \frac{f_{\underset{\sim}{n}}}{\underset{\sim}{n} \cdot \underset{\sim}{\omega}_f} \left(e^{i\underset{\sim}{n}\cdot\underset{\sim}{\omega}_f t} - 1\right), \tag{B.2}$$

with $|\underset{\sim}{n}|_\infty := \max\{|n_1|, \ldots, |n_B|\}$, the l^∞ norm on $\mathbb{Z}^B$, and define

$$\sigma_n(t) \equiv \sigma_n[F](t) := \frac{F_1(t) + \cdots + F_n(t)}{n}. \tag{B.3}$$

By the hypothesis, σ_n converges to F uniformly on $\mathbb{R}$ (see e.g. [12]). We also know that $|F_n| \leq D$, uniformly on n and t, where D is the left hand side of (5.2). Therefore $|\sigma_n(t)| \leq D$ uniformly on n

and t. Hence, for n large enough,

$$
\begin{aligned}
|\exp(iF(t)) - P_n(i\sigma_n(t))| &\leq |\exp(iF(t)) - \exp(i\sigma_n(t))| + |\exp(i\sigma_n(t)) - P_n(i\sigma_n(t))| \\
&\leq |\exp(iF(t)) - \exp(i\sigma_n(t))| + \left(1 - \frac{|\sigma_n(t)|}{n+2}\right)^{-1} \frac{|\sigma_n(t)|^{n+1}}{(n+1)!}.
\end{aligned}
$$

From the fact that $|\sigma_n(t)|$ is uniformly bounded on n and t and from the uniform continuity of the exponential function on compact sets we conclude that $|\exp(iF(t)) - P_n(i\sigma_n(t))|$ can be made smaller than any prescribed $\epsilon > 0$, for all $t \in \mathbb{R}$, by choosing n large enough.

The Fourier coefficients $W(\xi)$, $\xi \in \mathbb{R}$, of the almost-periodic function $\exp(iF(t))$ are given by (see, e.g. [12])

$$
W(\xi) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \exp(iF(\tau)) e^{-i\xi\tau} \, d\tau. \tag{B.4}
$$

According to our previous remark, we have

$$
|W(\xi)| \leq \epsilon + \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} P_n(i\sigma_n(\tau)) e^{-i\xi\tau} \, d\tau \right|, \tag{B.5}
$$

for any prescribed $\epsilon > 0$, for n large enough. Now, for each n fixed, $\int_{-T}^{T} P_n(i\sigma_n(\tau)) e^{-i\xi\tau} \, d\tau$ is bounded in T if ξ is not of the form $\underset{\sim}{n} \cdot \underset{\sim}{\omega}_f$, with $\underset{\sim}{n} \in \mathbb{Z}^B$.

Hence, for ξ not of the form $\underset{\sim}{n} \cdot \underset{\sim}{\omega}_f$, with $\underset{\sim}{n} \in \mathbb{Z}^B$, we have $W(\xi) = 0$. This completes the proof that $\exp(iF(t))$ (and hence $q(t)$) is quasi-periodic with $\beta(\exp(iF(t))) = \beta(F) = \beta(f)$.

C The Relation Between q and f

Since f is real and quasi-periodic we write

$$
f(t) = f_0 + \sum_{\substack{\underset{\sim}{n} \in \mathbb{Z}^B \\ \underset{\sim}{n} \neq \underset{\sim}{0}}} f_{\underset{\sim}{n}} e^{i\underset{\sim}{n}\cdot\underset{\sim}{\omega}_f t},
$$

with $f_0 = M(f) \in \mathbb{R}$ and $\overline{f_{\underset{\sim}{n}}} = f_{-\underset{\sim}{n}}$.

To simplify our analysis we will consider here the case where the sum above is a finite sum. This situation is physically more realistic anyway.

To fix the notation we write

$$
f(t) = f_0 + \sum_{a=1}^{2J} f_a e^{i\underset{\sim}{n}_a\cdot\underset{\sim}{\omega}_f t}, \tag{C.1}
$$

with the convention $\underset{\sim}{n}_a = -\underset{\sim}{n}_{2J-a+1} \neq \underset{\sim}{0}$, $1 \leq a \leq J$, with $f_a \equiv f_{\underset{\sim}{n}_a}$ and where $J \geq 1$. Clearly $\overline{f_a} = f_{2J-a+1}$, $1 \leq a \leq J$.

We get

$$
q(t) = e^{i\gamma_f} e^{if_0 t} \prod_{a=1}^{2J} \exp\left(\frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} e^{i\underset{\sim}{n}_a\cdot\underset{\sim}{\omega}_f t} \right), \tag{C.2}
$$

with

$$\gamma_f \;:=\; i \sum_{a=1}^{2J} \frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f}. \tag{C.3}$$

As one easily sees, $\gamma_f \in \mathbb{R}$.

Expanding the exponential functions inside of the product we have

$$q(t) \;=\; e^{i\gamma_f} \sum_{p_1,\,\ldots,\,p_{2J}=0}^{\infty} \left\{ \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right)^{p_a} \right] \right\} \exp\left(i \left(f_0 + \underset{\sim}{\omega}_f \cdot \sum_{b=1}^{2J} p_b \, \underset{\sim}{n}_b \right) t \right). \tag{C.4}$$

The function q^2 is obtained by replacing $f \to 2f$, what gives

$$q(t)^2 \;=\; e^{i2\gamma_f} \sum_{p_1,\,\ldots,\,p_{2J}=0}^{\infty} \left\{ \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{2f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right)^{p_a} \right] \right\} \exp\left(i \left(2f_0 + \underset{\sim}{\omega}_f \cdot \sum_{b=1}^{2J} p_b \, \underset{\sim}{n}_b \right) t \right). \tag{C.5}$$

Hence,

$$M(q^2) \;=\; e^{i2\gamma_f} \sum_{\substack{p_1,\,\ldots,\,p_{2J}\,=\,0 \\ 2f_0 + \underset{\sim}{\omega}_f \cdot \sum_{b=1}^{2J} p_b \, \underset{\sim}{n}_b = 0}}^{\infty} \prod_{a=1}^{2J} \frac{1}{p_a!} \left(\frac{2f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right)^{p_a}. \tag{C.6}$$

Depending on the values of f_0 and of $\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f$, $1 \leq a \leq 2J$, it may be impossible to find a solution for

$$2f_0 + \underset{\sim}{\omega}_f \cdot \sum_{b=1}^{2J} p_b \, \underset{\sim}{n}_b = 0 \tag{C.7}$$

with $p_1, \ldots, p_{2J} \geq 0$. If that happens we have $M(q^2) = 0$.

Note that (C.7) has always solutions in the case $f_0 = 0$. There we have

$$\underset{\sim}{\omega}_f \cdot \sum_{b=1}^{2J} p_b \, \underset{\sim}{n}_b = \underset{\sim}{\omega}_f \cdot \sum_{b=1}^{J} (p_b - p_{2J-b+1}) \, \underset{\sim}{n}_b \tag{C.8}$$

and this can be made equal to zero by taking $p_b = p_{2J-b+1}$ for all b with $1 \leq b \leq J$.

From (C.4) we conclude that q has a Fourier decomposition

$$q(t) = \sum_{\underline{m} \in \mathbb{Z}^A} Q_{\underline{m}} e^{i\underline{m}\cdot\underline{\omega} t}$$

with $\underline{\omega}$ defined in (1.27) and with

$$Q_{\underline{m}} \;=\; e^{i\gamma_f} \sum_{p_1,\,\ldots,\,p_{2J}=0}^{\infty} \delta\left(\underline{P},\ \underline{m}\right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right)^{p_a} \right], \tag{C.9}$$

where

$$\underline{P} \;\equiv\; \underline{P}(p_1, \ldots, p_{2J}, \underset{\sim}{n}_1, \ldots, \underset{\sim}{n}_{2J}) \;:=\; \begin{cases} \displaystyle\sum_{b=1}^{2J} p_b \underset{\sim}{n}_b \in \mathbb{Z}^B, & \text{if } \underset{\sim}{f_0} = 0, \\[2ex] \left(\displaystyle\sum_{b=1}^{2J} p_b \underset{\sim}{n}_b,\ 1 \right) \in \mathbb{Z}^{B+1}, & \text{if } \underset{\sim}{f_0} \neq 0. \end{cases} \tag{C.10}$$

and where $\delta(\underline{P},\ \underline{m})$ is the Krönecker delta:

$$\delta(\underline{P},\ \underline{m}) \ := \ \begin{cases} 1, & \text{if } \underline{P} = \underline{m}, \\ 0, & \text{else.} \end{cases}$$

The presence of the factor $\delta(\underline{P},\ \underline{m})$ remembers the fact that $Q_{\underline{m}}$ is trivially zero if $\underline{P} = \underline{m}$ has no solutions for $p_1, \ldots, p_{2J} \in \mathbb{N}$. We say that $Q_{\underline{m}}$ is *non-trivial* if $\underline{P} = \underline{m}$ has solutions for some p_b's in $\mathbb{N}$

D The Exponential Decay of $Q_{\underline{m}}$

After the remarks of Appendix C we are now prepared for the proof of inequality (5.18). Of course, we have to concentrate on the non-trivial $Q_{\underline{m}}$'s. In this case the vectors $\underline{m}$ are either of the form $\underline{m} = (\underset{\sim}{m},\ 1) \in \mathbb{Z}^{B+1}$ for $\underset{\sim}{f_0} \neq 0$ or of the form $\underline{m} = \underset{\sim}{m} \in \mathbb{Z}^B$ for $\underset{\sim}{f_0} = 0$, where $\underset{\sim}{m} \in \mathbb{Z}^B$ satisfies

$$\underset{\sim}{m} \ = \ \sum_{b=1}^{2J} p_b \underset{\sim}{n}_b \tag{D.1}$$

for some $p_1, \ldots, p_{2J} \in \mathbb{N}$.

Let us denote by m^k (respectively, by n_b^k) the k-th component of $\underset{\sim}{m}$ (respectively, of $\underset{\sim}{n}_b$) with $1 \leq k \leq B$. The condition (D.1) says that

$$m^k \ = \ \sum_{b=1}^{2J} p_b n_b^k$$

with $1 \leq k \leq B$. Hence,

$$|m^k| \ \leq \ \left(\max_{1\leq b\leq 2J} p_b\right) \sum_{b=1}^{2J} |n_b^k|$$

and, after summing over all k, we get

$$\left(\max_{1\leq b\leq 2J} p_b\right) \ \geq \ \mathcal{N}^{-1}|\underset{\sim}{m}|, \tag{D.2}$$

where $|\underset{\sim}{m}| = \|\underset{\sim}{m}\|_1$ is the l_1 norm of $\underset{\sim}{m}$ and where

$$\mathcal{N} \ := \ \sum_{b=1}^{2J} |\underset{\sim}{n}_b|.$$

Note that, since the $\underset{\sim}{n}_b$'s are fixed by the choice of f, $\mathcal{N}$ is non-zero and will not be changed.

Let us now return to (C.9). It says that for the non-trivial $Q_{\underline{m}}$'s we have

$$|Q_{\underline{m}}| \ \leq \ \sum_{\substack{p_1,\ \ldots,\ p_{2J} = 0 \\ \max\limits_{1\leq b\leq 2J} p_b \geq \mathcal{N}^{-1}|\underset{\sim}{m}|}}^{\infty} \frac{\varphi^{p_1+\cdots+p_{2J}}}{p_1! \cdots p_{2J}!}$$

$$
\begin{aligned}
&\leq \; 2J \sum_{p_1,\ldots,p_{2J-1}=0}^{\infty} \; \sum_{p_{2J}\geq \mathcal{N}^{-1}|\underset{\sim}{m}|}^{\infty} \frac{\varphi^{p_1+\cdots+p_{2J}}}{p_1!\cdots p_{2J}!} \\
&= \; 2Je^{(2J-1)\varphi} \sum_{n\geq \lceil \mathcal{N}^{-1}|\underset{\sim}{m}|\rceil}^{\infty} \frac{\varphi^n}{n!},
\end{aligned}
\tag{D.3}
$$

where

$$
\varphi \; := \; \max_{1\leq a\leq 2J} \left| \frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right| .
$$

We conclude that

$$
|Q_{\underline{m}}| \; \leq \; \left(2Je^{(2J-1)\varphi}\right) \frac{\varphi^{\lceil \mathcal{N}^{-1}|\underset{\sim}{m}|\rceil}}{\lceil \mathcal{N}^{-1}|\underset{\sim}{m}|\rceil !} \left(1 - \frac{\varphi}{\lceil \mathcal{N}^{-1}|\underset{\sim}{m}|\rceil + 1}\right)^{-1}, \tag{D.4}
$$

for $\lceil \mathcal{N}^{-1}|\underset{\sim}{m}|\rceil + 1 > \varphi$. Above $\lceil x \rceil$ is the lowest integer larger or equal to x. From this, and since $|\underset{\sim}{m}| \leq |\underline{m}| \leq 1 + |\underset{\sim}{m}|$ for the sort of vectors $\underline{m} \in \mathbb{Z}^A$ in consideration, the inequality (5.18) follows immediately, for convenient choices of $\mathcal{Q}$ and $\chi > 0$.

Note that (D.4) implies an actually even stronger decay in $|\underline{m}|$ for $Q_{\underline{m}}$ than that given in (5.18). Unfortunately, however, this does not seem to help to improve our estimates in a way to change the main results of this paper, specially those concerning the large-n behaviour of the constants $\mathcal{K}_n$ and $\mathcal{L}_n$, important for the proof of quasi-periodicity of the solution g.

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