A Classical Large N Hierarchical Vector Model in Three Dimensions: A Nonzero Fixed Point and Canonical Decay of Correlation Functions

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Hierarchical. . .

Abstract

We consider a hierarchical N-component classical vector model on a three-dimensional lattice $\mathbb{Z}^3$ for large N. The model differs from the usual one in that the kernel of the inverse Laplace operator is non-translational invariant but has matrix elements which are positive and exhibit the same falloff as the inverse Laplacian in $\mathbb{Z}^3$. We introduce a renormalization group transformation and for $N = \infty$, corresponding to the leading order of the $1/N$ expansion, we construct explicitly a nonzero fixed point for this transformation and also obtain some correlation functions. The two-point function has canonical decay. For $1 \ll N < \infty$, we obtain the fixed point and the two-point function in the first $1/N$ approximation. Canonical decay is still obtained, in contrast to what is reported for the full model.

I. Introduction

Hierarchical models are formulated to mimic systems of interest in statistical mechanics and euclidean quantum field theories. In particular, they have proved to be a good laboratory to test ideas and to develop tools that allow the rigorous analysis of the continuum and the thermodynamic limits in such systems.

In the case of boson lattice field theories, the hierarchical versions are defined by replacing, in the Gaussian (free) measure, the Laplace operator Δ in the inverse covariance by a hierarchical long-range Laplacian Δ_h. In general, although $(\Delta_h)^{-1}$ is not translational invariant on the lattice, it has matrix elements with the same canonical decay properties as for Δ^{-1}.

The main simplification involved in hierarchical models is of a technical nature and has to do with renormalization group transformations (RGT), which preserve locality of the interacting potential.

In [1], a given hierarchical version for the lattice real boson vector model, with a large number $N \in I\!N$ of components and in three euclidean dimensions, was considered that led to a recursion of the RGT which corresponds to the Wilson's (approximate) recursion when the value of the associated block-spin length, $L \in I\!N, L > 1$, in the lattice $\Lambda \subset \mathbb{Z}^3$, equals two for each

RGT step. In dimension $d \in I\!N$, the corresponding RGT is given by

$$e^{-\mathcal{T}V(\vec{\phi})} = const \int e^{-\frac{1}{2}L^d \left[V\left(L^{-\frac{d-2}{2}}\vec{\phi}+\vec{z}\right)+V\left(L^{-\frac{d-2}{2}}\vec{\phi}-\vec{z}\right)\right]} d\nu\left(\vec{z}\right) \qquad \text{(I.1)}$$

where $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_N) \in \mathbb{R}^N$ denotes the N-component field (a random process), V is the interacting potential, $\vec{z}$ describes the fluctuation field (another random process) and where

$$d\nu\left(\vec{z}\right) = \prod_{x \in \Lambda} e^{-\frac{1}{2}|\vec{z}(x)|^2} d\vec{z}(x)$$

for $d\vec{z}(x)$, at each lattice site x, being the product of Lesbesgue measures $dz_1(x) \ldots dz_N(x)$; is a normalized and mean zero Gaussian measure with unit covariance on $\Lambda \subset \mathbb{Z}^d$. In (I.1), the *const* factor is a normalization and $\mathcal{T}V$ describes the effective potential generated by the RGT.

In [1], a nonzero fixed point solution of the above RGT was constructed first for the $N = \infty$ regime, given by the leading order in the $1/N$ expansion for the model, and was then extended to the case $1 \ll N < \infty$. Physical stability, asymptotics and other properties of this solution have also been analyzed.

Recently, in a companion paper [2], we reconsidered the previous analysis of [1] for the finite N fixed point and improved our understanding of this model by constructing the thermodynamic limit of some correlation functions. Even though the fixed point is nonzero, we have shown that the two-point function has canonical (non-anomalous) decay at large distances. The canonical decay is also seen for other correlations studied there. This behavior differs from what is reported to occur in the full model (see e.g. [3, 4]).

In this paper, we consider another hierarchical version for the same problem for which, as will be shown in the sequel, instead of the Wilson's transformation, the RGT in dimension d is given by (see [5])

$$e^{-\mathcal{T}V(\vec{\phi})} \;=\; const \int e^{-L^d\left[V\left(L^{-\frac{d-2}{2}}\vec{\phi}+\vec{z}\right)\right]} d\nu\left(\vec{z}\right) \;. \qquad \text{(I.2)}$$

Due to the non-symmetrical form of this transformation, in contrast with the Wilson's case given in equation (I.1), we will refer to the present model as the *non-symmetrical* hierarchical model hereafter.

The point we will be focusing on is the decay of correlations at large distances. Precisely, we want to determine whether or not the correlations of the non-symmetrical hierarchical model also have canonical decay.

The reason that justifies our motivation in performing this analysis lays in the fact that, while sharing the "simplicity" of a hierarchical model, it presents desirable features not found in the Wilson's symmetrical model (I.1), namely:

- the associated inverse hierarchical Laplacian ${\Delta_h}^{-1}$, on the lattice, has positive matrix elements as for the Laplacian Δ^{-1} in the full case;

- as it will be shown in subsection I.3, in the case of a scalar quadratic perturbation $V(\phi) = \mu\phi^2/2, \mu \geq 0$, associated with a nonzero Gaussian fixed point, non-canonical or anomalous behavior of the two-point function appears discontinuously in μ, at $\mu = 0$;

- working with the non-symmetrical hierarchical classical vector model in dimension $d = 3$ or higher, the Goldstone picture has been confirmed in [6, 7] by showing the change from canonical to non-canonical decay of the two-point function, depending on the considered field direction.

These facts make the non-symmetrical model a better candidate to mimic the full model than Wilson's. Next, we give a detailed description of this model.

I.1 The Model

Formally, the partition function for the N-component vector model on a lattice $\mathbb{Z}^d$ is taken as

$$Z = \int e^{-\frac{1}{2}\langle\vec{\phi},\Delta\vec{\phi}\rangle - V(\vec{\phi})} \prod_x d\vec{\phi}(x) \tag{I.3}$$

where $\Delta = -\Delta_L$, Δ_L being the lattice Laplacian, $\vec{\phi} = (\phi_1, \ldots, \phi_N) \in \mathbb{R}^N$, $x \in \mathbb{Z}^d$.

We now define what we call the non-symmetrical hierarchical model. With a view to our renormalization group (RG) analysis, it is convenient to introduce a sequence of unit lattices $\{\Lambda_j,\ j = 0,1,\ldots,\ell\}$, where $\Lambda_j = \left[-\frac{L^{\ell-j}}{2}, \frac{L^{\ell-j}}{2}\right]^d \cap \mathbb{Z}^d$, $L > 1$ odd. We also introduce N-component vector fields $\vec{\phi}^{(j)}(x^{(j)}) \in {\rm I\!R}^N$, $x^{(j)} \in \Lambda_j$ and Gaussian measures $d\mu_j(\vec{\phi}^{(j)})$. $d\mu_j$ are normalized and mean-zero measures and have covariances G_j which are taken to be diagonal in the components of the vector field, being defined by

$$
\begin{aligned}
&G_j\left(x^{(j)} = Lx^{(j+1)} + u, y^{(j)} = Ly^{(j+1)} + \bar{u}\right) \\
&\qquad\qquad = L^{-(d-2)} G_{j+1}(x^{(j+1)}, y^{(j+1)}) + \delta_{j+1}(x^{(j+1)}, y^{(j+1)})
\end{aligned}
\tag{I.4}
$$

where the symbol δ denotes the Kronecker delta function, $G_\ell \equiv G_\ell(x^{(\ell)}, y^{(\ell)}) = \left[1 - L^{-(d-2)}\right]^{-1}$ and $u, \bar{u} \in B_0$, B_0 being the lattice block $\left[-\frac{L}{2}, \frac{L}{2}\right]^d \cap \mathbb{Z}^d$ containing L^d points of Λ_j.

Letting the site 0 denote the origin for the coordinate system on the lattice and e_1 the lattice unit vector for any given direction, it can be checked that, upon iteration, we have

$$
G_0\left(0, L^k e_1\right) = \frac{1}{(1 - L^{-(d-2)})} \, \frac{1}{L^{k(d-2)}} \quad ; \quad k = 0,1,\ldots,\ell-1 \tag{I.5}
$$

that is, canonical decay for the covariances.

The partition function for the non-symmetrical hierarchical vector model in the finite volume Λ_0 is taken as in (I.3) with the Gaussian part being given by $d\mu_0$, and for $x \equiv x^{(0)}$ and $\vec{\phi}(x) \equiv \vec{\phi}^{(0)}(x^{(0)})$, we have $V(\vec{\phi}) = \sum_{x^{(0)}\in\Lambda_0} V(\vec{\phi}^{(0)}(x^{(0)}))$. It reads

$$Z_0 \equiv Z_0(\ell) = \int e^{-V(\vec{\phi})}\, d\mu_0(\vec{\phi}) \tag{I.6}$$

The definition of the normalized p-point correlation functions follows in a similar way and is given by

$$\begin{aligned} &\left\langle \phi_{i_1}(x_1)\ \phi_{i_2}(x_2)\ldots\phi_{i_p}(x_p)\right\rangle_\ell = \\ &\qquad = (Z_0)^{-1}\int \phi_{i_1}(x_1)\ \phi_{i_2}(x_2)\ldots\phi_{i_p}(x_p)\ e^{-V(\vec{\phi})}\, d\mu_0(\vec{\phi}) \end{aligned} \tag{I.7}$$

with $i_1,\ldots,i_p = 1,\ldots,N$, and with Z_0 given in (I.6).

I.2 The Non-Symmetrical RGT

In the context of the RG method, we now present some identities for the partition function, introduce the RGT, effective actions and give representations for correlation functions in terms of effective actions and effective spin (field) variables.

Corresponding to the decomposition (I.4) of the covariance, we introduce the decomposition of the fields by

$$\vec{\phi}^{(j)}(x^{(j)} = Lx^{(j+1)} + u) = L^{-(d-2)/2}\vec{\phi}^{(j+1)}(x^{(j+1)}) + \vec{\eta}^{(j+1)}(x^{(j+1)})$$

which gives the fundamental identity, for any given function f_x,

$$\int \prod_{x\in\Lambda_j} f_x(\vec{\phi}^{(j)})\ d\mu_j(\vec{\phi}^{(j)}) = \int \prod_{x\in\Lambda_{j+1}} \bar{f}_x(\vec{\phi}^{(j+1)})\ d\mu_{j+1}(\vec{\phi}^{(j+1)}) \tag{I.8}$$

with

$$\bar{f}_x(\vec{\phi}) = \int \prod_{u\in B_0} f_{Lx+u}(L^{-(d-2)/2}\vec{\phi} + \vec{\zeta})\ e^{-\vec{\zeta}^{\ 2}/2}\ d\vec{\zeta}$$

where we adopt the convention that the $\vec{\zeta}$ integral includes the normalization factor $(2\pi)^{-\frac{N}{2}}$.

The fundamental identity (I.8) follows from the decomposition identity for the generating function of Gaussian measures. As it can be seen, repeated applications of (I.8) to the partition function $Z_0 \equiv Z_0(\ell)$ gives

$$\begin{aligned} Z_0 &\equiv \int \prod_{x^{(0)}\in\Lambda_0} e^{-V\left(\vec{\phi}^{(0)}(x^{(0)})\right)} d\mu_0 = \int \prod_{x^{(1)}\in\Lambda_1} e^{-RV\left(\vec{\phi}^{(1)}(x^{(1)})\right)} d\mu_1 \\ &= \cdots = \int e^{-R^{\ell}V\left(\vec{\phi}^{(\ell)}(x^{(\ell)})\right)} d\mu_\ell \equiv Z_\ell \end{aligned}$$

where, up to a normalization, the RGT is defined, in agreement with (I.2), by

$$e^{-RW(\vec{\phi})} = \int e^{-L^d W\left(L^{-(d-2)/2}\vec{\phi}+\vec{\eta}\right)}\ e^{-\vec{\eta}^{\ 2}/2} d\vec{\eta} \qquad \text{(I.9)}$$

Thus, it is clear that the effective actions remain local under this transformation.

Applying the fundamental identity (I.8) to both, the numerator and the denominator of the expression (I.7), we can represent the p-point functions in terms of effective spin variables. Doing this for the two-point function, and letting $I_i(\vec{\phi}) = \phi_i$, we get

$$
\begin{aligned}
&\langle \vec{\phi}_i(0)\ \vec{\phi}_j(L^k e_1)\rangle_\ell = \\
&= (Z_1)^{-1} \int M_V I_i\left(\vec{\phi}^{(1)}(0)\right)\ M_V I_j\left(\vec{\phi}^{(1)}\left(L^{k-1}e_1\right)\right) \\
&\quad \times \prod_{x^{(1)}\in\Lambda_1} e^{-RV\left(\vec{\phi}^{(1)}(x^{(1)})\right)}\ d\mu_1 \\
&= \cdots = (Z_k)^{-1} \int [(M_{R^{k-1}V}\ldots M_V I_i)\left(\vec{\phi}^{(k)}(0)\right) \\
&\quad \times (M_{R^{k-1}V}\ldots M_V I_i)]\left(\vec{\phi}^{(k)}(e_1)\right) \prod_{x^{(k)}\in\Lambda_k} e^{-R^k V\left(\vec{\phi}^{(k)}(x^{(k)})\right)}\ d\mu_k \\
&= \cdots = (Z_{k+1})^{-1} \int [M_{R^k V}[(M_{R^{k-1}V}\ldots M_V I_i)\ (M_{R^{k-1}V} \\
&\quad \times \ldots M_V I_i)]]\left(\vec{\phi}^{(k+1)}(0)\right) \prod_{x^{(k+1)}\in\Lambda_{k+1}} e^{-R^{k+1}V\left(\vec{\phi}^{(k+1)}(x^{(k+1)})\right)}\ d\mu_{k+1} \\
&= \cdots = (Z_\ell)^{-1} \int [M_{R^{\ell-1}V}\ldots M_{R^k V} \\
&\quad \times [(M_{R^{k-1}V}\ldots M_V I_i)\ (M_{R^{k-1}V}\ldots M_V I_i)]]\left(\vec{\phi}^{(\ell)}\right)\ e^{-R^{\ell-1}V\left(\vec{\phi}^{(\ell)}\right)}\ d\mu_\ell
\end{aligned}
\tag{I.10}
$$

where all the partition functions also carry an ℓ dependence and M_W is the linear operator

$$
[M_W f]\left(\vec{\phi}\right) = \frac{\int f\left(L^{-(d-2)/2}\vec{\phi}+\vec{\eta}\right) e^{-W\left(L^{-(d-2)/2}\vec{\phi}+\vec{\eta}\right)}\ e^{-\frac{1}{2}\vec{\eta}^2} d\vec{\eta}}{\int e^{-W\left(L^{-(d-2)/2}\vec{\phi}+\vec{\eta}\right)}\ e^{-\frac{1}{2}\vec{\eta}^2} d\vec{\eta}} \tag{I.11}
$$

Remarks:

1. In (I.10), after $(k-1)$ iterations, the points 0 and $L^k e_1$ belong to the same block, thus reducing the two corresponding fields to the same

variable. After this, the action of M is global, on both fields.

2. We note that in (I.11), apart from a normalization, when acting on a field $\vec{\phi}$, M_W gives the associated effective spin (field) variable obtained by integration over the corresponding fluctuation against an interacting potential W.

3. Also, it is clear from equation (I.11) that, at a given point W, the operator M_W is L^{-d} times the linearized RGT associated with RW at this point. This observation is helpful in understanding the scaling behavior of effective spin variables.

The above formalism appears in [6, 7, 8], and was used in [2] to compute the $N = \infty$ correlation functions. When $N = \infty$, at the nonzero fixed point, the spectral properties of the operator M_V turn out to be completely known for the symmetrical case.

The same method is, in principle, successful to treat any case whenever the properties of M_V can be obtained. Below, we illustrate how it works in another model. For $d > 2$, we consider the case of a non-symmetrical hierarchical scalar (one-component) model with a quadratic perturbation. This consists of an example of how an anomalous decay appears within a non-symmetrical model.

I.3 A Nonzero Fixed Point and Anomalous Decay: A Non-Symmetrical Hierarchical Model with a Quadratic Potential

We are going to consider the case of a quadratic perturbation $V(\phi) = \frac{1}{2}\mu\phi^2$, $\mu \geq 0$, of a non-symmetrical hierarchical model with a one-component field in dimension $d > 2$. The RGT here reduces to [see equation (I.9)]

$$e^{-RV(\phi)} = \int e^{-\frac{1}{2}\mu L^d \left(L^{-(d-2)/2}\phi+\eta\right)^2} e^{-\eta^2/2} d\eta \tag{I.12}$$

Let us first assume $\mu > 0$. Simply by computing the Gaussian integral above, we see that, under (I.12), $V(\phi) \mapsto V_1(\phi) \equiv [RV](\phi) = \frac{1}{2}\left(\mu L^2/\left(1+\mu L^d\right)\right)\phi^2$. That is, $\mu_0 \equiv \mu \mapsto \mu_1 \equiv \mu L^2/\left(1+\mu L^d\right)$. Obviously, iteration of the RGT (I.12) gives $V_n(\phi) \equiv [R^n V](\phi) = \frac{1}{2}\mu_n\phi^2 = \frac{1}{2}\frac{\mu_{n-1}L^2}{1+\mu_{n-1}L^d}\phi^2$, with $\mu_j \equiv \frac{\mu_{j-1}L^2}{\left(1+\mu_{j-1}L^d\right)}$, $j = 1, \ldots, n-1$. Hence, for $\mu \equiv \mu_0 > 0$, we have the nonzero fixed point solution

$$\tilde{V}(\phi) \equiv \lim_{n\to\infty} V_n(\phi) = \frac{1}{2}\tilde{\mu}\phi^2$$

where

$$\tilde{\mu} = \frac{L^2-1}{L^d} \tag{I.13}$$

corresponds to the $n \longrightarrow \infty$ limit of

$$\mu_n = \frac{L^{2n}\,\mu_0\tilde{\mu}}{\tilde{\mu} + \left(L^{2n}-1\right)\mu_0} \quad . \tag{I.14}$$

Now, we will compute the infinite volume two-point function

$$\langle \phi(0)\ \phi(L^k e_1)\rangle_\infty \equiv \lim_{\ell \longrightarrow \infty} \langle \phi(0)\ \phi(L^k e_1)\rangle_\ell \quad .$$

This will be done using (I.10). To compute the action of the M-operator strings appearing in (I.10), taking the two points to the same block, we will use the formalism developed in [6]. It is shown there that the action of each of these operator strings can be expressed *uniquely* in terms of the last RG step action. For the generic N-component case, we have (compare with equation (1.13) of [6])

$$[M_{R^{n-1}V} \ldots M_V\ I_k]_\ell(\vec{\phi}) = L^{-n\frac{d-2}{2}}\phi_k - a_n$$

$\partial\phi_k$(I.15)where the constant $a_n \equiv a_n\,(d)$ is given by

$$a_n = L^{-n\left(\frac{(d+2)}{2}\right)}\left(L^{2n} - 1\right)\tilde{\mu}^{-1}. \tag{I.16}$$

When $N = 1$, (I.3) amounts to $[M_{R^{n-1}V} \ldots M_V\ I]_\ell(\phi) = L^{-n\frac{(d-2)}{2}}\phi - a_n \frac{\partial R^n V(\phi)}{\partial\phi}$, for $I(\phi) = \phi$, reducing to $\left[\left(L^{-n\frac{(d-2)}{2}} - a_n\mu_n\right)\phi\right]$ after evaluation on R^nV.

Applying this result to (I.10), setting $\phi^{(\ell)} \equiv \phi$, we then obtain

$$
\begin{aligned}
\langle \phi(0)\ \phi(L^k e_1) \rangle_\ell \ &= \ (Z_\ell)^{-1} \int \left[M_{R^{\ell-1}V} \ldots M_{R^k V} \ (M_{R^{k-1}V} \ldots M_V I) \right. \\
&\quad \times \ \left. (M_{R^{k-1}V} \ldots M_V I) \right] (\phi) \ e^{-R^{\ell-1}V(\phi)} \, d\mu_\ell \\
&= \ (Z_\ell)^{-1} \int M_{R^{\ell-1}V} \ldots M_{R^k V} \\
&\quad \times \left[\left(L^{-k\frac{d-2}{2}} - a_k \mu_k \right) I \right]^2 (\phi) \ e^{-R^{\ell-1}V(\phi)} \ d\mu_\ell \\
&= \ (Z_\ell)^{-1} \int \left[M_{R^{\ell-1}V} \ldots M_{R^k V} \right] \\
&\quad \times \left[\left(L^{-k\frac{(d+2)}{2}} + \left(L^{-k\frac{d+2}{2}} - L^{-k\frac{d-2}{2}} \right) \left(\tfrac{\mu_k}{\tilde{\mu}} - 1 \right) \right) I \right]^2 (\phi) \\
&\quad \times \ e^{-R^{\ell-1}V(\phi)} \ d\mu_\ell
\end{aligned}
\tag{I.17}
$$

where, at each step, the partition function $Z_\ell \equiv Z_\ell(\ell)$ is given by the same integral with $M_{R^{\ell-1}V} \ldots M_{R^k V} [\quad]^2 (\phi)$ replaced by the number one.

On the other hand, we can easily compute the action of the remaining M-string in (I.17) obtaining

$$
(M_{R^{\ell-1}V} \ldots M_{R^k V})\, \phi^2 = \gamma_{\ell-k} + \xi_{\ell-k}\, \phi^2 \tag{I.18}
$$

where

$$
\gamma_{\ell-k} = L^{-2} \sum_{j=k}^{\ell-1} \xi_j \, \frac{\mu_{j-1}}{\mu_j} \xi_{\ell-k} = L^{-(d+2)(\ell-2k)} \left(\frac{\mu_{\ell-k}}{\mu_k} \right)^2 \quad . \tag{I.19}
$$

With this, using (I.14) and performing the last integrals in (I.17), we get

$$
\begin{aligned}
\langle \phi(0)\ \phi(L^k e_1) \rangle_\ell \ &= \ \left[L^{-\frac{d+2}{2}k} \ + \ L^{-\frac{d-4}{2}k} \left(1 - L^{-3k} \right) \frac{\tilde{\mu} - \mu_0}{\tilde{\mu} + \left(L^{2k} - 1 \right) \mu_0} \right]^2 \\
&\qquad \times \quad \left(\gamma_{\ell-k} + \xi_{\ell-k} \, \frac{1}{1 - L^{-(d-2)}} \right)
\end{aligned}
\tag{I.20}
$$

As, from (I.13), $0 < \tilde{\mu} < 1$ and, according to (I.14), $\tilde{\mu} < \frac{\left(L^{2n}-1\right)\mu_0\tilde{\mu}}{\tilde{\mu}+(L^{2n}-1)\mu_0} < \mu_n < 2\tilde{\mu}$ the behavior of both $\gamma_{\ell-k}$ and $\xi_{\ell-k}$ in (I.19) are controlled by the $L^{-(d+2)(\ell-2k)}$ factor. For this reason, only the constant term of (I.18) matters in, both, the thermodynamic limit at fixed k and the double limit $k, \ell \to \infty$ with $k \leq \ell$, the later corresponding to the limit of infinite separation.

Letting γ_∞ be given as $\lim_{\ell\to\infty}\gamma_{\ell-k}$ [see (I.19)], then the last observation applied to (I.20) gives, after taking the thermodynamic limit,

$$\left\langle \phi(0)\ \phi(L^k e_1)\right\rangle_\infty = \gamma_\infty \left[L^{-\frac{d+2}{2}k} + L^{-\frac{d-4}{2}k}\left(1 - L^{-3k}\right)\frac{\tilde{\mu}-\mu_0}{\tilde{\mu}+\left(L^{2k}-1\right)\mu_0}\right]^2 \tag{I.21}$$

whenever $\mu_0 > 0$. As, for $\mu_0 = 0$ [check below (I.4)], corresponding to a starting model with no interacting potential, the model has a zero fixed point and the two-point function has canonical decay $L^{-(d-2)}$, we see in (I.21) that the two-point function for $\mu_0 > 0$ has a non-canonical decay given by $L^{(d+2)/2}$. This effect has thus a discontinuity at $\mu = 0$ and is associated with a change from one to γ_∞ in the wave-function renormalization (coefficient).

I.4 The Large N Non-Symmetrical Hierarchical Vector Model in Three Dimensions: Definition and Results.

Going back to the N-component vector model, hereafter we set $d = 3$ and assume that the initial action is rotational invariant, so that the system has an $O(N)$ symmetry. In order to analyze this model when N is a finite but large integer, it is suitable to introduce the reduced variable $\varphi \in \mathbb{R}$, and fluctuation variables $s \in \mathbb{R}$ and $u \geq 0$ verifying

$$\vec{\phi} = N^{1/2}(\varphi, 0, \ldots, 0) \tag{I.22}$$

and

$$\vec{z} = N^{1/2}(s, \zeta_1, \ldots, \zeta_{N-1}) \ , \ {\zeta_1}^2 + \ldots + {\zeta_{N-1}}^2 = u \ . \tag{I.23}$$

Then, $d\vec{z} = u^{(N-3)/2} du \, ds \, d\Omega_{N-1}$. Further upon making the substitutions $V\left(\vec{\phi}\right) = v\left(\vec{\phi}^2\right) \to \frac{N}{2}\ v\left(\varphi^2\right)$ and $RV\left(\vec{\phi}\right) = Rv\left(\vec{\phi}^2\right) \to \frac{N}{2} v_1(\varphi^2)$, the RGT for the non-symmetrical model (I.9) can be written as

$$e^{-\frac{N}{2}v_1} = N\Omega_{N-1} \int_{-\infty}^{\infty} ds \int_0^{\infty} du \ e^{-L^3 \frac{N}{2} v(\psi)} \ e^{-\frac{N}{2}\left(s^2 + u - \ln u\right)} \ e^{-\frac{3}{2}\ln u} \tag{I.24}$$

where $\psi \equiv \psi(\varphi, s, u) = (L^{-\frac{1}{2}}\varphi + s)^2 + u$ and Ω_{N-1} is the $(N-1)$-dimensional solid angle.

Corresponding to the leading order of the (formal) $1/N$-expansion for (I.24), and letting $v(\varphi^2) \equiv v_{N=\infty}(\varphi^2)$ be independent of N, we define the

$N = \infty$ RGT as

$$v_1 = \inf_{s,u} \left[L^3\, v(\psi) + s^2 + u - \ln u \right] \equiv L^3 v(\psi_0) + s_0^2 + u_0 - \ln u_0 \qquad \text{(I.25)}$$

for $s_0 \equiv s_0(\varphi^2)$ and $u_0 \equiv u_0(\varphi^2)$ being solutions of the variational equations

$$\begin{aligned} L^3 v'(\psi) \left(s + L^{-\frac{1}{2}} \varphi \right) + s \;&=\; 0 \\ L^3 v'\,(\psi) + 1 - 1/u = 0 & \end{aligned} \qquad \text{(I.26)}$$

where the prime denotes a derivative with respect to φ^2, and $\psi_0 \equiv \psi(\varphi, s_0, u_0)$.

Our first result describes a nonzero fixed point solution to (I.25). We have:

Theorem 1 *Let $\tau = v'$ and $t_0 = \frac{L}{L-1} \equiv (\varphi_0)^2$. Then, when $N = \infty$, there exists a fixed point $\widetilde{v}_{N=\infty}$ of (I.25) given, in terms of t, the inverse function of $\widetilde{v}'$, by*

$$\begin{aligned} t(\tau) \;&=\; t_0 + t_0\, L\, \tau \sum_{n=0}^{\infty} \tfrac{1}{L^n} \left[\tfrac{1-L^{-2}}{(1-L^{-2})-(1-L^{-2n})L\tau} \right]^2 \\ &\qquad\qquad \times \left[\tfrac{1-L^{-2}}{(1-L^{-2})-\left(1-L^{-2(n+1)}\right)L\tau} \right] \end{aligned} \qquad \text{(I.27)}$$

for $\tau \neq \tau_n \equiv \frac{1-L^{-2}}{L(1-L^{-2n})}$, $n = 0, 1, \ldots$. Thus, $t(\tau)$ extends to an analytic function in a disk centered at τ_0. Moreover, the potential $\widetilde{v}_{N=\infty}(\varphi^2)$ is stable whenever φ is real, that is $t'(\tau) > 0$ for $\tau < (1 - L^{-2})/L$, and $\widetilde{v}'_{N=\infty}(\varphi^2)$ is analytic there. The fixed point $\widetilde{v}_{N=\infty}$ is nonzero and has a unique minimum at $\varphi^2 = (\varphi_0)^2$, with $\widetilde{v}''_{N=\infty}((\varphi_0)^2) = \left(2L\varphi_0^2\right)^{-1}$. Finally, normalizing $\widetilde{v}_{N=\infty}$ so *that* $\widetilde{v}_{N=\infty}\,(\varphi^2)=0$ *the following global bounds hold*

$$a'+b'|\varphi|\leq\widetilde{v}_{N=\infty}(\varphi^2)\ \leq\ a+b\varphi^2\quad,\quad\varphi\in\mathbb{R}\tag{I.28}$$

for some real a *and* a' *and positive* b *and* b'.

Remark: The derivatives of the critical potential $\widetilde{v}_{N=\infty}(\varphi^2)$ can be all computed, as an application of the analytic inverse function theorem. An integral representation for the inverse function is given in [9], and in [10] explicit formulas for the Taylor coefficients are obtained.

Concerning the two-point function, we note that because of the $O(N)$ symmetry of the model,

$$\langle\phi_i(0)\phi_j(L^ke_1)\rangle_\ell=\frac{\delta_{ij}}{N}\ \langle\vec{\phi}(0)\cdot\vec{\phi}(L^ke_1)\rangle_\ell\tag{I.29}$$

the dot denoting the Euclidean inner product in $\mathbb{R}^n$.

Using in (I.29) the definition (I.10), and applying the relations $\sum_{i=1}^N\ \phi_i$ $(\partial V/\partial\phi_i)\,=\,2N\varphi^2\,v'(\varphi^2)$ and $\bigtriangledown V\cdot\bigtriangledown V\;=4N\varphi^2[v'(\varphi^2)]^2$, to pass to the reduced variables (I.22) and (I.23), we obtain, for $I(\varphi^2)=\varphi^2$,

$$\begin{aligned}\langle\phi_i(0)\phi_j(L^ke_1)\rangle_\ell\ \ &=\ \ \delta_{ij}\ (Z_\ell)^{-1}\int\left[m_{\ell-k}\left[\left(L^{-k/2}I-a_kv'(I)I\right)^2(\varphi^2)\right]\right]\\&\qquad\times\ \ e^{-\frac{N}{2}v(\varphi^2)}\ e^{-\frac{N}{2}\left(1-L^{-1}\right)\varphi^2}\ e^{(N-1)\ln\varphi}\ d\varphi\end{aligned}\tag{I.30}$$

where, as in (I.11), for rotational invariant $f(\varphi^2)$, we define

$$[m\,f]\,(\varphi^2)\;=\;\frac{\displaystyle\int_{-\infty}^{\infty}ds\int_0^{\infty}du\;f(\psi)\;e^{-\frac{N}{2}(s^2+u-\ln u)-\frac{3}{2}\ln u}\;\,e^{-\frac{N}{2}L^3 v(\psi)}}{\displaystyle\int_{-\infty}^{\infty}ds\int_0^{\infty}du\;e^{-\frac{N}{2}(s^2+u-\ln u)-\frac{3}{2}\ln u}\;\,e^{-\frac{N}{2}L^3 v(\psi)}} \qquad \text{(I.31)}$$

Note that, for $N = \infty$, we define the action of $m_\infty \equiv m_{N=\infty}$ of the operator m in (I.31), according to the Laplace method, as

$$[m_\infty\,f]\,(\varphi^2) = f\left(\psi_0 \equiv \left(L^{-1/2}\varphi + s_0\right)^2 + u_0\right) \qquad \text{(I.32)}$$

We can now state our second result, regarding the $N = \infty$ two-point functions:

Theorem 2 *For $N = \infty$, in the thermodynamic limit $\ell \to \infty$, we have*

$$\langle \phi_i(0)\phi_j(L^k e_1)\rangle_{N=\infty,\infty} \equiv \lim_{\ell\to\infty}\langle \phi_i(0)\phi_j(L^k e_1)\rangle_{N=\infty,\ell} = \;\delta_{ij}\;L^{-k}\varphi_0^2 \qquad \text{(I.33)}$$

Remarks:

1. As it will become clear in the sequel, our techniques apply as well to other correlation functions. In particular, for the "partially truncated" (P.T.) functions defined below, we have,

$$\begin{aligned}
&\frac{1}{N^{n+j}}\,\langle\left[\left(\vec\phi\cdot\vec\phi\right)(0)\right]^n\;\left[\left(\vec\phi\cdot\vec\phi\right)(L^k e_1)\right]^j\rangle^{P.T.}_{N=\infty,\infty}\\
&\quad\equiv \lim_{\ell\to\infty}\left(\frac{1}{N^{n+j}}\langle\left[\left(\vec\phi\cdot\vec\phi\right)(0)\right]^n\left[\left(\vec\phi\cdot\vec\phi\right)(L^k e_1)\right]^j\rangle_{N=\infty,\ell}\right.\\
&\quad\left.-\frac{1}{N^{n+j}}\langle\left[\left(\vec\phi\cdot\vec\phi\right)(0)\right]^n\rangle_\infty\langle\left[\left(\vec\phi\cdot\vec\phi\right)(L^k e_1)\right]^j\rangle_{N=\infty,\ell}\right)\\
&= L^{-2k(n+j)}(\varphi_0)^{2(n+j)}\;.
\end{aligned}$$

2. Though the coefficient φ_0^2 in I.33 show a change in the wave-function (re-)normalization, as compared to the free case, all the above correlation functions have canonical decay, as $L^{-k} = |x|^{-1}$.

3. We do not know if Wick's theorem holds for expectations of all monomials.

Let us now consider the $1 \ll N < \infty$ regime and the model as described by the $1/N$ expansion. In this case, by extending the preceding analysis to the first $1/N$ correction, using the Laplace method to write an expansion for the integrals in (I.10), we can state our last result:

Theorem 3 *Let $1 \ll N < \infty$. Up to order $1/N$:*

1. *the fixed point solution to (I.24) is given by*

$$\widetilde{v} = \widetilde{v}_{N=\infty} + \frac{1}{N}\,\widetilde{r} \tag{I.34}$$

where $\widetilde{v}_{N=\infty}$ is the $N = \infty$ nonzero fixed point solution of Theorem 1 and $\widetilde{r}$ is given in equation (III.4) below, so that $v \neq 0$ for sufficiently large N.

2. *the (finite-volume) two-point function for extremely separated points* 0 *and $L^{\ell-1}$ in the lattice Λ_ℓ, for fixed ℓ and at the fixed point v of (I.34),*

has canonical decay and is given by

$$\langle \phi_i(0)\phi_j(L^{\ell-1}e_1)\rangle_\ell = \delta_{ij}\ L^{-(\ell-1)}(\varphi_0)^2\ \left(1+\frac{c}{N}\right) \tag{I.35}$$

for some finite real constant c, independent of N and ℓ.

Remark: The extreme separation condition should not be taken as an a priori limitation of the last result for the two-point function. Besides the fact c does not depend on ℓ, as seen in the case of the quadratic perturbation treated above, though we see a change in the wave-function (re-)normalization, the two-point function has the same non-canonical decay there, no matter whether the thermodynamic limit is taken first, followed by the limit of infinite separation, or the separation taken to infinity, together with the volume.

In Section II., we treat the $N = \infty$ case and prove theorems 1 and 2. Theorem 3 will be proven in Section III.. We leave some final remarks for Section IV. and, for completeness regarding the non-symmetrical model, the spectral properties of the inverse Laplace operator $(\Delta_h)^{-1}$ as well as the falloff of the matrix elements of Δ_h are presented in the Appendix.

II. The $N = \infty$ Model: Fixed Point and Correlation Functions

In this section, we consider the $N = \infty$ model. We derive and solve a functional equation for t appearing in theorem 1, the inverse function of $\widetilde{v}'_{N=\infty} \equiv d\widetilde{v}_{N=\infty}/d\varphi^2$, where $\widetilde{v}_{N=\infty} \equiv \widetilde{v}_{N=\infty}(\varphi^2)$ is the fixed point solution of (I.25). Furthermore, we give bounds on the globally defined $\widetilde{v}$, and also obtain correlation functions.

¿From the variational equations (I.26), we have

$$s_0 = (u_0 - 1)\, L^{-1/2}\varphi \tag{II.1}$$

and taking the derivative of (I.25) gives

$$v_1' = h \cdot (v' \circ \mathcal{G})$$

where the dot here means multiplication, ψ_0 is given below (I.26) and where we define

$$h = L^2 + \frac{L^{5/2} s_0}{\varphi} \qquad ; \qquad \mathcal{G}(\varphi^2) = \psi_0 \quad .$$

At the fixed point

$$v' = h \cdot (v' \circ \mathcal{G}) \quad . \tag{II.2}$$

Multiplying (II.2) by $1/h$, applying t, using (II.1) and (I.26), we obtain

$$t(v'/h) = L^{-1}{u_0}^2 + u_0 \tag{II.3}$$

We now introduce the parameter $\tau \equiv v'$. From (I.26) and (II.1), we have $u_0 = 1 - L\tau$ which, upon substituting in (II.3) yields

$$t(\tau) - \frac{L}{(1-L\tau)^2}\ t\left(\frac{\tau}{L^2(1-L\tau)}\right) = -\frac{L}{1-L\tau} \quad . \tag{II.4}$$

¿From (II.4), we note that $t(0) = \varphi_0^2 \equiv L/(L-1)$, so that introducing the linear operator $\mathcal{M}$ by

$$\left(\mathcal{M}f\right)(\tau) = \frac{L}{(1-L\tau)^2}\ f\left(\frac{\tau}{L^2(1-L\tau)}\right) \quad ,$$

and after subtracting $t(0)$, we rewrite (II.4) as

$$\left[\left(1-\mathcal{M}\right)\left(t-t(0)\right)\right](\tau) = \frac{L^2\tau}{(L-1)(1-L\tau)} \quad . \tag{II.5}$$

Using in (II.5) the Neumann series for $(1-\mathcal{M})^{-1}$, we then obtain

$$t(\tau) = \frac{L}{L-1}\ \frac{1}{1-L\tau} + \frac{L^2}{L-1}\sum_{n=1}^{\infty}\left[\mathcal{M}^n\left(\frac{\cdot}{1-L\cdot}\right)\right](\tau) \quad . \tag{II.6}$$

It can be easily verified that

$$\begin{aligned}\left[\mathcal{M}^n\left(\tfrac{\cdot}{1-L\cdot}\right)\right](\tau) \;&=\; \tfrac{\tau}{L^n}\ \frac{1}{\left[1-L\left(1+L^{-2}+\ldots+L^{-2(n-1)}\right)\right]^2} \\ &\quad\times\ \frac{1}{\left[1-L\left(1+L^{-2}+\ldots+L^{-2n}\right)\right]}\end{aligned} \tag{II.7}$$

which establishes (I.27). Thus, from (II.6) and (II.7), we see that $t(\tau)$ is meromorphic with poles at $\tau_n = L^{-1}\left[1+L^{-2}+\ldots+L^{-2n}\right]^{-1}$, $n = 0, 1, \ldots$, with an accumulation point at $\tau_\infty \equiv L^{-1}(1-L^{-2})$. Thus, the analytic inverse function theorem can be used to show analyticity of $\widetilde{v}'$, as well as to compute $\widetilde{v}''(\varphi_0^2)$.

It is clear from representation (II.6) that $t' \equiv dt(\tau)/d\tau$ is positive for $\tau \in [0, \tau_\infty)$, but we also have

Lemma II..1 *The function t' satisfies $t'(\tau) > 0$, for $\tau \in (-\infty, \tau_\infty)$.*

Proof: Notice that for

$$A_0 \equiv 1 \quad ; \quad A_n(\tau) \equiv \frac{\sigma}{\sigma - (1 - L^{-2n} L\tau)} \ , \ n = 1, \ldots \quad ,$$

with $\sigma = 1 - L^{-2}$, and with t_0 of Theorem 1, (II.6) can be written as

$$t(\tau) = t_0 + t_0\, L\, \tau \sum_{n \geq 0} \frac{1}{L^n} \left([A_n(\tau)]^2 A_{n+1}(\tau) \right) \quad . \tag{II.8}$$

Next, using

$$\tau\, A_n'(\tau) = \frac{(1 - L^{-2n})\ L\tau}{\sigma - (1 - L^{-2n})\, L\tau}\ A_n(\tau) \quad ,$$

we have from (II.8), by direct computation,

$$\frac{t'(\tau)}{t_0 L} = [A_1(\tau)]^2 + \sum_{n \geq 1} \frac{[A_n(\tau)]^2 A_{n+1}(\tau)}{L^n}\ [2A_n(\tau) + A_{n+1}(\tau) - 2] \quad . \tag{II.9}$$

Next, we notice that A_n, $n > 0$, has simple poles at p_n/L, where $\{p_n \equiv \frac{\sigma}{1-L^{-2n}}, n \in I\!N\}$ is a decreasing sequence converging to $0 < \sigma < 1$. This ensures that the sequence $\{A_n, n \in I\!N\} = \{1/[1 - p_n^{-1}]\, , n \in I\!N\}$ converges to $A_\infty \equiv \sigma/(\sigma - L\tau)$, monotonically, being decreasing for $\tau < 0$ and increasing for $0 < \tau < \sigma/2$. This result can be used to get lower bounds on (II.9), taking the first or the last term of the sum, depending on the sign, and achieve the proof of the Lemma II..1. □

By Lemma II..1, the analytic inverse function theorem, and the asymptotic behavior $\lim_{\tau\to-\infty} t(\tau) = -L$ and $\lim_{\tau\to\infty} t(\tau) = \infty$, there exists $v'(\varphi^2)$, monotone increasing and analytic in $(-L,\infty)$ with $v'(\varphi_0^2) = 0$ and $\lim_{\varphi^2\to\infty} v'(\varphi^2) = (1 - L^{-2})/L$. Normalizing v such that $v(\varphi_0^2) = 0$, then $v(\varphi^2)$ has a Taylor expansion about φ_0^2 starting with $\frac{1}{2L\varphi_0^2}(\varphi^2 - \varphi_0^2)^2$. Moreover, we also have global upper and lower bounds for v given in:

Proposition 1 *Set $a \equiv \varphi_0^2(2 - L^{-2})\sigma^{-1}$ and let $\Phi^2 = \varphi^2 - \varphi_0^2$. Then*

$$\frac{\sigma}{L}\left[\Phi^2 - \frac{3a^{1/3}}{2}\left((\varphi^2)^{2/3} - (\varphi_0^2)^{2/3}\right)\right] \leq \tilde{v}_{N=\infty}(\varphi^2) \leq \frac{1}{L}\left[\Phi^2 - \varphi_0^2 \ln\frac{\varphi^2}{\varphi_0^2}\right] \tag{II.10}$$

Proof: We note that if $f(\tau) > t(\tau)$ then, setting $h \equiv t^{-1}$, $v'(\varphi^2) < h(\varphi^2)$ which upon integration gives $v(\varphi^2) < \int_{\varphi_0^2}^{\varphi^2} h(\phi)d\phi$ and similarly for the lower bound. The upper bound in (II.10) follows by taking for f the first term in (II.6). For the lower bound, we note that the first term of $t(\tau)$ in (II.6) is $\varphi_0^2(1 - L\tau)^{-1}$ and that the sum is bounded by $(\varphi^2\sigma)^{-1}\tau(1 - \tau_\infty^{-1}\tau)^{-1}$. For f, we take $f = a(1 - \tau_\infty^{-1}\tau)^{-3}$.

This concludes Proposition 1 and, as (II.10) gives (I.28), also the proof of Theorem 1. □

We now turn to the correlation functions. Iterating (II.2) p times gives

$$v' = H_p v' \circ \mathcal{G}^{(1)} \circ \ldots \circ \mathcal{G}^{(p)} \tag{II.11}$$

where the upper indices count the number of compositions, and

$$H_p = \prod_{j=0}^{p-1} h_j$$

with $h_0 \equiv h$ and

$$h_j(\varphi^2) = \left(h \circ \mathcal{G}^{(1)} \circ \ldots \circ \mathcal{G}^{(j)}\right)(\varphi^2) \quad .$$

Multiplying (II.11) by $1/H_p$ and applying t yields

$$t\left(\frac{1}{H_p} v'\right) = \mathcal{G}^{(1)} \circ \ldots \circ \mathcal{G}^{(p)} \quad . \tag{II.12}$$

Now, the linear operator m_∞ [see (I.32)] acting on $f(\varphi^2)$ gives

$$(m_\infty f)(\varphi^2) = f(\psi_0) \equiv (f \circ \mathcal{G})(\varphi^2) \tag{II.13}$$

so that $(m_\infty I)(\varphi^2) = \mathcal{G}(\varphi^2)$ and $m_\infty^n I = \mathcal{G}^{(1)} \circ \ldots \circ \mathcal{G}^{(n+1)}$. Thus, using (II.12)

$$m_\infty^n I = t\left(\frac{1}{H_{n+1}} v'\right)$$

and then, going back to (I.30), we obtain

$$\begin{aligned} \langle \phi_i(0)\phi_j(L^k e_1)\rangle_{N=\infty,\infty} &\equiv \lim_{\ell\to\infty} \langle \phi_i(0)\phi_j(L^k e_1)\rangle_{N=\infty,\ell} \\ &= \delta_{ij}\ L^{-k} \lim_{n\to\infty} \left[t\left(\tfrac{1}{H_{n+1}} v'(\varphi^2)\right)\right] \\ &= \delta_{ij}\ L^{-k} t(0) = \delta_{ij}\ L^{-k}\varphi_0^2 \quad . \end{aligned} \tag{II.14}$$

which corresponds to (I.33). Taking more general f's in (II.13) allows us to compute other correlation functions. We have now concluded the proof of Theorem 2. □

III. The First $1/N$ Correction

In this section, we extend some of the $N = \infty$ results of section II. up to the first $1/N$ correction and prove Theorem 3. In the framework of the $1/N$ expansion, we write

$$v(\varphi^2) = v_{N=\infty}(\varphi^2) + \frac{1}{N}\, r(\varphi^2) \tag{III.1}$$

where $v_{N=\infty}$ is the N-independent leading order potential of section II. [see above equation (I.25)]. Doing the analogous decomposition for v_1 in (I.24), letting $w \equiv (s, u) \in \mathbb{R} \times [0, \infty)$ and for ψ defined below (I.24), we have

$$e^{-\frac{N}{2}(v_{N=\infty,1} + \frac{1}{N} r_1)} = N\Omega_{N-1} \int dw \; e^{-Nf(w)} \; e^{-g(w)} \tag{III.2}$$

where, with an abuse of notation, we have

$$\begin{aligned} f(w) &\equiv f(s,u) = \tfrac{1}{2} L^3 v_{N=\infty}(\psi) + \tfrac{1}{2}(s^2 + u - \ln u) \\ g(w) &\equiv g(s,u) = \tfrac{1}{2} L^3 r(\psi) + \tfrac{3}{2} \ln u \end{aligned} \tag{III.3}$$

Applying the steepest descent method, we can compute an expansion for the integral in (III.2) about $w_0 = (s_0, u_0)$, s_0 and u_0 being solutions of (I.26).

Comparing terms of the same $1/N$ order gives back (I.25) and shows that, under the RGT (I.24), r transforms according to

$$r_1(\varphi^2) = L^3 r(\psi_0) + 3\ln u_0 + \ln\left[\left(detD^2 f\right)_0\right]$$

with ψ_0 defined below (I.24) and where $(detD^2 f)_0$ is the Hessian of f at w_0. At the fixed point, we thus have

$$\tilde{r}(\varphi^2) = L^3 \tilde{r}(\psi_0) + 3\ln u_0 + \ln\left[\left(detD^2 f\right)_0\right] \tag{III.4}$$

which can already be used to calculate any derivative of $\tilde{r}$ at φ_0^2, assuming $\tilde{r}$ analytic. From (III.4), we also see that the order $1/N$ fixed point solution

$$\tilde{v}(\varphi^2) = \tilde{v}_{N=\infty}(\varphi^2) + \frac{1}{N}\tilde{r}(\varphi^2)$$

is nonzero for large enough N.

This shows part one of Theorem 3. To prove the second part, we go back to the two-point function (I.30) and, for technical simplification, we assume the points to have an extreme separation in the lattice. This is enough for our purposes here. The general case can be treated as well, following what is presented below.

Similar to f and g in (III.3), we define the functions $\hat{f}$ and $\hat{g}$ by

$$\begin{aligned} \hat{f}(\hat{w}) &\equiv \hat{f}(s,u,\varphi) = f(w) + \tfrac{1-L^{-2}}{2}\varphi^2 - \tfrac{1}{2}\ln\varphi^2 \\ \hat{g}(\hat{w}) &\equiv \hat{g}(s,u,\varphi) = g(w) + \ln\varphi^2 \end{aligned} \tag{III.5}$$

with $\hat{w} \equiv (s, u, \varphi) \in \mathrm{I\!R} \times [0, \infty) \times \mathrm{I\!R}$. Now, letting

$$h(\hat{w}) = \psi \left[L^{-\ell/2}\varphi - a_\ell \frac{dv(\psi)}{d\psi} \right]^2 \tag{III.6}$$

the two-point function (I.30), for extremely separated points 0 and $x = L^{\ell-1}e_1$ can be expressed as

$$\langle \phi_i(0)\phi_j(L^{\ell-1}e_1)\rangle_\ell = \delta_{ij}\ (Z_\ell)^{-1} \int d\,\hat{w}\ \ h(\hat{w})\ e^{-N\hat{f}(\hat{w})}\ e^{-\hat{g}(\hat{w})} \tag{III.7}$$

with Z_ℓ given by the same integral with $h(\hat{w})$ replaced by one.

Direct calculation shows that the function $\hat{f}$ has a minimum at $w_0 = (s_0, u_0, \varphi_0)$ with the same s_0 and u_0 of (I.26), and $\varphi_0 = \sqrt{L/(L-1)}$ given in Theorem 1. Hence, applying the steepest descent method to the integrals in (III.7), with $y = \sqrt{N}\,(w - w_0)$, we obtain

$$\begin{aligned}
\langle \phi_i(0)\phi_j(L^{\ell-1}e_1)\rangle_\ell &= \delta_{ij}\ h_0 + \tfrac{h_0'}{\sqrt{N}} \int y\, d\rho(y) \\
&\quad - \tfrac{h_0'}{\sqrt{N}} \left(\int y\, d\rho(y) \right) \left(\int \left[\left(T_0^{(3)} \hat{f} \right) + \frac{\hat{g}_0'}{\sqrt{N}} y \right] d\rho(y) \right) \\
&\quad + \int \left[-\frac{h_0'}{\sqrt{N}} y \left[\left(T_0^{(3)} \hat{f} \right) + \frac{\hat{g}_0'}{\sqrt{N}} y \right] + \frac{h_0''}{2N} y^2 \right] d\rho(y) + \mathcal{O}\left(N^{-3/2} \right)
\end{aligned} \tag{III.8}$$

where the subscript 0 means evaluation at $w = [w_0 + y/\sqrt{N}]_{y=0} = w_0$, and $T^{(3)}$ is the third order Taylor operator, so that when acting on $\hat{f}$ produces an $\mathcal{O}\left(N^{-1/2}\right)$ term. All the primes indicates derivatives within the multivariable context regarding $\hat{w} = (s, u, \varphi)$ entering in y, and $d\rho(y)$ is a normalized measure with Gaussian density $\exp\left(-\hat{f}_0'' y^2/2 - \hat{g}_0' y/\sqrt{N} \right)$.

Applying a parity argument, all the linear terms in $s - s_0$ and $\varphi - \varphi_0$ in the $\int y\, d\rho(y)$ integrals vanish. Paying with an exponentially small factor $\mathcal{O}\left(e^{-\epsilon N}\right)$, $\epsilon > 0$, the same is verified for the integral of the linear $u - u_0$ term. This also ensures that only diagonal or product of diagonal terms (denoted below by $\mathcal{D}$) survive in the integrals of even powers of y so that (III.8) yields

$$\begin{aligned} \langle \phi_i(0)\phi_j(L^{\ell-1}e_1)\rangle_\ell \;&=\; \delta_{ij}\; h_0 - \tfrac{\delta_{ij}}{N}\int \left[(h_0' y)\left[\hat{g}_0' y + \left(T_0^{(3)}\hat{f}\right) y^3\right] + \frac{h_0''}{2} y^2 \right]_{\mathcal{D}} \\ &\qquad\qquad \times \;\; d\rho(y) \;+\; \mathcal{O}\left(N^{-3/2}\right) \end{aligned} \tag{III.9}$$

Using (III.1), (III.5) and (III.6), and for a_n in (I.16), putting $b_n \equiv L^{n/2} a_n = L^3\left(1 - L^{-2\ell}\right)\left(L^2 - 1\right)^{-1}$, we have

$$\begin{aligned} h_0 \;&=\; \varphi_0^2\, L^{-\ell}\left(1 - \frac{2b_\ell}{N}\left(\frac{dr(\psi)}{d\psi}\right)_0\right) \\ h_0' \;&=\; L^{-\ell}\left(\frac{\partial\, \psi}{\partial\, y}\right)_0 \left[1 - 2\varphi_0^2\, b_\ell\, v_\infty''(\varphi_0^2)\right] \\ h_0'' \;&=\; L^{-\ell}\left\{\left(\frac{\partial^2\, \psi}{\partial\, y^2}\right)_0 \left[1 - 4\, b_\ell\, v_\infty''(\varphi_0^2) - 2\,\varphi_0^2\, b_\ell\, v_\infty''(\varphi_0^2)\right]\right. \\ &\qquad \left. +\, 2\varphi_0^2\, b_\ell\left(\frac{\partial\, \psi}{\partial\, y}\right)_0^2 \left[b_\ell\left(v_\infty''(\varphi_0^2)\right)^2 - v_\infty'''(\varphi_0^2)\right]\right\} \quad . \end{aligned} \tag{III.10}$$

As $\hat{g}_0'$ and $\hat{f}_0'''$ do not depend on ℓ, we see that the two-point function (III.9) has the structure

$$\langle \phi_i(0)\phi_j(L^{\ell-1}e_1)\rangle_\ell = \delta_{ij}\; L^{-\ell}\,\varphi_0^2\;\left[1 + \frac{1}{N}\left(K_1 b_\ell + K_2 b_\ell^2\right) + \mathcal{O}\left(N^{-3/2}\right)\right]$$

for some constants K_1 and K_2, fixed essentially by equations (III.10) and independent of ℓ and N. Hence, in the thermodynamic limit $\ell \to \infty$, we obtain

$$L^{\ell} \left\langle \phi_i(0)\phi_j(L^{\ell-1}e_1)\right\rangle_{\ell} \;\to\; \delta_{ij}\,\varphi_0^2\,\left[1+\frac{K}{N}+\mathcal{O}\left(N^{-3/2}\right)\right]$$

for some finite K, uniform in N.

This shows (I.35) and concludes the proof of Theorem 3. □

Remark: With more work, the above analysis can be used to treat a generic two-point function and may be extended to compute other $2p-$point functions. It can also be used obtain higher orders in $1/N$. It is likely that only even powers of $\sqrt{1/N}$ appear in the expansion for these functions.

IV. Final Remarks

In the case of finite N, we have considered the fixed point and correlation functions only to order $1/N$, obtaining canonical decay for the two-point function. We expect this result to hold also without this restriction. The reason for this is that, due to the good properties of the $N=\infty$ fixed point solution, such as global stability and analyticity, we do not see any obstacles to carrying out a rigorous construction of a fixed point for N large, as in [1] and obtain the correlation functions as done in [2], for the symmetrical model.

There is the question of whether or not there exists some finite value N_0 of N where anomalous behavior sets in. For $N=1$, the existence of a nonzero fixed point in a hierarchical model is shown in [11], but the behavior of correlation functions is not determined. As our results show absence of anomalous decay (even though the fixed points are nonzero and the wave-function renormalization is different from one) in both, the symmetrical and the non-symmetrical $O(N)$ vector hierarchical models, contrarily to what happens in the case of a one-component quadratic perturbation, it is certainly interesting to understand the full model from the point of view of configurations in a real space renormalization group approach.

Appendix: Infinite Lattice Covariance

Here we obtain the spectral decomposition of the infinite lattice covariance operator G which is used to determine the long-range behavior of G^{-1}. G satisfies the recursion relation [see (I.4) and (I.5)], with $x, y \in \mathbb{Z}^d$, $u_\alpha, \bar{u}_\alpha \in [0, L)$, $\alpha = 1, \ldots, d$,

$$G\left(Lx + u, Ly + \bar{u}\right) = L^{-(d-2)} G(x, y) + \delta(x, y) \quad .$$

We define the normalized block average operator $C : \ell_2(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)$ by

$$\left(Cf\right)(x) = L^{-d/2} \sum_{0 \leq u_\alpha < L} f(Lx + u) \quad ,$$

with $\sum_u$ meaning the multiple sum over the d components of u. We find, for the adjoint C^*,

$$(C^* g)(Lx + u) = L^{-d/2} g(x) \ , \ 0 \le u_\alpha < L$$

and easily check that $CC^* = I$.

Now let, with $C_0 = I$ and $C_n \equiv C^n$, $n \ge 1$,

$$\Gamma_n = C_n^* C_n - C_{n+1}^* C_{n+1} \quad . \tag{A.1}$$

We verify that

$$\begin{aligned} \Gamma_n \Gamma_m &= \delta_{nm} \Gamma_n \ , \ \Gamma_n^* = \Gamma_n \\ C_n^* \Gamma_n C_n &= \Gamma_{n+1} \qquad\qquad (A.2) \\ C_{n+1} \Gamma_n &= 0 \quad . \qquad\qquad (A.3) \end{aligned}$$

¿From (A.1), we see that the functions in the range $R(\Gamma_n)$ of Γ_n are constant on the L^n blocks and, from (A.3), we see that the average of $f \in R(\Gamma_n)$ over a L^{n+1} block is zero. Furthermore, $\{\Gamma_n\}$ furnishes a resolution of the identity, i.e. $I = \sum_{n=0}^{\infty} \Gamma_n$, and will be used in the spectral decomposition of G in $R(\Gamma_0)^\perp$ given by:

Proposition 2 *Let* $\lambda_n = L^d L^{2(n-1)} \frac{1-L^{-2n}}{1-L^{-2}}$. *Then*

$$G = \sum_{n=0}^{\infty} \lambda_n \Gamma_n \tag{A.4}$$

in the subspace $R(\Gamma_0)^\perp$. *Each* λ_n *has infinite multiplicity.*

Proof: It is easily shown that

$$G = L^2 C^* G C + L^d C^* C \quad . \tag{A.5}$$

Write $G = \sum_{n=0}^{\infty} \beta_n \Gamma_n$, substitute in (A.5) with $L^d C^* C = L^d \left[-\Gamma_0 + \sum_{n=0}^{\infty} \Gamma_n\right]$ and use (A.2). Equating coefficients of Γ_n we have

$$\beta_n = L^2 \beta_{n-1} + L^d \quad . \tag{A.6}$$

Iterating (A.6) with $\beta_0 = 0$ gives

$$\beta_n = L^d \sum_{j=0}^{n-1} L^{2j} \; = \; L^d L^{2(n-1)} \frac{1 - L^{2n}}{1 - L^2} \quad ,$$

from which (A.4) follows. □

¿From Proposition 2, the hierarchical Laplacian $\Delta_h = G^{-1}$ has a positive point spectrum $\{\lambda_n^{-1}\}$ with zero as the only point of accumulation.

We now determine the long-range behavior of

$$\Delta_h \left(0, L^k e\right) \; = \; \sum_n \lambda_n^{-1} \Gamma_n \left(0, L^k e\right) \tag{A.7}$$

where we recall that $\Gamma_n = C_n^* C_n - C_{n+1}^* C_{n+1}$. Let $f(x) = \delta(x, 0)$ and $g(y) = \delta(y, L^k e)$. Since $C_n f(x) = L^{-nd/2} \sum_{0 \le u < L^n} f(L^n x + u)$, $C_n f(x)$ only has support at $x = 0$, and thus we only need $C_n g(0) = L^{-nd/2} \chi(k < n)$, where χ is the characteristic function. Substituting in (A.7) gives

$$\begin{aligned} \Delta_h \left(0, L^k e\right) \;&=\; \frac{L^d}{L^2 - 1} \sum_n \left(L^{2n} - 1\right) \\ &\quad \times \left[L^{-nd} \chi\left(k < n\right) - L^{-(n+1)d} \chi(k < (n+1))\right] \end{aligned}$$

which, for large k, has a dominant *const* $L^{-k(d+2)}$ behavior.

The spectral decomposition of G can also be used to calculate directly the matrix elements of $(\Delta_h + \mu I)^{-1} = (G^{-1} + \mu I)^{-1}$, i.e. the two-point function of the hierarchical model with quadratic perturbation.

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Note Added in Proof: After the submission of this paper, we found that K. Gawedzki and A. Kupiainen [see J. Stat. Phys. 35, 267-284 (1984)] had already proven the main result of our companion work [2]. Neither did their method make use of our basic equations (I.10) and (I.3), nor did they solved the spectral properties of the linearized RGT at $N = \infty$.