On the continuous limit of integrable lattices III. Kupershmidt systems and $sl(N+1)$ KdV theories

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Abstract

We discuss the connection between the zero-spacing limit of the N fields Kupershmidt lattice and the KdV type theory corresponding to the Lie algebra $sl(N+1)$. The case $N=2$ is worked out in most detail, recovering from the limit process the Boussinesq theory with its infinitely many commuting vector fields, their Lax pairs and Hamiltonian formulations. The "recombination method" proposed here to derive the Boussinesq hierarchy from the limit of the $N=2$ Kupershmidt system works in principle for arbitrary N.

1 Introduction

In this paper we conclude an analysis of the relations between certain integrable lattices and the KdV type theories.
In [2], we have considered a class of lattices, referred to as *Volterra systems*; in the "interpolated version", which is the starting point for the continuous limit, the phase space of the N-fields Volterra (V_N) system is a set of N-tuples $\mathbf{a} = (a_1, ..., a_N)$, where each element a_p is a real smooth function of a continuous variable x. The Lax operator is

$$L^{\epsilon}(\mathbf{a}) = \frac{1}{2}\sum_{p=1}^{N}(a_p\Delta_{(2p-1)\epsilon} + a_{p[(-2p+1)\epsilon]}\Delta_{(-2p+1)\epsilon}) \ . \tag{1.1}$$

Here, for each displacement η and each function f of x, we indicate with $f_{[\eta]}$ the shifted function $x \mapsto f(x+\eta)$ (so, the notation $a_{p[\eta]}$ stands for the function $x \mapsto a_p(x+\eta)$). Also, we denote with Δ_η the shift operator sending any function f into $f_{[\eta]}$. All the shifts η considered in Eq.(1.1) are integer multiples of a fundamental shift ϵ, representing the lattice spacing.
The limit $\epsilon \mapsto 0$, in which $(\Delta_\epsilon - 1)/\epsilon \mapsto \partial_x$, has been analyzed in [2]; in this way, a strict relation has been pointed out between the V_N system and a KdV type theory in N fields. The limiting KdV theory in the fields $\mathbf{u} = (u_1, ..., u_N)$ is associated (in the Drinfeld-Sokolov sense [5]) to the Lie algebra $sp(N)$, and rests on the Lax operator

$$L^{sp(N)}(\mathbf{u}) := \partial^{2N} + \frac{1}{2}\sum_{l=1}^{N}(u_l\partial^{2N-2l} + \partial^{2N-2l}u_l) \ . \tag{1.2}$$

In the case $N = 2$, the main structures describing the integrability of the $sp(2)$ KdV (Lax pairs, Hamiltonian structure, hierarchies of conserved functionals and commuting vector fields) have been fully reconstructed from the $\epsilon \mapsto 0$ limit of homologous V_2 objects. The limit process for the case $N = 1$ had also been analyzed in [1]; the V_1 theory is well known in the literature on integrable lattices as the Kac-Moerbeke system [6], and its zero-spacing limit is the ordinary KdV theory.
In the present paper we analyze along the same lines another class of integrable lattices, introduced in [4], to which we refer as the N-fields Kupershmidt (or K_N) systems ($N = 1, 2, 3, ...$). Working directly in the interpolated version, we can describe the K_N phase space as a set of N-tuples

$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)$; each α_k is a real smooth function of the variable x, ranging over the torus $\mathbf{T} := \mathbf{R}/\mathbf{Z}$. The Lax operator, depending on the lattice spacing ϵ, is

$$L^\epsilon(\boldsymbol{\alpha}) = \Delta_\epsilon + \sum_{k=1}^{N} \alpha_{2k-1} \Delta_{(-2k+1)\epsilon} \ . \tag{1.3}$$

This operator gives rise to a hierarchy of infinitely many commuting vector fields, which are Hamiltonian w.r.t. a quadratic Poisson tensor (this could be interpreted as the reduction of a conveniently defined quadratic R-matrix Poisson tensor on the algebra of differential-difference operators).

If $N = 1$, the corresponding theory is equivalent to the one based on the symmetric operator (1.1); in our terminology, the K_1 and V_1 systems are two isomorphic realizations of the Kac-Moerbeke theory. On the contrary, the V_N and K_N systems are essentially different for $N > 1$.

Such a difference can also be traced back considering the $\epsilon \mapsto 0$ limit. It was proved in [4] that the limit of the K_N Lax operator under a suitable field rescaling $\boldsymbol{\alpha} \mapsto \mathbf{u} = (u_1, ..., u_N)$ is

$$L(\mathbf{u}) = \partial_x{}^{N+1} + \sum_{k=1}^{N} u_k \partial_x{}^{N-k} \ . \tag{1.4}$$

This is the Lax operator of the KdV type (or Gelfand-Dickey) theory corresponding to the Lie algebra $sl(N+1)$, namely the ordinary KdV for $N = 1$, the Boussinesq theory for $N = 2$, etc. .

In spite of these results on the limit of the K_N Lax operator (1.3), it is still worthy to discuss the $\epsilon \mapsto 0$ procedure, so as so reconstruct the full structure characterizing the integrability of the limiting KdV type theory. In particular, it is of interest to recover its infinitely many commuting vector fields (a topic not discussed in [4]) with the associated Lax pairs and Hamiltonian formulations.

In this paper we illustrate a method to carry on this program. The proposed approach will be worked in detail for the Boussinesq case $N = 2$; the case $N = 1$ will be employed to introduce the main technique, and the case $N = 3$ will also be considered in relation with the Poisson tensor. A distinguished feature of these constructions is the necessity to recombine linearly the vector fields of the hierarchy, the Hamiltonian functions and the companions of the Lax operator in order to recover the corresponding KdV type structures for

$\epsilon \mapsto 0$; the constant coefficients appearing in these linear combinations are determined algorithmically.
The algorithm employed here is different from the recursive one proposed in [3] for recombining the vector fields in the $N = 1$ case (and independently developed in [1], starting from the V_1 Lax operator). This iterative method is based on the $\epsilon \mapsto 0$ limit of the biHamiltonian recursion relations; its extension from the K_1 to the general K_N system is not possible, because only one (local) Hamiltonian structure is known for $N > 1$.
To overcome this difficulty, we will give alternative characterizations of the recombination coefficients, named C_s conditions (for better clarity, $C_s^{(1)}$ for the $N = 1$ theory, $C_s^{(2)}$ for $N = 2$, etc.); these conditions rest on the ϵ-expansion of the Kupershmidt Lax pairs and Hamiltonian functions at a particular point of the phase space. A compatibility argument allows to characterize the global behaviour of the recombined objects on the grounds of the C_s prescriptions at the particular point; in this way the recombination problem, living in principle in a functional space, is reduced to the solution of a linear system in a finite number of numerical unknowns. A similar idea was employed in [2] for the V_N systems.
The paper is organized as follows. In Sect.2 we use the K_1 (or Kac-Moerbeke) system to exemplify the $C_s^{(1)}$ method. In Sect.3 we work in detail the K_2 system and recover the Boussinesq theory in the $\epsilon \mapsto 0$ limit; the main theorem on the recombination scheme (Prop. 3.3), which justifies the $C_s^{(2)}$ algorithm, is proved in Sect.4. In Sect.5 we briefly discuss the K_N system for N arbitrary, and show by direct computation that the $N = 3$ Poisson tensor gives, in the $\epsilon \mapsto 0$ limit, the Poisson structure of the $sl(4)$ KdV type theory.
Even though the framework in which we work is well defined on purely theoretical grounds, the explicit constructions require non little efforts from the computational viewpoint; so, much of them have been realized using the MATHEMATICA package.

2 The K_1 system and its continuous limit.

Our description of this system and its continuous limit will be concise, because this topic was already considered in the two known formulations, both with a Kupershmidt-type Lax operator [3] and with a symmetric one [1] [2]. From the viewpoint of the present work, this system is the $N = 1$ case of the Kupershmidt N-fields theory; we briefly discuss it to make the paper self-contained. In comparison with [3], we add some facts on the continuous limit of the Hamiltonians and the companions of the Lax operator.
The Lax pairs and the Hamiltonian formulation of the K_1 hierarchy are summarized in Table I. We denote with $\boldsymbol{a}$ the phase space of the system, each point of which is a smooth function $\alpha = \alpha(x)$.

Table I: K_1 (or Kac-Moerbeke) system.

Lax formulation for the s^{th} vector field X^ϵ_s : $\quad dL^\epsilon/dt_s = [A^\epsilon_s, L^\epsilon]$,

$$L^\epsilon(\alpha) := \Delta_\epsilon + \alpha\Delta_{-\epsilon} , \qquad A^\epsilon_s(\alpha) := \frac{1}{2}(L^\epsilon)^{2s}_+(\alpha) \qquad (s = 0, 1, 2, ...) .$$

Poisson tensor at a point α: $\quad Q^\epsilon_\alpha : T^*_\alpha \boldsymbol{a} \to T_\alpha\ \boldsymbol{a}$,

$$Q^\epsilon_\alpha = \frac{1}{2}(\ \alpha\alpha_{[\epsilon]}\Delta_\epsilon - \alpha\alpha_{[-\epsilon]}\Delta_{-\epsilon}\ ) .$$

Hamiltonian formulation: $X^\epsilon_s = Q^\epsilon df^\epsilon_s \qquad (s = 0, 1, 2, ...)$.

Hamiltonian functions:

$$f^\epsilon_0(\alpha) := \frac{1}{2}\int dx\ \log\alpha , \qquad f^\epsilon_s(\alpha) := \frac{1}{2s}\mathrm{tr}(L^\epsilon)^{2s}(\alpha) \qquad (s = 1, 2, 3, ...) .$$

First companions of the Lax operator:
$A^\epsilon_0(\alpha) = \frac{1}{2}$,
$A^\epsilon_1(\alpha) = \frac{1}{2}\Delta_{2\epsilon} + \frac{1}{2}(\alpha + \alpha_{[\epsilon]})$,
$A^\epsilon_2(\alpha) = \frac{1}{2}\Delta_{4\epsilon} + \frac{1}{2}(\alpha + \alpha_{[2\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{2\epsilon} + \frac{1}{2}(\alpha^2 + \alpha\alpha_{[-\epsilon]} + 2\alpha\alpha_{[\epsilon]}$
$\quad + \alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha^2_{[\epsilon]})$,
$A^\epsilon_3(\alpha) := \frac{1}{2}\Delta_{6\epsilon} + \frac{1}{2}(\alpha + \alpha_{[5\epsilon]} + \alpha_{[2\epsilon]} + \alpha_{[4\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{4\epsilon}$

$$+\tfrac{1}{2}(\alpha^2+\alpha\alpha_{[2\epsilon]}+\alpha_{[2\epsilon]}^2+\alpha\alpha_{[-\epsilon]}+\alpha\alpha_{[3\epsilon]}+2\alpha_{[2\epsilon]}\alpha_{[3\epsilon]}+\alpha_{[4\epsilon]}\alpha_{[3\epsilon]}+\alpha_{[3\epsilon]}^2$$
$$+2\alpha\alpha_{[\epsilon]}+2\alpha_{[2\epsilon]}\alpha_{[\epsilon]}+\alpha_{[3\epsilon]}\alpha_{[\epsilon]}+\alpha_{[\epsilon]}^2)\Delta_{2\epsilon}+\tfrac{1}{2}(\alpha^3+2\alpha^2\alpha_{[-\epsilon]}+\alpha\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}$$
$$+\alpha\alpha_{[-\epsilon]}^2+3\alpha^2\alpha_{[\epsilon]}+2\alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]}+\alpha_{[2\epsilon]}^2\alpha_{[\epsilon]}+2\alpha\alpha_{[-\epsilon]}\alpha_{[\epsilon]}+\alpha_{[2\epsilon]}\alpha_{[3\epsilon]}\alpha_{[\epsilon]}$$
$$+3\alpha\alpha_{[\epsilon]}^2+2\alpha_{[2\epsilon]}\alpha_{[\epsilon]}^2+\alpha_{[\epsilon]}^3)\ .$$

First Hamiltonians after f_0^ϵ:

$f_1^\epsilon(\alpha)=\int dx\ \alpha\ ,$
$f_2^\epsilon(\alpha)=\frac{1}{2}\int dx\ (\alpha^2+2\alpha\alpha_{[\epsilon]})\ ,$
$f_3^\epsilon(\alpha)=\frac{1}{3}\int dx\ (\alpha^3+3\alpha^2\alpha_{[\epsilon]}+3\alpha\alpha_{[\epsilon]}^2+3\alpha\alpha_{[\epsilon]}\alpha_{[2\epsilon]})\ .$

First vector fields :
$X_0^\epsilon(\alpha)=0\ ,$
$X_1^\epsilon(\alpha)=\frac{1}{2}\alpha(\alpha_{[\epsilon]}-\alpha_{[-\epsilon]})\ ,$
$X_2^\epsilon(\alpha)=\frac{1}{2}\alpha(-\alpha\alpha_{[-\epsilon]}-\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}-\alpha_{[-\epsilon]}^2+\alpha\alpha_{[\epsilon]}+\alpha_{[2\epsilon]}\alpha_{[\epsilon]}+\alpha_{[\epsilon]}^2)\ ,$

$$X_3^\epsilon(\alpha)=\tfrac{1}{2}\alpha(-\alpha^2\alpha_{[-\epsilon]}-\alpha\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}-\alpha_{[-2\epsilon]}^2\alpha_{[-\epsilon]}-\alpha_{[-2\epsilon]}\alpha_{[-3\epsilon]}\alpha_{[-\epsilon]}-2\alpha\alpha_{[-\epsilon]}^2$$
$$-2\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}^2-\alpha_{[-\epsilon]}^3+\alpha^2\alpha_{[\epsilon]}+\alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]}+\alpha_{[2\epsilon]}^2\alpha_{[\epsilon]}+\alpha_{[2\epsilon]}\alpha_{[3\epsilon]}\alpha_{[\epsilon]}$$
$$+2\alpha\alpha_{[\epsilon]}^2+2\alpha_{[2\epsilon]}\alpha_{[\epsilon]}^2+\alpha_{[\epsilon]}^3)\ .$$

In the definition of A_s^ϵ, the subscript $_+$ denotes the projection on the shift operators of non-negative order; the trace tr of a differential-shift operator is the integral over x of the zero order term in the shift ([1]); these notations will also be employed in the rest of the paper.
The Poisson tensor, the vector fields and the Hamiltonians reported in the previous Table coincide *exactly* (also in the normalizations) with the homologous objects of [2]; these were expressed in terms of another field variable a, related to the present one by $\alpha=\frac{1}{4}a_{[-\epsilon]}^2$. With appropriate specifications, one could show that this transformation sets up a gauge equivalence between the symmetric Lax operator considered in [2] and the Kupershmidt-type operator employed here.

[1] More precisely, consider an operator $G:=\sum_p g_p\Delta_{p\epsilon}$, where the index p ranges over the relative integers and each coefficient is a smooth function $g_p(x)$. Then, by definition,

$$G_+:=\sum_{p\geq 0}g_p\Delta_{p\epsilon}\ ;\qquad \mathrm{tr}G:=\int dx\ g_0(x)\ .$$

As for the KdV theory, we adopt all notations and normalization conventions of [1] [2]. In particular, $\mathcal{U}$ denotes the phase space of the theory, each point of which is a smooth function $u = u(x)$; the Hamiltonians of the hierarchy are the functionals $h_s^{KdV}(u) := \frac{4^s}{2s+1}\mathrm{Tr}(L^{KdV})^{s+\frac{1}{2}}(u)$, and the companions of $L^{KdV}(u) = \partial_{xx} + u$ are the operators $B_s^{KdV}(u) := 4^{s-1}(L^{KdV})_+^{s-\frac{1}{2}}(u)$ $(s = 0, 1, 2, ...)$, etc. . Of course, when speaking of any KdV type theory we denote with $_+$ the projection on nonnegative powers of ∂_x, and with Tr the well known Adler trace for pseudo-differential operators.
Now, following [4], we introduce the transformation

$$\Theta^\epsilon : \mathcal{U} \to \boldsymbol{a} \, , \qquad u \mapsto \alpha = 1 + \epsilon^2 u \tag{2.1}$$

and employ it to pull back to the phase space $\mathcal{U}$ the geometrical structures and the Lax scheme of the K_1 system living on $\boldsymbol{a}$. Let us consider, in particular, the Lax operator

$$L^\epsilon(u) := L^\epsilon(\alpha)\Big|_{\alpha=1+\epsilon^2 u} \tag{2.2}$$

and the pull back Q_u^ϵ of the Poisson tensor. Then, in the $\epsilon \mapsto 0$ limit, we obtain:

$$L^\epsilon(u) = 2 + \epsilon^2(\partial_{xx} + u) + O(\epsilon^3) = 2 + \epsilon^2 L^{KdV}(u) + O(\epsilon^3) \ , \tag{2.3}$$

$$Q_u^\epsilon = \frac{1}{\epsilon^3}\partial_x + O(\frac{1}{\epsilon^2}) = \frac{1}{\epsilon^3}Q^{KdV}(u) + O(\frac{1}{\epsilon^2}) \ . \tag{2.4}$$

The K_1 system also possesses a second Poisson tensor, and a recombination of the latter with Q^ϵ produces in the $\epsilon \mapsto 0$ limit the second KdV Poisson tensor; we give no more details about this fact, falling outside the purposes of the present paper.
All the Hamiltonians, the vector fields and the companion operators of KdV theory can be obtained as continuous limits of homologous K_1 objects. The main point in this construction is that the K_1 objects must be conveniently recombined before sending ϵ to zero; as observed in the Introduction, the numerical coefficients of these recombinations could be determined recursively with the methods described in [3], but a different approach will be preferred here, which is more suitable for the extension to the K_N systems with $N > 1$.
Let us start with the following

2.1 ***Definition.*** *Let z be an indeterminate, and*

$$\lambda(z) := z + \frac{1}{z} \ . \tag{2.5}$$

For $s = 1, 2, 3, ...$ we will put

$$q_s(z) := \frac{1}{2}\Big(\lambda^{2s}(z)\Big)_+ \ , \tag{2.6}$$

the subscript $_+$ denoting the projection on the nonnegative powers of z. Also, we will put

$$p_0(\epsilon) := \frac{1}{2}\epsilon^2 \ , \qquad p_s(\epsilon) := \frac{1}{2s}\begin{pmatrix} 2s \\ s \end{pmatrix}(1 + s\epsilon^2) \qquad (s = 1, 2, 3, ...) \ . \tag{2.7}$$

$\diamond$

In order to make clear the motivations of the previous definition, let us consider the K_1 Lax operator and its companions at the point $u = 0$ of the space $\mathcal{U}$ (corresponding to $\alpha = 1$ in Eq.(2.1)); it is easily found that

$$(L^\epsilon)^s(0) = \lambda^s(\Delta_\epsilon) \ , \qquad A^\epsilon_s(0) = q_s(\Delta_\epsilon) \ . \tag{2.8}$$

Furthermore, let us evaluate the K_1 Hamiltonians at $u = \text{const.} = 1$ (i.e., $\alpha = 1 + \epsilon^2$); the traces appearing in their definition are easily computed, and it is found that

$$f^\epsilon_0(1) = \frac{1}{2}\log(1 + \epsilon^2) = p_0(\epsilon) + O(\epsilon^3) \ , \tag{2.9}$$

$$f^\epsilon_s(1) = \frac{1}{2s}\begin{pmatrix} 2s \\ s \end{pmatrix}(1 + \epsilon^2)^s = p_s(\epsilon) + O(\epsilon^3) \qquad (s = 1, 2, 3...) \ ; \tag{2.10}$$

so, the polynomials $p_0(\epsilon)$, $p_1(\epsilon)$, ... turn out to be the second order expansions of the Hamiltonians at $u = 1$.

2.2 ***Definition.*** *Let $s \in \{0, 1, 2, ...\}$ be a fixed integer; consider two systems of coefficients $(c_{sj})_{j=-1,...,s}$ and $(d_{sj})_{j=0,...,s-1}$ (intending the second one to be empty if $s = 0$). These coefficients are said to satisfy the $C^{(1)}_s$ conditions if the following holds:*

$s = 0$:

$$c_{0,-1} + c_{00} p_0(\epsilon) = \frac{1}{2}\epsilon^2 \ ; \tag{2.11}$$

$s \geq 1$:

$$\sum_{j=1}^{s} c_{sj} q_j(e^\epsilon) + \sum_{j=0}^{s-1} d_{sj} \lambda^j(e^\epsilon) = 4^{s-1}\epsilon^{2s-1} + O(\epsilon^{2s}) \tag{2.12}$$

and

$$c_{s,-1} + \sum_{j=0}^{s} c_{sj} p_j(\epsilon) = 0 \ . \tag{2.13}$$

$\diamond$

The superscript (1) in $C_s^{(1)}$ marks the relations between the above prescriptions and the K_1 system, as already explained in the Introduction. (For avoiding too many indexes, this superscript has been omitted in the coefficients c_{sj}, d_{sj}).
Clearly, for $s = 0$ the condition (2.11) yields two equations for the unknowns $c_{0,-1}$ and c_{00}. For $s = 1, 2, 3, ...$, the $C_s^{(1)}$ conditions (2.12) (2.13) give rise to a system of $2s+2$ linear, algebraic equations in the $2s+2$ unknowns c_{sj} and d_{sj}; the equations are obtained equating the coefficients of $1, \epsilon, \epsilon^2, ..., \epsilon^{2s-1}$ in the two sides Eq.(2.12), and setting to zero the coefficients of 1 and ϵ^2 in the l.h.s. of Eq.(2.13).
For a given integer s, let us consider a system of coefficients satisfying the $C_s^{(1)}$ conditions, and employ them to construct the linear combinations of operators, vector fields and Hamiltonians according to the following prescriptions:

$$B_s^\epsilon := \sum_{j=1}^{s} c_{sj} A_j^\epsilon + \sum_{j=0}^{s-1} d_{sj} (L^\epsilon)^j \ , \tag{2.14}$$

$$Z_s^\epsilon := \sum_{j=1}^{s} c_{sj} X_j^\epsilon \ , \tag{2.15}$$

$$h_s^\epsilon := c_{s,-1} + \sum_{j=0}^{s} c_{sj} f_j^\epsilon \tag{2.16}$$

(intending $B_0^\epsilon := 0$, $Z_0^\epsilon := 0$). We can regard these objects to be defined on either the $\boldsymbol{a}$ or the $\mathcal{U}$ phase space, thanks to the one-to-one correspondence (2.1). Let us work in particular on $\mathcal{U}$, and perform expansions in powers of ϵ; where a shift operator $\Delta_{k\epsilon}$ occurs, we replace it with the expansion of $e^{k\epsilon\partial}$. With these prescriptions, the recombinations (2.14–2.16) give rise to power series in ϵ; the coefficients are differential operators in the case of A_s^ϵ and differential polynomials (in u) in the case of Z_s^ϵ. In the case of h_s^ϵ, each coefficient of the ϵ-expansion also involves a differential polynomial, integrated over x.

Let us return to the $C_s^{(1)}$ conditions, and interpret them from the viewpoint of these expansions. For $s \geq 1$, comparing Eq.(2.8) with Eq.(2.12) (and replacing formally ϵ with $\epsilon\partial$), we see that Eq.(2.12) means $B_s^\epsilon(0) = 4^{s-1}\epsilon^{2s-1}\partial^{2s-1} + O(\epsilon^{2s})$; on the other hand, $4^{s-1}\partial^{2s-1}$ is just the s^{th} KdV companion operator at $u = 0$, so we can write

$$B_s^\epsilon(0) = \epsilon^{2s-1} B_s^{KdV}(0) + O(\epsilon^{2s}) \; . \tag{2.17}$$

As for the second $C_s^{(1)}$ condition (Eq.(2.13)), by comparison with (2.9) (2.10) (2.16) we see that it amounts to the prescription

$$h_s^\epsilon(1) = O(\epsilon^3) \tag{2.18}$$

for the recombined Hamiltonian.

Finally, Eq.(2.11) means

$$h_0^\epsilon(1) = \frac{1}{2}\epsilon^2 = \epsilon^2 h_0^{KdV}(1) \; . \tag{2.19}$$

Even though Eq.s (2.17–2.19) simply envolve the behaviour of the recombined K_1 operators and Hamiltonians at a particular point of the phase space $\mathcal{U}$, they allow to infer the following, much stronger statement:

2.3 *Proposition.* *Let $s \in \{0, 1, 2, ...\}$ be a fixed integer, and consider the recombinations (2.14–2.16), with coefficients satisfying the $C_s^{(1)}$ conditions. Then at each point $u \in \mathcal{U}$ it is:*

$$B_s^\epsilon(u) = \epsilon^{2s-1} B_s^{KdV}(u) + O(\epsilon^{2s}) \; , \tag{2.20}$$

$$Z_s^\epsilon(u) = \epsilon^{2s-1} Z_s^{KdV}(u) + O(\epsilon^{2s}) \; , \tag{2.21}$$

$$h_s^\epsilon(u) = \epsilon^{2s+2} h_s^{KdV}(u) + O(\epsilon^{2s+3}) \; . \tag{2.22}$$

$\diamond$

The proof of these statements is omitted for brevity; it is similar to the (even more technical) proof of Prop. 3.3, given explicitly in Sect.4 for the K_2 system.
Let us illustrate the recombination schemes for $s = 0, 1, 2, 3$, depending on

$$q_1(z) = 1 + \frac{1}{2}z^2 \ , \qquad q_2(z) = 3 + 2z^2 + \frac{1}{2}z^4 \ , \tag{2.23}$$

$$q_3(z) = 10 + \frac{15}{2}z^2 + 3z^4 + \frac{1}{2}z^6 \ ;$$

$$\lambda^2(z) = z^2 + 2 + \frac{1}{z^2} \ , \qquad \lambda^3(z) = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \tag{2.24}$$

and

$$p_0(\epsilon) = \frac{1}{2}\epsilon^2 \ , \quad p_1(\epsilon) = 1+\epsilon^2 \ , \quad p_2(\epsilon) = \frac{3}{2}+3\epsilon^2 \ , \quad p_3(\epsilon) = \frac{10}{3}+10\epsilon^2 \ . \tag{2.25}$$

Eq.(2.11) has the unique solution

$$c_{0,-1} = 0 \ , \qquad c_{00} = 1 \ . \tag{2.26}$$

For $s = 1, 2, 3$ Eq.s (2.12) (2.13) have unique solutions, given respectively by

$$c_{1,-1} = -1, \qquad c_{10} = -2 \ , \qquad c_{11} = 1, \qquad d_{10} = -\frac{3}{2} \ ; \tag{2.27}$$

$$c_{2,-1} = -\frac{9}{2}, \ c_{20} = 6 \ , c_{21} = -6, \ c_{22} = 1 \ , \quad d_{20} = \frac{15}{2}, \ d_{21} = -2 \ ; \tag{2.28}$$

$$c_{3,-1} = -\frac{55}{3}, \ c_{30} = -20 \ , c_{31} = 30, \ c_{32} = -10 \ , c_{33} = 1 \ , \tag{2.29}$$

$$d_{30} = -45 \ , \quad d_{31} = 32 \ , \quad d_{32} = -\frac{15}{2} \ .$$

The recombinations (2.14–2.16) with these coefficients behave as foreseen by the previous Proposition; for example, let us consider the case $s = 3$ in which

$$B_3^\epsilon = 30A_1^\epsilon - 10A_2^\epsilon + A_3^\epsilon - 45 + 32L^\epsilon - \frac{15}{2}(L^\epsilon)^2 \ ; \tag{2.30}$$

$$Z_3^\epsilon = 30X_1^\epsilon - 10X_2^\epsilon + X_3^\epsilon \ ; \tag{2.31}$$

$$h_3^\epsilon = -\frac{55}{3} - 20 f_0^\epsilon + 30 f_1^\epsilon - 10 f_2^\epsilon + f_3^\epsilon \ . \tag{2.32}$$

Starting from Table I, one can apply the transformation (2.1) to the above recombinations and expand in ϵ; to the lowest orders, it is

$$B_3^\epsilon(u) = 16\epsilon^5 \Big[\partial_{xxxxx} + \frac{5}{2} u \partial_{xxx} + \frac{15}{4} u_x \partial_{xx} + \frac{5}{8}(5u_{xx} + 3u^2)\partial_x + \tag{2.33}$$

$$+ \frac{15}{16}(u_{xxx} + 2uu_x) \Big] + O(\epsilon^6) = \epsilon^5 B_3^{KdV}(u) + O(\epsilon^6) \ ;$$

$$Z_3^\epsilon(u) = \epsilon^5 (u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x) + O(\epsilon^6) = \epsilon^5 Z_3^{KdV}(u) + O(\epsilon^6) \ ; \tag{2.34}$$

$$h_3^\epsilon(u) = \frac{1}{2}\epsilon^8 \int dx (5u^4 - 10uu_x^2 + u_{xx}^2) + O(\epsilon^9) = \epsilon^8 h_3^{KdV}(u) + O(\epsilon^9) \ . \tag{2.35}$$

3 The K_2 system and Boussinesq theory.

The main facts about the K_2 system are described in Table II. The interpolated version of this system is a theory in two field variables $\alpha_1 := \alpha$ and $\alpha_2 := \rho$. The phase space $\boldsymbol{a}$ is a set of such pairs $\boldsymbol{\alpha} = (\alpha, \rho)$; at each point the tangent and cotangent spaces $T_{\boldsymbol{\alpha}}\boldsymbol{a}$ and $T^*_{\boldsymbol{\alpha}}\boldsymbol{a}$ are represented as sets of pairs $\dot{\boldsymbol{\alpha}} = (\dot\alpha, \dot\rho)$ and $\delta\boldsymbol{\alpha} = (\delta\alpha, \delta\rho)$, with the duality form

$$< \delta\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}} >:= \int dx(\delta\alpha\ \dot\alpha + \delta\rho\ \dot\rho)\ . \tag{3.1}$$

Table II reports the Lax formulation, the Poisson structure and the explicit expressions of the first elements in the associated hierarchy.

Table II: K_2 **system.**

Lax formulation for the s^{th} vector field X^ϵ_s: $dL^\epsilon/dt_s = [A^\epsilon_s, L^\epsilon]$,

$$L^\epsilon(\boldsymbol{\alpha}) := \Delta_\epsilon + \alpha\Delta_{-\epsilon} + \rho\Delta_{-3\epsilon}\ , \qquad A^\epsilon_s(\boldsymbol{\alpha}) := \frac{1}{3}(L^\epsilon)^{2s}_+(\boldsymbol{\alpha}) \qquad (s = 0, 1, 2, ...)\ .$$

Poisson tensor at a point $\boldsymbol{\alpha}$: $\qquad Q^\epsilon_{\boldsymbol{\alpha}} : T^*_{\boldsymbol{\alpha}}\boldsymbol{a} \to T_{\boldsymbol{\alpha}}\ \boldsymbol{a}$,

$$\begin{pmatrix} \dot\alpha \\ \dot\rho \end{pmatrix} = \frac{1}{3} \begin{pmatrix} Q^\epsilon_{\alpha\alpha} & Q^\epsilon_{\alpha\rho} \\ Q^\epsilon_{\rho\alpha} & Q^\epsilon_{\rho\rho} \end{pmatrix} \begin{pmatrix} \delta\alpha \\ \delta\rho \end{pmatrix}\ ,$$

$Q^\epsilon_{\alpha\alpha} := \rho_{[2\epsilon]}\Delta_{2\epsilon} + \alpha\alpha_{[\epsilon]}\Delta_\epsilon - \alpha\alpha_{[-\epsilon]}\Delta_{-\epsilon} - \rho\Delta_{-2\epsilon}$,
$Q^\epsilon_{\alpha\rho} := \alpha\rho_{[3\epsilon]}\Delta_{3\epsilon} + \alpha\rho_{[2\epsilon]}\Delta_{2\epsilon} - \alpha\rho - \alpha\rho_{[-\epsilon]}\Delta_{-\epsilon}$,
$Q^\epsilon_{\rho\alpha} := \alpha_{[\epsilon]}\rho\Delta_\epsilon + \alpha\rho - \alpha_{[-2\epsilon]}\rho\Delta_{-2\epsilon} - \alpha_{[-3\epsilon]}\rho\Delta_{-3\epsilon}$,
$Q^\epsilon_{\rho\rho} := \rho\rho_{[3\epsilon]}\Delta_{3\epsilon} + \rho\rho_{[2\epsilon]}\Delta_{2\epsilon} + \rho\rho_{[\epsilon]}\Delta_\epsilon - \rho\rho_{[-\epsilon]}\Delta_{-\epsilon} - \rho\rho_{[-2\epsilon]}\Delta_{-2\epsilon} - \rho\rho_{[-3\epsilon]}\Delta_{-3\epsilon}$.

Hamiltonian formulation: $X^\epsilon_s = Q^\epsilon df^\epsilon_s \qquad (s = 0, 1, 2, ...)$.

Hamiltonian functions:

$$f^\epsilon_0(\boldsymbol{\alpha}) := \frac{1}{2}\int dx\ \log(-3\rho)\ , \qquad f^\epsilon_s(\boldsymbol{\alpha}) := \frac{1}{2s}\mathrm{tr}(L^\epsilon)^{2s}(\boldsymbol{\alpha}) \qquad (s = 1, 2, 3, ...)\ .$$

First companions of the Lax operator:
$A^\epsilon_0(\boldsymbol{\alpha}) = \frac{1}{3}$,

$$A_1^\epsilon(\boldsymbol{\alpha}) = \tfrac{1}{3}\Delta_{2\epsilon} + \tfrac{1}{3}(\alpha + \alpha_{[\epsilon]}) \ ,$$
$$\begin{aligned}A_2^\epsilon(\boldsymbol{\alpha}) = {} & \tfrac{1}{3}\Delta_{4\epsilon} + \tfrac{1}{3}(\alpha + \alpha_{[2\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{2\epsilon} \\ & + \tfrac{1}{3}(\alpha^2 + \alpha\alpha_{[-\epsilon]} + 2\alpha\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[\epsilon]}^2 + \rho + \rho_{[2\epsilon]} + \rho_{[3\epsilon]} + \rho_{[\epsilon]}) \ ,\end{aligned}$$
$$\begin{aligned}A_3^\epsilon(\boldsymbol{\alpha}) = {} & \tfrac{1}{3}\Delta_{6\epsilon} + \tfrac{1}{3}(\alpha + \alpha_{[5\epsilon]} + \alpha_{[2\epsilon]} + \alpha_{[4\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{4\epsilon} \\ & + \tfrac{1}{3}(\alpha^2 + \alpha\alpha_{[2\epsilon]} + \alpha_{[2\epsilon]}^2 + \alpha\alpha_{[-\epsilon]} + \alpha\alpha_{[3\epsilon]} + 2\alpha_{[2\epsilon]}\alpha_{[3\epsilon]} + \alpha_{[4\epsilon]}\alpha_{[3\epsilon]} + \alpha_{[3\epsilon]}^2 \\ & + 2\alpha\alpha_{[\epsilon]} + 2\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[3\epsilon]}\alpha_{[\epsilon]} + \alpha_{[\epsilon]}^2 + \rho + \rho_{[5\epsilon]} + \rho_{[2\epsilon]} + \rho_{[4\epsilon]} + \rho_{[3\epsilon]} + \rho_{[\epsilon]})\Delta_{2\epsilon} \\ & + \tfrac{1}{3}(\alpha^3 + 2\alpha^2\alpha_{[-\epsilon]} + \alpha\alpha_{[-\epsilon]}^2 + 3\alpha^2\alpha_{[\epsilon]} + 2\alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}^2\alpha_{[\epsilon]} \\ & + 2\alpha\alpha_{[-\epsilon]}\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[3\epsilon]}\alpha_{[\epsilon]} + 3\alpha\alpha_{[\epsilon]}^2 + 2\alpha_{[2\epsilon]}\alpha_{[\epsilon]}^2 + \alpha_{[\epsilon]}^3 + 2\alpha\rho + \alpha_{[-\epsilon]}\rho + 2\alpha_{[\epsilon]}\rho \\ & + 2\alpha\rho_{[2\epsilon]} + \alpha_{[2\epsilon]}\rho_{[2\epsilon]} + \alpha_{[-\epsilon]}\rho_{[2\epsilon]} + \alpha_{[3\epsilon]}\rho_{[2\epsilon]} + 2\alpha_{[\epsilon]}\rho_{[2\epsilon]} + \alpha\rho_{[-\epsilon]} + \alpha_{[\epsilon]}\rho_{[4\epsilon]} \\ & + 2\alpha\rho_{[3\epsilon]} + \alpha_{[2\epsilon]}\rho_{[3\epsilon]} + \alpha_{[4\epsilon]}\rho_{[3\epsilon]} + \alpha_{[3\epsilon]}\rho_{[3\epsilon]} + 2\alpha_{[\epsilon]}\rho_{[3\epsilon]} + 2\alpha\rho_{[\epsilon]} + \alpha_{[2\epsilon]}\rho_{[\epsilon]} \\ & + \alpha_{[-\epsilon]}\rho_{[\epsilon]} + 2\alpha_{[\epsilon]}\rho_{[\epsilon]} + \rho\alpha_{[-3\epsilon]} + \alpha\alpha_{[-\epsilon]}\alpha_{[-2\epsilon]} + \rho\alpha_{[-2\epsilon]} + \rho_{[\epsilon]}\alpha_{[-2\epsilon]}) \ .\end{aligned}$$

First Hamiltonians after f_0^ϵ:

$$f_1^\epsilon(\boldsymbol{\alpha}) = \int dx \ \alpha \ ,$$
$$f_2^\epsilon(\boldsymbol{\alpha}) = \tfrac{1}{2}\int dx \ (\alpha^2 + 2\alpha\alpha_{[\epsilon]} + 2\rho) \ ,$$
$$\begin{aligned}f_3^\epsilon(\boldsymbol{\alpha}) = {} & \tfrac{1}{3}\int dx \ (\alpha^3 + 3\alpha^2\alpha_{[\epsilon]} + 3\alpha\alpha_{[\epsilon]}^2 + 3\alpha\alpha_{[\epsilon]}\alpha_{[2\epsilon]} + 3\alpha\rho + 3\alpha_{[\epsilon]}\rho + 3\alpha\rho_{[\epsilon]} \\ & + 3\alpha\rho_{[2\epsilon]} + 3\alpha\rho_{[3\epsilon]}) \ ,\end{aligned}$$

First vector fields :

$$X_s^\epsilon(\boldsymbol{\alpha}) = \tfrac{1}{3}\begin{pmatrix} X_{s,\alpha}^\epsilon(\boldsymbol{\alpha}) \\ X_{s,\rho}^\epsilon(\boldsymbol{\alpha}) \end{pmatrix} \ ,$$

where, for $s = 0, 1, 2, 3$:

$$X_{0,\alpha}^\epsilon(\boldsymbol{\alpha}) := 0 \ , \qquad X_{0,\rho}^\epsilon(\boldsymbol{\alpha}) := 0 \ ,$$

$$X_{1,\alpha}^\epsilon(\boldsymbol{\alpha}) := -\alpha\alpha_{[-\epsilon]} + \alpha\alpha_{[\epsilon]} - \rho + \rho_{[2\epsilon]} \ ,$$
$$X_{1,\rho}^\epsilon(\boldsymbol{\alpha}) := \alpha\rho - \alpha_{[-2\epsilon]}\rho - \alpha_{[-3\epsilon]}\rho + \alpha_{[\epsilon]}\rho \ ,$$

$$\begin{aligned}X_{2,\alpha}^\epsilon(\boldsymbol{\alpha}) := {} & -\alpha^2\alpha_{[-\epsilon]} - \alpha\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]} - \alpha\alpha_{[-\epsilon]}^2 + \alpha^2\alpha_{[\epsilon]} + \alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha\alpha_{[\epsilon]}^2 \\ & -\alpha\rho - \alpha_{[-2\epsilon]}\rho - \alpha_{[-3\epsilon]}\rho - \alpha_{[-\epsilon]}\rho + \alpha\rho_{[2\epsilon]} + \alpha_{[2\epsilon]}\rho_{[2\epsilon]} + \alpha_{[3\epsilon]}\rho_{[2\epsilon]} \\ & +\alpha_{[\epsilon]}\rho_{[2\epsilon]} - \alpha\rho_{[-\epsilon]} + \alpha\rho_{[3\epsilon]} \ ,\end{aligned}$$
$$\begin{aligned}X_{2,\rho}^\epsilon(\boldsymbol{\alpha}) := {} & \alpha^2\rho - \alpha_{[-2\epsilon]}^2\rho - 2\alpha_{[-2\epsilon]}\alpha_{[-3\epsilon]}\rho - \alpha_{[-4\epsilon]}\alpha_{[-3\epsilon]}\rho - \alpha_{[-3\epsilon]}^2\rho + \alpha\alpha_{[-\epsilon]}\rho \\ & -\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}\rho + 2\alpha\alpha_{[\epsilon]}\rho + \alpha_{[2\epsilon]}\alpha_{[\epsilon]}\rho + \alpha_{[\epsilon]}^2\rho + \rho\rho_{[2\epsilon]} - \rho\rho_{[-2\epsilon]} - \rho\rho_{[-3\epsilon]} \\ & -\rho\rho_{[-\epsilon]} + \rho\rho_{[3\epsilon]} + \rho\rho_{[\epsilon]} \ ,\end{aligned}$$

$$X_{3,\alpha}^\epsilon(\boldsymbol{\alpha}) := Q_{\alpha\alpha}^\epsilon(\delta\alpha) + Q_{\alpha\rho}^\epsilon(\delta\rho) \ ,$$
$$X_{3,\rho}^\epsilon(\boldsymbol{\alpha}) := Q_{\rho\alpha}^\epsilon(\delta\alpha) + Q_{\rho\rho}^\epsilon(\delta\rho) \ ,$$

with

$\delta\alpha := \alpha^2 + 2\alpha\alpha_{[\epsilon]} + \alpha^2_{[-\epsilon]} + \alpha^2_{[\epsilon]} + 2\alpha_{[-\epsilon]}\alpha + \alpha_{[\epsilon]}\alpha_{[2\epsilon]} + \alpha_{[-\epsilon]}\alpha_{[\epsilon]} + \alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}$
$\quad +\rho + \rho_{[-\epsilon]} + \rho_{[\epsilon]} + \rho_{[2\epsilon]} + \rho_{[3\epsilon]}$,
$\delta\rho := \alpha + \alpha_{[\epsilon]} + \alpha_{[-\epsilon]} + \alpha_{[-2\epsilon]} + \alpha_{[-3\epsilon]}$
(these are the components of df^ϵ_3).

Concerning the Boussinesq theory in the field variables $u_1 := u$, $u_2 := v$, we collect hereafter the basic elements, mainly in order to fix some notational standards. We denote with $\mathcal{U}$ the phase space, whose points are pairs $\mathbf{u} = (u, v)$; with our notations, the familiar Boussinesq equation $u_{\tau\tau} + \frac{1}{3}u_{xxxx} + \frac{2}{3}(u^2)_{xx} = 0$ arises from the evolution equation corresponding to the vector field Z^{Bou}_3.

Table III: Boussinesq theory.

Lax formulation for the s^{th} vector field Z^{Bou}_s: $dL^{Bou}/d\tau_s = [B^{Bou}_s, L^{Bou}]$,

$$L^{Bou}(\mathbf{u}) := \partial_{xxx} + u\partial_x + v \ , \qquad B^{Bou}_s := 3^{[\frac{s}{2}]-1}(L^{Bou})_+^{\frac{1}{3}[\frac{3s}{2}]-\frac{2}{3}} \qquad (s = 0, 1, 2, ...) .$$

(of course, [.] stands for the integer part).

Poisson tensor at a point $\mathbf{u}$: $\qquad Q^{Bou}_{\mathbf{u}} : T^*_{\mathbf{u}}\mathcal{U} \to T_{\mathbf{u}}\,\mathcal{U}$,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}\begin{pmatrix} \delta u \\ \delta v \end{pmatrix} .$$

Hamiltonian formulation: $Z^{Bou}_s = Q^{Bou} dh^{Bou}_s \qquad (s = 0, 1, 2, ...)$.
Hamiltonian functions:

$$h^{Bou}_s(\mathbf{u}) := \frac{3^{[\frac{s}{2}]+1}}{[\frac{3s}{2}]+1}\mathrm{Tr}(L^{Bou})(\mathbf{u})^{\frac{1}{3}[\frac{3s}{2}]+\frac{1}{3}} .$$

First nonzero companions of the Lax operator:
$B^{Bou}_2(\mathbf{u}) = \partial_x$,
$B^{Bou}_3(\mathbf{u}) := \partial_{xx} + \frac{2}{3}u$.

First Hamiltonians:

$h_0^{Bou}(\mathbf{u}) = \int dx\ u\ ,$
$h_1^{Bou}(\mathbf{u}) = \int dx\ v\ ,$
$h_2^{Bou}(\mathbf{u}) = \int dx\ uv\ ,$
$h_3^{Bou}(\mathbf{u}) = \frac{1}{9}\int dx\ (-u^3 + 9v^2 - 9u_x v + 3u_x^2)\ ,$
$h_4^{Bou}(\mathbf{u}) = \frac{1}{9}\int dx\ (-u^4 + 9uu_x{}^2 - 3u_{xx}{}^2 - 18uu_x v - 9u_{xxx}v + 18uv^2 - 9v_x{}^2\ )\ .$

First nonzero vector fields :

$$Z_2^{Bou}(\mathbf{u}) = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

$$Z_3^{Bou}(\mathbf{u}) = \begin{pmatrix} -u_{xx} + 2v_x \\ -\frac{2}{3}u_{xxx} - \frac{2}{3}uu_x + v_{xx} \end{pmatrix}\ .$$

In order to connect the theories described in Tables II and III, let us consider the transformation $\Theta^\epsilon : \mathcal{U} \to \boldsymbol{a}$, $\mathbf{u} = (u, v) \mapsto \boldsymbol{\alpha} = (\alpha, \rho)$, where

$$\begin{cases} \alpha = 2 + \frac{2}{3}\epsilon^2 u \\ \rho = -\frac{1}{3} - \frac{2}{3}\epsilon^2 u + \frac{4}{3}\epsilon^3 v\ . \end{cases} \tag{3.2}$$

This is the specialization to the case $N = 2$ of the general transformation introduced in [4], in order to obtain the $sl(N+1)$ Lax operator from the limit of the K_N Lax operator. In fact, setting

$$L^\epsilon(\mathbf{u}) := L^\epsilon(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\Theta^\epsilon(\mathbf{u})} \tag{3.3}$$

one easily finds, in the $\epsilon \mapsto 0$ limit,

$$L^\epsilon(\mathbf{u}) = \frac{8}{3} + \frac{4}{3}\epsilon^3(\partial_{xxx} + u\partial_x + v) + O(\epsilon^4) = \frac{8}{3} + \frac{4}{3}\epsilon^3 L^{Bou}(\mathbf{u}) + O(\epsilon^4)\ . \tag{3.4}$$

Now, let us go on in the analysis of the $\epsilon \mapsto 0$ limit, and employ the transformation Θ^ϵ to pull back onto $\mathcal{U}$ the Poisson tensor of Table II; in this way we obtain, at each point $\mathbf{u}$, the Poisson tensor

$$Q^\epsilon_{\mathbf{u}} := (T_{\mathbf{u}}\Theta^\epsilon)^{-1} Q^\epsilon_{\boldsymbol{\alpha}}\Big|_{\boldsymbol{\alpha}=\Theta^\epsilon(\mathbf{u})} (T^*_{\mathbf{u}}\Theta^\epsilon)^{-1}\ . \tag{3.5}$$

The above formula contains the inverses of the tangent and cotangent maps $T_{\mathbf{u}}\Theta^\epsilon : T_{\mathbf{u}}\mathcal{U} \to T_{\boldsymbol{\alpha}}\boldsymbol{a}$ and $T^*_{\mathbf{u}}\Theta^\epsilon : T^*_{\boldsymbol{\alpha}}\boldsymbol{a} \to T^*_{\mathbf{u}}\mathcal{U}$; explicit computations give

$$Q^\epsilon_{\mathbf{u}} = \begin{pmatrix} \frac{3}{2\epsilon^2} & 0 \\ \frac{3}{4\epsilon^3} & \frac{3}{4\epsilon^3} \end{pmatrix} \begin{pmatrix} \frac{1}{3}Q^\epsilon_{\alpha\alpha} & \frac{1}{3}Q^\epsilon_{\alpha\rho} \\ \frac{1}{3}Q^\epsilon_{\rho\alpha} & \frac{1}{3}Q^\epsilon_{\rho\rho} \end{pmatrix} \begin{pmatrix} \frac{3}{2\epsilon^2} & \frac{3}{4\epsilon^3} \\ 0 & \frac{3}{4\epsilon^3} \end{pmatrix}\ , \tag{3.6}$$

where $Q^\epsilon_{\alpha\alpha}$, etc. are the matrix elements reported in Table II, with α and ρ given by Eq.(3.2). The ϵ-expansion of this tensor gives

$$Q^\epsilon_{\mathbf{u}} = \frac{1}{3\epsilon^4}\left(\begin{array}{cc} 0 & \partial_x \\ \partial_x & 0 \end{array}\right) + O(\frac{1}{\epsilon^3}) = \frac{1}{\epsilon^4} Q^{Bou}_{\mathbf{u}} + O(\frac{1}{\epsilon^3}) \ . \tag{3.7}$$

In order to obtain (for $\epsilon \mapsto 0$) the Hamiltonians, the vector fields and the companion operators of the Boussinesq theory, we recombine the homologous objects of the K_2 system with the method already described in the previous section for the case $N = 1$.

3.1 ***Definition.*** *Let z be an indeterminate, and*

$$\lambda(z) := z + \frac{2}{z} - \frac{1}{3z^3} \ , \quad \mu(z) := \frac{2}{3z} - \frac{2}{3z^3} \ . \tag{3.8}$$

For $s = 1, 2, 3, \ldots$, we will put

$$q_s(z) := \frac{1}{3}\Big(\lambda^{2s}(z)\Big)_+ \tag{3.9}$$

and

$$p_s(\epsilon) := \frac{1}{2s}\mathrm{rem}\Big(\lambda^{2s}(z)\Big) + \mathrm{rem}\Big(\mu(z)\lambda^{2s-1}(z)\Big)\epsilon^2 + \mathrm{rem}\Big(\frac{4\lambda^{2s-1}(z)}{3z^3}\Big)\epsilon^3 \ , \tag{3.10}$$

where the "reminder" rem is coefficient of the zero order term in z. Also, we will put

$$p_0(\epsilon) := \epsilon^2 - 2\epsilon^3 \ . \tag{3.11}$$

$\diamond$

Let us consider the Lax operator for the Kupershmidt system and its companions, evaluated at $u = v = 0$ (i.e., $\alpha = 2$, $\rho = -\frac{1}{3}$ according to Eq.(3.2)); with the notations just defined, we can write

$$(L^\epsilon)^s(0,0) = \lambda^s(\Delta_\epsilon) \ ; \qquad A^\epsilon_s(0,0) = q_s(\Delta_\epsilon) \ . \tag{3.12}$$

The cubic polynomials in Eq.(3.10) (3.11) are obtained evaluating the K_2 Hamiltonians at $u = v = 1$ (i.e., $\alpha = 2 + \frac{2}{3}\epsilon^2$, $\rho = -\frac{1}{3} - \frac{2}{3}\epsilon^2 + \frac{4}{3}\epsilon^3$) and expanding up to order 3 in ϵ:

$$f_s^\epsilon(1,1) = p_s(\epsilon) + O(\epsilon^4) \qquad (s = 0, 1, 2, ...) \ . \tag{3.13}$$

In order to prove this statement we observe that, writing z in place of Δ_ϵ, for $s = 1, 2, 3, ...$ we have

$$f_s^\epsilon(1,1) = \frac{1}{2s}\mathrm{rem}\Big(\lambda(z) + \mu(z)\epsilon^2 + \frac{4}{3z^3}\epsilon^3\Big)^{2s} \ . \tag{3.14}$$

But

$$\Big(\lambda(z)+\mu(z)\epsilon^2+\frac{4}{3z^3}\epsilon^3\Big)^{2s} = \lambda^{2s}(z)+2s\mu(z)\lambda^{2s-1}(z)\epsilon^2+\frac{8s\lambda^{2s-1}(z)}{3z^3}\epsilon^3+O(\epsilon^4) \tag{3.15}$$

and so, taking the reminder, we obtain Eq.(3.13); the validity of this equation for $s = 0$ can also be checked easily.
The recombination rules in which we are interested stand on the following

3.2 *Definition.* *Let $s \in \{0, 1, 2, ...\}$ be a given integer, and consider two systems of coefficients $(c_{sj})_{j=-1,...,s}$ and $(d_{sj})_{j=0,...,[\frac{s}{2}]-1}$ (intending the second one to be empty if $s = 0$ or $s = 1$). These coefficients are said to satisfy the $C_s^{(2)}$ conditions if the following holds:*
$s = 0$:

$$c_{0,-1} + c_{00}p_0(\epsilon) = \epsilon^2 + O(\epsilon^3) \ , \tag{3.16}$$

$s = 1$:

$$c_{1,-1} + c_{10}p_0(\epsilon) + c_{11}p_1(\epsilon) = \epsilon^3 \ , \tag{3.17}$$

$s \geq 2$:

$$\sum_{j=1}^{s} c_{sj}q_j(e^\epsilon) + \sum_{j=0}^{[\frac{s}{2}]-1} d_{sj}\lambda^j(e^\epsilon) = 3^{[\frac{s}{2}]-1}\epsilon^{[\frac{3s}{2}]-2} + O(\epsilon^{[3s/2]-1}) \tag{3.18}$$

and

$$c_{s,-1} + \sum_{j=0}^{s} c_{sj}p_j(\epsilon) = 0 \ . \tag{3.19}$$

◇

For each integer s, the $C_s^{(2)}$ conditions correspond to a system of as many equations as the unknown coefficients. This is clear for $s = 0$ and 1. For

$s \geq 2$, we must equate the coefficients of all powers of ϵ from 0 to $[\frac{3s}{2}]-2$ in Eq.(3.18), and set to zero the coefficients of 1, ϵ^2, ϵ^3 in Eq. (3.19); so, the total number of equations to be satisfied is $1+([\frac{3s}{2}]-2)+3=[\frac{3s}{2}]+2$, which is just the number of coefficients (c_{sj}) and (d_{sj}).
From a given system of coefficients satisfying the $C^{(2)}_s$ conditions for some s, let us define the following linear combinations of operators, vector fields and Hamiltonians:

$$B^\epsilon_s := \sum_{j=1}^{s} c_{sj} A^\epsilon_j + \sum_{j=0}^{[\frac{s}{2}]-1} d_{sj} (L^\epsilon)^j \ , \tag{3.20}$$

$$Z^\epsilon_s := \sum_{j=1}^{s} c_{sj} X^\epsilon_j \ , \tag{3.21}$$

$$h^\epsilon_s := c_{s,-1} + \sum_{j=0}^{s} c_{sj} f^\epsilon_j \tag{3.22}$$

(intending $B^\epsilon_0 := 0$, $Z^\epsilon_0 := 0$). Then, we have by construction

$$\frac{dL^\epsilon}{d\theta_s} = [B^\epsilon_s, L^\epsilon] \ , \tag{3.23}$$

where $d/d\theta_s$ is the derivative along the vector field Z^ϵ_s, and

$$Z^\epsilon_s = Q^\epsilon dh^\epsilon_s \ . \tag{3.24}$$

If $s \geq 2$, let us evaluate the recombined operator at $\mathbf{u}=(0,0)$ and expand it in powers of ϵ. Recalling that $\Delta_{k\epsilon} = e^{k\epsilon\partial}$, and replacing ϵ with $\epsilon\partial$ in Eq.(3.18), we recognize this equation to be equivalent to

$$B^\epsilon_s(0,0) = 3^{[\frac{s}{2}]-1} \epsilon^{[\frac{3s}{2}]-2} \partial^{[\frac{3s}{2}]-2} + O(\epsilon^{[\frac{3s}{2}]-1}) \ = \epsilon^{[\frac{3s}{2}]-2} B^{Bou}_s(0,0) + O(\epsilon^{[\frac{3s}{2}]-1}) \ . \tag{3.25}$$

This clarifies the meaning of the first $C^{(2)}_s$ condition; the second one, corresponding to Eq.(3.19), means that the recombined Hamiltonian behaves as follows at $\mathbf{u}=(1,1)$:

$$h^\epsilon_s(1,1) = O(\epsilon^4) \ . \tag{3.26}$$

The $C^{(2)}_s$ conditions for $s=0$ and 1 consist only of Eq.s (3.16) (3.17), whose meaning is

$$h^\epsilon_0(1,1) = \epsilon^2 + O(\epsilon^3) = \epsilon^2 h^{Bou}_0(1,1) + O(\epsilon^3) \ , \tag{3.27}$$

$$h_1^\epsilon(1,1) = \epsilon^3 + O(\epsilon^4) = \epsilon^3 h_1^{Bou}(1,1) + O(\epsilon^4)\ . \tag{3.28}$$

Just as in the $N=1$ theory considered in the previous section, the above statements on the recombined operator and Hamiltonian at a particular point of the space $\mathcal{U}$ control the global behaviour of the recombinations (3.20–3.22):

3.3 *Proposition.* *Let $s \in \{0,1,2,...\}$ be a fixed integer. The $C_s^{(2)}$ conditions imply that at each point $\boldsymbol{u} \in \mathcal{U}$ it is:*

$$B_s^\epsilon(\boldsymbol{u}) = \epsilon^{[\frac{3s}{2}]-2} B_s^{Bou}(\boldsymbol{u}) + O(\epsilon^{[\frac{3s}{2}]-1})\ . \tag{3.29}$$

$$Z_s^\epsilon(\boldsymbol{u}) = \epsilon^{[\frac{3s}{2}]-2} Z_s^{Bou}(\boldsymbol{u}) + O(\epsilon^{[\frac{3s}{2}]-1})\ . \tag{3.30}$$

$$h_s^\epsilon(\boldsymbol{u}) = \epsilon^{[\frac{3s}{2}]+2} h_s^{Bou}(\boldsymbol{u}) + O(\epsilon^{[\frac{3s}{2}]+3})\ . \tag{3.31}$$

◇

The next section will be fully devoted to the proof of this Proposition. Independent of the general argument proposed therein, statements (3.29–3.31) can be checked by direct computation for the lowest values of s, say up to $s=4$. To perform calculations, it is necessary to know that

$$q_1(z) = \frac{1}{3}z^2 + \frac{4}{3}\ , \quad q_2(z) = \frac{1}{3}z^4 + \frac{8}{3}z^2 + \frac{68}{9}\ , \tag{3.32}$$

$$q_3(z) = \frac{1}{3}z^6 + 4z^4 + \frac{58}{3}z^2 + \frac{140}{3}\ ,$$

$$q_4(z) = \frac{1}{3}z^8 + \frac{16}{3}z^6 + \frac{328}{9}z^4 + \frac{1232}{9}z^2 + \frac{8092}{27}\ ;$$

$$p_0(\epsilon) = \epsilon^2 - 2\epsilon^3\ , \quad p_1(\epsilon) = 2 + \frac{2}{3}\epsilon^2\ , \quad p_2(\epsilon) = \frac{17}{3} + \frac{10}{3}\epsilon^2 + \frac{4}{3}\epsilon^3\ , \tag{3.33}$$

$$p_3(\epsilon) = \frac{70}{3} + \frac{170}{9}\epsilon^2 + \frac{40}{3}\epsilon^3\ , \qquad p_4(\epsilon) = \frac{2023}{18} + \frac{1022}{9}\epsilon^2 + \frac{980}{9}\epsilon^3\ .$$

The unique system of coefficients satisfying the $C_s^{(2)}$ condition for $s=0$ (i.e., Eq.(3.16)) is

$$c_{0,-1} = 0\ , \qquad c_{00} = 1\ ; \tag{3.34}$$

for $s=1$, we must satisfy Eq.(3.17), yielding the unique solution

$$c_{1,-1} = -\frac{3}{2}\ , \qquad c_{10} = -\frac{1}{2}\ , \qquad c_{11} = \frac{3}{4}\ . \tag{3.35}$$

Let us consider the recombined Hamiltonians

$$h_0^\epsilon = f_0^\epsilon \ , \tag{3.36}$$

$$h_1^\epsilon = -\frac{3}{2} - \frac{1}{2} f_0^\epsilon + \frac{3}{4} f_1^\epsilon \ . \tag{3.37}$$

Expressing the summands as in Table II, performing the substitution (3.2) and expanding in ϵ, we find

$$h_0^\epsilon(\mathbf{u}) = \frac{1}{2}\int dx \ \log(1+2\epsilon^2 u - 4\epsilon^3 v) = \epsilon^2 \int dx \ u + O(\epsilon^3) = \epsilon^2 h_0^{Bou}(\mathbf{u}) + O(\epsilon^3) \ , \tag{3.38}$$

$$h_1^\epsilon(\mathbf{u}) = -\frac{3}{2} - \frac{1}{4}\int dx \ \log(1+2\epsilon^2 u - 4\epsilon^3 v) + \frac{3}{4}\int dx \ (2 + \frac{2}{3}\epsilon^2 u) \tag{3.39}$$

$$= \epsilon^3 \int dx \ v + O(\epsilon^4) = \epsilon^3 h_1^{Bou}(\mathbf{u}) + O(\epsilon^4) \ ,$$

in agreement with Prop. 3.3. The other statements in this Proposition are trivially satisfied for $s = 0$ and 1 (note that $B_s^{Bou}(\mathbf{u}) = 0$ and $Z_s^{Bou}(\mathbf{u}) = 0$ in these cases).
For $s \geq 2$, the $C_s^{(2)}$ conditions are represented by Eq.s (3.18) (3.19); we report the solutions up to $s = 4$, which are unique. For $s = 2$, it is

$$c_{2,-1} = \frac{19}{8}, \ c_{20} = \frac{1}{4} \ , c_{21} = -\frac{9}{4} \ , c_{22} = \frac{3}{8} \ , d_{20} = -\frac{5}{24} \ ; \tag{3.40}$$

for $s = 3$, we find

$$c_{3,-1} = -\frac{37}{8}, \ c_{30} = -\frac{1}{4}, \ c_{31} = \frac{57}{8}, \ c_{32} = -\frac{21}{8}, \ c_{33} = \frac{9}{40}, \ d_{30} = \frac{1}{120}; \tag{3.41}$$

finally, the solution for $s = 4$ is

$$c_{4,-1} = \frac{147}{32}, \ c_{40} = \frac{1}{8}, \ c_{41} = -\frac{45}{4}, \ c_{42} = \frac{57}{8}, \ c_{43} = -\frac{27}{20}, \ c_{44} = \frac{9}{112}, \tag{3.42}$$

$$d_{40} = \frac{67}{48}, \qquad d_{41} = -\frac{18}{35}.$$

These coefficients can be employed to construct the recombinations (3.20) - (3.22); for example, for $s = 3$ we have

$$B_3^\epsilon = \frac{57}{8} A_1^\epsilon - \frac{21}{8} A_2^\epsilon + \frac{9}{40} A_3^\epsilon + \frac{1}{120} \ , \tag{3.43}$$

$$Z_3^\epsilon = \frac{57}{8} X_1^\epsilon - \frac{21}{8} X_2^\epsilon + \frac{9}{40} X_3^\epsilon \ , \tag{3.44}$$

$$h_3^\epsilon = -\frac{37}{8} - \frac{1}{4} f_0^\epsilon + \frac{57}{8} f_1^\epsilon - \frac{21}{8} f_2^\epsilon + \frac{9}{40} f_3^\epsilon \ . \tag{3.45}$$

On application of the transformation (3.2), the lowest order expansions in ϵ are

$$B_3^\epsilon(\mathbf{u}) = \epsilon^2 B_3^{Bou}(\mathbf{u}) + O(\epsilon^3) \ , \tag{3.46}$$

$$Z_3^\epsilon(\mathbf{u}) = \epsilon^2 Z_3^{Bou}(\mathbf{u}) + O(\epsilon^3) \ , \tag{3.47}$$

$$h_3^\epsilon(\mathbf{u}) = \epsilon^6 h_3^{Bou}(\mathbf{u}) + O(\epsilon^7) \tag{3.48}$$

(the Boussinesq objects being as in Table III).
Similar statements hold for $s = 4$; for example, the recombined Hamiltonian

$$h_4^\epsilon = \frac{147}{32} + \frac{1}{8} f_0^\epsilon - \frac{45}{4} f_1^\epsilon + \frac{57}{8} f_2^\epsilon - \frac{27}{20} f_3^\epsilon + \frac{9}{112} f_4^\epsilon \tag{3.49}$$

has the expansion

$$h_4^\epsilon(\mathbf{u}) = \epsilon^8 h_4^{Bou}(\mathbf{u}) + O(\epsilon^9) \ . \tag{3.50}$$

The justification of Eq.s (3.46– 3.48) and (3.50) by direct computation (rather than by the theoretical arguments of Sect.4) is a task of a great computational complexity. Using an authomatic manipulator is essential for this test; this is what we actually did ([2]) .

[2] The explicit expressions of the objects to be recombined for $s \leq 3$ are the ones written in Table II. For the sake of brevity, the printout for f_4^ϵ has not been reported in the Table.

4 Proof of Proposition 3.3

For $s = 0$ and 1, the verification of Eq.s (3.29–3.31) rests on the direct computation sketched in Sect.3. From now on, and up to the end of the present section, s is an arbitrary integer ≥ 2. The proof will be divided in three steps; in spite of technical differences, some basic ideas already employed in the previous paper [2] can be recognized.

Step 1: ϵ-expansions of the K_2 structures and their recombinations. It is evident that only nonnegative powers of ϵ will appear in the developments of the companion operators A^ϵ_s, the powers of L^ϵ and the recombinations (3.20). Let $q \geq 0$ be the exponent of the lowest order term in the ϵ-expansion of B^ϵ_s; then we can write

$$B^\epsilon_s(\mathbf{u}) = \sum_{m=q}^{+\infty} \epsilon^m B_{s,m}(\mathbf{u}) \ , \tag{4.1}$$

where $B_{s,m}(\mathbf{u})$ is, for each m, a differential operator with coefficients depending polynomially on the fields $\mathbf{u} = (u, v)$ and their x-derivatives. The nonnegative integer $q = q(s)$ will be determined in the following step; note that, by definition, the operator $B_{s,q}(\mathbf{u})$ is non zero, at least at some point $\mathbf{u} \in \mathcal{U}$. For the K_2 Lax operator, we have the expansion

$$L^\epsilon(\mathbf{u}) = \frac{8}{3} + \sum_{m=3}^{+\infty} \epsilon^m L_m(\mathbf{u}) \ , \tag{4.2}$$

where each term $L_m(\mathbf{u})$ is also a differential operator, depending polynomially on $\mathbf{u}$ and its derivatives; in particular

$$L_3(\mathbf{u}) = \frac{4}{3} L^{Bou}(\mathbf{u}) \tag{4.3}$$

(recall Eq.(3.4)). Now let us consider the Hamiltonians: $f^\epsilon_0(\boldsymbol{\alpha})$ is (the integral over x of) $\log(-3\rho)$ while $f^\epsilon_j(\boldsymbol{\alpha})$, for $j \geq 1$, is (the integral of) a polynomial expression in α, ρ and their shifts.
Expressing α and ρ via Eq.(3.2), we see that their shifts of any order k are

$$\begin{cases} \alpha_{[k\epsilon]} = 2 + \frac{2}{3}\epsilon^2 u_{[k\epsilon]} = 2 + \frac{2}{3}\epsilon^2 u + \frac{2}{3}\epsilon^3 k u_x + O(\epsilon^4) \\ \rho_{[k\epsilon]} = -\frac{1}{3} - \frac{2}{3}\epsilon^2 u_{[k\epsilon]} + \frac{4}{3}\epsilon^3 v_{[k\epsilon]} = -\frac{1}{3} - \frac{2}{3}\epsilon^2 u - \frac{2}{3}\epsilon^3 k u_x + \frac{4}{3}\epsilon^3 v + O(\epsilon^4) \ . \end{cases} \tag{4.4}$$

This implies that the expansion up to order 3 of any of the Hamiltonians f^ϵ_j is of the form

$$\text{const.} + \text{const.}\ \epsilon^2 \int dx\ u + \text{const.}\ \epsilon^3 \int dx\ v + \text{const.}\ \epsilon^3 \int dx\ u_x + O(\epsilon^4)\ ;$$

the third integral vanishes due to the presence of a total x-derivative. Similar conclusions hold for the recombination h^ϵ_s, so there are three real constants η_s, κ_s, χ_s such that

$$h^\epsilon_s(\mathbf{u}) = \eta_s + \kappa_s \epsilon^2 \int dx\ u + \chi_s \epsilon^3 \int dx\ v + O(\epsilon^4) \tag{4.5}$$

Comparing this result with Eq.(3.26) (expressing the second $C^{(2)}_s$ condition) we easily infer $\eta_s = \kappa_s = \chi_s = 0$, i.e. $h^\epsilon_s(\mathbf{u}) = O(\epsilon^4)$. More explicitly, we can write

$$h^\epsilon_s(\mathbf{u}) = \sum_{m=4}^{+\infty} \epsilon^m h_{s,m}(\mathbf{u})\ , \tag{4.6}$$

where each term $h_{s,m}(\mathbf{u})$ is (the integral over x of) a polynomial density in the fields $\mathbf{u}$ and their x-derivatives.
We go on in expanding and consider the K_2 Poisson tensor. Recalling Eq.(3.7), we can write

$$Q^\epsilon_{\mathbf{u}} = \sum_{m=-4}^{+\infty} \epsilon^m Q_{m,\mathbf{u}}\ , \tag{4.7}$$

where each term is a matrix differential operator, the first one being the Boussinesq Poisson tensor:

$$Q_{-4,\mathbf{u}} = Q^{Bou}_{\mathbf{u}}\ . \tag{4.8}$$

Now, let us consider the recombined vector field Z^ϵ_s; from the Hamiltonian formulation (3.23), from the previous expansion of Q^ϵ and Eq. (4.6), we infer

$$Z^\epsilon_s(\mathbf{u}) = \sum_{m=0}^{\infty} \epsilon^m Z_{s,m}(\mathbf{u})\ , \tag{4.9}$$

where $Z_{s,m}(\mathbf{u})$ depends polynomially on $\mathbf{u}$ and its derivatives. All these expansions will be used in the rest of the proof.

Step 2. Refinement in the expansions of B^ϵ_s, Z^ϵ_s. From now on, we denote by $\mathcal{L}_W$ the Lie derivative along any vector field W; with this notation, the Lax formulation (3.23) of Z^ϵ_s is written as

$$\mathcal{L}_{Z^\epsilon_s} L^\epsilon = [B^\epsilon_s, L^\epsilon] \ . \tag{4.10}$$

We insert in this equation the expansions (4.9) (4.2) (4.1), and infer that

$$Z_{s,m}(\mathbf{u}) = 0 \qquad \text{for } m < q \ . \tag{4.11}$$

This is obvious if $q = 0$; if $q > 0$ (which is indeed the case, as we will see later on), Eq.(4.11) follows from comparison between the two sides of Eq.(4.10). In principle, the l.h.s. contains all powers of ϵ of exponent ≥ 3; on the other hand the r.h.s. is clearly $O(\epsilon^{q+3})$, so the coefficients of $\epsilon^3, \epsilon^4, ..., \epsilon^{q+2}$ in the l.h.s must be zero, yielding the equations

$$\mathcal{L}_{Z_{s,0}} L_3 = 0 \tag{4.12}$$

$$\mathcal{L}_{Z_{s,0}} L_4 + \mathcal{L}_{Z_{s,1}} L_3 = 0 \tag{4.13}$$

$$\dots \qquad \dots \qquad \dots$$

$$\mathcal{L}_{Z_{s,0}} L_{q+2} + \mathcal{L}_{Z_{s,1}} L_{q+1} + ...\mathcal{L}_{Z_{s,q-1}} L_3 = 0 \ . \tag{4.14}$$

From the explicit expression of $L_3 = \frac{4}{3} L^{Bou}$ it is evident that $\mathcal{L}_W L_3 = 0$ for a vector field W iff W is identically zero. So, Eq.(4.12) implies $Z_{s,0} = 0$; inserting this result in Eq.(4.13) we obtain $\mathcal{L}_{Z_{s,1}} L_3 = 0$, which implies in turn $Z_{s,1} = 0$; iteration of this argument yields Eq.(4.11).
Now, let us return to Eq.(4.10), and equate the coefficients of ϵ^{q+3} in the two sides; the operator $L_3 = \frac{4}{3} L^{Bou}$ is still involved, and we obtain

$$\mathcal{L}_{Z_{s,q}} L^{Bou} = [B_{s,q}, L^{Bou}] \ . \tag{4.15}$$

This means that, at each point $\mathbf{u} \in \mathcal{U}$, the differential operator $B_{s,q}(\mathbf{u})$ is compatible with the Boussinesq Lax operator (i.e., that its commutator with $L^{Bou}(\mathbf{u})$ is tangent to the Lax submanifold in the space of differential operators). From here, and from a known result of Drinfeld and Sokolov [5], it follows that $B_{s,q}(\mathbf{u})$ is a linear combination of the form

$$B_{s,q}(\mathbf{u}) = \sum_i \gamma_{s,i} (L^{Bou})^{\frac{i}{3}}_{+}(\mathbf{u}) \ , \tag{4.16}$$

where i is an integer index, running over a finite set of values; in principle, the real coefficients $\gamma_{s,i}$ could be functionals of the variables $\mathbf{u}$, but the features of the B^ϵ_s expansion ensure that they are constant ([3]); we observe that it cannot be $\gamma_{s,i} = 0$ for each i, for this would contradict the assumption $B_{s,q} \neq 0$. Let us apply Eq.(4.16) at $\mathbf{u} = (0,0)$; in this way we obtain $B_{s,q}(0,0) = \sum_i \gamma_{s,i}\partial^i$, so that we can write

$$B^\epsilon_s(0,0) = \epsilon^q \sum_i \gamma_{s,i}\partial^i + O(\epsilon^{q+1}) \ . \tag{4.17}$$

Comparing this equation with Eq.(3.25) (expressing the first $C^{(2)}_s$ condition), we infer

$$q = [\frac{3s}{2}] - 2 \tag{4.18}$$

$$\gamma_{s,i} = \begin{cases} 3^{[\frac{s}{2}]-1} & \text{if } i = [\frac{3s}{2}] - 2 \\ 0 & \text{otherwise} \ . \end{cases} \tag{4.19}$$

From here and from Eq.(4.16) it follows that

$$B_{s,q}(\mathbf{u}) = 3^{[\frac{s}{2}]-1}(L^{Bou})_+^{\frac{1}{3}[\frac{3s}{2}]-\frac{2}{3}}(\mathbf{u}) = B^{Bou}_s(\mathbf{u}) \tag{4.20}$$

at any $\mathbf{u} \in \mathcal{U}$. Inserting this result into Eq.(4.15), we also obtain

$$Z_{s,q}(\mathbf{u}) = Z^{Bou}_s(\mathbf{u}) \ . \tag{4.21}$$

Eq.s (4.18) (4.20) (4.21) yield statements (3.29–3.30) in Prop. 3.3. To conclude the proof, we must derive Eq.(3.31), which will be attained in the sequel.

Step 3. ϵ-expansion of h^ϵ_s. Let us insert into Eq.(3.24) the expansions (4.7) (4.6) (4.9); recalling that $Z_{s,m} = 0$ for $0 \le m \le q-1$, we infer that the coefficients of ϵ^m in the r.h.s must also be zero for $0 \le m \le q-1$, yielding the equations

$$Q_{-4,\mathbf{u}} d\mathbf{u} h_{s,4} = 0 \ , \tag{4.22}$$

$$Q_{-4,\mathbf{u}} d\mathbf{u} h_{s,5} + Q_{-3,\mathbf{u}} d\mathbf{u} h_{s,4} = 0 \ , \tag{4.23}$$

$$\cdots \qquad \cdots \qquad \cdots$$

[3] Otherwise, the coefficients of the differential operator $B_{s,q}(\mathbf{u})$ would not be differential polynomials in $\mathbf{u}$.

$$Q_{-4,\mathbf{u}} d_{\mathbf{u}} h_{s,q+3} + Q_{-3,\mathbf{u}} d_{\mathbf{u}} h_{s,q+2} + \dots + Q_{q-5,\mathbf{u}} d_{\mathbf{u}} h_{s,4} = 0 \ . \tag{4.24}$$

Eq.(4.22) tells us that $h_{s,4}$ is a Casimir of the Poisson tensor $Q^{Bou} = Q_{-4}$; on the other hand, the only Casimirs of the Boussinesq theory which are (x-integrals of) differential polynomial densities in $\mathbf{u}$ are (up to constants) linear combinations with constant coefficients of h_0^{Bou} and h_1^{Bou}, so we can write

$$h_{s,4}(\mathbf{u}) = \eta_s + \chi_s \int dx \ u + \kappa_s \int dx \ v \ . \tag{4.25}$$

We claim that the coefficients η_s, χ_s and κ_s are zero; indeed by examination of the transformation (3.2) one infers that neither constants nor terms proportional to $\int dx \ u$ or $\int dx \ v$ can appear in the expansion of h_s^ϵ ([4]). In conclusion, we have $h_{s,4} = 0$; inserting this result into Eq. (4.23) we obtain $Q_{-4} dh_{s,5} = Q^{Bou} dh_{s,5} = 0$, and this again implies $h_{s,5} = 0$. Iterating this argument, we infer

$$h_{s,m} = 0 \qquad \text{for } m \leq q + 3 \ . \tag{4.26}$$

Now, let us return to Eq.(3.24) and equate the coefficients of ϵ^q in the expansions of the two sides; since $Z_{s,q} = Z_s^{Bou}$ and $Q_{-4} = Q^{Bou}$, we obtain

$$Z_s^{Bou}(\mathbf{u}) = Q_{\mathbf{u}}^{Bou} d_{\mathbf{u}} h_{s,q+4} \ , \tag{4.27}$$

which implies

$$h_{s,q+4} = h_s^{Bou} \tag{4.28}$$

up to the addition of a Casimir; the same argument employed above shows that this additive term is necessarily zero. The last equation (with the result (4.18) for q) leads to Eq.(3.31) of Prop. 3.3, whose proof is now concluded.

◇

[4] A constant term could appear at most at order 0 in ϵ; $\int dx \ u$ and $\int dx \ v$ could appear at most at orders 2 and 3, respectively. On the other hand, we know that $h_{s,0} = h_{s,2} = h_{s,3} = 0$.

5 A sketch of the continuous limit for the K_N system.

The Lax operator for the K_N system with any number N of fields $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)$ is written in the Introduction, see Eq.(1.3). The vector fields for this sistem have the Lax formulations $dL^\epsilon/dt_s = [A_s^\epsilon, L^\epsilon]$, where $A_s^\epsilon(\boldsymbol{\alpha}) := \frac{1}{N+1}(L^\epsilon)_+^{2s}(\boldsymbol{\alpha})$ for $s = 0, 1, 2, \ldots$. With our notations, the K_N Poisson tensor at any point $\boldsymbol{\alpha}$ sends an N-tuple $\delta\boldsymbol{\alpha} = (\delta\alpha_k)_{k=1,...,N}$ into $\dot{\boldsymbol{\alpha}} = (\dot{\alpha}_j)_{j=1,...,N}$, where

$$\dot{\alpha}_j = \frac{1}{N+1}\sum_{j=1}^{N} Q_{jk}^\epsilon\ \delta\alpha_k\ , \tag{5.1}$$

$$Q_{jk}^\epsilon := \alpha_{k+j[2k\epsilon]}\Delta_{2k\epsilon} - \alpha_{k+j}\Delta_{-2j\epsilon} + \Big(\sum_{l=0}^{2k-1} - \sum_{l=1-2j}^{2k-2j}\Big)\alpha_{k[l\epsilon]}\alpha_j\Delta_{l\epsilon} \tag{5.2}$$

$$+\Big(\sum_{l=1}^{k-1} - \sum_{l=k-N}^{k-j-1}\Big)\alpha_{k-l}\alpha_{j+l[2l\epsilon]}\Delta_{2l\epsilon}\ .$$

In the above sums, it is intended $\alpha_i := 0$ if $i < 1$ or $i > N$ and $\sum_{l=a}^{b} := 0$ if $a > b$.
Each vector field of the K_N hierarchy is Hamiltonian with respect to Q^ϵ, the Hamiltonians being the traces of even powers of L^ϵ.
It was shown in [4] that, under an appropriate field rescaling, the leading term in the ϵ expansion of $L^\epsilon(\boldsymbol{\alpha})$ is the $sl(N+1)$ KdV-type Lax operator. With our notations, the rescaling is the map $\mathbf{u} \mapsto \boldsymbol{\alpha} = \Theta^\epsilon(\mathbf{u})$ given by

$$\alpha_k = \frac{(-1)^{k-1}}{(2k-1)}\left(\begin{array}{c} N \\ k \end{array}\right) + \frac{(-1)^{k-1}}{2N+2}\sum_{m=2}^{k+1}(-2)^m \left(\begin{array}{c} N+1-m \\ k+1-m \end{array}\right)\epsilon^m u_{m-1}\ . \tag{5.3}$$

For $N = 1$, $\alpha_1 := \alpha$, $u_1 := u$ and for $N = 2$, $\alpha_1 := \alpha, \alpha_2 := \rho$, $u_1 := v$, $u_2 := u$ this trasformation gives, respectively, the mappings (2.1) and (3.2). For N arbitrary, the analysis of [4] shows that

$$L^\epsilon(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\Theta^\epsilon(\mathbf{u})} = \Big(1 + \sum_{l=1}^{N}\frac{(-1)^{l-1}}{2l-1}\left(\begin{array}{c} N \\ l \end{array}\right)\ \Big) + \frac{2^{N+1}}{2N+2}\epsilon^{N+1}L(\mathbf{u}) + O(\epsilon^{N+2})\ , \tag{5.4}$$

where $L(\mathbf{u})$ is the Lax operator of the $sl(N+1)$ KdV theory (see Eq.(1.4)). Combining this result with the method of Sect.s 2–3, one could show that

appropriate recombinations of the companion operators A^ϵ_s and of the K_N vector fields give, in the $\epsilon \mapsto 0$ limit, the homologous objects of the $sl(N+1)$ KdV theory.
The Hamiltonian formalism of this theory (with a recombination scheme for the Hamiltonians) could also be reconstructed, provided that one proves the following: if the transformation (5.3) is applied to the Kupershmidt Poisson tensor (5.2), the leading term in the ϵ expansion is the first $sl(N+1)$ Poisson structure. In principle, the verification of this statement could be performed directly, using the explicit expressions for the K_N Poisson tensor and the field rescaling. As a matter of fact, the computation is very difficult for N arbitrary, so we have limited the verification to the case $N=3$.
Let us introduce the notations $\alpha_1 := \alpha$, $\alpha_2 := \rho$, $\alpha_3 := \sigma$. The K_3 Poisson tensor has the matrix representation

$$Q^\epsilon_{\boldsymbol{\alpha}} = \frac{1}{4}\begin{pmatrix} Q^\epsilon_{\alpha\alpha} & Q^\epsilon_{\alpha\rho} & Q^\epsilon_{\alpha\sigma} \\ Q^\epsilon_{\rho\alpha} & Q^\epsilon_{\rho\rho} & Q^\epsilon_{\rho\sigma} \\ Q^\epsilon_{\sigma\alpha} & Q^\epsilon_{\sigma\rho} & Q^\epsilon_{\sigma\sigma} \end{pmatrix} , \tag{5.5}$$

$$\begin{aligned}
Q^\epsilon_{\alpha\alpha} &:= \rho_{[2\epsilon]}\Delta_{2\epsilon} + \alpha\alpha_{[\epsilon]}\Delta_\epsilon - \alpha\alpha_{[-\epsilon]}\Delta_{-\epsilon} - \rho\Delta_{-2\epsilon} \ , \\
Q^\epsilon_{\alpha\rho} &:= \sigma_{[4\epsilon]}\Delta_{4\epsilon} + \alpha\rho_{[3\epsilon]}\Delta_{3\epsilon} + \alpha\rho_{[2\epsilon]}\Delta_{2\epsilon} - \alpha\rho - \alpha\rho_{[-\epsilon]}\Delta_{-\epsilon} - \sigma\Delta_{-2\epsilon} \ , \\
Q^\epsilon_{\alpha\sigma} &:= \alpha\sigma_{[5\epsilon]}\Delta_{5\epsilon} + \alpha\sigma_{[4\epsilon]}\Delta_{4\epsilon} - \alpha\sigma - \alpha\sigma_{[-\epsilon]}\Delta_{-\epsilon} \ , \\
Q^\epsilon_{\rho\alpha} &:= \sigma_{[2\epsilon]}\Delta_{2\epsilon} + \alpha_{[\epsilon]}\rho\Delta_\epsilon + \alpha\rho - \alpha_{[-2\epsilon]}\rho\Delta_{-2\epsilon} - \alpha_{[-3\epsilon]}\rho\Delta_{-3\epsilon} - \sigma\Delta_{-4\epsilon} \ , \\
Q^\epsilon_{\rho\rho} &:= \rho_{[3\epsilon]}\rho\Delta_{3\epsilon} + (\alpha\sigma_{[2\epsilon]} + \rho_{[2\epsilon]}\rho)\Delta_{2\epsilon} + \rho_{[\epsilon]}\rho\Delta_\epsilon - \rho_{[-\epsilon]}\rho\Delta_{-\epsilon} \\
&\quad -(\alpha_{[-2\epsilon]}\sigma + \rho_{[-2\epsilon]}\rho)\Delta_{-2\epsilon} - \rho_{[-3\epsilon]}\rho\Delta_{-3\epsilon} \ , \\
Q^\epsilon_{\rho\sigma} &:= \rho\sigma_{[5\epsilon]}\Delta_{5\epsilon} + \rho\sigma_{[4\epsilon]}\Delta_{4\epsilon} + \rho\sigma_{[3\epsilon]}\Delta_{3\epsilon} + \rho\sigma_{[2\epsilon]}\Delta_{2\epsilon} - \rho\sigma - \rho\sigma_{[-\epsilon]}\Delta_{-\epsilon} - \rho\sigma_{[-2\epsilon]}\Delta_{-2\epsilon} \\
&\quad -\rho\sigma_{[-3\epsilon]}\Delta_{-3\epsilon} \ , \\
Q^\epsilon_{\sigma\alpha} &:= \alpha_{[\epsilon]}\sigma\Delta_\epsilon + \alpha\sigma - \alpha_{[-4\epsilon]}\sigma\Delta_{-4\epsilon} - \alpha_{[-5\epsilon]}\sigma\Delta_{-5\epsilon} \ , \\
Q^\epsilon_{\sigma\rho} &:= \rho_{[3\epsilon]}\sigma\Delta_{3\epsilon} + \rho_{[2\epsilon]}\sigma\Delta_{2\epsilon} + \rho_{[\epsilon]}\sigma\Delta_\epsilon + \rho\sigma - \rho_{[-2\epsilon]}\sigma\Delta_{-2\epsilon} - \rho_{[-3\epsilon]}\sigma\Delta_{-3\epsilon} \\
&\quad -\rho_{[-4\epsilon]}\sigma\Delta_{-4\epsilon} - \rho_{[-5\epsilon]}\sigma\Delta_{-5\epsilon} \ , \\
Q^\epsilon_{\sigma\sigma} &:= \sigma_{[5\epsilon]}\sigma\Delta_{5\epsilon} + \sigma\sigma_{[4\epsilon]}\Delta_{4\epsilon} + \sigma\sigma_{[3\epsilon]}\Delta_{3\epsilon} + \sigma_{[2\epsilon]}\sigma\Delta_{2\epsilon} + \sigma\sigma_{[\epsilon]}\Delta_\epsilon - \sigma_{[-\epsilon]}\sigma\Delta_{-\epsilon} \\
&\quad -\sigma_{[-2\epsilon]}\sigma\Delta_{-2\epsilon} - \sigma_{[-3\epsilon]}\sigma\Delta_{-3\epsilon} - \sigma_{[-4\epsilon]}\sigma\Delta_{-4\epsilon} - \sigma_{[-5\epsilon]}\sigma\Delta_{-5\epsilon} \ .
\end{aligned}$$

Setting $u_1 := u$, $u_2 := v$, $u_3 := w$, the transformation (5.3) takes the form

$$\begin{cases} \alpha = 3 + \frac{1}{2}\epsilon^2 u \\ \rho = -1 - \epsilon^2 u + \epsilon^3 v \\ \sigma = \frac{1}{5} + \frac{1}{2}\epsilon^2 u - \epsilon^3 v + 2\epsilon^4 w \ . \end{cases} \tag{5.6}$$

The pullback of the Poisson tensor (5.5) along this transformation is described again by Eq.(3.5). Explicitly, at each point $\mathbf{u} = (u, v, w)$ we have

$$Q^\epsilon_{\mathbf{u}} = \begin{pmatrix} \frac{2}{\epsilon^2} & 0 & 0 \\ \frac{2}{\epsilon^3} & \frac{1}{\epsilon^3} & 0 \\ \frac{1}{2\epsilon^4} & \frac{1}{2\epsilon^4} & \frac{1}{2\epsilon^4} \end{pmatrix} \begin{pmatrix} \frac{1}{4}Q^\epsilon_{\alpha\alpha} & \frac{1}{4}Q^\epsilon_{\alpha\rho} & \frac{1}{4}Q^\epsilon_{\alpha\sigma} \\ \frac{1}{4}Q^\epsilon_{\rho\alpha} & \frac{1}{4}Q^\epsilon_{\rho\rho} & \frac{1}{4}Q^\epsilon_{\rho\sigma} \\ \frac{1}{4}Q^\epsilon_{\sigma\alpha} & \frac{1}{4}Q^\epsilon_{\sigma\rho} & \frac{1}{4}Q^\epsilon_{\sigma\sigma} \end{pmatrix} \begin{pmatrix} \frac{2}{\epsilon^2} & \frac{2}{\epsilon^3} & \frac{1}{2\epsilon^4} \\ 0 & \frac{1}{\epsilon^3} & \frac{1}{2\epsilon^4} \\ 0 & 0 & \frac{1}{2\epsilon^4} \end{pmatrix} . \tag{5.7}$$

Expanding in ϵ, we have found

$$Q^\epsilon_{\mathbf{u}} = \frac{1}{5\epsilon^5} \begin{pmatrix} 0 & 0 & 4\partial_x \\ 0 & 4\partial_x & 2\partial_{xx} \\ 4\partial_x & -2\partial_{xx} & 2\partial_{xxx} + 2u\partial_x + u_x \end{pmatrix} + O(\frac{1}{\epsilon^4}) . \tag{5.8}$$

So, the continuous limit of the K_3 Poisson structure has the expected behaviour: the coefficient of $(1/\epsilon^5)$ is recognized to be the first Poisson tensor of the $sl(4)$ KdV theory.

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