An Adiabatic Theorem with Soft Photons

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Abstract

The adiabatic theorem of Quantum Mechanics disregards the soft photons of QED. Why, then, is it so successful? To understand the role of soft photons we prove an adiabatic theorem for the ground state of a simple model Hamiltonians that allows soft photons. We show that for weak electron-photon coupling, the adiabatic time scale is close to the time scale in a theory without the quantized radiation field up to a Lamb shift. The inclusion of photons also leads to a logarithmic correction in three dimensions coming from an infrared singularity characteristic of QED.

1 Introduction

The adiabatic theorem of Quantum Mechanics disregards the soft photons of QED. Its success suggests that doing so must be, at least, a very good approximation. In fact, there is evidence that this must be more than just a very good approximation: In the theory of the integral quantum Hall effect [23, 4] the Quantum Mechanics adiabatic theorem is used to explain the precise quantization of the Hall conductance, a quantization that, experimentally, is accurate to a high order of α, the fine structure constant.

¿From a mathematical and structural point of view QED does not seem all that close to Quantum Mechanics. In fact, there is good a-priori reason to even doubt the existence of a useful adiabatic theorem in QED. This is because, unlike Quantum Mechanics QED does not have a large gap in its spectrum (because of far away red photons) to protect the ground state. In the absence of such a gap, it is not clear that there is a candidate for an adiabatic theorem. Even if there was a gap, it is most probably much smaller than the Quantum Mechanics gap. Since the size of the gap enters in adiabatic estimates the success of the Quantum Mechanics adiabatic theorem is a mystery.

Let us elaborate on the significance of gaps in the common formulation of adiabatic theorems. Adiabatic theorems need a time scale to fix how slow is really slow. In the simplest of the adiabatic theorems of classical mechanics, say for the Harmonic oscillator [1], this time scale is set by the natural frequency of the (time independent) oscillator. A perturbation is adiabatic if the oscillator undergoes many oscillations during the time that the perturbation varies appreciably. In quantum mechanics the analog time scale is set by the gaps in the (unperturbed) spectrum. In the case of the Harmonic oscillator classical and quantum mechanics actually have the same time scale: $1/\omega$, classically, and $\hbar/\hbar\omega$ quantum mechanically[1]. The role of gaps in the adiabatic theorem has a natural and appealing physical interpretation: An adiabatic perturbation acts on the quantum system like *external* soft photons. A gap in the spectrum provides a protection against low energy excitations. This intuition is indeed supported by the general adiabatic theorems of quantum mechanics [12, 19, 3].

In the absence of a gap, several things can happen to the adiabatic theorem. The first is that there is no reasonable formulation of a useful adiabatic theorem. This is the case for example, for a free particle on the line perturbed by, say, a slow gauge field. Another thing that can happen is that the formulation of an adiabatic theorem can be elusive. An example for that is the adiabatic theorem for a quantum system with dense pure point (and singular continuous) spectrum studied in [2]. But, in practice, the most interesting thing that can happen, is that there is an adiabatic theorem, but the error is different than the error when there is a gap. An example to this is the adiabatic theorem for crossing eigenvalues studied first by Born and Fock. In the absence of crossing they showed that the error in the adiabatic theorem was $O\left(\frac{1}{\tau}\right)$ where τ is the adiabatic time scale set by the gap. In the case of linear crossing, they showed that the error is $O\left(\sqrt{\frac{1}{\tau}}\right)$.

There is an analog to this phenomenon in the classical adiabatic theorem, where a rather radical change in a scenario manifests itself in error estimate: Integrable and chaotic Hamiltonians lead to different errors in adiabatic theorem[25, 24].

Here we study a class of examples of adiabatic theorems for gapless Hamiltonians, inspired and motivated by low energy QED. In these models the ground state is at the threshold of the (absolutely) continuous spectrum. One such model is the Friedrichs model [15]. The second, and more interesting model is a model patterned after the Dicke model [14]. More precisely, it is the spin-boson model in the rotating wave approximation [18]. The model retains some of the infrared difficulties of low energy QED but is sufficiently simple so that various things can be explicitly calculated. Our aim is to show that the adiabatic theorem for the model essentially reproduce the Quantum Mechanics[2] results up to a small corrections that renormalizes the effective gap by a power of α–the order of (the analog of) the Lamb shift in the model. This explains why the Quantum Mechanics adiabatic theorem is right after-all.

We conclude the introduction with a classical argument, which we owe to Amos Ori, that suggests why radiation should not harm the Quantum Mechanics adiabatic theorem. The errors in the adiabatic theorem are of two sources: As the Hamiltonian evolves in time, the ground state may tunnel to the excited electronic state and it may radiate soft photons. Tunneling is essentially Quantum Mechanics without photons. This process contributes an error of order $1/\tau$, as we know from Born and Fock. To estimate photon emission, consider a slowly rotating dipole with frequency of rotation $1/\tau$. Since dipole radiation is proportional to the square of acceleration, the power radiated by the dipole is, classically, $O\left(\frac{e^2}{c\tau^4}\right)$, the total energy radiated in time τ is $O\left(\frac{e^2}{c\tau^3}\right)$ and the number of radiated photons $O\left(\frac{\alpha}{\tau^2}\right)$. Classically, radiation is always sub-dominant to quantum tunneling in the adiabatic limit.

This argument correctly predicts that there should be an adiabatic theorem. However, it does

[1] This relation between classical frequency and the distance between eigenvalues holds, in fact, for any integrable system as one can see from the following argument that we learned from S. Graffi: Semi-classically, the distance between eigenvalues $E_{j+\ell} - E_j \approx \hbar\omega\ell$ where the frequency is the classical frequency $\omega = \frac{\partial H}{\partial I}$, where H is the classical Hamiltonian and I is the action [11]

[2] without intrinsic photons

not predict the correct error in the quantum model. As we shall see an infrared divergence of QED lurks in the background and makes the error $\frac{\log\tau}{\tau}$ and not $\frac{1}{\tau}$ as this the classical argument suggests. Second, and perhaps more important, this argument misses an important role played by α. In this classical argument the number of emitted photons is sub-dominant to quantum tunneling in the adiabatic limit for any value of α. The quantum result, in contrast, turns out to say that only when α is small, when the Lamb shift is small on the scale of the distance between eigenvalues, is there agreement between the adiabatic theorem with and without photons.

2 The Adiabatic Theorem and A Commutator Equation

In this section we explain what we mean by "adiabatic theorem", and give a condition for an adiabatic theorem to hold. This condition is that the commutator equation, Eq. (4) below, has solutions X, Y which are bounded operators[3]. We also introduce notation, terminology, and collect known facts that we need. To simplify the presentation, we shall stay away from making optimal assertions.

We consider Hamiltonians that are bounded from below, and choose the origin of the energy axis so that the spectrum begins at zero. Let $H(s) \geq 0$ be a family of such self-adjoint Hamiltonians. The unitary evolution generated by the Hamiltonian, $U_\tau(s)$, is the solution of the initial value problem:

$$i\,\dot{U}_\tau(s) = \tau H(s) U_\tau(s), \quad U_\tau(0) = 1, \quad s \in [0,1]. \tag{1}$$

τ is the adiabatic time scale, and we are concerned with the limit of large τ. The physical time is $t = \tau s \in [0,\tau]$. Since τ is large $H(s) = H(t/\tau)$ varies adiabatically. We assume that all operators are defined on some fixed dense domain in the Hilbert space.

The (instantaneous) ground state is in the range of the kernel of $H(s)$ and we assume that the kernel is smooth and one-dimensional. Let $P(s) \neq 0$ be the projection on the kernel of $H(s)$, i.e. $H(s)\,P(s) = 0$, $dim\,P = Tr\,P = 1$. By smoothness we mean that $\dot{P}(s)$ a bounded operator.

The adiabatic theorems we consider are are concerned with the large time behavior of the evolution of the ground state where $t = O(\tau)$ or, equivalently, $s = O(1)$. The smoothness of the kernel implies that there is a natural candidate for an adiabatic theorem for the ground state, which is independent of whether $H(s)$ does or does not have a gap in it spectrum. Namely, that if $\psi(0) \in Range\,P(0)$ at time $s = 0$, then it evolves in time so that, $\psi_\tau(s) = U_\tau(s)\;\psi(0)$ lies in $Range\,P(s)$ at time s in the adiabatic limit, $\tau \to \infty$.

To formulate the adiabatic theorem with error estimates we need to get hold of *adiabatic phases* [9]. To do that we introduce the adiabatic evolution of Kato [19]: Let $U_A(s)$ be the solution of the evolution equation

$$\dot{U}_A(s) = [\dot{P}(s), P(s)]\,U_A(s), \quad U_A(0) = 1, \quad s \in [0,1]. \tag{2}$$

It is known that

$$U_A(s)\,P(0) = P(s)\,U_A(s). \tag{3}$$

That is $U_A(s)$ maps *Range* $P(0)$ onto *Range* $P(s)$. We can now formulate the basic adiabatic theorem :

[3] for X we also need that its derivative is bounded

Theorem 2.1 *Let $H(s)P(s) = 0$ for all $0 \le s \le 1$, with P differentiable projection on the ground state, with $\|\dot P(s)\| \le D$. Suppose that the commutator equation*

$$[\dot P(s), P(s)] = [H(s), X(s)] + Y(s), \tag{4}$$

has operator valued solutions, $X(s)$ and $Y(s)$ so that for $\varepsilon \searrow 0$

$$\|X(s)\| \ + \|\dot X(s)\| \ \le \ C \left\{ \begin{array}{c} \varepsilon^{-\nu} \\ |\log \varepsilon| \end{array} \right. , \quad \|Y(s)\| \le \hat C \, \varepsilon^{\mu}, \tag{5}$$

with $\mu, \nu \ge 0$. Then

$$\|(U_\tau(s) - U_A(s))P(0)\| \le \tilde C \left\{ \begin{array}{c} \tau^{-\frac{\mu}{\nu+\mu}} \\ \frac{\log \tau}{\tau} \end{array} \right. , \quad s \in [0,1]. \tag{6}$$

Remarks: 1. In the case that there is a gap in the spectrum, one can always find $X(s)$ bounded so $\nu = 0$, and $Y = 0$, see [3]. X, and therefor also $\tilde C$, is of the order of $(\text{gap})^{-1}$. This gives error of $1/\tau$, and generalizes the adiabatic theorem of Born and Fock and Kato for discrete spectra, to more complicated spectra provided there is a gap.

2. The theorem says that the physical evolution clings to the instantaneous spectral subspace. In particular, if P is one dimensional, it says that the physical evolution of the ground state remains close to the instantaneous ground state.

3. Here, and throughout, we are concerned only with the adiabatic theorem to lowest order. If s is chosen outside the support of $\dot P$ then much stronger results can be obtained. See e.g. [20].

4. The adiabatic time scale τ_0 set by this theorem is $\tau_0 = O((2+D)C)$.

Proof: Let $W(s) = U_A^\dagger(s)U_\tau(s))$, with $W(0) = 1$. From the equation of motion, and the commutator equation, Eq. (4),

$$\begin{aligned} P(0)\,\dot W(s) &= -P(0)U_A^\dagger(s)\Big(i\,\tau\,H(s) + [\dot P(s), P(s)]\Big)U_\tau(s) \\ &= -U_A^\dagger(s)\,P(s)\Big(i\,\tau\,H(s) + [\dot P(s), P(s)]\Big)U_\tau(s) \\ &= -U_A^\dagger(s)\,P(s)\,[\dot P(s), P(s)]\,U_\tau(s) \\ &= -U_A^\dagger(s)\,P(s)\,\Big([H(s), X(s)] + Y(s)\Big)\,U_\tau(s) \\ &= -U_A^\dagger(s)\,P(s)\,\Big(-X(s)\,H(s) + Y(s)\Big)\,U_\tau(s) \\ &= \frac{i}{\tau}\,P(0)\,U_A^\dagger(s)\,X(s)\,\dot U_\tau(s) - P(0)\,U_A^\dagger(s)\,Y(s)\,U_\tau(s). \end{aligned} \tag{7}$$

To get rid of derivatives of U_τ, which are large by the equation of motion, we rewrite the first term on the rhs (up to the $P(0)$ on the right) as :

$$\begin{aligned} U_A^\dagger(s)\,X(s)\,\dot U_\tau(s) &= \dot{\Big(U_A^\dagger(s)\,X(s)U_\tau(s)\Big)} - U_A^\dagger(s)\,\dot X(s)\,U_\tau(s) - \dot U_A^\dagger(s)\,X(s)\,U_\tau(s) \\ &= \dot{\Big(U_A^\dagger(s)\,X(s)\,U_\tau(s)\Big)} - U_A^\dagger(s)\,\dot X(s)\,U_\tau(s) + U_A^\dagger(s)\,[\dot P(s), P(s)]\,X(s)\,U_\tau(s). \end{aligned} \tag{8}$$

From this it follows, by integrating, that for $s \in [0,1]$

$$\begin{aligned} \|(U_\tau(s) - U_A(s))\,P(0)\| &= \|P(0)(U_\tau^\dagger(s) - U_A^\dagger(s))\,\| \\ \|P(0)(1 - W(s))\,\| &\le \hat C\varepsilon^{\mu} + \frac{(2+D)\,C}{\tau} \left\{ \begin{array}{c} \varepsilon^{-\nu} \\ |\log \varepsilon|. \end{array} \right. \end{aligned} \tag{9}$$

Choosing $\varepsilon = \tau^{-\frac{1}{\mu+\nu}}$ gives

$$\|(U_\tau(s) - U_A(s))\, P(0)\| \leq \tilde{C} \left\{ \begin{array}{c} \tau^{-\frac{\mu}{\nu+\mu}} \\ \frac{\log \tau}{\tau} \end{array} \right. . \tag{10}$$

This concludes the proof of the theorem. □

It is convenient to rewrite this solvability condition in a way that one needs to solve for a fixed X and Y rather than functions $X(s)$ and $Y(s)$. This is accomplished by

Corollary 2.1 *Let $P(s)$ be the family*

$$P(s) = V(s)\, P\, V^\dagger(s), \quad V(s) = \exp(i\, s\, \sigma). \tag{11}$$

It is enough to solve for the commutator equation

$$i\, K = [H, X] + Y, \quad K = \{\sigma, P\} - 2P\sigma P, \tag{12}$$

for fixed X and Y so that for $\varepsilon \searrow 0$

$$\|X\| \ \leq \ C \left\{ \begin{array}{c} \varepsilon^{-\nu} \\ |\log \varepsilon| \end{array} \right. , \quad \|Y\| \leq \hat{C}\, \varepsilon^\mu, \tag{13}$$

with $\mu, \nu \geq 0$, and $\|\dot{P}(s)\| \leq D$. $X(s)$ and $Y(s)$ are then determined by the obvious unitary conjugation.

Proof: Since $P(s) = V(s)\, P\, V^\dagger(s)$, we have

$$\dot{P}(s) = i\, V(s)\, [\sigma, P]\, V^\dagger(s), \tag{14}$$

and

$$\begin{aligned} [\dot{P}(s), P(s)] &= i\, V(s) \Big[[\sigma, P], P \Big] V^\dagger(s) \\ &= i\, V(s) \Big(\{\sigma, P\} - 2P\, \sigma\, P \Big) V^\dagger(s). \end{aligned} \tag{15}$$

□

3 An Adiabatic Theorem for Ground State at Threshold

As a warmup, let us prove an adiabatic theorem in a situation where the ground state lies at the threshold of the ac spectrum[4] of a one particle Schrödinger operator. The model we consider is a simple version of the Friedrichs model, and we do not make any pretense that it describes an interesting physical situation. Rather, the simplicity of its mathematical structure and the fact that it displays the basic elements relevant to adiabatic theorem without gaps is its motivation.

There is an inherent difficulty in the situation of a bound state at threshold in general, and in the Friedrichs model [15, 16, 17] in particular, namely, that a bound state at threshold is not a stable situation. Under a small deformation of the Hamiltonian, the ground state will, generically, split away from the ac spectrum and a gap develops. Since our aim is to study adiabatic theorem in the absence of a gap we force the ground state to stay at threshold of $H(s)$ for all s. A rather trivial way of implementing this is to consider $H(s) = V(s)\, H\, V^\dagger(s)$ where H has a bound state at threshold and $V(s)$ is a smooth family of unitaries.

[4] ac=absolutely continuous

3.1 The Friedrichs Model

We shall consider a family of Hamiltonians, closely related to the standard Friedrichs model [15], parameterized by the scaled time s, a real number $d > 0$ that plays the role of dimension, and a function f that describes the deformation of the family. Since we are only interested in the low energy behavior of the family we shall introduce an "ultraviolet cutoff" to avoid inessential.

The Hilbert space of the Friedrichs model (with an ultraviolet cutoff) is $\mathcal{H} = \mathbb{C} \oplus L^2([0,1], k^{d-1}\, dk)$. A vector $\psi \in \mathcal{H}$ is normalized by

$$\psi = \begin{pmatrix} \beta \\ f(k) \end{pmatrix} \qquad \|\psi\|^2 = |\beta|^2 + \int_0^1 |f(k)|^2 k^{d-1}\, dk, \quad \beta \in \mathbb{C}. \tag{16}$$

We choose a special, and trivial, case of a diagonal Hamiltonian whose action on a vector ψ is as follows:

$$H\psi = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \beta \\ f(k) \end{pmatrix} = \begin{pmatrix} 0 \\ k\, f(k) \end{pmatrix}. \tag{17}$$

H has a ground state at zero energy with projection

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{18}$$

The rest of the spectrum is the unit interval $[0,1]$, and is absolutely continuous with formal (delta normalized) eigenvectors

$$\psi_E = k^{\frac{d-1}{2}} \begin{pmatrix} 0 \\ \delta(k-E) \end{pmatrix}, \quad 0 < E < 1. \tag{19}$$

The density of states in this model is proportional to E^{d-1}.

We construct the family $H(s)$ by conjugating H with a family of unitaries:

$$V_f(s) = \exp is\sigma(f), \quad \sigma(f) = \begin{pmatrix} 0 & \langle f| \\ |f\rangle & 0 \end{pmatrix}, \tag{20}$$

where f is a vector in $L^2([0,1], k^{d-1}\, dk)$.

Theorem 3.1 *Let $H(s;d,f)$ be the family of Friedrichs models with a ground state at threshold for all s*

$$H(s;d,f) = V_f(s)\, H\, V_f^\dagger(s). \tag{21}$$

Suppose that

$$g(k) = i\, k^{-1}\, f(k) \in L^2([0,1], k^{d-1} dk), \quad V_f(s) = \exp\, i\, s\, \sigma(f) \tag{22}$$

then the quantum evolution of the ground state of $H(s;d,f)$ is adiabatic and its deviation from the instantaneous ground state is, at most, $O(1/\tau)$.

Remarks: 1. Note that if the conditions in the theorem hold in dimension d_0, then they hold in all dimensions $d \geq d_0$. The physical interpretation of that is that the density of states at low energies decreases with d. So, even though there is spectrum near zero, there is only very little of it.

2. If g is not in L^2 there may still be an adiabatic theorem with slower falloff in τ by accommodating $Y \neq 0$. An example will be discussed in the next section.

3. The Friedrichs model is vanilla: H has no interesting energy scale to fix the adiabatic time scale. The scale is set by the perturbation alone: $\tau_0 = O((1 + \|f\|^2)\,\|g\|)$. This is quite unlike the case in the usual adiabatic theorem.

Proof: In this case K of Corollary 2.1 is $K = \sigma(f)$. With $g \in L^2$, $\sigma(g)$ is a bounded (in fact, finite rank) operator and an easy calculation gives

$$[H, \sigma(g)] = \begin{pmatrix} 0 & \langle -kg| \\ |kg\rangle & 0 \end{pmatrix} = i\sigma(f). \tag{23}$$

Hence

$$X = \sigma(g), \quad Y = 0 \tag{24}$$

solve the commutator equation, Eq. (12), with a bounded $X(s)$ and $Y(s) = 0$. □

If g is not in L^2, $\sigma(s)$ and hence $X(s)$ are not bounded operators. Even then one can often get an adiabatic theorem but with a slower decay rate.

4 A Two Level System in a Photon Field

In this section we describe an adiabatic theorem for a model patterned after the Dicke model [14]. The Schrödinger operator is replaced by a two level system and the photons are considered as a field theory in Fock space. In the standard Dicke model one considers a single mode for the photon field. The model we consider allow all modes. The model has a conservation law that makes it amenable to explicit analysis. We also makes other sacrifices which are less important, such as the fact that real photons, in three dimensions, come with two polarization. The helicity of the photon does not play an interesting role in the questions we want to study, but clutters the notation. We disregard such realistic aspects. For the Dicke model, we prove an adiabatic theorem that says that the evolution of the ground state adheres to the instantaneous ground state and the time scale, at least in three dimensions, is essentially the time scale fixed by Quantum Mechanics without photons, but with a logarithmic correction.

4.1 The Dicke Model

The Dicke model is the Spin-Boson Hamiltonian in the rotating wave approximation. The rotating wave approximation, can indeed be motivated in the single-mode Dicke model. In the multi-mode case we consider the rotating wave approximation is not really motivated by an argument. It is a name that describes which terms in the Hamiltonian are kept and which are not. For a mathematical discussion of the spin-boson Hamiltonian, and the rotating wave model see [18, 13, 22].

The model describes a two level system coupled to a massless boson field in d dimensions. For the kind of applications we are interested in, i.e. low energy QED, it is natural to use atomic units where $e = \hbar = 1$ and the energy scale is Rydberg. In these units $\alpha = 1/c$, where c is the velocity of light, is the small parameter of the theory. The Hamiltonian is:

$$H(m, d, f, \alpha) = m\,(1 - P) \otimes \mathbf{1} + \alpha^{-1}\,\mathbf{1} \otimes E + \sqrt{\alpha}\,\sigma_+ \otimes a^\dagger(f) + \sqrt{\alpha}\,\sigma_- \otimes a(f), \tag{25}$$

acting on the Hilbert space $\mathbb{C}^2 \otimes \mathcal{F}$ with $\mathcal{F}$ being the symmetric Fock space over $L^2(R^d, d^d k)$. Here

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad E = \int |k|\, a^\dagger(k) a(k) d^d k. \tag{26}$$

$m > 0$ is the gap in the quantum Hamiltonian (without photons). $a(f)$ and $a^\dagger(f)$ are the usual creation and annihilation operators on $\mathcal{F}$ obeying the canonical commutation relations

$$[a(f), a^\dagger(g)] = \langle f|g\rangle. \tag{27}$$

We denote by $|0\rangle$ the field vacuum and by Ω the projection on the vacuum.

It may be worthwhile to explain where the various powers of α in H come from. For the radiation field the α^{-1} comes from $\hbar\omega = \hbar c|k|$ which explains why the field energy comes with a large coupling constant. The $\sqrt{\alpha}$ has one inverse power of c from minimal coupling, $\frac{e}{2mc}(p \cdot A + A \cdot p)$. Half a power comes from the standard formula for the vector potential

$$A(x) := \int d^3k \sqrt{\frac{2\pi c}{|k|}} \left(e^{-ik\cdot x}\, a^\dagger(k) + e^{ik\cdot x} a(k) \right). \tag{28}$$

Compare e.g. [21].

A useful formula we shall need is

$$E\, a^\dagger(g)\Omega = a^\dagger(|k| g)\Omega. \tag{29}$$

The model has the pleasant feature, not shared by the general spin-boson model, that photon emission is associated with the electron cascading down, and photon absorption with the electron excitation.

It is instructive to trace the origin of f in the model so that one can get a feeling for what are reasonable assumptions to impose on it. The electron-photon interaction is $\alpha\,(p \cdot A + A \cdot p)$ with $p = -i\,\nabla$. If we denote by $\psi_1(x)$ and $\psi_2(x)$ the atomic wave functions of the electronic two level system, then the matrix element of the vector potential between the two states gives f as

$$f(k) = -i \sqrt{\frac{2\pi}{|k|}} \int \left(\psi_1^*(x)\, e^{-ik\cdot x} (\nabla\psi_2)(x) \; - (\nabla\psi_1)^*(x)\, e^{-ik\cdot x}\, \psi_2(x) \right) d^d x. \tag{30}$$

Clearly, with reasonable atomic eigenfunctions, $f(k)$ has fast decay at infinity and the model is ultraviolet regular. In the infrared limit this behaves like

$$f(k) \to -i \sqrt{\frac{2\pi}{|k|}} \int \left(\psi_1^*(x)\, (\nabla\psi_2)(x) \; - (\nabla\psi_1)^*(x)\, \psi_2(x) \right) d^d x. \tag{31}$$

In particular we see that for small k

$$f(k) = K \sqrt{\frac{1}{|k|}}. \tag{32}$$

The square root singularity is a characteristic infrared divergence of QED, and it has consequences for the adiabatic theorem as we shall see. Note that with f having a square root singularity the model makes sense (as an operator) provide $d > 1$, for otherwise $a^\dagger(f)$ is ill defined since f is not in L^2.

An important parameter in the model is

$$\mathcal{E} = \left\langle f \left| \frac{1}{|k|} \right| f \right\rangle . \tag{33}$$

Bearing in mind the square root singularity of f we see that

$$\mathcal{E} \sim \int \frac{d^d k}{|k|^2}, \tag{34}$$

is finite for all $d > 2$.

4.2 Spectral Properties

What makes the Dicke model simple is that it has a constant of motion [18]. If we let $N = \int a^\dagger(k)\, a(k) d^d k$ be the photon number operator, then one checks that $\mathcal{N}$ commutes with H where

$$\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & N+1 \end{pmatrix} = \mathbf{1} \otimes N + P \otimes \mathbf{1}. \tag{35}$$

The spectrum of $\mathcal{N}$ is the non-negative integers. The spectral properties of $H(m, d, f, \alpha)$ can be studied by restricting to subspaces of $\mathcal{N}$.

$\mathcal{N} = 0$: The kernel of $\mathcal{N}$ is one dimensional and is associate with the projection

$$P = \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}. \tag{36}$$

Ω is the projection on the field vacuum. It is easy to see that $P\, H(\nu, d, f) P = 0$, so the model always has a state at zero energy. This state may or may not be the ground state. It is the ground state if $\alpha^2 \mathcal{E} < m$ [18].

$\mathcal{N} = 1$: The space is basically $\mathcal{H}$ of the Friedrichs model. The correspondence of vectors in the two spaces is

$$\begin{pmatrix} a^\dagger(g) \\ \beta \end{pmatrix} |0\rangle \leftrightarrow \begin{pmatrix} g \\ \beta \end{pmatrix} \tag{37}$$

The Hamiltonian action in the Friedrichs model language is:

$$H(m, d, f, \alpha) \leftrightarrow \begin{pmatrix} \frac{|k|}{\alpha} & |\sqrt{\alpha}\, f\rangle \\ \langle \sqrt{\alpha}\, f| & m \end{pmatrix}. \tag{38}$$

It is a standard fact about the Friedrichs model [15, 16] that provided

$$\alpha^2 \mathcal{E} < m, \tag{39}$$

the model has no bound state, and the spectrum is $[0, \infty)$ and is absolutely continuous. Since f has square root singularity at the origin, (and has fast decay at infinity), this condition holds for

$d \geq 3$ if α (or f) is not too large. In three dimensions, provided that the level spacing $m >> \alpha^2$ in atomic units, (about 10^{-3} eV), the inequality holds. In two dimensions the left hand side is log divergent, and the spectrum in the $\mathcal{N} = 1$ sector has a bound state at negative energy. This state lies below the bound state of the $\mathcal{N} = 0$ sector. We do not consider this situation and henceforth stick to $d \geq 3$.

$\mathcal{N} \geq 2$: It is known [18] that the bottom of the spectrum in all these sectors is at zero if (39) holds.

4.3 Resonance and Lamb Shift

The $\mathcal{N} = 1$ sector has a resonance that serves to define the Lamb shift in the model and plays a role in the adiabatic theorem. The resonance is a solutions of the analytically extended eigenvalue equation, given below, which for $\alpha \to 0$ converge to the eigenvalue m and lies in the lower half plane, see [15, 17, 22]. The real part of the shift is, by definition, the Lamb shift of the model, while the imaginary shift is the life time. It turns out that, for $d \geq 3$, the Lamb shift is dominant and the life time is a higher order in α. For the application to the adiabatic theorem we need only the dominant contribution, i.e. only the Lamb shift. Computing the Lamb shift is easy. Computing the life time is harder. For the sake of completeness we compute both, even though we only need one.

The eigenvalue equation is

$$E - m = \alpha^2\, G(\alpha E) \tag{40}$$

where $G(e)$ is defined as the analytic continuation from the upper half plane of

$$G(e) = \int_{\mathbb{R}^d} \frac{|f|^2}{e - |k|}\, d^d k, \quad \Im\, e \geq 0. \tag{41}$$

By taking the imaginary part, it is easy to see that Eq (40) has no solution in the upper half plane as it should (since the Hamiltonian is self-adjoint). To solve the equation in the lower half plane one needs an explicit expression, at least for small α, and e near αm, of this analytic continuation.

Now, suppose we found this analytic continuation, then, we can solve Eq. (40) by iteration, and to lowest order we have

$$E_r \approx m + \alpha^2\, G(\alpha m). \tag{42}$$

Clearly $G(\alpha m) \to -\mathcal{E}$, in the limit $\alpha \to 0$, so to leading order

$$E_r \approx m - \alpha^2\, \mathcal{E}. \tag{43}$$

To this order, one does not see the imaginary part of the resonance energy. The real part of the shift in the energy, $\alpha^2\, \mathcal{E}$ is, by convention,[21], the Lamb shift of the model. As we shall see $m - \alpha^2 \mathcal{E}$ will play the role of the gap (m in the theory without photons) in the adiabatic theorem of the model. It may be worthwhile to point out that the Lamb shift for the Hydrogen atom, [10], is actually of *higher* order, namely, $\alpha^3 \log(\alpha^{-1})$. Since the Lamb shift of Hydrogen also involves an ultraviolet regularization, while the present model is ultraviolet regular, it is not surprising that the order of the two is different. What is surprising is that the order of Hydrogen is higher rather than lower.

Estimating the life time is, as we noted, considerably harder, and, because of this, irrelevant to the adiabatic theorem. So a reader will loose little by skipping the rest of this section. However, for the benefit of the reader who is interested in how the computation of the life time goes, it is given below.

We shall show below that for $d \geq 3$, and $|e - \alpha m| < \alpha m$, the analytic continuation of $G(e)$ to the lower half plane, and to the next relevant order in α, is given by

$$G(e) = -\mathcal{E} - i\pi\, K\Omega^d e^{d-2}\,, \quad \Im\, e \leq 0, \tag{44}$$

where K is as in Eq. (32), and Ω^d is the surface area of the unit ball in d dimensions. ¿From Eq. (42), and taking into account Eq. (32), we get for the Lamb shift and the life-time:

$$\begin{aligned} E_r &\approx m - \alpha^2\mathcal{E} - i\alpha^2\pi\, K\,\Omega^d (m\alpha)^{d-2} \\ &= m - \alpha^2\mathcal{E} - i\alpha^2\pi\Omega^d (m\alpha)^{d-1}\, |f(\alpha m)|^2\,. \end{aligned} \tag{45}$$

The life time is higher order in α than the Lamb shift, and is of order α^d. For $d = 3$ this is, indeed, the order of the life time of atomic levels that decay by dipole transition. For small α the Lamb shift dominates the life time, both in the Dicke model and in Hydrogen.

It remains to show that the analytic continuation of $G(e)$ to the lower half plane in a neighborhood of $m\alpha$, is indeed given by Eq. (44). This can be done as follows: Let B_r be a ball of radius $r = 2m\alpha$. Then, in the upper half plane

$$G(e) = \left(\int_{B_r} + \int_{B_r^c}\right) \frac{|f|^2}{e - |k|}\, d^d k = G_r(e) + G_r^c(e). \tag{46}$$

Clearly, $G_r^c(e)$ extends analytically to a half circle in the lower half plane $|e - \alpha m| < \alpha m$. In the limit of $\alpha \to 0$, by continuity,

$$G_r^c(0) \to -\mathcal{E}. \tag{47}$$

This is the dominant piece, and it is real.

Consider the analytic continuation of $G_r(e)$ for $|e - \alpha m| \leq \alpha m$. Since, for small argument $f(k)$ is given by Eq. (32), one has (in the upper half plane)

$$G_r(e) = K\Omega^d \int_0^{2m\alpha} \frac{k^{d-2}}{e - k}\, dk = K\Omega^d \int_\gamma \frac{k^{d-2}}{e - k}\, dk, \tag{48}$$

where γ is the obvious semi-circle in the complex k plane and Ω^d the surface area of the unit ball in d dimensions. The right hand side is analytic in e in the lower half plane provided $|e - \alpha m| < \alpha m$, and so gives the requisite analytic continuation. Since e is small, and of order α, to leading order, we have

$$\begin{aligned} G_r(e) &= K\Omega^d \int_\gamma \frac{(k - e + e)^{d-2}}{e - k}\, dk \\ &= -K\Omega^d \sum_{j=0}^{d-2} \begin{pmatrix} d-2 \\ j \end{pmatrix} e^{d-j-2} \int_\gamma (k - e)^{j-1}\, dk \\ &\approx -K\Omega^d\, e^{d-2} \int_\gamma \frac{dk}{k - e} \\ &= -\,K\Omega^d\, e^{d-2} \left(i\,\pi + \log\left(\frac{2\alpha m}{e}\right) + O(\alpha \log \alpha)\right) \end{aligned} \tag{49}$$

and the error term in approximation that we did not compute is real and being sub-dominant to $\mathcal{E}$ is irrelevant.

Adiabatic Deformations Suppose, that the two level system of the Dicke model describes two Zeeman split energy levels of an atom in constant external magnetic field B. In this case, m is proportional to the $|B|$ (and so of order α in atomic units. Note that in this case Eq. (39) holds if α is small). Following Berry [9], let us consider adiabatically rotating the magnetic field keeping $|B|$ constant, then the family of Dicke models that we obtain is related by unitary transformation describing the effect of rotation in the two dimensional Hilbert space of the atom. This motivates looking at the family of Hamiltonians obtained by conjugating $H(m, d, f, \alpha)$ with a unitary $V(s)$. We choose

$$V(s) = \exp\left(i\, s\, \sigma\right), \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}, \tag{50}$$

which corresponds to a rotation of the atom, leaving the (transverse) photons alone. This choice of σ does not commute with $\mathcal{N}$,

$$[\mathcal{N}, \sigma] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J. \tag{51}$$

Had we chosen σ that commuted with $\mathcal{N}$ the adiabatic theorem for the ground state would hold for trivial reasons: One can reduce the problem to the $\mathcal{N} = 0$ sector where the ground state is an isolated eigenvalue. To have an interesting adiabatic theorem we need to choose σ that fails to commute with $\mathcal{N}$. This is the choice we have made.

4.4 The Adiabatic Theorem

Theorem 4.1 *Let $H(s; m, d, f, \alpha) = V(s) H(m, d, f, \alpha) V^\dagger(s)$, $s \in [0, 1]$ be the family of time dependent Dicke models with f square integrable, with square root singularity at $k = 0$; $m > \alpha^2 \langle f | \frac{1}{|k|} | f \rangle$; $d \geq 3$ and $V(s) = \exp\left(i\, s\, \sigma\right)$ as in Eq. (50). Then, U_A, the adiabatic evolution associated with the kernel of $H(s; m, d, f, \alpha)$, and U_τ, the Schrödinger evolution. are close in the sense that*

$$\|(U_A(s) - U_\tau(s)) P(0)| \leq C \begin{cases} \frac{1}{\tau} & \text{if } d > 3 \\ \frac{\log \tau}{\tau} & d = 3 \end{cases} \tag{52}$$

The effective gap is renormalized to $m\left(1 + O(\alpha^2)\right)$, and coincides with the gap without photons up to a correction of the order of the Lamb shift.

Proof: From Corollary 2.1 we find $K = \sigma \otimes \Omega$. We will first show that a solution of the commutator equation, Eq. (12), for $d > 0$, is

$$X = \frac{i\, X_1 - X_2(g)}{m - \alpha^2\, \mathcal{E}}, \quad Y = 0, \tag{53}$$

where

$$X_1 = J \otimes \Omega, \quad X_2(g) = P \otimes \left(a^\dagger(g)\, \Omega + \text{h.c}\right), \quad g = i\, \alpha^{\frac{3}{2}}\, \frac{f}{|k|}. \tag{54}$$

Note that the gap of the two level system m is renormalized to $m + i\,\alpha\,\langle f|g\rangle$, which is just the Lamb shift. This is a small correction, of order α^2.

Let us compute the commutators of X_1, X_2 with H:

$$\begin{aligned}
[H, X_1] &= \left[\begin{pmatrix} \frac{E}{\alpha} & \sqrt{\alpha}\, a^\dagger(f) \\ \sqrt{\alpha}\, a(f) & m + \frac{E}{\alpha} \end{pmatrix}, \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}\right] \\
&= m\,\sigma \otimes \Omega + \sqrt{\alpha}\, P \otimes \left(a^\dagger(f)\Omega + \Omega a(f)\right). \qquad (55)
\end{aligned}$$

For the second commutator

$$\begin{aligned}
[H, X_2] &= \left[\begin{pmatrix} \frac{E}{\alpha} & \sqrt{\alpha}\, a^\dagger(f) \\ \sqrt{\alpha}\, a(f) & m + \frac{E}{\alpha} \end{pmatrix}, \begin{pmatrix} a^\dagger(g)\,\Omega + \text{h.c} & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \frac{1}{\alpha} P \otimes (a^\dagger(|k|g)\Omega - \Omega a(|k|g) + \sqrt{\alpha} \begin{pmatrix} 0 & -\langle g|f\rangle \\ \langle f|g\rangle & 0 \end{pmatrix} \otimes \Omega. \qquad (56)
\end{aligned}$$

So, if we take g of Eq. (54) then

$$[H, i\,X_1 - X_2] = i\,\left(m - \alpha^2\,\mathcal{E}\right)\sigma \otimes \Omega. \qquad (57)$$

We see that we can formally solve for the commutator equation, Eq. (12) provided $\mathcal{E}$ is finite.

This is, however, not the only condition. X is a bounded operator in the Hilbert space provided $g \in L^2$, for otherwise $a^\dagger(g)$ is ill defined:

$$\int \frac{|f|^2}{|k|^2} d^d k \sim \int \frac{1}{|k|^3} d^d k < \infty. \qquad (58)$$

The integral is finite if $d \geq 4$ but is logarithmically divergent in $d = 3$.

So the $d > 3$ part of the theorem is done. For $d = 3$ we need to squeeze X back to the bounded operators. We do that by allowing for $Y \neq 0$.

Let χ_ε be the characteristic function of a ball of radius ε and $\chi^c_\varepsilon = 1 - \chi_\varepsilon$ and let $g^c_\varepsilon = \chi^c_\varepsilon g$ and $g_\varepsilon = \chi_\varepsilon g$. Let us take $X_2(g^c_\varepsilon)$ is well defined and it norm is $O(\alpha^{\frac{3}{2}} \log \varepsilon)$. For X we take, as before,

$$X = \frac{i\,X_1 - X_2(g^c_\varepsilon)}{m + i\sqrt{\alpha}\langle f|g^c_\varepsilon\rangle}. \qquad (59)$$

From this

$$\|X\| = O\left(\frac{1 + \alpha^{\frac{3}{2}}|\log \varepsilon|}{|m - \alpha^2 \mathcal{E}|}.\right) \qquad (60)$$

For Y we take

$$\begin{aligned}
(m + i\,\sqrt{\alpha}\langle f|g^c_\varepsilon\rangle)\,Y &= [H, X_2(g) - X_2(g^c_\varepsilon)] = [H, X_2(g_\varepsilon)] \qquad (61) \\
&= \frac{1}{\alpha} P \otimes \left(a^\dagger(|k|g_e)\Omega - \Omega a(|k|g_e)\right) + \sqrt{\alpha} \begin{pmatrix} 0 & -\langle g_e|f\rangle \\ \langle f|g_e\rangle & 0 \end{pmatrix} \otimes \Omega,
\end{aligned}$$

and we used the computation of the commutator Eq. (56). With f having a square root singularity,

$$\|Y\| = O\left(\frac{\sqrt{\alpha}\,\varepsilon + \alpha^2\,\varepsilon}{|m - \alpha^2 \mathcal{E}|}\right). \qquad (62)$$

This puts us in the frame of theorem 2.1 and establishes the main result. □

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