New Structures in the Theory of the Laser Model II: Microscopic Dynamics and a Non-Equilibrium Entropy Principle

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Abstract.

In a recent article, Alli and Sewell formulated a new version of the Dicke-Hepp-Lieb laser model in terms of quantum dynamical semigroups, and thereby extended the macroscopic picture of the model. In the present article, we complement that picture with a corresponding microscopic one, which carries the following new results. (a) The local microscopic dynamics of the model is piloted by the classical, macroscopic field, generated by the collective action of its components; (b) the global state of the system carries no correlations between its constituent atoms after transient effects have died out; and (c) in the latter situation, the state of the system at any time t maximises its entropy density, subject to the constraints imposed by the instantaneous values of its macroscopic variables.

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I. Introduction

The classic work by Hepp and Lieb[1] (HL) on the theory of the Dicke laser model[2] represents a landmark in the constructive quantum theory of phase transitions, far from equilibrium, in open quantum systems. Most notably, it demonstrated how the model undergoes a transition from normal to coherent radiation when the optical pumping strength reaches a critical value. This substantiated ideas that had been proposed by Graham and Haken[3] at a more heuristic level.

In a recent article, which we shall refer to as (I), Alli and Sewell[4] constructed a new version[5] of the Dicke model, in which the dynamics of the matter and radiation is governed by a quantum dynamical semigroup, which incorporates the dissipative action of an array of pumps and sinks: by contrast, the HL model represented a conservative system, comprising the matter, radiation and these reservoirs. The principle new results of (I) were (a) a generalisation of the HL form of the dynamics of the matter-radiation composite, (b) a demonstration that the model exhibits optically chaotic phases, in addition to the normal and coherent ones, and (c) a generalisation of the theory of quantum dynamical semigroups[6] to ones with unbounded generators.

The theories of both HL and (I) were centred on the macroscopic equations of motion, derived from the underlying quantum dynamics of their respective models in a limit where their sizes become infinite. In fact, they did not provide any treatment of the microdynamics in this limit. The object of the present article is to complement the macroscopic picture of (I) with a corresponding microscopic one.

In seeking a microscopic-cum-macroscopic picture of the dynamics of the model, we note that this has already been achieved for certain *conservative* systems with long range forces. For such systems, Morchio and Strocchi[7] have shown that the evolutes, b_t, of the local microscopic observables, b, depend not only on b and the time t, but also on certain macroscopic observables, i.e. 'observables at infinity': this represents a picture completely different from that of the Haag- Kastler scheme[8], in which the dynamics of the micro- observables is autonomous. Furthermore, the explicit structure of the equations of motion for the microscopic and macroscopic observables, b_t and B_t, has been obtained by Bagarello and Morchio[9] (BM) for a class of conservative, mean field theoretic models in the following form.

$$\frac{db_t}{dt} = f(b_t, B_t);\ \frac{dB_t}{dt} = F(B_t). \tag{1.1}$$

Thus, the microscopic dynamics is piloted by the macroscopic observables, which, for their part, evolve according to a self-contained law.

The question now arises as to whether a similar state of affairs prevails in the case of open dissipative systems, such as the laser model; and the first thing to say about that is that the methods of BM, which led to equn. (1.1), are not applicable there. The reason is simply that those methods depended heavily on the fact that, in a conservative system, the dynamics is automorphic and so preserves the algebraic structure of the observables. In an open dissipative system, on the other hand, the dynamics is governed by a semigroup[4] of transformations of the observables that does not preserve that structure, i.e., in general, $(ab)_t \neq a_t b_t$.

Thus, any attempt to derive equations of the form (1.1) for the laser model must be based on methods that are applicable to dissipative systems. Here, the essential problem is to pass to a limit where the size of the system becomes infinite, since the dynamics of the finite version of the model has already been formulated in (I). In fact, we surmount this problem by means of a compactness argument, that lends itself to the application of the Arzela-Ascoli theorem. As a result, we do indeed obtain a microscopic-cum-macroscopic dynamics of the form (1.1), though now, of course, the functions f and F incorporate the dissipative action of the pumps and sinks on the system.

The principal new results that we obtain in this way are of a very simple character, and may be summarised as follows.

(1) The microscopic observables of *this particular model* are those of the matter only, since the radiation field reduces to a classical macroscopic one.

(2) The dynamics of the local microscopic observables is piloted by that classical radiation field.

(3) The microdynamics contains certain transients, whose lifetimes are of the order of that of an atomic excited state. This dynamics simplifies considerably after these transients have died out, and leads to the decorrelation of the observables of different atoms. Furthermore, the resultant global microstate of the system is stationary and space-translationally invariant in the normal phase, and periodic with respect to both space and time in the coherent one. Evidently, it is very complicated in the chaotic phase.

(4) Following the decay of the transients, the instantaneous state of the model is determined by a non-equilibrium variational principle. Specifically, this state maximises the global entropy density, subject to the constraints imposed by the prevailing values of the macroscopic observables. As far as we know, this is the first example for which a principle of this kind has been proved for the non-equilibrium regime.

We shall present our arguments as follows. In Sections II and III, we shall provide a brief summary of (I), re-expressed in a form suitable for our present purposes. Specifically, Section II will be devoted to a formulation of the model and Section III to a resume of its macroscopic properties. In Section IV, we shall derive the above results (1) and (2) by passing to the infinite size limit of the quantum dynamics of the finite version of the model. We shall then derive the results (3) and (4) in Sections V and VI, respectively. We shall conclude, in Section VII, with a brief discussion of our main results and their possible ramifications.

II. The Model.

Our model, as formulated in (I), is a dissipative quantum system, $\Sigma^{(N)}$, consisting of a chain of $2N+1$ identical two-level atoms interacting with an $n-$mode radiation field. We label it by N only, since we shall be concerned with its properties in a limit where

N tends to infinity, while n remains fixed and finite. We formulate $\Sigma^{(N)}$ as a quantum dynamical system $(\mathcal{B}^{(N)}, T^{(N)}, \mathcal{S}^{(N)})$, where $\mathcal{B}^{(N)}$ is a $\star-$algebra of observables, $T^{(N)}$ is a one-parameter semigroup of completely positive (CP) transformations of $\mathcal{B}^{(N)}$, and $\mathcal{S}^{(N)}$ is a family of normal states on $\mathcal{B}^{(N)}$, that is stable under the dual of $T^{(N)}$. Most importantly, $T^{(N)}$ incorporates the dissipative action of pumps and sinks, which do not appear explicitly in the model, on the matter-radiation system. We build the model from its constituent parts as follows.

The Single Atom. This is assumed to be a two-state atom or spin, Σ_{at}. Its algebra of observables, $\mathcal{A}_{at}$, is that of the two-by-two matrices, and is thus the linear span of Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ and the identity, I. Its algebraic structure is provided by the relations

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I;\ \sigma_x\sigma_y = i\sigma_z,\ etc. \tag{2.1}$$

We define the spin raising and lowering operators

$$\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y). \tag{2.2}$$

We assume that the atom is coupled to a pump and a sink, and that accordingly (cf. (I)) its dynamics is given by a one-parameter semigroup $\{T_{at}(t)|t\in\mathbf{R}_+\}$ of completely positive, identity preserving contractions of $\mathcal{A}_{at}$, whose generator, L_{at}, is of the following form.

$$L_{at}\sigma_{\pm} = -(\gamma_1 \mp i\epsilon)\sigma_{\pm};\ L_{at}\sigma_z = -\gamma_2(\sigma_z - \eta I), \tag{2.3}$$

where $\epsilon(> 0)$ is the energy difference between the ground and excited states of an atom, and the γ's and η are constants whose values are determined by the atomic coupling to the energy source and sink, and are subject to the restrictions that

$$0 < \gamma_2 \leq 2\gamma_1;\ -1 \leq \eta \leq 1. \tag{2.4}$$

In particular, η is positive or negative according to whether the coupling of the atom to the pump or the sink is the stronger. In the former case, these couplings drive the atom to a terminal mixed state with *inverted population*, i.e. with higher occupation probability for the excited state than for the ground state.

The Matter. This consists of $2N + 1$ non-interacting copies of Σ_{at}, located at the sites $r = -N, . ., N$ of the one-dimensional lattice $\mathbf{Z}$. Thus, at each site, r, there is a copy, Σ_r, of Σ_{at}, whose algebra of observables, $\mathcal{A}_r$, and dynamical semigroup, T_r, are isomorphic with $\mathcal{A}_{at}$ and T_{at}, respectively. We denote by $\sigma_{r,u}$ the copy of σ_u at r, for $u = x, y, z, \pm$.

We define the algebra of observables, $\mathcal{A}^{(N)}$, and the dynamical semigroup, $T_{mat}^{(N)}$, of the matter to be $\otimes_{r=-N}^{N}\mathcal{A}_r$ and $\otimes_{r=-N}^{N}T_r$, respectively. Thus, $\mathcal{A}^{(N)}$ is the algebra of linear transformations of $\mathbf{C}^{4N+2}$. We identify elements A_r of $\mathcal{A}_r$ with those of $\mathcal{A}^{(N)}$ given by their tensor products with the identity operators attached to the remaining sites. Under this identification, the commutant, $\mathcal{A}_r'$, of $\mathcal{A}_r$ is the tensor product $\otimes_{s\neq r}\mathcal{A}_s$.

It follows from these specifications that the generator, $L_{mat}^{(N)}$, of $T_{mat}^{(N)}$ is given by the formula

$$L_{mat}^{(N)} = \sum\nolimits_{r=-N}^{N} L_r, \tag{2.5}$$

where

$$L_r \sigma_{r,\pm} = -(\gamma_1 \mp i\epsilon)\sigma_{r,\pm};\ L_r \sigma_{r,z} = -\gamma_2(\sigma_{r,z} - \eta I);$$
$$and\ L_r(A_r A_r') = (L_r A_r)A_r'\ \forall A_r \in \mathcal{A}_r,\ A_r' \in \mathcal{A}_r' \tag{2.6}$$

The Radiation. We assume that the radiation field consists of $n(<\infty)$ modes, represented by creation and destruction operators $\{a_l^\star, a_l | l = 0, .., n-1\}$ in a Fock-Hilbert space $\mathcal{H}_{rad}$, as defined by the standard specifications that (a) these operators satisfy the CCR,

$$[a_l, a_m^\star] = \delta_{lm} I;\ [a_l, a_m] = 0, \tag{2.7}$$

and (b) $\mathcal{H}_{rad}$ contains a (vacuum) vector Φ, that is annihilated by each of the a's and is cyclic w.r.t. the algebra of polynomials in the $a^\star$'s.

We define the Weyl map $z = (z_0, .., z_{n-1}) \to W(z)$, of C^n into $\mathcal{L}(\mathcal{H}_{rad})$ by the formula

$$W(z) \equiv W(z_0, .., z_{n-1}) = \exp[i(z.a + (z.a)^\star)],\ with\ z.a = \sum\nolimits_0^{n-1} z_l a_l \tag{2.8}$$

We then define $\mathcal{R}$ to be the $^\star-$algebra of polynomials in the a's, $a^\star$'s and the Weyl operators $W(z)$, with z running through $\mathbf{C}^n$. Thus, in view of the CCR (2.7), this algebra is the linear span of terms of the form $W(z)a_{l_1}^\sharp . .a_{l_m}^\sharp$, where $a^\sharp$ denotes a or $a^\star$. Equivalently, it is the linear span of the derivatives, of all orders, of the operators $W(t_0 z_0, .., t_{n-1} z_{n-1})$ w.r.t. the real variables $(t_0, .., t_{n-1})$.

We assume that the radiation dynamics is given by the canonical extension to $\mathcal{R}$ of Vanheuverszwijn semigroup[10], T_{rad}, of quasi-free CP transformations of the Weyl algebra of linear combinations of $\{W(z) | z \in \mathbf{C}^n\}$. The formal generator of this semigroup is

$$L_{rad} = \sum\nolimits_{l=0}^{n-1} \big(i\omega_l [a_l^\star a_l, .] + 2\kappa_l a_l^\star (.) a_l - \kappa_l \{a_l^\star a_l, .\}\big), \tag{2.9}$$

where $\{.,.\}$ denotes anticommutator, and the frequencies, ω_l, and the damping constants, κ_l, are positive.

The Composite System. This is the coupled system, $\Sigma^{(N)}$, comprising the matter and radiation. We assume that its algebra of observables, $\mathcal{B}^{(N)}$, is the tensor product $\mathcal{A}^{(N)} \otimes \mathcal{R}$. Thus, $\mathcal{B}^{(N)}$, like $\mathcal{R}$, is an algebra of both bounded and unbounded operators in the Hilbert space $\mathcal{H}^{(N)} := \mathbf{C}^{4N+2} \otimes \mathcal{H}_{rad}$. We shall identify elements A, R, of $\mathcal{A}^{(N)}$, $\mathcal{R}$, with $A \otimes I_{rad}$ and $I_{mat} \otimes R$, respectively.

We assume that the matter-radiation coupling is dipolar and is given by the interaction Hamiltonian

$$H_{int}^{(N)} = i(2N+1)^{-1/2} \sum\nolimits_{r=-N}^{N} \sum\nolimits_{l=0}^{n-1} \lambda_l (\sigma_{r,-} a_l^\star \exp(-2\pi i l r/n) - h.c.), \tag{2.10}$$

where the λ's are real-valued, $N-$independent coupling constants. Correspondingly, we define the radiation field, $\phi^{(N)}$, of the model by the stipulation that its value at the site r is the coefficient of $\sigma_{r,+}$ in this formula. Thus,

$$\phi_r^{(N)} = -i(2N+1)^{-1/2}\sum_{l=0}^{n-1}\lambda_l a_l \exp(2\pi ilr/n). \tag{2.11}$$

We now need to define the state space, $\mathcal{S}^{(N)}$, and the dynamical semigroup, $T^{(N)}$, in a way that takes account of the unboundedness both of $H_{int}^{(N)}$ and of some of the elements of $\mathcal{B}^{(N)}$. To this end, we start by defining $\mathcal{D}_0^{(N)}$ to be the space of density matrices in $\mathcal{H}^{(N)}$, with the trace norm topology, and $\mathcal{D}_1^{(N)}$ to be the subset of its elements, $\rho^{(N)}$, for which $Tr(B^\star\rho^{(N)}B)$ is finite for all B in $\mathcal{B}^{(N)}$. We define $\mathcal{S}_1^{(N)}$ to be the set of positive, normalised, linear functionals, $\psi^{(N)}$, on $\mathcal{B}^{(N)}$, that are in the one-to-one correspondence with the $\mathcal{D}_1^{(N)}-$class density matrices, given by the standard formula $\psi^{(N)}(B) = Tr(\rho^{(N)}B)$. We then adopt the formulation of $T^{(N)}$ and $\mathcal{S}^{(N)}$ provided by (I), in which $T^{(N)}$ is constructed as the modification of $T_{mat}^{(N)}\otimes T_{rad}$ due to the interaction $H_{int}^{(N)}$, and $\mathcal{S}^{(N)}$ is a 'large' subset of $\mathcal{S}_1^{(N)}$ that is stable w.r.t. this dynamical semigroup. Specifically, the construction yields the following results[4].

(1) The density matrices corresponding to $\mathcal{S}^{(N)}$ form a dense subset of $\mathcal{D}_0^{(N)}$, in the topology corresponding to the trace norm.

(2) $\mathcal{S}^{(N)}$ is stable under the transformations $\psi^{(N)}\rightarrow\psi^{(N)}\circ T^{(N)}(t) := \psi_t^{(N)}$.

(3)

$$\frac{d}{dt}\psi_t^{(N)}(B) = \psi_t^{(N)}(L^{(N)}B)\ \forall B\in\mathcal{B}^{(N)},\ t\in\mathbf{R}_+, \tag{2.12}$$

where

$$L^{(N)} = L_{mat}^{(N)} + L_{rad} + i[H_{int}^{(N)}, .]. \tag{2.13}$$

This completes our specification of $\Sigma^{(N)} = (\mathcal{B}^{(N)}, T^{(N)}, \mathcal{S}^{(N)})$.

The Sequence of Systems $\{\Sigma^{(N)}\}$. Since we shall be concerned with the properties of the above model in a limit where N tends to infinity and n remains fixed, we need to consider the sequence of systems $\{\Sigma^{(N)}|N\in\mathbf{N}\}$. Specifically, we shall investigate the evolution of these systems from initial states, $\psi^{(N)}$, for which the mean photon number is not super-extensive, i.e., its ratio to N is bounded. Thus, we assume that, for some finite constant C,

$$\psi^{(N)}(a_l^\star a_l) < CN,\ \forall N\in\mathbf{N}, l\in[0, n-1]. \tag{2.14}$$

This exclusion of super-extensivity was shown in (I) to persist, in a uniform way, at later times, in that

$$\psi_t^{(N)}(a_l^\star a_l) < DN,\ \forall N\in\mathbf{N}, t\in\mathbf{R}_+, l\in[0, n-1], \tag{2.15}$$

where D is a finite constant.

III. Macroscopic Dynamics and Phase Structure

We formulate the macroscopic description of the model, as in (I), in terms of the global intensive observables

$$s_l^{(N)} = (2N+1)^{-1}\sum_{r=-N}^{N}\sigma_{r,-}\exp(-2\pi ilr/n);\ l=0,..\,.,n-1 \tag{3.1}$$

and

$$p_l^{(N)} = (2N+1)^{-1}\sum_{r=-N}^{N}\sigma_{r,z}\exp(-2\pi ilr/n);\ l=0,..\,.,n-1, \tag{3.2}$$

together with the operators

$$\alpha_l^{(N)} = (2N+1)^{-1/2}a_l;\ l=0,..\,.,n-1, \tag{3.3}$$

corresponding to a scaling of the number operators $a_l^\star a_l$ in units of $2N+1$. We note that the set of $p^{(N)}$'s is the same as that of their adjoints, since

$$p_0^{(N)\star} = p_0^{(N)};\ and\ p_l^{(N)\star} = p_{n-l}^{(N)}\ for\ l=1,.\,.,n-1. \tag{3.4}$$

The set $M^{(N)} := \{s_l^{(N)}, s_l^{(N)\star}, p_l^{(N)}, \alpha_l^{(N)}, \alpha_l^{(N)\star}\}$ of macroscopic variables is a Lie algebra w.r.t. commutation, and, by equns. (2.10), (2.11) and (3.3), the radiation field and the interaction Hamiltonian are the algebraic functions of them given by

$$\phi_r^{(N)} = -i\sum_{l=0}^{n-1}\lambda_l\alpha_l^{(N)}\exp(2\pi irl/n) \tag{3.5}$$

and

$$H_{int}^{(N)} = i(2N+1)\sum_{l=0}^{n-1}\lambda_l(s_l^{(N)}\alpha_l^{(N)\star} - h.c.). \tag{3.6}$$

In (I), the dynamics of $M^{(N)}$ was extracted from the microscopic equation of motion (2.12), in a limit where $N\to\infty$ and n remains fixed and finite, subject to the assumptions of Section 2. As one might anticipate from the fact that the commutators of the elements of $M^{(N)}$ tend normwise to zero in this limit, this macrodynamics is classical.

To describe this dynamics, we simplify our notation by defining $x^{(N)}\equiv(x_1^{(N)},.\,.,x_{5n}^{(N)})$ to be the $5n$ self-adjoint operators, which, by equns. (3.1)-(3.4), comprise the Hermitian and anti-Hermitian parts of the elements of $M^{(N)}$. Correspondingly, in anticipation of a classical limit for the dynamics of these observables, as $N\to\infty$, we introduce a phase space, $X=\mathbf{R}^{5n}$ and, for each point $x=(x_1,.\,.,x_{5n})\in X$, we define $\{\alpha_l, s_l, p_l | l=0,.\,.,n-1\}$ to be the complex numbers related to x in precisely the same way that the elements of $M^{(N)}$ are to $x^{(N)}$. We define $\mathcal{C}_0(X)$ to be the space of bounded continuous functions on X, that tend to zero at infinity.

We formulate the macroscopic dynamics of $\Sigma^{(N)}$ in terms of the time-dependent quantum characteristic function $\mu_t^{(N)}$ on X, defined by the formula

$$\mu_t^{(N)}(y) = \psi_t^{(N)}\big(\exp(iy.x^{(N)})\big)\ \forall y\in X, t\in\mathbf{R}_+. \tag{3.7}$$

Further (cf. (I)), we relate the limiting form of this function, as $N\rightarrow\infty$, to the following classical dynamical system.

Definition 3.1. We define $\mathcal{K}$ to be the flow in X, whose equation of motion,

$$\frac{dx_t}{dt} = F(x_t), \tag{3.8}$$

is of the explicit form

$$\frac{d\alpha_{l,t}}{dt} = -(i\omega_l + \kappa_l)\alpha_{l,t} + \lambda_l s_{l,t} \tag{3.9a}$$

$$\frac{ds_{l,t}}{dt} = -(i\epsilon + \gamma_1)s_{l,t} + \sum_{m=0}^{n-1} \lambda_m p_{[l-m],t}\alpha_{m,t} \tag{3.9b}$$

and

$$\frac{dp_{l,t}}{dt} = -\gamma_2(p_{l,t} - \eta\delta_{l,0}) - 2\sum_{m=0}^{n-1} \lambda_m(\overline{\alpha}_{m,t}s_{[l+m],t} + \alpha_{m,t}\overline{s}_{[m-l],t}), \tag{3.9c}$$

where $[l\pm m] = l\pm m \ (mod\ n)$.

The following Propositions were proved in (I).

Proposition 3.2. *The equation of motion (3.8) (i.e. (3.9)) has a unique global solution, corresponding to a one- parameter semigroup of transformations $x\rightarrow\tau_t x(\equiv x_t)$ of X, that maps the compacts into compacts.*

Proposition 3.3. *(i) Under the above assumptions, $\mu_t^{(N)}$ tends pointwise, as $N\rightarrow\infty$, to the characteristic function, μ_t, of a probability measure, m_t, on X, i.e.*

$$\lim_{N\rightarrow\infty}\mu_t^{(N)}(y) = \mu_t(y) = \int_X \exp i(x.y)dm_t(x), \tag{3.10}$$

convergence being uniform on the $X-$compacts.

(ii) The evolution of m_t is induced by the dynamical semigroup, τ, of $\mathcal{K}$, according to the formula

$$\int_X dm_t f = \int_X dm_0 f\circ\tau(t) \ \forall f\in\mathcal{C}_0(X), \ t\in\mathbf{R}_+. \tag{3.11}$$

(iii) In the particular case of a pure phase, where m_0 is a Dirac measure, δ_x, with support at some point x, then $m_t = \delta_{x_t}$.

Comment. This last Proposition signifies that, in the limit $N\rightarrow\infty$, the macroscopic variables $(\alpha^{(N)}, s^{(N)}, p^{(N)})$ reduce to classical ones (α, s, p). Correspondingly, the radiative field, $\phi^{(N)}$, as given by equn. (3.5), reduces to a classical one, ϕ, whose value at the site r and time t is

$$\phi_{r,t} = -i\sum_{l=0}^{n-1} \lambda_l\alpha_{t,l}\exp(2\pi irl/n). \tag{3.12}$$

Phase Structure. By Def. 3.1, Prop. 3.2 and Prop. 3.3(iii), the properties of the pure phase are governed by the classical dynamical system $\mathcal{K}$. A study of this system yields the following results (cf. (I)).

(1) For η less than a certain specified critical value, η_1 (>0), the system has a unique stable stationary state corresponding to the fixed point, $x_0(\in X)$, given by

$$\alpha_l = 0;\ s_l = 0;\ p_l = \eta\delta_{l0}\ \forall l\in[0, n-1]. \tag{3.13}$$

(2) As η increases through a certain positive value, η_1, the fixed point x_0 becomes unstable, and, by Hopf bifurcation, gives way to a periodic orbit of the form

$$\alpha_{l,t} = \alpha_l^{(0)}\exp(-i(\nu t + \theta^{(0)}))\delta_{lk};\ s_{l,t} = s_l^{(0)}\exp((-i\nu t + \theta^{(0)}))\delta_{lk};\ p_{l,t} = \eta_1\delta_{l0}, \tag{3.14}$$

where the selection of the k'th mode and the values of the constants $\alpha^{(0)},\ s^{(0)}$ and ν are determined by the parameters of the model, while the phase angle $\theta^{(0)}$ is indeterminate. Thus, in optical terms, there is a transition from normal to coherent radiation as η passes through η_1. This entails a breakdown of the gauge symmetry, represented by the transformations in which the $\sigma_{r,-}, a_l$, and correspondingly s_l, α_l, are all rephased by the same factor $\exp(i\theta)$.

(3) The model generically undergoes a further transition from coherent to chaotic radiation, as η passes through a second critical value, $\eta_2(>\eta_1)$. Specifically, the following two scenarios, both of which involve gauge symmetry breakdown, are feasible.

(a) There can be a bifurcation, at $\eta = \eta_2$, from a periodic orbit to a strange attractor, as specified in the Ruelle-Takens scheme(11). This scenario is actually realised in the single mode case(4).

(b) For large n, there could be a succession of bifurcations, corresponding to the activation of different modes, leading to a chaotic phase along the lines of Landau's theory of hydrodynamical turbulence(12).

IV. The Microscopic Dynamics in Pure Phases.

Our objective now is to obtain the microscopic dynamics of the model, in the limit $N\to\infty$. We take the local microscopic observables to be those of the matter only, since, as noted in the Comment following Prop. 3.3, the radiation field reduces to a classical macroscopic one in this limit, and therefore this field carries no local microscopic observables within the terms of this model(13).

We formulate the $C^\star-$algebra of the observables of the matter, in the limit $N\to\infty$, as the standard quasi-local one for the infinite chain of atoms of the given species, occupying the sites of the lattice $\mathbf{Z}$. Thus, in order to construct this algebra, we define L to be the family of all finite point subsets of $\mathbf{Z}$, and we assign to each $\Lambda\in L$ the $C^\star-$algebra $\mathcal{A}_\Lambda := \otimes_{r\in\Lambda}\mathcal{A}_r$, with $\mathcal{A}_r$ as defined in Section 2. For $\Lambda\subset\Lambda'$, we identify elements A of $\mathcal{A}_\Lambda$ with $A\otimes I_{\Lambda'\backslash\Lambda}$ $(\in\mathcal{A}_{\Lambda'})$, thereby rendering $\mathcal{A}_\Lambda$ isotonic w.r.t. Λ. We then define $\mathcal{A}$ to be the completion of the local algebra $\mathcal{A}_L := \bigcup_{\Lambda \in L} \mathcal{A}_\Lambda$ w.r.t. the norm it inherits from the $\mathcal{A}_\Lambda$'s. Thus, the algebras $\mathcal{A}^{(N)}$ of the material observables of the systems $\Sigma^{(N)}$ are just the local subalgebras $\mathcal{A}_{[-N,N]}$ of $\mathcal{A}$.

We denote by $\mathcal{S}(\mathcal{A})$ the set of all states on $\mathcal{A}$, and we represent the space translation group $\mathbf{Z}$ by the automorphisms V of this algebra, defined by the formula

$$V(m)\sigma_{r,u} = \sigma_{r+m,u}, \ \forall r, m \in \mathbf{Z}, \ u = \pm, z. \tag{4.1}$$

We note that it follows from our specifications of the local structure of $\mathcal{A}$ that the semigroup $T_{mat}^{(N)}$, defined in Section 2, is just the restriction to $\mathcal{A}^{(N)}$ of the global one-parameter semigroup $T_{mat} := \otimes T_r$ of CP transformations of $\mathcal{A}$. The generator of T_{mat} is therefore

$$L_{mat} = \sum_{r \in \mathbf{Z}} L_r, \tag{4.2}$$

where L_r is given by equn. (2.6). The effects of the matter-field coupling on the properties of $\Sigma^{(N)}$, in the limit $N \to \infty$, will be treated presently.

Our aim now will be to obtain the microdynamics of the matter induced by that of the systems $\Sigma^{(N)}$ in the limit $N \to \infty$, subject to the following conditions, characteristic of pure phases, on their initial states.

(P1) The restrictions of the initial states, $\psi^{(N)}$, of the systems $\Sigma^{(N)}$, to the algebras $\mathcal{A}^{(N)} \equiv \mathcal{A}_{[-N,N]}$ are the same as those of a certain primary element, ω, of $\mathcal{S}(\mathcal{A})$, i.e.,

$$\omega_{|\mathcal{A}^{(N)}} = \psi^{(N)}_{|\mathcal{A}^{(N)}}, \ \forall N \in \mathbf{N}.$$

(P2) $\omega(s_l^{(N)})$ and $\omega(p_l^{(N)})$ converge to values s_l and p_l, respectively, for each $l \in [0, n-1]$, as $N \to \infty$.

(P3) The mean and dispersion of $\alpha_l^{(N)}$, for the state $\psi^{(N)}$, converge to values α_l and zero, respectively, for each $l \in [0, n-1]$, as $N \to \infty$.

The following lemma serves to confirm that the assumptions (P1- 3), together with equn. (3.7), imply that the conditions of Prop. 3.3(iii) prevail.

Lemma 4.1. *Under the above assumptions, the macroscopic probability measure, m_t, is simply δ_{x_t}, where $x_t = \tau(t)x$ and x is the initial phase point corresponding to (α, s, p) of conditions (P2,3).*

Proof. By equn. (3.7) and the Schwartz inequality,

$$|\mu_0^{(N)}(y) - \exp(iy.x)| \equiv |\psi^{(N)}\big(\exp(iy.(x^{(N)} - x)) - I\big)| \leq 2\big[\psi^{(N)}\big(\sin^2(\frac{y.(x^{(N)} - x)}{2})\big)\big]^{1/2}$$

$$\leq \big[\psi^{(N)}\big(y.(x^{(N)} - x)\big)^2\big]^{1/2},$$

and, in view of (P2,3), this tends to zero as $N\to\infty$. Hence, by Prop. 3.3(i), $\mu_0(y) = \exp(iy.x)$, i.e., $m_0 = \delta_x$; and consequently, by Prop. 3.3(iii), $m_t = \delta_{x_t}$, as required.

The following Proposition signifies that, in the limit $N\to\infty$, the microscopic dynamics is precisely that of the matter, under the action of the classical macroscopic radiation field, ϕ.

Proposition 4.2. *(i) Assuming the conditions (P1-3), the microscopic dynamics of the model corresponds, in the limit $N\to\infty$, to a two-parameter family,*

$$\{\mathcal{T}(s,t|\phi)|0\leq s\leq t; \mathcal{T}(s,u|\phi)\mathcal{T}_\mu(u,t|\phi) = \mathcal{T}(s,t|\phi)\},$$

of CP contractions of $\mathcal{A}$, in that

$$\lim_{N\to\infty}\psi^{(N)}\big(T^{(N)}(t)A\big) = \omega\big(\mathcal{T}(0,t|\phi)A\big)\ \forall A\in\mathcal{A}_L, t\in\mathbf{R}_+. \tag{4.3}$$

Furthermore, the generator of $\mathcal{T}$ is

$$\mathcal{L}(t|\phi) := \frac{\partial}{\partial t}\mathcal{T}(s,t|\phi)_{|s=t} = L_{mat} + \sum\nolimits_{r\in\mathbf{Z}}[\phi_{r,t}\sigma_{r,+} - h.c., .]. \tag{4.4}$$

Thus,

$$\mathcal{T}(s,t|\phi) = \mathbf{T}_a\exp\Big(\int_s^t du\mathcal{L}(u|\phi)\Big), \tag{4.5}$$

where $\mathbf{T}_a$ is the antichronological operator. We define

$$\omega_t := \omega\circ\mathcal{T}(0,t). \tag{4.6}$$

(ii) $\mathcal{T}$ factorises into a product of single site contributions according to the formula

$$\mathcal{T}(s,t|\phi) = \otimes_{r\in\mathbf{Z}}\mathcal{T}_r(s,t|\phi_r), \tag{4.7}$$

where

$$\mathcal{T}_r(s,t|\phi_r) = \mathbf{T}_a\exp\Big(\int_s^t du\mathcal{L}_r(u|\phi)\Big), \tag{4.8}$$

and

$$\mathcal{L}_r(t|\phi) = L_r + [\phi_{r,t}\sigma_{r,+} - h.c., .]. \tag{4.9}$$

Comment. This result is similar to that obtained by Bagarello and Morchio[9] for a class of conservative systems, in that the microscopic evolution is governed by a classical field, formed by the time-dependent macroscopic observables.

The proof of the Proposition requires the following lemmas.

Lemma 4.3. *Defining*

$$\tilde{\alpha}_{l,t}^{(N)} := \alpha_l^{(N)} - \alpha_{l,t}, \tag{4.10}$$

$$\lim_{N\to\infty}\psi_t^{(N)}(\tilde{\alpha}_{l,t}^{(N)\star}\tilde{\alpha}_{l,t}^{(N)}) = \lim_{N\to\infty}\psi_t^{(N)}(\tilde{\alpha}_{l,t}^{(N)}\tilde{\alpha}_{l,t}^{(N)\star}) = 0, \tag{4.11}$$

the convergence being uniform on the $t-$compacts.

Lemma 4.4. *Defining*

$$\tilde{s}_{l,t}^{(N)} := s_l^{(N)} - s_{l,t}, \tag{4.12}$$

$$\lim_{N\to\infty}\psi_t^{(N)}(\tilde{s}_{l,t}^{(N)}\tilde{s}_{l,t}^{(N)\star}) = 0, \tag{4.13}$$

the convergence being uniform on the t-compacts.

Proof of Prop. 4.2, assuming Lemma 4.3. (i) By equns. (2.5), (2.6), (2.9), (2.10), (2.12) and (2.13),

$$\psi_t^{(N)}(A) - \psi_0^{(N)}(A) = \int_0^t ds\psi_s^{(N)}(L_{mat}A) +$$

$$\int_0^t ds\sum_{l=0}^{n-1}\psi_s^{(N)}(\alpha_l^{(N)}\delta_l^+ A + \alpha_l^{(N)\star}\delta_l^- A)\ \forall A\in\mathcal{A}_L,\ t\in\mathbf{R}_+, \tag{4.14}$$

where

$$\delta_l^{\pm} = \pm\lambda_l\sum_{r\in\mathbf{Z}}\exp(\pm 2\pi irl/n)[\sigma_{r,\pm},.]. \tag{4.15}$$

We note now that $\mathcal{A}_L$ is stable under $\delta_l^{\pm}$, and that, by equns. (2.14), (2.15) and (3.3), $\psi_t^{(N)}(\alpha_l^{(N)\star}\alpha_l^{(N)})$ is bounded, uniformly w.r.t. N and t. It therefore follows from the Schwartz inequality that, for each $A\in\mathcal{A}_L$, the norm of the r.h.s. of equn. (4.14) is likewise uniformly bounded. Hence, by the Arzela-Ascoli theorem, $\psi_t^{(N)}(A)$ converges to a limit, uniformly over the $t-$compacts, as $N\to\infty$ over some subsequence of $\mathbf{N}$. Therefore, as $\mathcal{A}$ is norm-separable, it follows from the standard diagonalisation procedure that there is a state ω_t' on $\mathcal{A}$, such that, over some subsequence of the integers,

$$\lim_{N\to\infty}\psi_t^{(N)}(A) = \omega_t'(A)\ \forall A\in\mathcal{A}_L, t\in\mathbf{R}_+. \tag{4.16}$$

Further, by equns. (2.7), (4.2), (4.4), (4.8) and (4.10), we may re-express equn. (4.14) in the form

$$\psi_t^{(N)}(A) - \psi_0^{(N)}(A) = \int_0^t ds\psi_s^{(N)}\big(\mathcal{L}(s|\phi)A\big) + \int_0^t ds\sum_{l=0}^{n-1}\psi_s^{(N)}(\tilde{\alpha}_{l,s}^{(N)}\delta_l^+ A + \tilde{\alpha}_{l,s}^{(N)\star}\delta_l^- A). \tag{4.17}$$

By Schwartz's inequality and Lemma 4.3, the last integral in this equation tends to zero, uniformly on the compacts, as $N\to\infty$. Hence, by equn. (4.16), we see that equn. (4.17) reduces to the following form in this limit.

$$\omega_t'(A) - \omega_0'(A) = \int_0^t ds\omega_s'\big(\mathcal{L}(s|\phi)A\big),$$

and hence
$$\frac{d}{dt}\omega_t'(A) = \omega_t'\big(\mathcal{L}(t|\phi)A\big). \tag{4.18}$$
Consequently, since equn. (4.5) implies that
$$\frac{\partial}{\partial s}\mathcal{T}(s,t|\phi) = -\mathcal{L}(s|\phi)\mathcal{T}(s,t|\phi),$$
it follows from equn. (4.18) that
$$\frac{\partial}{\partial s}\omega_s'(\mathcal{T}(s,t|\phi)A) = 0 \ \forall 0\leq s\leq t, \ A\in\mathcal{A}_L.$$
Hence, $\omega_s'\big(\mathcal{T}(s,t|\phi)A\big)$ is independent of s, and consequently, by assumption (P1) and equn. (4.16),
$$\omega_t'(A) = \omega_0'\big(\mathcal{T}(0,t|\phi)A\big)\equiv\omega\big(\mathcal{T}(0,t|\phi)A\big),$$
i.e., by equn. (4.6),
$$\omega_t' = \omega_t = \omega\circ\mathcal{T}(0,t|\phi). \tag{4.19}$$
Equations (4.3) and (4.4) now follow, at least for convergence over a subsequence of integers, from equns. (4.16), (4.18) and (4.19). The removal of the restriction to a subsequence is achieved by noting that, in view of the uniqueness of the evolution $\omega\to\omega_t$, specified by equn. (4.6), the compactness argument underlying the derivation of equn. (4.19) may be re-employed to show that all subsequential limits of $\{\psi^{(N)}(T^{(N)}(t)A)|N\in\mathbf{N}\}$ are equal to $\omega_t(A)$.

(ii) The factorisation property is an immediate consequence of the fact that, by equns. (4.4) and (4.9),
$$\mathcal{L}(t|\phi) = \sum\nolimits_{r\in\mathbf{Z}}\mathcal{L}_r(t|\phi_r).$$

Proof of Lemma 4.3, assuming Lemma 4.4. Since, by equns. (2.7), (3.3) and (4.10), $[\tilde{\alpha}_{l,t}^{(N)},\tilde{\alpha}_{l,t}^{(N)\star}] = N^{-1}$, it suffices to establish that the l.h.s. of equn. (4.11) tends to zero, uniformly on the $t-$compacts, as $N\to\infty$.

To this end, we define
$$J_{l,t}^{(N)} := \psi_t^{(N)}(\tilde{\alpha}_{l,t}^{(N)\star}\tilde{\alpha}_{l,t}^{(N)}), \tag{4.20}$$
and then infer from this equation and equns. (2.12) and (4.10) that
$$\frac{dJ_{l,t}^{(N)}}{dt} = \psi_t^{(N)}\big((L^{(N)}+\frac{\partial}{\partial t})\tilde{\alpha}_{l,t}^{(N)\star}\tilde{\alpha}_{l,t}^{(N)}\big).$$
It follows from this equation, together with equns. (2.13), (3.1)-(3.3), (4.10), (4.12) and (4.20) that
$$\frac{dJ_{l,t}^{(N)}}{dt} = -2\kappa_l J_{l,t}^{(N)} + 2\lambda_l Re\big(\psi_t^{(N)}(\tilde{\alpha}_{l,t}^{(N)\star}\tilde{s}_{l,t}^{(N)})\big), \tag{4.21}$$

and consequently,

$$J_{l,t}^{(N)}=J_{l,0}^{(N)}\exp(-2\kappa_lt)+2\lambda_lRe\int_0^t dt'\exp(-2\kappa_l(t-t'))\psi_{t'}^{(N)}(\tilde{\alpha}_{l,t'}^{(N)\star}\tilde{s}_{l,t'}^{(N)}). \tag{4.22}$$

Therefore, defining

$$K_{l,t}^{(N)}:=\psi_t^{(N)}(\tilde{s}_{l,t}^{(N)}\tilde{s}_{l,t}^{(N)\star}), \tag{4.23}$$

it follows from the Schwartz inequality and equn. (4.22) that

$$J_{l,t}^{(N)}\leq J_{l,0}^{(N)}\exp(-2\kappa_lt)+\int_0^t dt'|\lambda_l|\exp(-2\kappa_l(t-t'))[K_{l,t'}^{(N)}J_{l,t'}^{(N)}]^{1/2}. \tag{4.24}$$

For any fixed $T(>0)$, we now define

$$C_{l,T}^{(N)}:=sup_{t\in[0,T]}J_{l,t}^{(N)} \tag{4.25}$$

and

$$D_{l,T}^{(N)}:=sup_{t\in[0,T]}K_{l,t}^{(N)}, \tag{4.26}$$

noting that these are both finite, in view of the equn. (2.15) and the uniform boundedness of $s^{(N)}$. Hence, by equns. (4.24)-(4.26),

$$J_{l,t}^{(N)}\leq J_{l,0}^{(N)}\exp(-2\kappa_lt)+2|\lambda_l|\int_0^t dt'\exp(-2\kappa_l(t-t'))[C_{l,T}^{(N)}D_{l,T}^{(N)}]^{1/2}\ \forall\ t\in[0,T]. \tag{4.27}$$

¿From this formula it follows easily that

$$J_{l,t}^{(N)}\leq J_{l,0}^{(N)}+\frac{|\lambda_l|}{\kappa_l}[C_{l,T}^{(N)}D_{l,T}^{(N)}]^{1/2},$$

and hence that

$$C_{l,T}^{(N)}\leq J_{l,0}^{(N)}+\frac{|\lambda_l|}{\kappa_l}[C_{l,T}^{(N)}D_{l,T}^{(N)}]^{1/2}, \tag{4.28}$$

which in turn implies that

$$C_{l,T}^{(N)}\leq\frac{1}{4}\Big[\frac{|\lambda_l|}{\kappa_l}(D_{l,T}^{(N)})^{1/2}+(\frac{\lambda_l^2}{\kappa_l^2}D_{l,T}^{(N)}+4J_{l,0}^{(N)})^{1/2})\Big]^2. \tag{4.29}$$

Further, by definition (4.20) and assumption (P3), $J_{l,0}^{(N)}$ converges to zero as $N\to\infty$; and by Lemma 4.4, the same is true of $D_{l,T}^{(N)}$. Hence, by equn. (4.29), $C_{l,T}^{(N)}$ tends to zero as $N\to\infty$. By equns. (4.20) and (4.25), this is just the required result.

Proof of Lemma 4.4. Let $q_l^{(N)}$ denote the Hermitian or anti-Hermitian part of $s_l^{(N)}$; and, correspondingly, let $q_{l,t}$ denote the real or imaginary part of $s_{l,t}$. Then, defining

$$c_{l,t}^{(N)}(v):=\psi_t^{(N)}(\exp(ivq_l^{(N)}))\ \forall v\in\mathbf{R}, \tag{4.30}$$

we see from equn. (3.7) that $c_{l,t}^{(N)}$ is just the restriction of the characteristic function $\mu_t^{(N)}$ to one matter mode. Thus, by Prop. 3.3(iii) and Lemma 4.1,

$$\lim_{N\to\infty} c_{l,t}^{(N)}(v) = \exp(ivq_{t,l}), \ \forall v\in\mathbf{R}, t\in\mathbf{R}_+, \tag{4.31}$$

convergence being uniform on the compacts. On integrating this equation against the Fourier transform, $\hat{f}(v)$, of any element, f, of the Schwartz space $\mathcal{D}(\mathbf{R})$, we see that

$$\lim_{N\to\infty} \psi_t^{(N)}(f(q_l^{(N)})) = f(q_{l,t}) \ \forall f\in\mathcal{D}(\mathbf{R}), t\in\mathbf{R}_+, \tag{4.32}$$

convergence being uniform on the $t-$compacts for given f. We shall now use this formula to prove the required result, in the form of the equation

$$\lim_{N\to\infty} \psi_t^{(N)}((q_l^{(N)})^m) = (q_{l,t})^m, \ for\ m = 1, 2, \tag{4.33}$$

convergence being uniform on the compacts.

To establish this formula, we first note that, by equns. (2.2) and (3.1), together with our definition of $q_l^{(N)}$, the norm of this operator is 1/2. Thus, if f is any $\mathcal{D}(\mathbf{R})-$class function with support disjoint from $[-1/2, 1/2]$, then $f(q_l^{(N)}) = 0$, and hence, by equn. (4.32), $f(q_{l,t}) = 0$. This implies that $q_{l,t}$ must lie in the interval $[-1/2, 1/2]$. We now re-employ equn. (4.32), choosing f to be of the form $f(q) = q^m h(q)$, where $m = 1$ or 2, and h is an element of $\mathcal{D}(\mathbf{R})$, whose value is unity over the interval $[-1/2, 1/2]$. This choice yields precisely the required result (4.33).

V. Asymptotic form of the Dynamics and States of Pure Phases

We shall now show that the microscopic dynamics of the model simplifies greatly when transient effects, which decay in times of the order of the lifetime of the excited states of its atoms, are discarded.

We start by noting that, by Prop. 4.2, the microscopic dynamics of the model is completely determined by the action of the CP contractions, $\mathcal{T}_r$, on the single site algebras $\mathcal{A}_r$. Thus, defining

$$\sigma_{r,u}(t) := \mathcal{T}_r(0,t)\sigma_{r,u}, \ for\ u = +, -, x, y, z, \tag{5.1}$$

we see from equations (2.1), (4.5) and (4.7)-(4.9) that

$$\frac{d\sigma_{r,-}(t)}{dt} + (i\epsilon + \gamma_1)\sigma_{r,-}(t) - i\phi_{r,t}\sigma_{r,z}(t) = 0 \tag{5.2}$$

and

$$\frac{d\sigma_{r,z}(t)}{dt} + \gamma_2\sigma_{r,z}(t) - 2i(\overline{\phi}_{r,t}\sigma_{r,-}(t) - h.c.) = \gamma_2\eta I. \tag{5.3}$$

Thus, the spin vector $\sigma_r(t) = (\sigma_{r,x}(t), \sigma_{r,y}(t), \sigma_{r,z}(t))$ evolves according to a linear inhomogeneous equation

$$\frac{d\sigma_r(t)}{dt} = b_r(t)\sigma_r(t) + cI, \tag{5.4}$$

where $b_r(t)$ is a linear transformation of $\mathbf{C}^3$, whose explicit form is obtained from the equivalence of equn. (5.4) to the pair of equations (5.2) and (5.3), and

$$c = (0, 0, \gamma_2\eta). \tag{5.5}$$

The solution of equn. (5.4) may be expressed in the form

$$\sigma_r(t) = g_r(t,0)\sigma_r(0) + \int_0^t dt' g_r(t,t')cI, \tag{5.6}$$

where g_r is the Green function for b_r, determined by the formula

$$\frac{\partial}{\partial t} g_r(t,t') = b_r(t)g_r(t,t') \ \forall t \geq t' \geq 0; \ g_r(t,t) = I_3, \tag{5.7}$$

and I_3 is the identity operator in $\mathbf{C}^3$.

Lemma 5.1. *For fixed, t', $g(t,t')$ decays to zero, with increasing t, at least as fast as* $\exp(-\gamma t)$, *where* $\gamma = min(\gamma_1, \gamma_2)$.

Proof. Let ξ_r be an arbitrary element of $\mathbf{R}^3$, and, for fixed $t'(\in\mathbf{R}_+)$ and any $t > t'$, let $\xi_r(t) = (\xi_{r,x}(t), \xi_{r,y}(t), \xi_{r,z}(t)) := g_r(t,t')\xi_r$. Then, by equn. (5.7), $\xi_r(t)$ satisfies the homogeneous linear equation obtained by replacing σ_r by ξ_r in equn. (5.4) and discarding the term cI on the r.h.s. Hence, defining $\xi_{r,\pm} = 1/2(\xi_{r,x} \pm i\xi_{r,y})$, it follows from the equivalence between equn. (5.4) and the pair of equations (5.2) and (5.3) that the equation of motion for $\xi_r(t)$ is also given by replacing σ by ξ in the latter equations and discarding the r.h.s. of (5.3). Thus,

$$\frac{d}{dt}\xi_{r,-}(t) + (\gamma_1 - i\epsilon)\xi_{r,-}(t) - i\phi_{r,t}\xi_{r,z}(t) = 0$$

and

$$\frac{d}{dt}\xi_{r,z}(t) + \gamma_2\xi_{r,z}(t) - 2i(\overline{\phi}_{r,t}\xi_{r,-}(t) - c.c.) = 0.$$

Hence,

$$\frac{d}{dt}|\xi_r(t)|^2 \equiv \frac{d}{dt}(4|\xi_{r,-}(t)|^2 + \xi_{r,z}(t)^2) = -8\gamma_1|\xi_{r,-}(t)|^2 - 2\gamma_2\xi_{r,z}(t)^2 =$$

$$-2[\gamma_1\xi_{r,x}(t)^2 + \gamma_1\xi_{r,y}(t)^2 + \gamma_2\xi_{r,z}(t)^2] \leq -2min(\gamma_1,\gamma_2)|\xi_r(t)|^2.$$

Consequently, defining $\gamma := min(\gamma_1, \gamma_2)$,

$$\frac{d}{dt}(|\xi_r(t)|^2\exp(2\gamma t)) \leq 0,$$

which implies that $|\xi_r(t)|$, and hence $g_r(t,t')$, decay to zero at least as fast as $\exp(-\gamma t)$, with increasing t.

Comment. This lemma signifies that the first term on the r.h.s. of equn. (5.6) is merely a transient, which decays in a time, γ^{-1}, of the order of the lifetime of an atomic excited state. Consequently, by (5.5), the asymptotic form for $\sigma_r(t)$, obtained by discarding the transients, is

$$\sigma_r(t) = \theta_r(t)I, \tag{5.8}$$

where

$$\theta_r(t) = \int_0^t dt' g_r(t,t')(0,0,\gamma_2\eta). \tag{5.9}$$

Hence, since the state ω_t is given by its action on the finite monomials in the components of the σ_r's, the following Proposition is a simple consequence of these last two equations, together with Prop. 4.1.

Proposition 5.2. *The asymptotic form of the time- dependent state ω_t carries no correlations between the observables at the different sites and is therefore of the form*

$$\omega_t = \otimes_{r\in\mathbf{Z}}\omega_{r,t}, \tag{5.10}$$

where the atomic state $\omega_{r,t}$ is given by

$$\omega_{r,t}(\sigma_r) = \theta_r(t). \tag{5.11}$$

Comment. It follows immediately from this Proposition that the asymptotic state of the model is completely determined by the form of θ_r. We shall denote the components of θ_r analogously with those of σ_r : thus $\theta_r = (\theta_{r,x}, \theta_{r,y}, \theta_{r,z})$ and $\theta_{r,\pm} = \frac{1}{2}(\theta_{r,x} \pm \theta_{r,y})$. The following Proposition provides an explicit formula for θ_r in terms of the asymptotic forms of the macroscopic variables s and p.

Proposition 5.3. *The function θ_r is given by the equations*

$$\theta_{r,-}(t) = \sum_{l=0}^{n-1} s_{l,t}\exp(2\pi irl/n) \tag{5.12}$$

and

$$\theta_{r,z}(t) = \sum_{l=0}^{n-1} p_{l,t}\exp(2\pi irl/n). \tag{5.13}$$

Proof. By equns. (5.2), (5.3), (5.7) and (5.9), the equations of motion for θ_r are formally the same as those for σ_r, i.e.,

$$\frac{d\theta_{r,-}(t)}{dt} + (i\epsilon + \gamma_1)\theta_{r,-}(t) - i\phi_{r,t}\theta_{r,z}(t) = 0 \tag{5.14}$$

and

$$\frac{d\theta_{r,z}(t)}{dt} + \gamma_2\theta_{r,z}(t) - 2i(\overline{\phi}_{r,t}\theta_{r,-}(t) - c.c.) = \gamma_2\eta. \tag{5.15}$$

Further, defining

$$\hat{s}_r(t) = \sum\nolimits_{l=0}^{n-1} s_{l,t}\exp(2\pi irl/n) \tag{5.16}$$

and

$$\hat{p}_r(t) = \sum\nolimits_{l=0}^{n-1} p_{l,t}\exp(2\pi irl/n), \tag{5.17}$$

we see, by elementary Fourier analysis, that equns. (3.9b) and (3.9c) are equivalent to

$$\frac{d\hat{s}_r(t)}{dt} + (i\epsilon + \gamma_1)\hat{s}_r(t) - i\phi_{r,t}\hat{p}_r(t) = 0 \tag{5.18}$$

and

$$\frac{d\hat{p}_r(t)}{dt} + \gamma_2\hat{p}_r(t) - 2i(\overline{\phi}_{r,t}\hat{s}_r(t) - c.c.) = \gamma_2\eta, \tag{5.19}$$

respectively. Hence, the equations of motion (5.14) and (5.15) for $\theta_{r,-}$ and $\theta_{r,z}$ are equivalent to those given by (5.18) and (5.19) for $\hat{s}_r$ and $\hat{p}_r$. Consequently, in the asymptotic regime, where the initial conditions are irrelevant, $\theta_{r,-} = \hat{s}_r$ and $\theta_{r,z} = \hat{p}_r$. By equations (5.16) and (5.17), this is the required result.

The following Corollary is an immediate consequence of Props. 5.2 and 5.3.

Corollary 5.4. *The asymptotic state ω_t is spatially periodic, with periodicity n, i.e.,*

$$\omega_t = \omega_t \circ V(n), \tag{5.20}$$

where V is the group of spatial translational automorphisms of $\mathcal{A}$ defined by equn. (4.1).

Explicit Asymptotic Form of ω_t. Equations (5.10)-(5.13) serve to express the time-dependent microstate ω_t in terms of the macroscopic variables. We shall now present its explicit form for both the normal and coherent phases. We have no means of doing the same thing for the chaotic phase, since we have no explicit solution for the macroscopic dynamics there.

1. The Normal Phase. Here, by equns. (3.13) and (5.10)- (5.13), ω_t takes the stationary asymptotic value ω_∞, defined by the equations

$$\omega_\infty = \otimes_{r\in\mathbf{Z}}\omega_{\infty,r} \tag{5.21a}$$

and

$$\omega_{\infty,r}(\sigma_{r,\pm}) = 0;\ \omega_{\infty,r}(\sigma_{r,z}) = \eta. \tag{5.21b}$$

2. The Coherent Phase. In this phase, it follows from equns. (3.14) and (5.10)-(5.13) that the asymptotic form, ω_t^{coh}, of ω_t is given by the formula

$$\omega_t^{coh} = \otimes_{r\in\mathbf{Z}}\omega_{r,t}^{coh}, \tag{5.22a}$$

where

$$\omega_{r,t}^{coh}(\sigma_{r,-}) = s_l^{(0)}\exp\big(i(\frac{2\pi ikr}{n} - \nu t - \theta_0)\big);\ \omega_{r,t}^{coh}(\sigma_{r,z}) = \eta. \tag{5.22b}$$

Comment. By contrast with the thermal equilibrium states of pure phases, the stable states ω_t^{coh} are time-dependent. The same is true for the pure chaotic phases, since the macrosopic variables s_t and p_t are time-dependent there too[14].

VI. A Non-Equilibrium Maximum Entropy Principle

We shall now establish that, as a further consequence of the results of Section 5, ω_t maximises the global entropy density of the matter, as defined on the states with spatial periodicity n, subject to the condition that the limiting value, as $N\to\infty$, of the expectation values of the macroscopic variables $s^{(N)}, p^{(N)}$ are s_t, p_t, respectively.

For this purpose, we start by defining $\mathcal{S}_n$ to be the set of spatially periodic states, ψ, on $\mathcal{A}$, with period n, and $\mathcal{S}_{n,x_t}$ to be the subset of these states for which

$$\lim_{N\to\infty}\psi(s_l^{(N)}) = s_{l,t} \ and \ \lim_{N\to\infty}\psi(p_l^{(N)}) = p_{l,t}, \ \forall l\in[0,n], t\in\mathbf{R}_+.$$

Equivalently, by equns. (3.1) and (3.2), $\mathcal{S}_{n,x_t}$ consists of the states, ψ, of spatial periodicity n, which satisfy the conditions

$$n^{-1}\sum_{r=0}^{n-1}\psi(\sigma_{r,-})\exp(-2\pi ilr/n) = s_{l,t} \tag{6.1}$$

and

$$n^{-1}\sum_{r=0}^{n-1}\psi(\sigma_{r,z})\exp(-2\pi ilr/n) = p_{l,t}, \tag{6.2}$$

for $l = 0, . ., n-1$ and $t\in\mathbf{R}_+$.

We formulate the global entropy density functional, s on $\mathcal{S}_n$ in the following standard way[15]. For $\psi\in\mathcal{S}_n$ and $\Lambda\in L$, we define ρ_Λ^ψ to be the density matrix in $\mathcal{H}(\Lambda)$ representing the restriction of ψ to $\mathcal{A}(\Lambda)$, and $S_\Lambda(\psi)$ to be the corresponding local entropy, i.e.,

$$S_\Lambda(\psi) = -kTr(\rho_\Lambda^\psi ln\rho_\Lambda^\psi), \tag{6.3}$$

k being Boltzmann's constant. We define the global entropy density functional, s, on $\mathcal{S}_n$ by the formula

$$s(\psi) = \lim_{\Lambda\uparrow\mathbf{Z}}\frac{S_\Lambda(\psi)}{|\Lambda|}, \tag{6.4}$$

the convergence being guaranteed by the subadditivity of entropy[16] and the periodicity of ψ.

Proposition 6.1. *The asymptotic time-dependent state ω_t maximises the functional s, as restricted to $\mathcal{S}_{n,x_t}$.*

Proof. We first note that, by Cor. 5.4, $\omega_t\in\mathcal{S}_n$, and that, by equations (5.10)-(5.13), it satisfies the conditions (6.1) and (6.2). Hence, $\omega_t\in\mathcal{S}_{n,x_t}$.

Next, we observe that, if ψ is an arbitrary element of S_{n,x_t}, then it follows by elementary Fourier analysis from equns. (5.12), (5.13), (6.1) and (6.2) that $\psi(\sigma_r) = \theta_r(t)$, for $r\in[0, n-1]$, and consequently, by the periodicity of ψ, for all $r\in\mathbf{Z}$. Therefore, by equn. (5.11), the restrictions of ψ and ω_t to each single site algebra $\mathcal{A}_r$ are identical. In other words, *all the elements of* $\mathcal{S}_{n,x_t}$ *coincide on each of the single site algebras.*

Further, by the subadditivity of entropy,

$$S_\Lambda(\psi)\leq\sum\nolimits_{r\in\Lambda}S_{\{r\}}(\psi),\ \forall\psi\in\mathcal{S}_{n,x_t},$$

while, by equns. (5.10) and (6.3),

$$S_\Lambda(\omega_t) = \sum\nolimits_{r\in\Lambda}S_{\{r\}}(\omega_t).$$

Consequently, as ψ and ω_t coincide on the single site algebras,

$$S_\Lambda(\psi)\leq S_\Lambda(\omega_t),$$

and therefore, by equn. (6.4),

$$s(\psi)\leq s(\omega_t)\ \forall\psi\in\mathcal{S}_{n,x_t},$$

which proves the Proposition.

VII. Concluding Remarks.

We have provided a microscopic picture of the multi-mode laser model, that is complementary to the macroscopic one of (I). Our principal new results are the following.

(1) The microscopic dynamics is governed by the classical, macroscopic field, ϕ, which in turn is generated by the collective action of the atoms on the radiation (Prop. 4.1).

(2) This microdynamics serves to drive the system into states in which the observables of different atoms are uncorrelated (Prop. 5.2). These states have particularly simple forms in both the normal and coherent phases, being stationary and space-translationally invariant in the former and periodic w.r.t. both space and time, in the latter.

(3) In each phase, the eventual time-dependent microstate of the system maximises the specific entropy of the model, subject to the constraints imposed by the instantaneous values of the macroscopic variables. To the best of our knowledge, this is the first example where such a variational principle has been established in a regime far from thermal equilibrium.

These results concern (1) an interplay between microphysics and macrophysics, (2) the decorrelation of certain micro-observables in the course of time, and (3) a maximum entropy principle, corresponding to a conditional thermodynamic stability far from thermal equilibrium. Although the model treated here is but a caricature of a laser, one might envisage that counterparts of these results might have at least some measure of validity in the physics of real dissipative systems. For example, one might expect a local version of the maximum entropy principle (3) to prevail in hydrodynamics, in such a way as to impose local equilibrium conditions on the states of fluids.

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(5) Apart from the difference in the their mathematical settings, this model differs from that of HL in the following physical respect. The atoms of HL are pairs of Fermi oscillators, and can thus accommodate $0, 1$ or 2 electrons each; whereas those of (I) correspond to two-level atoms, each with precisely one electron.

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(13) In the present model, this is due to the fact that the radiation field is truncated to n modes. Note that here, as in the non-truncated model of the radiation, the Fourier

coefficients of $\phi^{(N)}$ are $N-$independent multiples of the $\alpha^{(N)}$'s, rather than the a's; and consequently the latter do not appear in the local field algebra when one passes to the limit $N \to \infty$.

(14) If s_t and p_t were constant, then it would follow from equn. (3.9a) that α_t, and hence x_t, would tend to a constant as t tended to infinity, and consequently that the macroscopic dynamics would not be chaotic.

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