# ON THE CONTINUOUS LIMIT OF INTEGRABLE LATTICES II. VOLTERRA SYSTEMS AND SP(N) THEORIES


Carlo Morosi *
Livio Pizzocchero **

* Dipartimento di Matematica, Politecnico di Milano
Piazza L. da Vinci 32, I-20133 Milano, Italy
e-mail carmor@ mate.polimi.it
** Dipartimento di Matematica, Università di Milano
Via C. Saldini 50, I-20133 Milano, Italy
and Istituto Nazionale di Fisica Nucleare,
Sezione di Milano, Italy
e-mail pizzocchero@ elanor.mat.unimi.it



Abstract. A connection is suggested between the zero-spacing limit of a generalized N-fields Volterra ($V_N$) lattice and the KdV-type theory which is associated, in the Drinfeld-Sokolov classification, to the simple Lie algebra $sp(N)$. As a preliminary step, the results of the previous paper [1] are suitably reformulated and identified as the realization for $N = 1$ of the general scheme proposed here. Subsequently, the case $N = 2$ is analyzed in full detail; the infinitely many commuting vector fields of the $V_2$ system (with their Hamiltonian structure and Lax formulation) are shown to give in the continuous limit the homologous $sp(2)$ KdV objects, through conveniently specified operations of field rescaling and recombination. Finally, the case of arbitrary $N$ is attacked, showing how to obtain the $sp(N)$ Lax operator from the continuous limit of the $V_N$ system.

## 1. Introduction

In a previous paper [1] we have reconstructed some essential features of ordinary KdV theory (the Lax pairs, the biHamiltonian formulation, the hierarchy of conserved functionals and the associated vector fields, the one-soliton solutions) from the zero-spacing limit of the so-called Kac-Moerbeke (KM) system (in fact, we were unaware that the results presented therein on the vector fields and the Poisson structures had also been obtained in [2]).

The mainstream in which these studies are inserted is the analysis of the procedures allowing to obtain the KdV-type theories from the zero-spacing limit of integrable Toda-type lattices. An introduction to the topic of integrable lattices and their continuous limits, together with some historical references, has been given in [1]; on this point, we take the occasion to cite also [3] [4].

The KM system considered in [1] [2] is a reduction of the Toda lattice: using the Flaschka coordinates $(a, b)$ (where $a = (a(i))$ and $b = (b(i))$ are sequences, the index $i$ running over the lattice sites) we can describe the KM flows as the restriction of (half of) the infinitely many commuting Toda flows to the submanifold $b = 0$.

Of course, it would be desirable to extend as much as possible the above results to appropriate classes of lattices, recovering from the continuous limit of each class a family of KdV-type theories. In this paper, we pursue this program in connection with a series $V_N$ $(N = 1, 2, ...)$ of integrable lattices, hereby denominated *Volterra systems*. We suggest a connection between the zero-spacing limit of these lattices and a class of KdV-type theories associated, in the Drinfeld-Sokolov classification [5], to the simple Lie algebras $sp(N)$ of the symplectic series[1] .

The KM system is just the Volterra system $V_1$, whereas the ordinary KdV appears to be connected with the rank 1 Lie algebra $sp(1)$ (in fact isomorphic to both algebras $sl(2)$ and $so(3)$, more frequently associated to this theory). In the present paper, we work out in full detail the $V_2$ system, yielding the $sp(2)$ KdV in the continuous limit. We also show how to obtain, for an arbitrary integer $N$, the Lax operator of the $sp(N)$ KdV theory from the zero-spacing limit of the $V_N$ Lax operator; this is a basic step for reconstructing the full scheme exploited in the cases $N = 1, 2$.

Before going into technical details about the $V_N$ and $sp(N)$ theories, let us introduce the general ideas on which the limiting process is based.

The first step in this framework is the interpolation of the lattice, i.e., the replacement of the sequences of phase space coordinates with smooth functions. For example, in the discrete version of KM theory a point of the phase space is a sequence $(a(i))$, indexed by the lattice sites; in the interpolated version, this sequence is replaced with a smooth function $a$ of a continuous variable $x$. To help intuition, one can think the lattice sites to be arranged at the points $x_i = i\epsilon$, where $\epsilon$ is a fundamental spacing which will be sent ultimately to zero. We can interpolate in this way both infinite and periodic lattices, letting the $x$ variable run over the real line $\mathbb{R}$ or the torus $\mathbb{R}/\mathbb{Z}$; in this paper, we will consider for definiteness the periodic case, but the results presented here could be adapted to the infinite case $x \in \mathbb{R}$

[1] For each $N$, $sp(N)$ is the Lie algebra of the symplectic group $Sp(N)$, formed by the $2N \times 2N$ matrices preserving the standard symplectic form, see for instance [6].

(with obvious technical adjustements, of the type described in [1]).
In this framework, we are naturally led to consider the differential-difference operators acting on (conveniently defined spaces of) smooth functions of $x$. For each displacement $\eta$ and each function $p$ of $x$, let us denote by $p_{[\eta]}$ the shifted function given by $p_{[\eta]}(x) := p(x+\eta)$; furthermore, let $\Delta_\eta$ be the shift operator sending $p$ into $p_{[\eta]}$. If $f$ is a given function of $x$, $f\Delta_\eta$ stands for the composition of the shift and the operator of multiplication by $f$. The Lax operators and the other relevant linear operators for the interpolated systems of interest can be represented within this scheme. For example the KM Lax operator, depending on the fundamental spacing $\epsilon$, is $L^\epsilon(a) = \frac{1}{2}(a\Delta_\epsilon + a_{[-\epsilon]}\Delta_{-\epsilon})$; this is obtained with the constraint $b=0$ from the Toda Lax operator $L^\epsilon(a,b) = \frac{1}{2}(a\Delta_\epsilon + a_{[-\epsilon]}\Delta_{-\epsilon}) + b$.
The subsequent step in this scheme is the introduction of a transformation ("field rescaling") relating the field variables of the interpolated lattice to another system of functions, to be ultimately identified with the field variables of the KdV-type theory associated to the lattice. The simplest example is the transformation $a(x) = 2 + \epsilon^2 u(x)$, giving the KM field variable $a$ in terms of a new variable $u$.
The final step is the $\epsilon \mapsto 0$ limit. In the particular case of KM theory, it is found that the differential-difference Lax operator goes into the KdV Lax operator $L^{sp(1)}(u) = \partial^2 + u$ (in the precise sense given by Eq.(2.10) of the next Section). As a matter of fact, the KdV biHamiltonian structure, the vector fields and Hamiltonian functions of the hierarchy, as well as the fractional powers of the Lax operator can also be obtained from the $\epsilon \mapsto 0$ limits of conveniently defined linear combinations of homologous KM objects; giving the rules to form these recombinations, in [1] we have pointed out some formal analogy with the Wick reordering, well known to field theorists.
In the present paper, the extension of the previous results to the $V_2$ system and the $sp(2)$ KdV is obtained through a new formulation of the recombination scheme, based on a system of real polynomials which are naturally associated to the problem under consideration. The need for this new formulation (also possible, but not indispensable for $V_1$) arises from a peculiar feature of the $V_N$ systems for each $N \geq 2$, i.e., the fact that these hierarchies, even though Hamiltonian and equipped with Lax pairs, do not seem to possess a second Hamiltonian structure (at least of a local type; this fact was already noticed for other integrable lattices, see [7]). Concerning the case of general $N$, we work out only the field rescaling and the continuous limit of the Lax operator.
After these considerations, let us give the necessary informations about the systems considered in this paper. Let $N$ be any positive integer; the phase space of the (interpolated) $V_N$ system is a set $\mathcal{A}$ of $N$-tuples $\mathbf{a} = (a_1, a_2, ..., a_N)$ where, for each $p$, $a_p$ is a real smooth function of the variable $x$. The Lax operator for this system at any point $\mathbf{a}$ is the symmetric operator[2]

$$L^\epsilon(\mathbf{a}) := \frac{1}{2}\sum_{p=1}^{N} \left(a_p \Delta_{(2p-1)\epsilon} + a_{p[(-2p+1)\epsilon]}\Delta_{(-2p+1)\epsilon}\right) . \qquad (1.1)$$

[2] Here and in the sequel, the notions of symmetry or skew-symmetry for differential-difference or differential operators are always referred to the pairing $<\psi,\psi'>:= \int dx\ \psi\psi'$, where $\int$ denotes integration over the torus $\mathbb{R}/\mathbb{Z}$.

The $V_N$ system entails infinitely many commuting vector fields $X^\epsilon_s$ $(s = 0, 1, 2, ...)$ on $\mathcal{A}$, admitting the Lax formulation

$$\frac{dL^\epsilon}{dt_s} = [A^\epsilon_s, L^\epsilon] \tag{1.2}$$

(where $d/dt_s$ stands for the Lie derivative along $X^\epsilon_s$). The "companion operators" $A^\epsilon_s$ in this equation are defined by

$$A^\epsilon_s(a) := \text{const.}(L^\epsilon)^{2s}_{\text{skew}}(a) \ , \tag{1.3}$$

where the normalization constant (to be fixed according to the specific needs) may depend on $N$ and $s$; in this equation, *skew* denotes the "skew-symmetric part": for each differential-difference operator $G = \sum_{j=-\infty}^{j_{\max}} g_j \Delta_{j\epsilon}$,

$$G_{\text{skew}} := \sum_{j=1}^{j_{\max}} (g_j \Delta_{j\epsilon} - g_{j\ [-j\epsilon]} \Delta_{-j\epsilon}) \ . \tag{1.4}$$

The vector fields $X^\epsilon_s$ are Hamiltonian, the Hamiltonian functions being the traces of the even powers of $L^\epsilon$. For $N = 1, 2$, the explicit expressions of the Poisson tensor and all the details about the Hamiltonians can be found in Tables Ia and IIa; for $N$ arbitrary, the $V_N$ Poisson tensor can be described in terms of $R$-matrix theory, as we will briefly explain hereafter.
The $V_N$ theory can be viewed as a reduction of a Toda-like theory in $2N$ fields $\mathbf{a} = (a_1, a_2, ..., a_N)$ and $\mathbf{b} = (b_1, b_2, ..., b_N)$, based on the Lax operator

$$L^\epsilon(\mathbf{a}, \mathbf{b}) := \frac{1}{2} \sum_{p=1}^{N} \left(a_p \Delta_{(2p-1)\epsilon} + a_{p[(-2p+1)\epsilon]} \Delta_{(-2p+1)\epsilon}\right) \tag{1.5}$$

$$+ \frac{1}{2} \sum_{p=1}^{N} \left(b_p \Delta_{(2p-2)\epsilon} + b_{p[(-2p+2)\epsilon]} \Delta_{(-2p+2)\epsilon}\right) \ ,$$

which becomes the $V_N$ operator for $\mathbf{b} = 0$. For $N = 1$, one has the standard Toda lattice; the description of KM system as a reduction of Toda can be found for example in [1]. It is well known that the Toda lattice can be described in terms of R-matrix theory; we will refer for definiteness to the $R$-matrix employed in [8, Sect. 5]. Here, the algebra of differential-difference operators is splitted as a direct sum of two Lie subalgebras (the skew-symmetric operators and the "nonpositive order" operators, formed with the shifts $\Delta_{j\epsilon}$ of orders $j \le 0$); the corresponding $R$-matrix gives in a standard way a linear and a quadratic Poisson structure [9] [10]. Both Poisson tensors can be employed for $N$ arbitrary to construct the biHamiltonian formulation of the theory with the Lax operator (1.5). It turns out that the linear tensor cannot be restricted to the submanifold $\mathbf{b} = 0$; on the contrary, the restriction is possible for the quadratic tensor, yielding the previously mentioned Hamiltonian formulation of the $V_N$ system.
We now pass to the $sp(N)$ KdV-type theories. Let the positive integer $N$ be arbitrary; the phase space of the corresponding theory is a set $\mathcal{U}$ of $N$-tuples $\mathbf{u} =$

$(u_1, ..., u_N)$ , where $u_l$ is a smooth function of $x$ for each $l$. The Lax operator at a point $\mathbf{u}$ is the symmetric differential operator

$$L^{sp(N)}(\mathbf{u}) := \partial^{2N} + \frac{1}{2}\sum_{l=1}^{N}(u_l\partial^{2N-2l} + \partial^{2N-2l}u_l) \ . \tag{1.6}$$

A family of infinitely many commuting vector fields $Z_s^{sp(N)}$ $(s = 0, 1, 2, ...)$ can be defined on $\mathcal{U}$ in terms of this operator; if $d/d\tau_s$ is the Lie derivative along $Z_s^{sp(N)}$, we have

$$\frac{dL^{sp(N)}}{d\tau_s} = [B_s^{sp(N)}, L^{sp(N)}] \ , \qquad B_s^{sp(N)}(\mathbf{u}) := \text{const.}(L^{sp(N)})_+^{\frac{2s+1-2N}{2N}}(\mathbf{u}), \tag{1.7}$$

where $+$ takes the non-negative powers of $\partial$ in the algebra of pseudo-differential operators. The standard splitting of this algebra into the differential and the purely integral operators (formed, respectively, with the nonnegative and the negative powers of $\partial$) gives a well known $R$-matrix. In this paper we will consider the Hamiltonian formulation of the vector fields $Z_s^{sp(N)}$, coming from the $R$-matrix linear Poisson structure (the second Hamiltonian formulation, coming from the quadratic Poisson structure, is less interesting for our purposes).

Further details about this theory, including its Lie-theoretical interpretation, can be found in the fundamental Drinfeld-Sokolov paper [5] (according to the classification of these authors, what we are calling here the $sp(N)$ KdV is the theory induced removing the vertex $c_0$ from the Dynkin diagram of the affine Lie algebra $C_N^{(1)} = sp(N)[\zeta, \zeta^{-1}]$: see Table 4, page 2022 to recover the Lax operator (1.6) ).

Let us summarize the organization of the paper. In Sect.2 we briefly review the continuous limit of the $KM = V_1$ system (here, some facts about KM and KdV theories are reported for conveniency in Tables Ia-b); we also give the new presentation of the recombination scheme already mentioned in this Introduction. In Sect.3 we describe the $V_2$ and $sp(2)$ theories; the explicit expressions of the Poisson tensors and of the first vector fields and Hamiltonians are collected in Tables IIa-b. In Sect.4 we construct the field rescaling for the $V_2$ theory, introduce the recombination scheme and state the main theorem (Prop.4.4) on the continuous limit of the Lax pairs and of the Hamiltonian scheme, recovering in this way the $sp(2)$ KdV. In Sect.5, we prove Prop.4.4 . In Sect.6, we consider the $V_N$ theory for arbitrary $N$, and obtain the $sp(N)$ Lax operator from its continuous limit, after an appropriate field rescaling.

Most of the explicit computations for Sect.s 2-4 have been performed using the MATHEMATICA symbolic manipulator. The rapid increase with $N$ of the complexity of the expressions for the $V_N$ vector fields is evident even from the comparison between the cases $N = 1$ and 2; the strength of the arguments employed in Sect. 5 to prove the main theorem is the fact that they are essentially independent from these explicit expressions, resting on general compatibility arguments for the Lax scheme.

## 2. The one-component case: a review of the KM system and its continuous limit

Let us start directly from the interpolated KM $= V_1$ system, in the sense explained in the Introduction. The phase space of this system is a set $\mathcal{A}$ of real functions $a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. At any point $a \in \mathcal{A}$, the tangent and cotangent spaces will be denoted by $T_a\mathcal{A}$ and $T_a^*\mathcal{A}$; their elements can be represented as real functions of $x$, written $\dot{a}$ and $\delta a$ respectively, and we have the pairing $< \delta a, \dot{a} >= \int dx\ \delta a\ \dot{a}$.
The KM system envolves infinitely many vector fields $X_s^\epsilon$ $(s = 0, 1, 2, ...)$ on $\mathcal{A}$, admitting the Lax formulation (1.2-3). The Lax operator is as in Eq.(1.1) (with $N = 1$ and $a_1 = a$); it is written in Table Ia, together with the compatible Poisson tensors $Q^\epsilon$, $P^\epsilon$ giving the biHamiltonian formulation of this system. At each point $a$, both of them are linear maps between the cotangent and the tangent space, sending any covector $\delta a$ into the vector $\dot{a} = Q_a^\epsilon \delta a$ or $\dot{a} = P_a^\epsilon \delta a$. The zero-th Hamiltonian function is a Casimir of $Q^\epsilon$ : $Q^\epsilon df_0^\epsilon = 0$; the Hamiltonians $f_s^\epsilon$ with $s \geq 1$ are traces of the even powers of the Lax operator. The traceform tr on differential-difference operators is defined setting $\mathrm{tr}(G) := \int dx\ \mathrm{rem}(G)$ for each $G$; the "reminder" $\mathrm{rem}(G)$ of an operator $G = \sum_{j=-\infty}^{j_{max}} g_j \Delta_{j\epsilon}$ is, by definition, the function $g_0$.
After the KM system, let us consider the ordinary KdV theory. The phase space is a set $\mathcal{U}$ of real smooth functions $u$, defined on the torus $\mathbb{R}/\mathbb{Z}$; writing Eq.(1.6) with $N = 1$ and $u_1 = u$, we obtain the familiar Lax operator. The main facts about KdV theory are summarized in Table Ib. Of course, the traceform Tr appearing in the definition of the Hamiltonians is the Adler trace for pseudo-differential operators ($\mathrm{Tr}(F) := \int dx\ \mathrm{Res}(F)$, where the residue $\mathrm{Res}(F)$ is the coefficient of $\partial^{-1}$ in the operator $F$). The zero-th Hamiltonian is a Casimir of the first Poisson tensor $Q^{sp(1)}$.

---

**Table Ia: KM $=$ V$_1$ system.**

---

Lax formulation for the $s^{\mathrm{th}}$ vector field $X_s^\epsilon$: $dL^\epsilon/dt_s = [A_s^\epsilon, L^\epsilon]$ $\quad (s = 0, 1, 2, ....)$

$$L^\epsilon(a) := \frac{1}{2}(a\Delta_\epsilon + a_{[-\epsilon]}\Delta_{-\epsilon}) \ , \qquad A_s^\epsilon(a) := (L^\epsilon)^{2s}_{\mathrm{skew}}(\mathbf{a}) \ .$$

Bihamiltonian structure at a point $a$. $Q_a^\epsilon, P_a^\epsilon : T_a^*\mathcal{A} \to T_a\ \mathcal{A}$ ,

$$Q_a^\epsilon = \frac{1}{8}(\ aa_{[\epsilon]}\Delta_\epsilon - aa_{[-\epsilon]}\Delta_{-\epsilon}\ ) \ ,$$

$$P_a^\epsilon = \frac{1}{32}\Big(\ aa_{[\epsilon]}^2 a_{[2\epsilon]}\Delta_{2\epsilon} + (a^3 a_{[\epsilon]} + aa_{[\epsilon]}^3)\Delta_\epsilon - (aa_{[-\epsilon]}^3 + a^3 a_{[-\epsilon]})\Delta_{-\epsilon} - aa_{[-\epsilon]}^2 a_{[-2\epsilon]}\Delta_{-2\epsilon}\Big) \ .$$

BiHamiltonian formulation:
$X_0^\epsilon = Q^\epsilon df_0^\epsilon = 0$ , $\quad X_s^\epsilon = Q^\epsilon df_s^\epsilon = P^\epsilon df_{s-1}^\epsilon \quad (s = 1, 2, 3, ...)$ .
Hamiltonian functions

$$f_0^\epsilon(a) := \int dx\ \log a \ , \qquad f_s^\epsilon(a) := \frac{1}{2s}\mathrm{tr}(L^\epsilon)^{2s}(a) \qquad (s = 1, 2, 3, ...) \ .$$

First companions of the Lax operator:
$A_0^\epsilon(a)=0\ ,\qquad A_1^\epsilon(a)=\frac{1}{4}(aa_{[\epsilon]}\Delta_{2\epsilon}-a_{[-\epsilon]}a_{[-2\epsilon]}\Delta_{-2\epsilon})$
$A_2^\epsilon(a)=\frac{1}{16}\{aa_{[2\epsilon]}a_{[3\epsilon]}a_{[\epsilon]}\Delta_{4\epsilon}+(a^3a_{[\epsilon]}+aa^2_{[2\epsilon]}a_{[\epsilon]}+aa^2_{[-\epsilon]}a_{[\epsilon]}+aa^3_{[\epsilon]})\Delta_{2\epsilon}$
$-(a^2a_{[-2\epsilon]}a_{[-\epsilon]}+a^3_{[-2\epsilon]}a_{[-\epsilon]}+a_{[-2\epsilon]}a^2_{[-3\epsilon]}a_{[-\epsilon]}+a_{[-2\epsilon]}a^3_{[-\epsilon]})\Delta_{-2\epsilon}-a_{[-2\epsilon]}a_{[-4\epsilon]}a_{[-3\epsilon]}a_{[-\epsilon]}\Delta_{-4\epsilon}\}\ .$

First Hamiltonians after $f_0^\epsilon$:

$f_1^\epsilon(\mathbf{a})=\frac{1}{4}\int dx\ a^2\ ,$
$f_2^\epsilon(\mathbf{a})=\frac{1}{32}\int dx\ (\ a^4+2a^2a^2_{[\epsilon]})\ ,$
$f_3^\epsilon(\mathbf{a})=\frac{1}{192}\int dx\ (a^6+3a^4a^2_{[\epsilon]}+3a^2a^2_{[2\epsilon]}a^2_{[\epsilon]}+3a^2a^4_{[\epsilon]})\ .$

First vector fields :
$X_0^\epsilon(a)=0\ ,\qquad X_1^\epsilon(a)=\frac{1}{16}a(-a^2_{[-\epsilon]}+a^2_{[\epsilon]})\ ,$
$X_2^\epsilon(a)=\frac{1}{64}a(-a^2a^2_{[-\epsilon]}-a^2_{[-2\epsilon]}a^2_{[-\epsilon]}-a^4_{[-\epsilon]}+a^2a^2_{[\epsilon]}+a^2_{[2\epsilon]}a^2_{[\epsilon]}+a^4_{[\epsilon]})\ ,$
$X_3^\epsilon(a)=\frac{1}{256}a(-a^4a^2_{[-\epsilon]}-a^2a^2_{[-2\epsilon]}a^2_{[-\epsilon]}-a^4_{[-2\epsilon]}a^2_{[-\epsilon]}-a^2_{[-2\epsilon]}a^2_{[-3\epsilon]}a^2_{[-\epsilon]}$
$\quad-2a^2a^4_{[-\epsilon]}-2a^2_{[-2\epsilon]}a^4_{[-\epsilon]}-a^6_{[-\epsilon]}+a^4a^2_{[\epsilon]}+a^2a^2_{[2\epsilon]}a^2_{[\epsilon]}$
$\quad+a^4_{[2\epsilon]}a^2_{[\epsilon]}+a^2_{[2\epsilon]}a^2_{[3\epsilon]}a^2_{[\epsilon]}+2a^2a^4_{[\epsilon]}+2a^2_{[2\epsilon]}a^4_{[\epsilon]}+a^6_{[\epsilon]})\ .$

---

**Table Ib: sp(1) KdV theory.**

---

Lax formulation for the $s^{\text{th}}$ vector field $Z_s^{sp(1)}$:
$dL^{sp(1)}/d\tau_s=[B_s^{sp(1)},L^{sp(1)}]\ ,\quad (s=0,1,2,...)$

$$L^{sp(1)}(u):=\partial^2+u\ ,\qquad B_s^{sp(1)}(u):=4^{s-1}(L^{sp(1)})_+^{s-\frac{1}{2}}(u)\ .$$

BiHamiltonian structure at a point $u$. $Q_u^{sp(1)},S_u^{sp(1)}:T_u^*\mathcal{U}\to T_u\,\mathcal{U}$ ,

$$Q_u^{sp(1)}=\partial\ ,\qquad S_u^{sp(1)}=\partial^3+4u\partial+2u_x\ .$$

BiHamiltonian formulation:
$Z_0^{sp(1)}=Q^{sp(1)}dh_0^{sp(1)}=0\ ,\quad Z_s^{sp(1)}=Q^{sp(1)}dh_s^{sp(1)}=S^{sp(1)}dh_{s-1}^{sp(1)}\quad (s=1,2,3,...)\ .$

Hamiltonian functions

$$h_s^{sp(1)}(u):=\frac{4^s}{2s+1}\,\mathrm{Tr}(L^{sp(1)})^{s+\frac{1}{2}}(u)\qquad (s=0,1,2,...)\ .$$

First companions of the Lax operator:
$B_0^{sp(1)}(u)=0,\qquad B_0^{sp(1)}(u)=\partial\ ,B_2^{sp(1)}(u)=4\partial^3+6u\partial+3u_x\ .$

First Hamiltonians:

$h_0^{sp(1)}(u):=\frac{1}{2}\int dx\ u\ ,\qquad h_1^{sp(1)}(u)=\frac{1}{2}\int dx\ u^2\ ,$
$h_2^{sp(1)}(u)=\frac{1}{2}\int dx(2u^3-u_x^2)\ ,\qquad h_3^{sp(1)}(u)=\frac{1}{2}\int dx(5u^4-10uu_x^2+u_{xx}^2).$

First vector fields :
$Z_0^{sp(1)}(u)=0\ ,\qquad Z_1^{sp(1)}(u)=u_x\ ,\qquad Z_2^{sp(1)}(u)=u_{xxx}+6uu_x\ ,$
$Z_3^{sp(1)}(u)=u_{xxxxx}+10uu_{xxx}+20u_xu_{xx}+30u^2u_x\ .$

---

The starting point to construct the continuous limit of the KM system is the introduction of a transformation $\Psi^\epsilon : \mathcal{U} \to \mathcal{A}$ , $u \mapsto a = \Psi^\epsilon(u)$, with

$$a := 2 + \epsilon^2 u \ . \tag{2.1}$$

With a suitable specification of the function spaces $\mathcal{A}$ and $\mathcal{U}$, this map is bijective with inverse $(\Psi^\epsilon)^{-1}(a) = (a-2)/\epsilon^2$, and allows to transport all the KM geometric structures of $\mathcal{A}$ onto $\mathcal{U}$. For example, each KM vector field $X^\epsilon_s$ on $\mathcal{A}$ induces a vector field on $\mathcal{U}$, denoted for simplicity with the same symbol and given explicitly by $X^\epsilon_s(u) := (T\Psi^\epsilon)^{-1}(u) X^\epsilon_s(a)|_{a=\Psi^\epsilon(u)}$ ($T$ standing for the tangent map).
The key point in the limit process illustrated in [1] is a *recombination method* for the KM structures (including the Hamiltonians and the Lax pairs, besides the the vector fields and Poisson tensors already considered in [2]). Let us define recursively a system of coefficients $(c_{s,j})$ $(s = 0, 1, 2, ...; \ j = 0, 1, ..., s)$, where

$$c_{0,0} := 1 \ , \tag{2.2}$$

$$c_{s+1,j} := \begin{cases} -\binom{2s+2}{s+1} - \sum_{i=1}^{s} \binom{2i}{i} \left( c_{s,i-1} - 4 c_{s,i} \right) & \text{for} \quad j = 0 \\ c_{s,j-1} - 4 c_{s,j} & \text{for} \quad j = 1, 2, ..., s \\ 1 & \text{for} \quad j = s+1 \end{cases} \tag{2.3}$$

(here $\binom{r}{k}$ denotes as usually the binomial coefficient, for each pair $r, k$). Furthermore, for $s = 0, 1, 2, ...$, let us put

$$c_{s,-1} := -c_{s,0} \log 2 - \sum_{j=1}^{s} \frac{1}{2j} \binom{2j}{j} c_{s,j} \ . \tag{2.4}$$

The first coefficients constructed in this way are:

$$\begin{array}{lllll} c_{0,-1} = -\log 2 \ , & c_{0,0} = 1 \ , & & & \\ c_{1,-1} = -1 + 2\log 2 \ , & c_{1,0} = -2 \ , & c_{1,1} = 1 \ , & & \\ c_{2,-1} = \frac{9}{2} - 6 \log 2 \ , & c_{2,0} = 6 \ , & c_{2,1} = -6 \ , & c_{2,2} = 1 \ , & \\ c_{3,-1} = -\frac{55}{3} + 20 \log 2 \ , & c_{3,0} = -20 \ , & c_{3,1} = 30 \ , & c_{3,2} = -10 \ , & c_{3,3} = 1 \ . \end{array} \tag{2.5}$$

For $s$ arbitrary, let us introduce the recombinations

$$h^\epsilon_s := c_{s,-1} + \sum_{j=0}^{s} c_{s,j} f^\epsilon_j \ , \qquad Z^\epsilon_s := \sum_{j=1}^{s} c_{s,j} X^\epsilon_j \ , \qquad B^\epsilon_s := \sum_{j=1}^{s} c_{s,j} A^\epsilon_j \ , \tag{2.6}$$

$$S^\epsilon := P^\epsilon - 4 Q^\epsilon \ . \tag{2.7}$$

We summarize the main results on the continuous limit in the following statements (see [1] for the proofs[3] ):

[3] For uniformity with the rest of the present paper, the normalizations for the KM structures considered here are slightly different from the ones adopted in the previous paper. Denoting with barred symbols the objects considered in [1], we have $L^\epsilon = \frac{1}{2}\overline{L^\epsilon}$, $f^\epsilon_s = \frac{1}{4^s}\overline{f^\epsilon_s}$, $X^\epsilon_s = \frac{1}{4^{s+1}}\overline{X^\epsilon_s}$, $A^\epsilon_s = \frac{1}{4^{s+1}}\overline{A^\epsilon_s}$, $Q^\epsilon = \frac{1}{4}\overline{Q^\epsilon}$, $P^\epsilon = \frac{1}{16}\overline{P^\epsilon}$, $c_{s,j} = 4^{j-s}\overline{c_{s,j}}$, $c_{s,-1} = -\frac{1}{4^s}\overline{c_s}$ $(s = 0, 1, 2, ...; \ j = 0, 1, ..., s)$. The normalizations adopted for the KdV theory have not been changed.

**2.1. Proposition.** The $\epsilon \mapsto 0$ limit of the KM structures considered above are the homologous KdV structures. More precisely,

$$\lim_{\epsilon\mapsto 0} \frac{1}{\epsilon^{2s+2}} h_s^\epsilon(u) = h_s^{sp(1)}(u) \ , \qquad \lim_{\epsilon\mapsto 0} \frac{1}{\epsilon^{2s-1}} Z_s^\epsilon(u) = Z_s^{sp(1)}(u) \ . \tag{2.8}$$

$$\lim_{\epsilon\mapsto 0} \epsilon^3 Q_u^\epsilon = Q_u^{sp(1)} \ , \qquad \lim_{\epsilon\mapsto 0} \epsilon S_u^\epsilon = S_u^{sp(1)} \ . \tag{2.9}$$

$$\lim_{\epsilon\mapsto 0} \frac{1}{\epsilon^2}(L^\epsilon(u) - 2) = L^{sp(1)}(u) \ , \qquad \lim_{\epsilon\mapsto 0} \frac{1}{\epsilon^{2s-1}} B_s^\epsilon(u) = B_s^{sp(1)}(u) \ . \tag{2.10}$$

$\diamondsuit$

Let us illustrate, for example, how to obtain the second KdV Hamiltonian function via expansion and recombination of the KM Hamiltonians $f_0^\epsilon, f_1^\epsilon, f_2^\epsilon$. To this purpose, we take from Table Ia the explicit expressions of these functions, make the field rescaling (2.1) and expand up to $O(\epsilon^8)$ in $\epsilon$; in this way, we find

$$f_0^\epsilon(u) = \int dx \log(2+\epsilon^2 u) = \log 2 + \frac{\epsilon^2}{2}\int dx\ u - \frac{\epsilon^4}{8}\int dx\ u^2 + \frac{\epsilon^6}{24}\int dx\ u^3 + O(\epsilon^8) \ ,$$

$$f_1^\epsilon(u) = \frac{1}{4}\int dx (2+\epsilon^2 u)^2 = 1 + \epsilon^2 \int dx\ u + \frac{1}{4}\epsilon^4 \int dx\ u^2 \ , \tag{2.11}$$

$$f_2^\epsilon(u) = \frac{1}{32}\int dx \Big( (2+\epsilon^2 u)^4 + 2(2+\epsilon^2 u_{(\epsilon)})^2 (2+\epsilon^2 u)^2 \Big)$$

$$= \frac{3}{2} + 3\epsilon^2 \int dx\ u + \frac{9}{4}\epsilon^4 \int dx\ u^2 + \frac{1}{4}\epsilon^6 \int dx\ (3u^3 - 2u_x^2) + O(\epsilon^8) \ .$$

Now, we recall Eq.s (2.5-6) and put

$$h_2^\epsilon := (\frac{9}{2} - 6\log 2) + 6 f_0^\epsilon - 6 f_1^\epsilon + f_2^\epsilon \ ; \tag{2.12}$$

the conclusion is

$$h_2^\epsilon(u) = \frac{1}{2}\epsilon^6 \int dx\ (2u^3 - u_x^2) + O(\epsilon^8) = \epsilon^6 h_2^{sp(1)}(u) + O(\epsilon^8) \ . \tag{2.13}$$

After this example, let us return to the general algorithm for finding the coefficients $c_{s,j}$. For uniformity with the method applied in the next section to the $V_2$ system, we give an alternative prescription for the determination of these coefficients.
To this purpose, let us consider the function $p_0(\xi)$ and the polynomials $p_1(\xi)$, $p_2(\xi), \ldots$, in one indeterminate $\xi$, defined setting

$$p_s(\xi) := \begin{cases} \log(2+2\xi) & \text{for} \quad s = 0 \\ \frac{1}{2s}\binom{2s}{s}(1+\xi)^{2s} & \text{for} \quad s = 1, 2, \ldots \ . \end{cases} \tag{2.14}$$

**2.2. Proposition.** For $s = 0, 1, 2, \ldots$, the coefficients $c_{s,j}$ $(j = -1, 0, \ldots, s)$ defined by Eq.s(2.2-4) are the unique $(s+2)$-tuple of real numbers such that

$$c_{s,-1} + \sum_{j=0}^{s} c_{s,j} p_j(\xi) = \frac{2^{3s+1}}{2s+1} \binom{s+\frac{1}{2}}{s+1} \xi^{s+1} + O(\xi^{s+2}) \ . \tag{2.15}$$

**Proof.** As it was proved [1], the coefficients defined by (2.2-4) are such that

$$c_{s,-1} + \sum_{j=0}^{s} c_{s,j} f_j^{\epsilon}(u) = \epsilon^{2s+2} h_s^{sp(1)}(u) + O(\epsilon^{2s+4}) \ ; \tag{2.16}$$

from here, we will infer Eq.(2.15). To this purpose, let us apply Eq.(2.16) with $u(x) = \text{const.} = 2$. We note that $f_0^{\epsilon}(2) = \log(2 + 2\epsilon^2) = p_0(\epsilon^2)$; to compute $f_s^{\epsilon}(2)$ for $s > 0$, we observe that

$$(L^{\epsilon})^{2s}(2) = (1+\epsilon^2)^{2s} (\Delta_{\epsilon} + \Delta_{-\epsilon})^{2s} = (1+\epsilon^2)^{2s} \sum_{k=0}^{2s} \binom{2s}{k} \Delta_{(2k-2s)\epsilon} \ ; \tag{2.17}$$

so, taking the trace, we find that

$$f_s^{\epsilon}(2) = \frac{1}{2s} \binom{2s}{s} (1+\epsilon^2)^{2s} = p_s(\epsilon^2) \qquad (s = 0, 1, 2, \ldots) \ . \tag{2.18}$$

On the other hand, $h_s^{sp(1)}(2) = \frac{4^s}{2s+1} \int dx \ \mathrm{Res}(L^{sp(1)})^{s+\frac{1}{2}}(2)$ ; but

$$(L^{sp(1)})^{s+\frac{1}{2}}(2) = (\partial^2 + 2)^{s+\frac{1}{2}} = \sum_{k=0}^{+\infty} \binom{s+\frac{1}{2}}{k} 2^k \partial^{2s+1-2k} \ . \tag{2.19}$$

Reading from this sum the coefficient of $\partial^{-1}$, we conclude

$$h_s^{sp(1)}(2) = \frac{2^{3s+1}}{2s+1} \binom{s+\frac{1}{2}}{s+1} \qquad (s = 0, 1, 2, \ldots) \ . \tag{2.20}$$

Inserting the results (2.18) and (2.20) into Eq.(2.16) (with $u = 2$), and setting $\epsilon^2 = \xi$, we obtain that the $c_{s,j}$ actually satisfy Eq.(2.15).
Let us now prove the uniqueness of the system of coefficients fulfilling this equation. To this purpose, we employ again the $c_{s,j}$ defined by Eq.s(2.2-4) (recalling that $c_{s,s} = 1$ for each $s$). Let us define

$$q_s(\xi) := c_{s,-1} + \sum_{j=0}^{s} c_{s,j} p_j(\xi) \qquad (s = 0, 1, 2, \ldots) \ . \tag{2.21}$$

From now on $s$ is fixed; we introduce the arrays

$$\mathbf{c} := (c_{s,-1}, c_{s,0}, \ldots, c_{s,s-1}, 1), \ \mathbf{p}(\xi) := \begin{pmatrix} 1 \\ p_0(\xi) \\ p_1(\xi) \\ \ldots \\ p_s(\xi) \end{pmatrix}, \ \mathbf{q}(\xi) := \begin{pmatrix} 1 \\ q_0(\xi) \\ q_1(\xi) \\ \ldots \\ q_s(\xi) \end{pmatrix} \tag{2.22}$$

and the $(s+2)\times(s+2)$ lower triangular matrix

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots \\ c_{0,-1} & 1 & 0 & 0 & \dots & .. \\ c_{1,-1} & c_{1,0} & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{s,-1} & c_{s,0} & c_{s,1} & \dots & \dots & 1 \end{pmatrix} . \tag{2.23}$$

Then we can write Eq.s(2.21) and (2.15) as

$$\mathbf{q}(\xi) = C \ \mathbf{p}(\xi) , \tag{2.24}$$

$$\mathbf{c} \ \mathbf{p}(\xi) = \frac{2^{3s+1}}{2s+1} \binom{s+\frac{1}{2}}{s+1} \xi^{s+1} + O(\xi^{s+2}) . \tag{2.25}$$

Let $\mathbf{c}' = \left( c'_{s,-1}, c'_{s,0}, ....c'_{s,s-1}, c'_{s,s} \right)$ be another system of coefficients satisfying Eq.(2.15); then

$$(\mathbf{c}' - \mathbf{c}) \ \mathbf{p}(\xi) = O(\xi^{s+2}) . \tag{2.26}$$

From Eq.s(2.24),(2.26), we infer that the array $\mathbf{b} = \left( b_{-1}, b_0, b_1 .... b_{s-1}, b_s \right)$ defined by $(\mathbf{c}' - \mathbf{c}) = \mathbf{b} \ C$ satisfies the condition

$$\mathbf{b} \ \mathbf{q}(\xi) = O(\xi^{s+2}) . \tag{2.27}$$

Let us recall that the elements of the array $\mathbf{q}(\xi)$ are 1 and $q_j(\xi) = \frac{2^{3j+1}}{2j+1} \binom{j+\frac{1}{2}}{j+1} \xi^{j+1} + O(\xi^{j+2})$ $(j = 0, 1, ..., s)$. Developing to order zero in $\xi$, we see that $\mathbf{b} \ \mathbf{q}(\xi) = b_{-1} + O(\xi)$, so Eq.(2.27) implies $b_{-1} = 0$. Taking this fact into account, and developing to order 1, we find $\mathbf{b} \ \mathbf{q}(\xi) = b_0 \xi + O(\xi^2)$; so, using again Eq.(2.27), we infer $b_0 = 0$. Iterating this argument, we conclude that each component of $\mathbf{b}$ is zero, which implies $\mathbf{c}' = \mathbf{c}$. Summing up, $\mathbf{c}$ is the unique system of coefficients satisfying Eq.(2.15), and the proof is concluded. $\Diamond$

## 3. The two-components Volterra ($V_2$) system and the $sp(2)$ theory

Each point in the phase space $\mathcal{A}$ of the $V_2$ system is a pair $\mathbf{a}$ of real smooth functions on $\mathbb{R}/\mathbb{Z}$; for simplicity of notation, we denote these functions with $a$ and $r$ instead of using the symbols $a_1$ and $a_2$ prescribed by the Introduction. A tangent vector $\dot{\mathbf{a}} \in T_{\mathbf{a}}\mathcal{A}$ and a covector $\delta\mathbf{a} \in T^*_{\mathbf{a}}\mathcal{A}$ are both identified with pairs of functions, denoted respectively with $(\dot{a}, \dot{r})$ and $(\delta a, \delta r)$, and we have the pairing $< \delta\mathbf{a}, \dot{\mathbf{a}} >:= \int dx\ (\delta a\ \dot{a} + \delta r\ \dot{r})$.
A sequence of vector fields $X^\epsilon_s$ $(s = 0, 1, 2, ...)$ is defined on the phase space $\mathcal{A}$; the Lax formulations are based on the operator (1.1) (with $N = 2$, $a_1 = a$, $a_2 = r$) and appear in Table IIa (where we use the notations $d/dt_s$ for the Lie derivative along $X^\epsilon_s$, and *skew* for the "skew-symmetric part" of a differential-difference operator, defined by Eq.(1.4) ). All these vector fields are Hamiltonian w.r.t. the Poisson tensor $Q^\epsilon$ and the Hamiltonians $f^\epsilon_s : \mathcal{A} \to \mathbb{R}$ introduced in Table IIa. It should be noted that $X^\epsilon_0 = Q^\epsilon df^\epsilon_0 = 0$, which means that $f^\epsilon_0$ is a Casimir of the Poisson structure. To the best of our knowledge, a second Hamiltonian structure for the $V_2$ system is not presently known; the methods employed successfully for the second KM Poisson structure (based on the Toda recursion operator [1] or the master symmetry approach [11]) could be applied in principle to the $V_2$ case; unfortunately, the "second Poisson tensor" obtained in this way is a highly formal object, involving inverses of differential-difference operators (and the same seems to happen for all the $V_N$ theories with $N \geq 2$). For this reason, we will leave aside the problem and confine our attention to the "first" Poisson tensor $Q^\epsilon$.
In the next Section, we will show that the $\epsilon \mapsto 0$ limit of the $V_2$ system is the $sp(2)$ theory; here, we review some the basic facts about the latter. A point in the phase space $\mathcal{U}$ of this theory is a pair $\mathbf{u}$, formed by two smooth functions on the torus; again we stray slightly from the notation of the Introduction and denote these functions with $u$, $v$ rather than $u_1$, $u_2$. Tangent vectors and covectors at $\mathbf{u}$ are represented as pairs of functions $\dot{\mathbf{u}} = (\dot{u}, \dot{v})$ and $\delta\mathbf{u} = (\delta u, \delta v)$, with the obvious pairing. A sequence of vector fields $Z^{sp(2)}_s$ $(s = 0, 1, 2, ...)$ is defined on $\mathcal{U}$ through their Lax formulations, based on the differential operator of Eq.(1.6) (with $N = 2$, $u_1 = u$, $u_2 = v$) and its fractional powers of exponents $1/4$, $3/4$, $5/4$, etc. . The details are given in Table IIb (where we write $d/d\tau_s$ for the Lie derivative along the vector field $Z^{sp(2)}_s$). The first nontrivial vector field of the hierarchy is $Z^{sp(2)}_2$. All vector fields are Hamiltonian w.r.t. the Poisson tensor $Q^{sp(2)}$ reported in the Table; the Hamiltonians $h^{sp(2)}_s$ are (Adler) traces of the fractional powers of $L^{sp(2)}$. The vanishing of $Z^{sp(2)}_0$ and $Z^{sp(2)}_1$ means that the first two Hamiltonians $h^{sp(2)}_0$ and $h^{sp(2)}_1$ are Casimirs of $Q^{sp(2)}$. The vector fields of the $sp(2)$ theory also admit a second Hamiltonian formulation $Z^{sp(2)}_s = S^{sp(2)} dh^{sp(2)}_{s-2}$; however, this fact is not considered in the rest of the paper, so we do not even report the explicit expression of the Poisson tensor $S^{sp(2)}$. As a final remark, we mention some facts related to the known isomorphism between $sp(2)$ and the Lie algebra $so(5)$ of the skew symmetric $5 \times 5$ matrices. Due to this isomorphism, the KdV theory discussed here could be as well interpreted in terms of $so(5)$; this alternative description can be found, for example, in [12], where we also give the second Poisson tensor[4].

[4] It is known that the flows and the biHamiltonian structure of a KdV-type theory are intrinsi-

**Table IIa: $\mathbf{V}_2$ system.**

Lax formulation for the $s^{\rm th}$ vector field $X^\epsilon_s$: $dL^\epsilon/dt_s = [A^\epsilon_s, L^\epsilon] \quad (s = 0,1,2,3,....)$

$$L^\epsilon(\mathbf{a}) := \frac{1}{2}(r\Delta_{3\epsilon} + a\Delta_\epsilon + a_{[-\epsilon]}\Delta_{-\epsilon} + r_{[-3\epsilon]}\Delta_{-3\epsilon}) \ , \qquad A^\epsilon_s(\mathbf{a}) := -\frac{3}{8}(L^\epsilon)^{2s}_{\rm skew}(\mathbf{a}) \ .$$

Poisson tensor at a point $\mathbf{a} = (a, r)$. $Q^\epsilon_{\mathbf{a}} : T^*_{\mathbf{a}}\mathcal{A} \to T_{\mathbf{a}}\ \mathcal{A}$

$$\begin{pmatrix} \delta a \\ \delta r \end{pmatrix} \mapsto \begin{pmatrix} \dot a \\ \dot r \end{pmatrix} := \frac{3}{16}\begin{pmatrix} Q^\epsilon_{aa} & Q^\epsilon_{ar} \\ Q^\epsilon_{ra} & Q^\epsilon_{rr} \end{pmatrix}\begin{pmatrix} \delta a \\ \delta r \end{pmatrix}$$

$$\begin{aligned}
Q^\epsilon_{aa} &= -2rr_{[\epsilon]}\Delta_{3\epsilon} - 2a_{[\epsilon]}r\Delta_{2\epsilon} - aa_{[\epsilon]}\Delta_\epsilon + aa_{[-\epsilon]}\Delta_{-\epsilon} + 2a_{[-\epsilon]}r_{[-2\epsilon]}\Delta_{-2\epsilon} + 2r_{[-2\epsilon]}r_{[-3\epsilon]}\Delta_{-3\epsilon} \ , \\
Q^\epsilon_{ar} &= -ar_{[\epsilon]}\Delta_\epsilon - ar + ar_{[-2\epsilon]}\Delta_{-2\epsilon} + ar_{[-3\epsilon]}\Delta_{-3\epsilon} \ , \\
Q^\epsilon_{ra} &= -a_{[3\epsilon]}r\Delta_{3\epsilon} - a_{[2\epsilon]}r\Delta_{2\epsilon} + ar + a_{[-\epsilon]}r\Delta_{-\epsilon} \ , \\
Q^\epsilon_{rr} &= -rr_{[3\epsilon]}\Delta_{3\epsilon} + rr_{[-3\epsilon]}\Delta_{-3\epsilon} \ .
\end{aligned}$$

Hamiltonian formulation: $X^\epsilon_s = Q^\epsilon df^\epsilon_s \qquad (s = 0,1,2,3,...)$

$$f^\epsilon_0(\mathbf{a}) := \int dx\ \log r \ , \qquad f^\epsilon_s(\mathbf{a}) := \frac{1}{2s}\,{\rm tr}(L^\epsilon)^{2s}(\mathbf{a}) \qquad (s = 1,2,3,...) \ .$$

First companions of the Lax operator: $A^\epsilon_0(\mathbf{a}) = 0$ ,

$$\begin{aligned}
A^\epsilon_1(\mathbf{a}) = -\tfrac{3}{32}\{&rr_{[3\epsilon]}\Delta_{6\epsilon} + (a_{[3\epsilon]}r + ar_{[\epsilon]})\Delta_{4\epsilon} + (aa_{[\epsilon]} + a_{[2\epsilon]}r + a_{[-\epsilon]}r_{[-\epsilon]})\Delta_{2\epsilon} \\
&-(a_{[-2\epsilon]}a_{[-\epsilon]} + ar_{[-2\epsilon]} + a_{[-3\epsilon]}r_{[-3\epsilon]})\Delta_{-2\epsilon} - (a_{[-\epsilon]}r_{[-4\epsilon]} + a_{[-4\epsilon]}r_{[-3\epsilon]})\Delta_{-4\epsilon} - r_{[-6\epsilon]}r_{[-3\epsilon]}\Delta_{-6\epsilon}\} \ .
\end{aligned}$$

First Hamiltonians after $f^\epsilon_0$:

$$f^\epsilon_1(\mathbf{a}) = \tfrac{1}{4}\int dx\ (a^2 + r^2) \ ,$$

$$\begin{aligned}
f^\epsilon_2(\mathbf{a}) = \tfrac{1}{32}\int dx\ (\ &a^4 + 2a^2a^2_{[\epsilon]} + 4aa_{[2\epsilon]}a_{[\epsilon]}r + 2a^2r^2 + 2a^2_{[2\epsilon]}r^2 + 2a^2_{[3\epsilon]}r^2 + r^4 + 2r^2r^2_{[3\epsilon]} \\
&+ 4aa_{[3\epsilon]}rr_{[\epsilon]} + 2a^2r^2_{[\epsilon]}\ ) \ ,
\end{aligned}$$

$$\begin{aligned}
f^\epsilon_3(\mathbf{a}) = \tfrac{1}{192}\int dx\ (&a^6 + 3a^4a^2_{[\epsilon]} + 3a^2a^2_{[2\epsilon]}a^2_{[\epsilon]} + 3a^2a^4_{[\epsilon]} + 6a^3a_{[2\epsilon]}a_{[\epsilon]}r + 6aa^3_{[2\epsilon]}a_{[\epsilon]}r \\
&+6aa_{[2\epsilon]}a^2_{[3\epsilon]}a_{[\epsilon]}r + 6aa_{[2\epsilon]}a^3_{[\epsilon]}r + 3a^4r^2 + 3a^2a^2_{[2\epsilon]}r^2 + 3a^4_{[2\epsilon]}r^2 + 3a^2a^2_{[3\epsilon]}r^2 + 6a^2_{[2\epsilon]}a^2_{[3\epsilon]}r^2 \\
&+3a^2_{[4\epsilon]}a^2_{[3\epsilon]}r^2 + 3a^4_{[3\epsilon]}r^2 + 3a^2a^2_{[\epsilon]}r^2 + 3a^2_{[2\epsilon]}a^2_{[\epsilon]}r^2 + 6aa_{[2\epsilon]}a_{[\epsilon]}r^3 + 3a^2r^4 + 3a^2_{[2\epsilon]}r^4 \\
&+3a^2_{[3\epsilon]}r^4 + r^6 + 6aa_{[4\epsilon]}a_{[3\epsilon]}a_{[\epsilon]}rr_{[2\epsilon]} + 6a_{[2\epsilon]}a_{[4\epsilon]}a_{[3\epsilon]}r^2r_{[2\epsilon]} + 3a^2a^2_{[\epsilon]}r^2_{[2\epsilon]} + 6aa_{[2\epsilon]}a_{[\epsilon]}rr^2_{[2\epsilon]} \\
&+3a^2_{[2\epsilon]}r^2r^2_{[2\epsilon]} + 3a^2_{[3\epsilon]}r^2r^2_{[4\epsilon]} + 6a_{[5\epsilon]}a_{[4\epsilon]}a_{[3\epsilon]}r^2r_{[3\epsilon]} + 6aa_{[5\epsilon]}a_{[\epsilon]}rr_{[2\epsilon]}r_{[3\epsilon]} + 6a_{[5\epsilon]}a_{[2\epsilon]}r^2r_{[2\epsilon]}r_{[3\epsilon]} \\
&+6a_{[6\epsilon]}a_{[3\epsilon]}r^2r_{[4\epsilon]}r_{[3\epsilon]} + 6aa_{[2\epsilon]}a_{[\epsilon]}rr^2_{[3\epsilon]} + 3a^2r^2r^2_{[3\epsilon]} + 3a^2_{[5\epsilon]}r^2r^2_{[3\epsilon]} + 6a^2_{[2\epsilon]}r^2r^2_{[3\epsilon]} \\
&+3a^2_{[6\epsilon]}r^2r^2_{[3\epsilon]} + 6a^2_{[3\epsilon]}r^2r^2_{[3\epsilon]} + 3r^4r^2_{[3\epsilon]} + 3r^2r^2_{[6\epsilon]}r^2_{[3\epsilon]} + 3r^2r^4_{[3\epsilon]} + 6a^2a_{[2\epsilon]}a_{[3\epsilon]}a_{[\epsilon]}r_{[\epsilon]} \\
&+6a^3a_{[3\epsilon]}rr_{[\epsilon]} + 6aa^2_{[2\epsilon]}a_{[3\epsilon]}rr_{[\epsilon]} + 6aa^2_{[4\epsilon]}a_{[3\epsilon]}rr_{[\epsilon]} + 6aa^3_{[3\epsilon]}rr_{[\epsilon]} + 6aa_{[3\epsilon]}a^2_{[\epsilon]}rr_{[\epsilon]} \\
&+6a_{[2\epsilon]}a_{[3\epsilon]}a_{[\epsilon]}r^2r_{[\epsilon]} + 6aa_{[3\epsilon]}r^3r_{[\epsilon]} + 6a^2a_{[4\epsilon]}a_{[\epsilon]}r_{[2\epsilon]}r_{[\epsilon]} + 6aa_{[2\epsilon]}a_{[4\epsilon]}rr_{[2\epsilon]}r_{[\epsilon]}
\end{aligned}$$

---

cally associated to the underlying Lie algebra, while the Lax operator depends on the choice of a matrix representation of that algebra. As a matter of fact, $sp(2)$ and $so(5)$ give two inequivalent matrix representations of the same abstract Lie algebra, and this is the reason for the difference between the Lax operator of [12] and the one reported in Table IIb.

$$+6aa_{[3\epsilon]}rr^2_{[4\epsilon]}r_{[\epsilon]}+6aa_{[5\epsilon]}a_{[4\epsilon]}rr_{[3\epsilon]}r_{[\epsilon]}+6aa_{[6\epsilon]}rr_{[4\epsilon]}r_{[3\epsilon]}r_{[\epsilon]}+6aa_{[3\epsilon]}rr^2_{[3\epsilon]}r_{[\epsilon]}$$
$$+3a^4r^2_{[\epsilon]}+3a^2a^2_{[4\epsilon]}r^2_{[\epsilon]}+3a^2a^2_{[3\epsilon]}r^2_{[\epsilon]}+6a^2a^2_{[\epsilon]}r^2_{[\epsilon]}+6aa_{[2\epsilon]}a_{[\epsilon]}rr^2_{[\epsilon]}+3a^2r^2r^2_{[\epsilon]}$$
$$+3a^2_{[3\epsilon]}r^2r^2_{[\epsilon]}+3a^2r^2_{[4\epsilon]}r^2_{[\epsilon]}+6aa_{[3\epsilon]}rr^3_{[\epsilon]}+3a^2r^4_{[\epsilon]}\ ) \ .$$

First vector fields :

$$X^\epsilon_0(\mathbf{a})=0\ , \qquad X^\epsilon_1(\mathbf{a})=\tfrac{3}{32}\begin{pmatrix}X^\epsilon_{1,a}(\mathbf{a})\\ X^\epsilon_{1,r}(\mathbf{a})\end{pmatrix}\ , \qquad X^\epsilon_2(\mathbf{a})=\tfrac{3}{128}\begin{pmatrix}X^\epsilon_{2,a}(\mathbf{a})\\ X^\epsilon_{2,r}(\mathbf{a})\end{pmatrix}\ ,$$

$$\begin{aligned}
X^\epsilon_{1,a}(\mathbf{a}) := {}& aa^2_{[-\epsilon]}-aa^2_{[\epsilon]}-2a_{[2\epsilon]}a_{[\epsilon]}r-ar^2+2a_{[-2\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}+ar^2_{[-2\epsilon]}\\
&+2a_{[-3\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}+ar^2_{[-3\epsilon]}-2a_{[3\epsilon]}rr_{[\epsilon]}-ar^2_{[\epsilon]}\ ,\\
X^\epsilon_{1,r}(\mathbf{a}) := {}& a^2r-a^2_{[2\epsilon]}r+a^2_{[-\epsilon]}r-a^2_{[3\epsilon]}r+rr^2_{[-3\epsilon]}-rr^2_{[3\epsilon]}\ ,
\end{aligned}$$

$$\begin{aligned}
X^\epsilon_{2,a}(\mathbf{a}) := {}& a^3a^2_{[-\epsilon]}+aa^2_{[-2\epsilon]}a^2_{[-\epsilon]}+aa^4_{[-\epsilon]}-a^3a^2_{[\epsilon]}-aa^2_{[2\epsilon]}a^2_{[\epsilon]}-aa^4_{[\epsilon]}-2a^2a_{[2\epsilon]}a_{[\epsilon]}r\\
&-2a^3_{[2\epsilon]}a_{[\epsilon]}r-2a_{[2\epsilon]}a^2_{[3\epsilon]}a_{[\epsilon]}r-2a_{[2\epsilon]}a^3_{[\epsilon]}r-a^3r^2-aa^2_{[2\epsilon]}r^2-aa^2_{[3\epsilon]}r^2\\
&-2aa^2_{[\epsilon]}r^2-2a_{[2\epsilon]}a_{[\epsilon]}r^3-ar^4-2a_{[4\epsilon]}a_{[3\epsilon]}a_{[\epsilon]}rr_{[2\epsilon]}-aa^2_{[\epsilon]}r^2_{[2\epsilon]}-2a_{[2\epsilon]}a_{[\epsilon]}rr^2_{[2\epsilon]}\\
&+2a^2a_{[-2\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}+2a^3_{[-2\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}+2a_{[-2\epsilon]}a^2_{[-3\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}+2a_{[-2\epsilon]}a^3_{[-\epsilon]}r_{[-2\epsilon]}\\
&+a^3r^2_{[-2\epsilon]}+aa^2_{[-2\epsilon]}r^2_{[-2\epsilon]}+aa^2_{[-3\epsilon]}r^2_{[-2\epsilon]}+2aa^2_{[-\epsilon]}r^2_{[-2\epsilon]}+2a_{[-2\epsilon]}a_{[-\epsilon]}r^3_{[-2\epsilon]}\\
&+ar^4_{[-2\epsilon]}+2a_{[-4\epsilon]}a_{[-3\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}r_{[-4\epsilon]}+aa^2_{[-\epsilon]}r^2_{[-4\epsilon]}+2a_{[-2\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}r^2_{[-4\epsilon]}\\
&+2aa_{[-2\epsilon]}a_{[-3\epsilon]}a_{[-\epsilon]}r_{[-3\epsilon]}+2a^2a_{[-3\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}+2a^2_{[-2\epsilon]}a_{[-3\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}\\
&+2a^2_{[-4\epsilon]}a_{[-3\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}+2a^3_{[-3\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}+2a_{[-3\epsilon]}a^2_{[-\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}\\
&+2aa_{[-4\epsilon]}a_{[-\epsilon]}r_{[-4\epsilon]}r_{[-3\epsilon]}+2a_{[-3\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}r^2_{[-6\epsilon]}+2r_{[-2\epsilon]}r_{[-3\epsilon]}a_{[-6\epsilon]}r_{[-6\epsilon]}r_{[-5\epsilon]}\\
&+2a_{[-2\epsilon]}a_{[-4\epsilon]}r_{[-2\epsilon]}r_{[-4\epsilon]}r_{[-3\epsilon]}+a^3r^2_{[-3\epsilon]}+aa^2_{[-4\epsilon]}r^2_{[-3\epsilon]}+aa^2_{[-3\epsilon]}r^2_{[-3\epsilon]}+2aa^2_{[-\epsilon]}r^2_{[-3\epsilon]}\\
&+2a_{[-2\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}r^2_{[-3\epsilon]}+2ar^2_{[-2\epsilon]}r^2_{[-3\epsilon]}+2a_{[-3\epsilon]}r_{[-2\epsilon]}r^3_{[-3\epsilon]}+ar^4_{[-3\epsilon]}-2a_{[-\epsilon]}a_{[\epsilon]}r^2r_{[-\epsilon]}\\
&+2a_{[-\epsilon]}a_{[\epsilon]}r^2_{[-2\epsilon]}r_{[-\epsilon]}+aa^2_{[-\epsilon]}r^2_{[-\epsilon]}-aa^2_{[\epsilon]}r^2_{[-\epsilon]}-2a_{[2\epsilon]}a_{[\epsilon]}rr^2_{[-\epsilon]}+2a_{[-2\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}r^2_{[-\epsilon]}\\
&-2a_{[2\epsilon]}a_{[\epsilon]}rr^2_{[3\epsilon]}-ar^2r^2_{[3\epsilon]}-2aa_{[2\epsilon]}a_{[3\epsilon]}a_{[\epsilon]}r_{[\epsilon]}-2a^2a_{[3\epsilon]}rr_{[\epsilon]}-2a^2_{[2\epsilon]}a_{[3\epsilon]}rr_{[\epsilon]}\\
&-2a^2_{[4\epsilon]}a_{[3\epsilon]}rr_{[\epsilon]}-2a^3_{[3\epsilon]}rr_{[\epsilon]}-2a_{[3\epsilon]}a^2_{[\epsilon]}rr_{[\epsilon]}-2a_{[3\epsilon]}r^3r_{[\epsilon]}-2aa_{[4\epsilon]}a_{[\epsilon]}r_{[2\epsilon]}r_{[\epsilon]}\\
&-2a_{[2\epsilon]}a_{[4\epsilon]}rr_{[2\epsilon]}r_{[\epsilon]}-2a_{[3\epsilon]}rr^2_{[4\epsilon]}r_{[\epsilon]}-2a_{[3\epsilon]}rr^2_{[3\epsilon]}r_{[\epsilon]}-a^3r^2_{[\epsilon]}-aa^2_{[4\epsilon]}r^2_{[\epsilon]}\\
&-aa^2_{[3\epsilon]}r^2_{[\epsilon]}-2aa^2_{[\epsilon]}r^2_{[\epsilon]}-2a_{[2\epsilon]}a_{[\epsilon]}rr^2_{[\epsilon]}-2ar^2r^2_{[\epsilon]}-ar^2_{[4\epsilon]}r^2_{[\epsilon]}-2a_{[3\epsilon]}rr^3_{[\epsilon]}\\
&-ar^4_{[\epsilon]}-2a_{[\epsilon]}rr_{[2\epsilon]}r_{[3\epsilon]}aa_{[5\epsilon]}-2a_{[4\epsilon]}rr_{[3\epsilon]}r_{[\epsilon]}aa_{[5\epsilon]}-2rr_{[4\epsilon]}r_{[3\epsilon]}r_{[\epsilon]}aa_{[6\epsilon]}\\
&+ar^2_{[-3\epsilon]}r^2_{[-6\epsilon]}+2a_{[-\epsilon]}r_{[-2\epsilon]}r_{[-4\epsilon]}a_{[-5\epsilon]}r_{[-5\epsilon]}+2a_{[-4\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}a_{[-5\epsilon]}r_{[-5\epsilon]}\\
&+2a_{[-2\epsilon]}a_{[-\epsilon]}r_{[-2\epsilon]}r^2_{[-5\epsilon]}+ar^2_{[-2\epsilon]}r^2_{[-5\epsilon]}+2a_{[-3\epsilon]}r_{[-2\epsilon]}r_{[-3\epsilon]}r^2_{[-5\epsilon]}+2a_{[-3\epsilon]}r^3_{[-2\epsilon]}r_{[-3\epsilon]}\ ,\\
X^\epsilon_{2,r}(\mathbf{a}) := {}& a^4r-a^4_{[2\epsilon]}r+2a^2a^2_{[-\epsilon]}r+a^2_{[-2\epsilon]}a^2_{[-\epsilon]}r+a^4_{[-\epsilon]}r-2a^2_{[2\epsilon]}a^2_{[3\epsilon]}r\\
&-a^2_{[4\epsilon]}a^2_{[3\epsilon]}r-a^4_{[3\epsilon]}r+a^2a^2_{[\epsilon]}r-a^2_{[2\epsilon]}a^2_{[\epsilon]}r+a^2r^3-a^2_{[2\epsilon]}r^3+2a_{[-4\epsilon]}a_{[-\epsilon]}rr_{[-4\epsilon]}r_{[-3\epsilon]}\\
&+a^2_{[-\epsilon]}r^3-a^2_{[3\epsilon]}r^3-2a_{[2\epsilon]}a_{[4\epsilon]}a_{[3\epsilon]}rr_{[2\epsilon]}-a^2_{[2\epsilon]}rr^2_{[2\epsilon]}+2aa_{[-2\epsilon]}a_{[-\epsilon]}rr_{[-2\epsilon]}\\
&+a^2rr^2_{[-2\epsilon]}+a^2_{[-\epsilon]}rr^2_{[-4\epsilon]}+2a_{[-2\epsilon]}a_{[-3\epsilon]}a_{[-\epsilon]}rr_{[-3\epsilon]}+2aa_{[-3\epsilon]}rr_{[-2\epsilon]}r_{[-3\epsilon]}\\
&+2a^2rr^2_{[-3\epsilon]}+a^2_{[-4\epsilon]}rr^2_{[-3\epsilon]}+a^2_{[-3\epsilon]}rr^2_{[-3\epsilon]}+2a^2_{[-\epsilon]}rr^2_{[-3\epsilon]}+r^3r^2_{[-3\epsilon]}\\
&+rr^4_{[-3\epsilon]}+2aa_{[-\epsilon]}a_{[\epsilon]}rr_{[-\epsilon]}-a^2_{[2\epsilon]}rr^2_{[-\epsilon]}+a^2_{[-\epsilon]}rr^2_{[-\epsilon]}-a^2_{[3\epsilon]}rr^2_{[4\epsilon]}\\
&-2a^2_{[2\epsilon]}rr^2_{[3\epsilon]}-2a^2_{[3\epsilon]}rr^2_{[3\epsilon]}-r^3r^2_{[3\epsilon]}-rr^4_{[3\epsilon]}-2a_{[2\epsilon]}a_{[3\epsilon]}a_{[\epsilon]}rr_{[\epsilon]}\\
&+a^2rr^2_{[\epsilon]}-a^2_{[3\epsilon]}rr^2_{[\epsilon]}-2a_{[4\epsilon]}a_{[3\epsilon]}rr_{[3\epsilon]}aa_{[5\epsilon]}-2a_{[2\epsilon]}rr_{[2\epsilon]}r_{[3\epsilon]}aa_{[5\epsilon]}-rr^2_{[3\epsilon]}aa^2_{[5\epsilon]}\\
&-2a_{[3\epsilon]}rr_{[4\epsilon]}r_{[3\epsilon]}aa_{[6\epsilon]}-rr^2_{[3\epsilon]}aa^2_{[6\epsilon]}+rr^2_{[-3\epsilon]}r^2_{[-6\epsilon]}-rr^2_{[3\epsilon]}rr^2_{[6\epsilon]}\ .
\end{aligned}$$

---

**Table IIb: sp(2) theory.**

Lax formulation for the $s^{\rm th}$ vector field $Z_s^{sp(2)}$: $dL^{sp(2)}/d\tau_s = [B_s^{sp(2)}, L^{sp(2)}]$ $(s = 0, 1, 2, ...)$

$$L^{sp(2)}(\mathbf{u}) = \partial^4 + u\partial^2 + u_x\partial + (\frac{1}{2}u_{xx} + v) \ , \qquad B_s^{sp(2)}(\mathbf{u}) := 4^{s-2}(L^{sp(2)})_+^{\frac{2s-3}{4}}(\mathbf{u}) \ .$$

Poisson tensor at a point $\mathbf{u} = (u, v)$. $Q_{\mathbf{u}}^{sp(2)} : T_{\mathbf{u}}^*\mathcal{U} \to T_{\mathbf{u}}\,\mathcal{U}$

$$\begin{pmatrix} \delta u \\ \delta v \end{pmatrix} \mapsto \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} := \begin{pmatrix} 0 & 4\partial \\ 4\partial & -\partial^3 + 2u\partial + u_x \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} \ .$$

Hamiltonian formulation: $Z_s^{sp(2)} = Q^{sp(2)} dh_s^{sp(2)} \qquad (s = 0, 1, 2, ...)$

$$h_s^{sp(2)}(\mathbf{u}) := \frac{2^{2s-2}}{2s+1}\,\mathrm{Tr}(L^{sp(2)})^{\frac{2s+1}{4}}(\mathbf{u}) \ .$$

First companions of the Lax operator :

$$B_0^{sp(2)}(\mathbf{u}) = B_1^{sp(2)}(\mathbf{u}) = 0, \qquad B_2^{sp(2)}(\mathbf{u}) = \partial, \qquad B_3^{sp(2)}(\mathbf{u}) := 4\partial^3 + 3u\partial + \frac{3}{2}u_x \ .$$

First Hamiltonians :

$$h_0^{sp(2)}(\mathbf{u}) = \frac{1}{16}\int dx\ u \ , \qquad h_1^{sp(2)}(\mathbf{u}) = \frac{1}{32}\int dx\ (-u^2 + 8v) \ ,$$

$$h_2^{sp(2)}(\mathbf{u}) = \frac{1}{32}\int dx\ (-u^3 + 8uv - u_x^2) \ ,$$

$$h_3^{sp(2)}(\mathbf{u}) = \frac{1}{128}\int dx\ (5u^4 - 48u^2v + 192v^2 - 4uu_x^2 + 32u_{xx}v + 4u_{xx}^2) \ .$$

First vector fields :

$$Z_0^{sp(2)}(\mathbf{u}) = Z_1^{sp(2)}(\mathbf{u}) = 0 \ , \quad Z_2^{sp(2)}(\mathbf{u}) = \begin{pmatrix} u_x \\ v_x \end{pmatrix} \ ,$$

$$Z_3^{sp(2)}(\mathbf{u}) = \begin{pmatrix} u_{xxx} - 3uu_x + 12v_x \\ -2v_{xxx} + 3u_xu_{xx} + \frac{3}{2}uu_{xxx} + 3uv_x \end{pmatrix} \ .$$

## 4. Field rescaling and continuous limit of the $V_2$ system

Let us keep all the notations of the previous Section concerning the $V_2$ system. We introduce a space $\mathcal{U}$ of pairs $\mathbf{u} = (u, v)$ and apply it into the phase space $\mathcal{A}$ of the Volterra system through the map $\Psi^\epsilon : \mathcal{U} \to \mathcal{A}$ , $\mathbf{u} \mapsto \mathbf{a} = \Psi^\epsilon(\mathbf{u})$, with

$$\begin{cases} a := -3 - \frac{1}{4}\epsilon^2 u \ , \\ r := \frac{1}{3} + \frac{1}{4}\epsilon^2 (u - u_{[\epsilon]} + u_{[2\epsilon]}) + \epsilon^4 v \ . \end{cases} \tag{4.1}$$

If $\mathcal{A}$ and $\mathcal{U}$ are conveniently chosen, the map $\Psi^\epsilon$ is one-to-one, with inverse $(\Psi^\epsilon)^{-1}$ described by the relations

$$\begin{cases} u = -\frac{4}{\epsilon^2}(3 + a) \ , \\ v = \frac{1}{\epsilon^4}(\frac{8}{3} + r) + \frac{1}{\epsilon^4}(a - a_{[\epsilon]} + a_{[2\epsilon]}) \ . \end{cases} \tag{4.2}$$

For $\mathbf{a} \in \mathcal{A}$ and $\mathbf{u} = (\Psi^\epsilon)^{-1}(\mathbf{a})$, the tangent map $T_{\mathbf{a}}(\Psi^\epsilon)^{-1} : T_{\mathbf{a}}\mathcal{A} \to T_{\mathbf{u}}\mathcal{U}$ sends $\dot{\mathbf{a}}$ into $\dot{\mathbf{u}}$, with

$$\begin{cases} \dot{u} = -\frac{4}{\epsilon^2}\dot{a} \ , \\ \dot{v} = \frac{1}{\epsilon^4}\dot{r} + \frac{1}{\epsilon^4}(\dot{a} - \dot{a}_{[\epsilon]} + \dot{a}_{[2\epsilon]}) \ . \end{cases} \tag{4.3}$$

The adjoint map $T^*_{\mathbf{a}}(\Psi^\epsilon)^{-1} : T^*_{\mathbf{u}}\mathcal{U} \to T^*_{\mathbf{a}}\mathcal{A}$ sends a covector $\delta\mathbf{u}$ into $\delta\mathbf{a}$, with

$$\begin{cases} \delta a := -\frac{4}{\epsilon^2}\delta u + \frac{1}{\epsilon^4}(\delta v - \delta v_{[-\epsilon]} + \delta v_{[-2\epsilon]}) \ , \\ \delta r := \frac{1}{\epsilon^4}\delta v \ . \end{cases} \tag{4.4}$$

Using this transformation, we can pull-back onto $\mathcal{U}$ all the $V_2$ structures carried by the phase space $\mathcal{A}$; we will keep the same notations for any object on $\mathcal{A}$ and its pull-back to $\mathcal{U}$. By so doing, we induce at each point $\mathbf{u}$ the following objects:
i) the operators

$$L^\epsilon(\mathbf{u}) := L^\epsilon(\mathbf{a})|_{\mathbf{a}=\Psi^\epsilon(\mathbf{u})}, \qquad A^\epsilon_s(\mathbf{u}) := A^\epsilon_s(\mathbf{a})|_{\mathbf{a}=\Psi^\epsilon(\mathbf{u})} \qquad (s = 0, 1, 2, ...); \tag{4.5}$$

the explicit expression of $L^\epsilon(\mathbf{u})$ is obtained from Table IIa and Eq.(4.1).
ii) The vector fields

$$X^\epsilon_s(\mathbf{u}) := T_{\mathbf{a}}(\Psi^\epsilon)^{-1} X^\epsilon_s(\mathbf{a})|_{\mathbf{a}=\Psi^\epsilon(\mathbf{u})} \qquad (s = 0, 1, 2, ...) \ . \tag{4.6}$$

iii) The Poisson tensor

$$Q^\epsilon_{\mathbf{u}} : T^*_{\mathbf{u}}\mathcal{U} \to T_{\mathbf{u}}\mathcal{U} \ , \quad Q^\epsilon_{\mathbf{u}} := T_{\mathbf{a}}(\Psi^\epsilon)^{-1} \circ Q^\epsilon_{\mathbf{a}} \circ T^*_{\mathbf{a}}(\Psi^\epsilon)^{-1}\big|_{\mathbf{a}=\Psi^\epsilon(\mathbf{u})} \ . \tag{4.7}$$

Explicitly, we have

$$Q^\epsilon_{\mathbf{u}} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \begin{pmatrix} -\frac{3}{\epsilon^4} Q^\epsilon_{uu} & \frac{3}{4\epsilon^6} Q^\epsilon_{uv} \\ \frac{3}{4\epsilon^6} Q^\epsilon_{vu} & -\frac{3}{16\epsilon^8} Q^\epsilon_{vv} \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} \ , \tag{4.8}$$

where the matrix elements have the following expressions (with $a$ and $r$ given by Eq.(4.1)):

$$
\begin{aligned}
Q^\epsilon_{uu} &:= 2rr_{[\epsilon]}\Delta_{3\epsilon}+2a_{[\epsilon]}r\Delta_{2\epsilon}+aa_{[\epsilon]}\Delta_\epsilon-aa_{[-\epsilon]}\Delta_{-\epsilon}-2a_{[-\epsilon]}r_{[-2\epsilon]}\Delta_{-2\epsilon}-2r_{[-2\epsilon]}r_{[-3\epsilon]}\Delta_{-3\epsilon}\ ,\\
Q^\epsilon_{uv} &:= 2rr_{[\epsilon]}\Delta_{3\epsilon}+2(a_{[\epsilon]}r-2rr_{[\epsilon]})\Delta_{2\epsilon}+(aa_{[\epsilon]}-2a_{[\epsilon]}r+ar_{[\epsilon]}+2rr_{[\epsilon]})\Delta_\epsilon\\
&\quad+(ar-aa_{[\epsilon]}+2a_{[\epsilon]}r)+(aa_{[\epsilon]}-aa_{[-\epsilon]})\Delta_{-\epsilon}+(aa_{[-\epsilon]}-ar_{[-2\epsilon]}-2a_{[-\epsilon]}r_{[-2\epsilon]})\Delta_{-2\epsilon}\\
&\quad+(2a_{[-\epsilon]}r_{[-2\epsilon]}-aa_{[-\epsilon]}-ar_{[-3\epsilon]}-2r_{[-2\epsilon]}r_{[-3\epsilon]})\Delta_{-3\epsilon}+(2r_{[-2\epsilon]}r_{[-3\epsilon]}-2a_{[-\epsilon]}r_{[-2\epsilon]})\Delta_{-4\epsilon}\\
&\quad-2r_{[-2\epsilon]}r_{[-3\epsilon]}\Delta_{-5\epsilon}\ ,\\
Q^\epsilon_{vu} &:= 2r_{[2\epsilon]}r_{[3\epsilon]}\Delta_{5\epsilon}+2(a_{[3\epsilon]}r_{[2\epsilon]}-r_{[2\epsilon]}r_{[\epsilon]})\Delta_{4\epsilon}\\
&\quad+(a_{[2\epsilon]}a_{[3\epsilon]}+a_{[3\epsilon]}r-2a_{[2\epsilon]}r_{[\epsilon]}+2rr_{[\epsilon]})\Delta_{3\epsilon}+(2a_{[\epsilon]}r-a_{[2\epsilon]}a_{[\epsilon]}+a_{[2\epsilon]}r)\Delta_{2\epsilon}\\
&\quad+(aa_{[\epsilon]}-a_{[2\epsilon]}a_{[\epsilon]})\Delta_\epsilon+(aa_{[\epsilon]}-ar-2a_{[\epsilon]}r)+(2ar_{[-\epsilon]}-aa_{[-\epsilon]}-a_{[-\epsilon]}r-2rr_{[-\epsilon]})\Delta_{-\epsilon}\\
&\quad+2(r_{[-2\epsilon]}r_{[-\epsilon]}-a_{[-\epsilon]}r_{[-2\epsilon]})\Delta_{-2\epsilon}-2r_{[-2\epsilon]}r_{[-3\epsilon]}\Delta_{-3\epsilon}\ ,\\
Q^\epsilon_{vv} &:= 2r_{[2\epsilon]}r_{[3\epsilon]}\Delta_{5\epsilon}+(2a_{[3\epsilon]}r_{[2\epsilon]}-2r_{[2\epsilon]}r_{[3\epsilon]}-2r_{[2\epsilon]}r_{[\epsilon]})\Delta_{4\epsilon}\\
&\quad+(a_{[2\epsilon]}a_{[3\epsilon]}+a_{[3\epsilon]}r-2a_{[3\epsilon]}r_{[2\epsilon]}+a_{[2\epsilon]}r_{[3\epsilon]}+rr_{[3\epsilon]}\\
&\quad+2r_{[2\epsilon]}r_{[3\epsilon]}-2a_{[2\epsilon]}r_{[\epsilon]}+2rr_{[\epsilon]}+2r_{[2\epsilon]}r_{[\epsilon]})\Delta_{3\epsilon}\\
&\quad+(2a_{[3\epsilon]}r_{[2\epsilon]}-a_{[\epsilon]}r_{[2\epsilon]}+2a_{[2\epsilon]}r_{[\epsilon]}-2rr_{[\epsilon]}-2r_{[2\epsilon]}r_{[\epsilon]}-a_{[2\epsilon]}a_{[3\epsilon]}-a_{[2\epsilon]}a_{[\epsilon]}\\
&\quad+a_{[2\epsilon]}r-a_{[3\epsilon]}r+2a_{[\epsilon]}r+a_{[2\epsilon]}r_{[2\epsilon]})\Delta_{2\epsilon}\\
&\quad+(a_{[2\epsilon]}a_{[3\epsilon]}+aa_{[\epsilon]}+a_{[3\epsilon]}r-a_{[2\epsilon]}r-2a_{[\epsilon]}r+ar_{[\epsilon]}-2a_{[2\epsilon]}r_{[\epsilon]}-a_{[\epsilon]}r_{[\epsilon]}+2rr_{[\epsilon]})\Delta_\epsilon\\
&\quad+(2a_{[\epsilon]}r+2ar_{[-\epsilon]}-a_{[2\epsilon]}r_{[-\epsilon]}+a_{[\epsilon]}r_{[-\epsilon]}-2rr_{[-\epsilon]}-aa_{[-\epsilon]}-a_{[2\epsilon]}a_{[\epsilon]}\\
&\quad+ar-a_{[-\epsilon]}r)\Delta_{-\epsilon}+\\
&\quad+(aa_{[-\epsilon]}+aa_{[\epsilon]}-ar-2a_{[\epsilon]}r+a_{[-\epsilon]}r+2rr_{[-\epsilon]}+2r_{[-2\epsilon]}r_{[-\epsilon]}\\
&\quad+a_{[\epsilon]}r_{[-2\epsilon]}-ar_{[-2\epsilon]}-2a_{[-\epsilon]}r_{[-2\epsilon]}-2ar_{[-\epsilon]})\Delta_{-2\epsilon}\\
&\quad+(2ar_{[-\epsilon]}-2rr_{[-\epsilon]}-2r_{[-2\epsilon]}r_{[-\epsilon]}-aa_{[-\epsilon]}-a_{[-\epsilon]}r-ar_{[-3\epsilon]}-rr_{[-3\epsilon]}\\
&\quad-2r_{[-2\epsilon]}r_{[-3\epsilon]}+2a_{[-\epsilon]}r_{[-2\epsilon]})\Delta_{-3\epsilon}\\
&\quad+(2r_{[-2\epsilon]}r_{[-3\epsilon]}+2r_{[-2\epsilon]}r_{[-\epsilon]}-2a_{[-\epsilon]}r_{[-2\epsilon]})\Delta_{-4\epsilon}-2r_{[-2\epsilon]}r_{[-3\epsilon]}\Delta_{-5\epsilon}\ .
\end{aligned}
\tag{4.9}
$$

iv) The Hamiltonians

$$
f^\epsilon_s(\mathbf{u}) := f^\epsilon_s(\mathbf{a})|_{\mathbf{a}=\Psi^\epsilon(\mathbf{u})} \qquad (s=0,1,2,\ldots)\ . \tag{4.10}
$$

We will expand the above objects as (formal) power series in $\epsilon$; any such series will be of the form $\sum_{k=m}^{+\infty}\epsilon^k y_k$, where the nature of the coefficients $y_k$ depends on the object under consideration (they can be real numbers, real functions, differential operators, and so on); it may happen that the integer $m$ be negative. From now on, for each integer $l$, $O(\epsilon^l)$ denotes a series where $\epsilon$ appears only with exponents $\geq l$.

Let us start from the expansion of the Lax operator and of the Poisson tensor. To this purpose, we write the $j^{\rm th}$ power of the shift operator as

$$
\Delta_{j\epsilon} = e^{j\epsilon\partial} = \sum_{k=0}^{+\infty}\frac{j^k}{k!}\epsilon^k\partial^k\ . \tag{4.11}
$$

Similarly, we express any shifted function $u_{[j\epsilon]}$ or $v_{[j\epsilon]}$ through its Taylor series. Both the Lax operator $L^\epsilon(\mathbf{u})$ and the tensor $Q^\epsilon_{\mathbf{u}}$ can be expanded as power series of differential operators; let us compute the lowest order terms.

**4.1. Proposition.** One has

$$
L^\epsilon(\mathbf{u}) = -\frac{8}{3}+\epsilon^4 L^{sp(2)}(\mathbf{u})+O(\epsilon^5)\ , \tag{4.12}
$$

$$Q^\epsilon_{\mathbf{u}} = \frac{1}{\epsilon^5} Q^{sp(2)}_{\mathbf{u}} + O(\frac{1}{\epsilon^4}) \ , \tag{4.13}$$

with $L^{sp(2)}(\mathbf{u})$ and $Q^{sp(2)}_{\mathbf{u}}$ as in Table IIb.
**Proof.** By definition, it is

$$L^\epsilon(\mathbf{u}) = \frac{1}{2}(re^{3\epsilon\partial} + ae^{\epsilon\partial} + a_{[-\epsilon]}e^{-\epsilon\partial} + r_{[-3\epsilon]}e^{-3\epsilon\partial}) \ , \tag{4.14}$$

with $a$ and $r$ as in Eq.(4.1). The determination of the lower order terms in this expansion amounts to a lenghty, but not difficult computation. For example, we have

$$\begin{aligned} r_{[-3\epsilon]} &= \frac{1}{3} + \frac{1}{4}\epsilon^2(u_{[-3\epsilon]} - u_{[-2\epsilon]} + u_{[-\epsilon]}) + \epsilon^4 v_{[-3\epsilon]} \\ &= \frac{1}{3} + \frac{1}{4}\epsilon^2 u - \frac{1}{2}\epsilon^3 u_x + \epsilon^4(\frac{3}{4}u_{xx} + v) + O(\epsilon^5) \ , \\ e^{-3\epsilon\partial} &= 1 - 3\epsilon\partial + \frac{9}{2}\epsilon^2\partial^2 - \frac{9}{2}\epsilon^3\partial^3 + \frac{27}{8}\epsilon^4\partial^4 + O(\epsilon^5) \ , \end{aligned} \tag{4.15}$$

so that

$$\begin{aligned} r_{[-3\epsilon]}e^{-3\epsilon\partial} &= \frac{1}{3} - \epsilon\partial + \frac{1}{4}\epsilon^2(6\partial^2 + u) - \frac{1}{4}\epsilon^3(6\partial^3 + 3u\partial + 2u_x) \\ &+ \frac{1}{16}\epsilon^4(18\partial^4 + 18u\partial^2 + 24u_x\partial + 12u_{xx} + 16v) + O(\epsilon^5) \ . \end{aligned} \tag{4.16}$$

The other summands in Eq.(4.13) are expanded similarly; as a final result, one finds that all terms in $\epsilon$, $\epsilon^2$ and $\epsilon^3$ mutually cancel, while the coefficient of $\epsilon^4$ is the $sp(2)$ Lax operator.
Let us come to the Poisson tensor; from the representation (4.8), we can expect the expansion of $Q^\epsilon_{\mathbf{u}}$ to contain negative powers of $\epsilon$, of exponents $\geq -8$. Explicit computation shows that the lowest order nonzero terms are in fact of order $-5$, and correspond to the $sp(2)$ Poisson tensor. Indeed, we have

$$\begin{aligned} Q^\epsilon_{uu} &= O(\frac{1}{\epsilon^4}) \ , \qquad Q^\epsilon_{uv} = \frac{4}{\epsilon^5}\partial + O(\frac{1}{\epsilon^4}) \ , \\ Q^\epsilon_{vu} &= \frac{4}{\epsilon^5}\partial + O(\frac{1}{\epsilon^4}) \ , \qquad Q^\epsilon_{vv} = \frac{1}{\epsilon^5}(-\partial^3 + 2u\partial + u_x) + O(\frac{1}{\epsilon^4}) \ , \end{aligned} \tag{4.17}$$

thus proving the thesis. ◊

Now we consider the Hamiltonians, the vector fields and the companion operators $A^\epsilon_s(\mathbf{u})$; to obtain from their expansions the homologous objects of the $sp(2)$ theory, we need a recombination scheme of the type already applied to the KM system. To illustrate this scheme, we will first give the constructive rule for the coefficients of the recombination, and subsequently state the general recombination theorem.

**4.2. Definition.** i) $p_0(\xi)$ is the function of a real parameter $\xi$ defined by

$$p_0(\xi) := \log(\frac{1}{3} + \frac{1}{2}\xi + \xi^2) \ ; \tag{4.18}$$

ii) $p_s(\xi)$ is the polynomial in $\xi$ defined by

$$p_s(\xi) := \frac{1}{2s}\mathrm{rem}(\ell(\xi,z)^{2s}) \ , \tag{4.19}$$

where the "reminder" rem is the coefficient of the zero order term in $z$, and one considers the $(2s)^{\mathrm{th}}$ power of the rational function

$$\ell(\xi,z) := \frac{1}{2}(\frac{1}{3}+\frac{1}{2}\xi+\xi^2)(z^3+\frac{1}{z^3}) - \frac{1}{2}(3+\frac{1}{2}\xi)(z+\frac{1}{z}) \ . \tag{4.20}$$

$\diamondsuit$

The relation between the above objects and the $V_2$ Hamiltonians is easily illustrated. First of all, let us note that, for $u(x) = \text{const.} = 2$, $v(x) = \text{const.} = 1$, Eq.(4.1) becomes $a = -3 - \frac{1}{2}\epsilon^2$, $r = \frac{1}{3} + \frac{1}{2}\epsilon^2 + \epsilon^4$. Let us evaluate the Hamiltonian $f_0^\epsilon$ at this point; comparing with Eq (4.18), we see that $f_0^\epsilon(2,1) = p_0(\xi)\big|_{\xi=\epsilon^2}$. Also, if we consider the $V_2$ Lax operator and compare with Eq.(4.20), we see that the expression of $L^\epsilon(2,1)$ is obtained from $\ell(\xi,z)$ with the substitutions $\xi \mapsto \epsilon^2$, $z \mapsto \Delta_\epsilon$. From Table IIa, Eq.(4.19) and the above result on $f_0^\epsilon$ we conclude that

$$f_s^\epsilon(2,1) = p_s(\epsilon^2) \qquad (s = 0,1,2,...) \ . \tag{4.21}$$

We remark that, for each $s \geq 1$, the polynomial $p_s$ has degree $4s$ in $\xi$. In particular, we have

$$\begin{aligned} p_1(\xi) &= \tfrac{41}{18} + \tfrac{5}{6}\xi + \tfrac{7}{24}\xi^2 + \tfrac{1}{4}\xi^3 + \tfrac{1}{4}\xi^4 \ , \\ p_2(\xi) &= \tfrac{2957}{432} + \tfrac{293}{72}\xi + \tfrac{133}{288}\xi^2 + \tfrac{271}{96}\xi^3 + \tfrac{1143}{256}\xi^4 + \tfrac{23}{16}\xi^5 + \tfrac{23}{64}\xi^6 + \tfrac{3}{16}\xi^7 + \tfrac{3}{32}\xi^8 \ , \\ p_3(\xi) &= \tfrac{2048495}{69984} + \tfrac{176315}{7776}\xi - \tfrac{54515}{31104}\xi^2 + \tfrac{3065}{144}\xi^3 + \tfrac{1945195}{41472}\xi^4 + \tfrac{315455}{13824}\xi^5 + \tfrac{1118665}{165888}\xi^6 \\ &\quad + \tfrac{77545}{9216}\xi^7 + \tfrac{8095}{1152}\xi^8 + \tfrac{255}{128}\xi^9 + \tfrac{5}{12}\xi^{10} + \tfrac{5}{32}\xi^{11} + \tfrac{5}{96}\xi^{12} \ . \end{aligned} \tag{4.22}$$

Futhermore, if we truncate the series $p_0(\xi)$, say to order 4, we get

$$p_0(\xi) = -\log 3 + \frac{3}{2}\xi + \frac{15}{8}\xi^2 - \frac{27}{8}\xi^3 + \frac{63}{64}\xi^4 + O(\xi^5) \ . \tag{4.23}$$

**4.3. Definition.** Let $s$ be any nonnegative integer and consider an $(s+2)$-tuple of real numbers $(c_{s,-1}, c_{s,0}, c_{s,1}, ...., c_{s,s})$. We say that the coefficients $(c_{s,j})$ $(j = -1, ..., s)$[5] satisfy the $(\mathbf{R}_s)$ condition if

$$c_{s,-1} + \sum_{j=0}^{s} c_{s,j}p_j(\xi) = \frac{2^{2s-2}}{2s+1}\binom{s+\frac{1}{2}}{s+1}\xi^{s+1} + O(\xi^{s+2}) \ . \tag{4.24}$$

$\diamondsuit$

We note that the determination of a system of coefficients satisfying the above condition amounts to solve a system of $(s+2)$ linear equations, in the same number

[5] We could simply write $(c_{-1}, c_0, c_1, ..., c_s)$ or $(c_j)$. The redundant notation, with an additional subscript $s$, is suggested by the context.

of unknowns; for $s = 0, 1, ..., 6$ we have checked that the system admits a unique solution. For $s = 0, 1, 2, 3$ the explicit expressions of the coefficients $c_{s,j}$ are:

$$\begin{aligned}
&c_{0,-1} = \tfrac{1}{12}\log 3 \ , \qquad c_{0,0} = \tfrac{1}{12} \ , \\
&c_{1,-1} = \tfrac{41}{108} + \tfrac{5}{54}\log 3 \ , \qquad c_{1,0} = \tfrac{5}{54} \ , \qquad c_{1,1} = -\tfrac{1}{6} \ , \\
&c_{2,-1} = -\tfrac{1037}{648} - \tfrac{77}{486}\log 3 \ , \qquad c_{2,0} = -\tfrac{77}{486} \ , \qquad c_{2,1} = \tfrac{37}{27} \ , \qquad c_{2,2} = -\tfrac{2}{9} \ , \\
&c_{3,-1} = -\tfrac{584669}{96228} - \tfrac{221}{729}\log 3 \ , \ c_{3,0} = -\tfrac{221}{729}, \ c_{3,1} = \tfrac{2675}{297}, \ c_{3,2} = -\tfrac{332}{99}, \ c_{3,3} = \tfrac{16}{55} \ .
\end{aligned} \tag{4.25}$$

We conjecture that a unique solution exists for each nonnegative $s$. This would be granted if one were able to prove that, for each $s$, the matrix of elements $p_{kj}$ := coefficient of $\xi^j$ in $p_k(\xi)$ $(k = 0, 1, ..., s; j = 1, 2, ..., s+1)$ has a nonzero determinant, which is a nontrivial combinatorial problem.
Let $s$ be any nonnegative integer and $(c_{s,-1}, c_{s,0}, ... c_{s,s})$ an $(s + 2)$-tuple of real numbers; define the recombinations

$$h^\epsilon_s := c_{s,-1} + \sum_{j=0}^{s} c_{s,j} f^\epsilon_j \ , \qquad Z^\epsilon_s := \sum_{j=1}^{s} c_{s,j} X^\epsilon_j \ , \qquad B^\epsilon_s := \sum_{j=1}^{s} c_{s,j} A^\epsilon_j \tag{4.26}$$

(intending $Z^\epsilon_0 := 0$ and $B^\epsilon_0 := 0$). We note that, by construction, it is

$$\frac{dL^\epsilon}{d\theta_s} = [B^\epsilon_s, L^\epsilon] \ , \qquad Z^\epsilon_s = Q^\epsilon dh^\epsilon_s \tag{4.27}$$

(denoting with $d/d\theta_s$ the derivative along $Z^\epsilon_s$).
**4.4. Proposition.** If the sequence $(c_{s,-1}, c_{s,0}, ... c_{s,s})$ satisfies the $(\mathbf{R}_s)$ condition, at each point $\mathbf{u} \in \mathcal{U}$ it is

$$h^\epsilon_s(\mathbf{u}) = \epsilon^{2s+2} h^{sp(2)}_s(\mathbf{u}) + O(\epsilon^{2s+3}) \ , \tag{4.28}$$

$$Z^\epsilon_s(\mathbf{u}) = \epsilon^{2s-3} Z^{sp(2)}_s(\mathbf{u}) + O(\epsilon^{2s-2}) \ , \tag{4.29}$$

$$B^\epsilon_s(\mathbf{u}) = \epsilon^{2s-3} B^{sp(2)}_s(\mathbf{u}) + O(\epsilon^{2s-2}) \ . \tag{4.30}$$

$\diamondsuit$

This Proposition tells us that the lowest order terms in the $\epsilon$-expansion of the recombined $V_2$ Hamiltonians and vector fields belong to the $sp(2)$ theory; the next Section will be fully devoted to the proof of this fact. As we will see, the $(\mathbf{R}_s)$ condition is, essentially, the statement of Eq.(4.28) at $\mathbf{u} = (2, 1)$; so, proving Prop.4.4 amounts to showing that Eq.s(4.28-30) hold *everywhere* on the phase space $\mathcal{U}$ if (4.28) is satisfied *at this particular point* .
As an exercise, let us check directly the statements of Prop.4.4 for the first Hamiltonian functions. If we consider the Hamiltonians of Table IIa, express $a$ and $r$ in terms of $u$ and $v$ via Eq.(4.1) and expand in $\epsilon$, we find:

$$\begin{aligned}
f^\epsilon_0(\mathbf{u}) :=& -\log 3 + \tfrac{3}{4}\epsilon^2 \textstyle\int dx \ u - \tfrac{3}{32}\epsilon^4 \int dx \ (3u^2 - 32v) \\
&+ \tfrac{9}{64}\epsilon^6 \textstyle\int dx \ (u^3 + 4u_x^2 - 16uv) - \tfrac{9}{4}\epsilon^7 \int dx \ u_x v \\
&- \tfrac{3}{1024}\epsilon^8 \textstyle\int dx \ (27u^4 + 288uu_x^2 + 112u_{xx}^2 - 576u^2 v + 1152u_{xx}v + 1536v^2) + O(\epsilon^9),
\end{aligned} \tag{4.31}$$

$$f_1^\epsilon(\mathbf{u}) := \tfrac{41}{18} + \tfrac{5}{12}\epsilon^2 \int dx\ u + \tfrac{1}{96}\epsilon^4 \int dx\ (3u^2 + 16v) - \tfrac{1}{32}\epsilon^6 \int dx\ (u_x^2 - 4uv)$$
$$+\tfrac{1}{8}\epsilon^7 \int dx\ u_x v + \tfrac{1}{384}\epsilon^8 \int dx\ (7u_{xx}^2 + 72u_{xx}v + 96v^2) + O(\epsilon^9),$$
$$f_2^\epsilon(\mathbf{u}) := \tfrac{2957}{432} + \tfrac{293}{144}\epsilon^2 \int dx\ u + \tfrac{1}{1152}\epsilon^4 \int dx\ (453u^2 - 1280v)$$
$$+\tfrac{1}{768}\epsilon^6 \int dx\ (31u^3 - 348u_x^2 + 960uv) + \tfrac{5}{4}\epsilon^7 \int dx\ u_x v$$
$$+\tfrac{1}{12288}\epsilon^8 \int dx (21u^4 - 2784uu_x^2 + 5072u_{xx}^2 + 3072u^2 v + 35328u_{xx}v + 42240v^2) + O(\epsilon^9),$$
$$f_3^\epsilon(\mathbf{u}) := \tfrac{2048495}{69984} + \tfrac{176315}{15552}\epsilon^2 \int dx\ u + \tfrac{5}{124416}\epsilon^4 \int dx\ (81429u^2 - 369328v)$$
$$+\tfrac{5}{41472}\epsilon^6 \int dx\ (5075u^3 - 30439u_x^2 + 67972uv) + \tfrac{84965}{10368}\epsilon^7 \int dx\ u_x v$$
$$+\tfrac{5}{1990656}\epsilon^8 \int dx\ (28485u^4 - 1432656uu_x^2 + 1576324u_{xx}^2 + 1334304u^2 v$$
$$+9825696u_{xx}v + 12880896v^2) + O(\epsilon^9)\ .$$

The coefficients satisfying the $(\mathbf{R}_s)$ condition for $s = 0, 1, 2, 3$ have been already computed; using their explicit expressions (4.25), together with the previous expansions, one sees that the recombinations $h_s^\epsilon$ $(s = 0, 1, 2, 3)$ with these coefficients give at the lowest orders the corresponding $sp(2)$ objects, in agreement with Prop.4.4 . For example,

$$\begin{aligned} h_3^\epsilon(\mathbf{u}) &= -\tfrac{584669}{96228} - \tfrac{221}{729}\log 3 - \tfrac{221}{729}\ f_0^\epsilon(\mathbf{u}) + \tfrac{2675}{297} f_1^\epsilon(\mathbf{u}) - \tfrac{332}{99} f_2^\epsilon(\mathbf{u}) + \tfrac{16}{55} f_3^\epsilon(\mathbf{u}) \\ &= \tfrac{1}{128}\epsilon^8 \int dx\ (5u^4 - 48u^2 v + 192v^2 - 4uu_x^2 + 32u_{xx}v + 4u_{xx}^2) + O(\epsilon^9) \\ &= \epsilon^8 h_3^{sp(2)}(\mathbf{u}) + O(\epsilon^9)\ . \end{aligned} \tag{4.32}$$

## 5. Proof of Proposition 4.4

Prop.4.4 is a straightforward outcome of the subsequent three Lemmas; most of the effort is required for proving the second one. Let us start evaluating the $sp(2)$ Hamiltonians at the particular point $\mathbf{u} = (2, 1)$.

**Lemma 5.1.** For $s = 0, 1, 2, ...$, it is

$$h_s^{sp(2)}(2,1) = \frac{2^{2s-2}}{2s+1} \binom{s+\frac{1}{2}}{s+1}\ . \tag{5.1}$$

**Proof.** The $sp(2)$ Lax operator at $\mathbf{u} = (2, 1)$ is $L^{sp(2)}(2, 1) = \partial^4 + 2\partial^2 + 1 = (\partial^2 + 1)^2$, and so

$$(L^{sp(2)})^{\frac{2s+1}{4}}(2,1) = (\partial^2 + 1)^{\frac{2s+1}{2}} = \sum_{k=0}^{+\infty} \binom{s+\frac{1}{2}}{k} \partial^{2s+1-2k}\ . \tag{5.2}$$

The term of order $-1$ in $\partial$ in the above sum corresponds to $k = s + 1$, and the relation

$$h_s^{sp(2)}(2,1) = \frac{2^{2s-2}}{2s+1} \mathrm{Res}(L^{sp(2)})^{\frac{2s+1}{4}}(2,1) \tag{5.3}$$

yields Eq.(5.1) . ◊

After these preliminaries, we fix a nonnegative integer, and consider an arbitrary $(s+2)$-tuple $(c_{s,-1}, c_{s,0}, c_{s,1}, ...., c_{s,s})$ of real coefficients (excluding that all of them are zero). The statements we will make in the forthcoming Lemma 5.2 do not

depend on the $(\mathbf{R}_s)$ condition, so for the moment we do not put it on the coefficients $c_{s,j}$.
Let us form on the phase space $\mathcal{U}$ the linear combinations $h^\epsilon_s$, $Z^\epsilon_s$ and $B^\epsilon_s$ as in Eq.(4.26). From the definitions of the Hamiltonians and the operators and from the explicit expression (4.1) of the field rescaling it is easily inferred that, at each point $\mathbf{u}$, there is an expansion $h^\epsilon_s(\mathbf{u}) = \gamma_s + \sum_m \epsilon^m h_{s,m}(\mathbf{u})$ where $\gamma_s$ is a constant, $m$ goes from 2 to $+\infty$, and the $h_{s,m}$ are (integrals of) differential polynomial densities in $u, v$, without constant parts. We note that it cannot be $h_{s,m}(\mathbf{u}) = 0$ identically in $\mathbf{u}$ for each $m$, because the relation $h^\epsilon_s(\mathbf{u}) = \gamma_s$ would imply the $V_2$ Hamiltonians $f^\epsilon_0, ..., f^\epsilon_s$ and the constant function 1 to be linearly dependent, which is clearly false.

**5.2. Definition.** Throughout the rest of the section, $q$ will denote the lowest integer such that the function $h_{s,q}$ is not identically zero on $\mathcal{U}$. $\Diamond$

Summing up, we have

$$h^\epsilon_s(\mathbf{u}) = \gamma_s + \sum_{m=q}^{+\infty} \epsilon^m h_{s,m}(\mathbf{u}) \ , \qquad q \geq 2, \qquad h_{s,q} \neq 0 \ ; \tag{5.4}$$

from here and from Eq.(4.13) we infer that the vector field $Z^\epsilon_s = Q^\epsilon dh^\epsilon_s$ has an expansion

$$Z^\epsilon_s(\mathbf{u}) = \sum_{m=q-5}^{+\infty} \epsilon^m Z_{s,m}(\mathbf{u}) \ , \qquad Z_{s,q-5} = Q^{sp(2)} dh_{s,q-5} \ . \tag{5.5}$$

As for the operator $B^\epsilon_s$, independently of any consideration envolving $q$ we can state that there is an expansion of the form

$$B^\epsilon_s(\mathbf{u}) = \sum_{m=1}^{+\infty} \epsilon^m B_{s,m}(\mathbf{u}) \ , \tag{5.6}$$

where the $B_{s,m}(\mathbf{u})$ are differential operators of order $\leq m$ in $\partial$. The skew-symmetry of $B^\epsilon_s(\mathbf{u})$ is inherited by all these operators (this also explains why the expansion (5.6) does not have a term of order zero in $\epsilon$: the zero order part is necessarily a constant, and the skew-symmetry forces the constant to be zero). Finally, for the Lax operator we can write, recalling Eq.(4.12),

$$L^\epsilon(\mathbf{u}) = -\frac{8}{3} + \epsilon^4 L^{sp(2)}(\mathbf{u}) + \sum_{m=5}^{+\infty} \epsilon^m L_m(\mathbf{u}) \ , \tag{5.7}$$

where the $L_m(\mathbf{u})$ are differential operators of order $\leq m$. The coefficients of the powers of $\partial$ in all the operators $B_{s,m}(\mathbf{u})$, $L_m(\mathbf{u})$ are differential polyomials in the $\mathbf{u}$ variables.

**Lemma 5.2.** i) The integer $q$ in Eq.(5.4) is even, so that we can write $q = 2p + 2$ with $p = 0, 1, 2, ...$;
ii) there is a nonzero constant $\kappa_s$ such that, at any point $\mathbf{u} \in \mathcal{U}$,

$$h_{s,q}(\mathbf{u}) = \kappa_s h^{sp(2)}_p(\mathbf{u}) \ ; \tag{5.8}$$

iii) it is also

$$Z_{s,q-5}(\mathbf{u}) = \kappa_s Z_p^{sp(2)}(\mathbf{u}) \ ; \tag{5.9}$$

iv) if $q \geq 6$, it is

$$B_{s,q-5}(\mathbf{u}) = \kappa_s B_p^{sp(2)}(\mathbf{u}) \ ; \tag{5.10}$$

furthermore, for $q > 6$ the lower order coefficients in the expansion (5.6) are zero:

$$B_{s,m}(\mathbf{u}) = 0 \qquad \text{for} \quad 1 \leq m \leq q-6 \ . \tag{5.11}$$

**Proof.** We will give *ad hoc* proofs for each value of $q$ lower than 6, and a general argument for the higher values. Our analysis will also show that $q$ cannot be odd. We start expanding the transformation (4.1) up to order 3 in $\epsilon$. It is easily found that, for $\mathbf{a} = \Psi^\epsilon(\mathbf{u})$ and each integer $k$,

$$\begin{cases} a_{[k\epsilon]} = -3 - \frac{1}{4}\epsilon^2 u - \frac{1}{4}k\epsilon^3 u_x + O(\epsilon^4) \ , \\ r_{[k\epsilon]} = \frac{1}{3} + \frac{1}{4}\epsilon^2 u + \frac{1}{4}(1+k)\epsilon^3 u_x + O(\epsilon^4) \ ; \end{cases} \tag{5.12}$$

this implies $h_s^\epsilon(\mathbf{u}) = \text{const.} + \text{ const. } \epsilon^2 \int dx \ u + \text{ const. } \epsilon^3 \int dx \ u_x + O(\epsilon^4)$. But $\int dx \ u = 16 h_0^{sp(2)}(\mathbf{u})$ and $\int dx \ u_x = 0$, so we can write

$$h_s^\epsilon(\mathbf{u}) = \text{const.} + \text{const. } \epsilon^2 h_0^{sp(2)}(\mathbf{u}) + O(\epsilon^4) \ . \tag{5.13}$$

Let us compare this result with Eq.(5.4). If $q = 2$, the constant multiplying $\epsilon^2$ in Eq.(5.13) is nonzero; denoting this constant with $\kappa_s$, we can write

$$h_{s,2}(\mathbf{u}) = \kappa_s h_0^{sp(2)}(\mathbf{u}) \ , \tag{5.14}$$

which is just the statement ii); statement iii) follows immediately from Eq.(5.14), from the second relation (5.5) and the fact that $Q^{sp(2)} dh_0^{sp(2)} = Z_0^{sp(2)}$ (incidentally, we recall that this vector field is zero).

So, the Lemma is proved for $q = 2$; since Eq.(5.13) has no term of order 3 in $\epsilon$, we also see that the case $q = 3$ cannot occur.

All cases with $q \geq 4$ can be attacked from the Lax formulation of the vector field $Z_s^\epsilon$: at any point $\mathbf{u}$, we write

$$\mathcal{L}_{Z_s^\epsilon} L^\epsilon(\mathbf{u}) = [B_s^\epsilon(\mathbf{u}), L^\epsilon(\mathbf{u})] \ , \tag{5.15}$$

where $\mathcal{L}$ is the Lie derivative (in view of the subsequent computations, the notation $d/d\theta_s$ of Eq.(4.27) is no more convenient). The strategy we employ is to insert the expansions (5.5) and (5.7) in the above Lax equation, and draw the appropriate conclusions.

Let $q = 4$; substituting the expansions in Eq.(5.15), we see that the coefficient of $\epsilon^3$ is $\mathcal{L}_{Z_{s,-1}} L^{sp(2)}(\mathbf{u})$ in the l.h.s, and zero in the r.h.s.. The equality $\mathcal{L}_{Z_{s,-1}} L^{sp(2)}(\mathbf{u}) = 0$ implies[6]

$$Z_{s,-1}(\mathbf{u}) = 0 \ . \tag{5.16}$$

[6] For any vector field $Y$ on the phase space $\mathcal{U}$, it is $(\mathcal{L}_Y L^{sp(2)})(\mathbf{u}) = \dot{u}\partial^2 + (\dot{v} + (1/2)\dot{u}_{xx})|_{\dot{\mathbf{u}}=Y(\mathbf{u})}$. From here, it is evident that $\mathcal{L}_Y L^{sp(2)} = 0$ iff $Y = 0$.

But, as we know, $Z_1^{sp(2)}(\mathbf{u}) = 0$, so statement iii) is trivially satisfied in the present case. In order to derive ii), we note that Eq.(5.16) and the second relation (5.5) imply $Q^{sp(2)} dh_{s,4} = 0$. On the other hand, any Casimir of the Poisson tensor $Q^{sp(2)}$ (coming from a differential polynomial density in $u, v$, without constant terms) is a linear combination with constant coefficients of the fundamental Casimirs $h_0^{sp(2)}(\mathbf{u}) = \frac{1}{16} \int dx \ u$ and $h_1^{sp(2)}(\mathbf{u}) = \frac{1}{32} \int dx \ (-u^2 + 8v)$ . Therefore, there are two constants $\chi_s, \kappa_s$ such that

$$h_{s,4}(\mathbf{u}) = \chi_s h_0^{sp(2)}(\mathbf{u}) + \kappa_s h_1^{sp(2)}(\mathbf{u}) \ ; \tag{5.17}$$

but, from the expansion of Eq.(4.1) up to order 4 in $\epsilon$, it is seen that no summand of the form const. $\int dx \ u$ can occur in the expression of $h_{s,4}$, and so $\chi_s = 0$. On the contrary, $\kappa_s$ cannot be zero, for this would contradict the assumption $h_{s,4} \neq 0$. Summing up, statement ii) is proved and the analysis of the case $q = 4$ is concluded. Let us show that the case $q = 5$ cannot occur. We assume by absurd the first nonzero term in Eq.(5.4) to be $h_{s,5}$. Expanding Eq.(5.15) and equating the coefficients of $\epsilon^4$ in the two sides, we obtain $\mathcal{L}_{Z_{s,0}} L^{sp(2)}(\mathbf{u}) = 0$. So, we have $0 = Z_{s,0} = Q^{sp(2)} dh_{s,5}$, and the characterization just recalled for the $sp(2)$ Casimirs implies $h_{s,5}(\mathbf{u}) = \chi_s h_0^{sp(2)}(\mathbf{u}) + \eta_s h_1^{sp(2)}(\mathbf{u})$ for some pair of constants $\chi_s$, $\eta_s$. On the other hand, analysing Eq.(4.1) up to order 5 in $\epsilon$, we see that no terms proportional to $\int dx \ u$ or $\int dx \ (-u^2 + 8v)$ can occur in $h_{s,5}$, yielding the conclusion $\chi_s = \eta_s = 0$. This clearly contradicts the initial assumption that $h_{s,5}$ does not vanish identically. The discussion of the case $q = 6$ starts again from Eq.(5.15). Equating the coefficients of $\epsilon^6$ in the two sides of its expansion, we find

$$\mathcal{L}_{Z_{s,1}} L^{sp(2)}(\mathbf{u}) = [B_{s,1}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] \ ; \tag{5.18}$$

the above relation expresses the compatibility of the operator $B_{s,1}(\mathbf{u})$ with the $L^{sp(2)}(\mathbf{u})$, at each point $\mathbf{u}$: the commutator of these operators is tangent to the "Lax submanifold" of the $sp(2)$ theory. On the other hand, it is known [5] that any differential operator compatible with $L^{sp(2)}(\mathbf{u})$ is a linear combination of the skew-symmetric operators $B_2^{sp(2)}(\mathbf{u}) = (L^{sp(2)})_+^{\frac{1}{4}}(\mathbf{u})$, $B_3^{sp(2)}(\mathbf{u}) = 4(L^{sp(2)})_+^{\frac{3}{4}}(\mathbf{u})$, $B_4^{sp(2)} = 16(L^{sp(2)})_+^{\frac{5}{4}}(\mathbf{u})$, ... (of orders $1, 3, 5, ...$ in $\partial$) plus a linear combination of the symmetric operators 1, $L^{sp(2)}(\mathbf{u})$, $(L^{sp(2)})^2(\mathbf{u})$, ... (of orders 0, 4, 8, ...). On the other hand, $B_{s,1}(\mathbf{u})$ is skew-symmetric and of order $\leq 1$, so the above considerations imply

$$B_{s,1}(\mathbf{u}) = \kappa_s B_2^{sp(2)}(\mathbf{u}) \tag{5.19}$$

for some constant $\kappa_s$[7]. This fact, together with Eq.(5.18), gives

$$Z_{s,1}(\mathbf{u}) = \kappa_s Z_2^{sp(2)}(\mathbf{u}); \tag{5.20}$$

since $Z_{s,1} = Q^{sp(2)} dh_{s,6}$ (see Eq.(5.15) ), we also find that $h_{s,6}$ is $\kappa_s$ times $h_2^{sp(2)}$ up to a Casimir, i.e.,

$$h_{s,6}(\mathbf{u}) = \kappa_s h_2^{sp(2)}(\mathbf{u}) + \chi_s h_0^{sp(2)}(\mathbf{u}) + \eta_s h_1^{sp(2)}(\mathbf{u}) \ . \tag{5.21}$$

[7] In principle, $\kappa_s$ could be a real functional of the variables $\mathbf{u}$ (say, the integral over $x$ of some $\mathbf{u}$-dependent density). In fact, this is not the case: such a kind of $\mathbf{u}$-dependence is excluded for all the differential operators $B_{s,m}(\mathbf{u})$. The same considerations apply to the other constants (denoted with $\kappa_s$, $\chi_s$, $\eta_s$, $\xi_{s,i}$) appearing in the rest of the proof.

Finally, the analysis of Eq.(4.1) up to order 6 in $\epsilon$ ensures that the constants $\chi_s$, $\eta_s$ are zero. Then $\kappa_s$ is necessarily nonzero, and Eq.s(5.19-21) give the statements ii), iii) and iv) for the present case $q = 6$.
Let us pass to the general case $q > 6$, $q$ even. Expanding Eq.(5.15), and equating the coefficients of $\epsilon^5$, $\epsilon^6$, $\epsilon^7$, ..., $\epsilon^{q-2}$ in the two sides, we obtain $q-6$ equations

$$\begin{cases} 0 = [B_{s,1}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] \ , \\ 0 = [B_{s,2}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] + [B_{s,1}(\mathbf{u}), L_5(\mathbf{u})] \ , \\ 0 = [B_{s,3}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] + [B_{s,2}(\mathbf{u}), L_5(\mathbf{u})] + [B_{s,1}(\mathbf{u}), L_6(\mathbf{u})] \ , \\ \dots \quad \dots \quad \dots \\ 0 = [B_{s,q-6}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] + [B_{s,q-7}(\mathbf{u}), L_5(\mathbf{u})] + \dots + [B_{s,1}(\mathbf{u}), L_{q-3}(\mathbf{u})] \end{cases} \tag{5.22}$$

whereas, equating the coefficients of $\epsilon^{q-1}$, we get

$$\mathcal{L}_{Z_{q-5}} L^{sp(2)}(\mathbf{u}) = [B_{s,q-5}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] + [B_{s,q-6}(\mathbf{u}), L_5(\mathbf{u})] + \dots + [B_{s,1}(\mathbf{u}), L_{q-2}(\mathbf{u})] \ . \tag{5.23}$$

The system (5.22) can be treated iteratively. The first equation tells us that $B_{s,1}(\mathbf{u})$ is in the centralizer of $L^{sp(2)}(\mathbf{u})$; on the other hand, the centralizer of the Lax operator consists of the linear combinations of the fractional powers $(L^{sp(2)})^{\frac{k}{4}}(\mathbf{u})$ ($k \in \mathbb{Z}$), and the *differential* operators commuting with $L^{sp(2)}(\mathbf{u})$ are the linear combinations of nonnegative integer powers, which are all symmetric. Since $B_{s,1}(\mathbf{u})$ is skew-symmetric, it is necessarily zero. Inserting this result into the second Eq.(5.22), we obtain $[B_{s,2}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] = 0$, and the same argument employed above gives $B_{s,2}(\mathbf{u}) = 0$. Iterating this procedure, we obtain

$$B_{s,1}(\mathbf{u}) = B_{s,2}(\mathbf{u}) = .... = B_{s,q-6}(\mathbf{u}) = 0 \ . \tag{5.24}$$

From here and Eq.(5.23), we infer

$$\mathcal{L}_{Z_{s,q-5}} L^{sp(2)}(\mathbf{u}) = [B_{s,q-5}(\mathbf{u}), L^{sp(2)}(\mathbf{u})] \ . \tag{5.25}$$

Now the characterization (recalled after Eq.(5.18)) for the operators compatible with $L^{sp(2)}(\mathbf{u})$ and the skew-symmetry of $B_{s,q-5}(\mathbf{u})$ imply that the latter is a finite linear combination with constant coefficients of the operators $B_i^{sp(2)}(\mathbf{u}) = 4^{i-2}(L^{sp(2)})_+^{\frac{2i-3}{4}}(\mathbf{u})$ ($i = 2, 3, 4, \dots$):

$$B_{s,q-5}(\mathbf{u}) = \sum_i \xi_{s,i} B_i^{sp(2)}(\mathbf{u}) \ . \tag{5.26}$$

In order to characterize the constants $\xi_{s,i}$, we apply Eq.(5.26) at the particular point $\mathbf{u} = (0,0)$. Let us consider the transformation (4.1) at this point, and recall that $B_{s,q-5}(0,0)$ is the coefficient of $\epsilon^{q-5}$ in the expansion of $B_s^\epsilon(0,0)$; also, let us explicitate $B_i^{sp(2)}(0,0)$. In this way we find

$$B_{s,q-5}(0,0) = \text{const.}\ \partial^{q-5} \ , \qquad B_i^{sp(2)}(0,0) = 4^{i-2}\ \partial^{2i-3} \ . \tag{5.27}$$

Inserting these results in Eq.(5.26), we obtain that $\xi_{s,i} = 0$ for $2i-3 \neq q-5$; if $p$ is defined by $q = 2p+2$, this implies $\xi_{s,i} = 0$ for $i \neq p$, and setting $\kappa_s := \xi_{s,p}$ we can write

$$B_{s,q-5}(\mathbf{u}) = \kappa_s B_p^{sp(2)}(\mathbf{u}) \ . \tag{5.28}$$

This result, together with Eq.(5.25), gives

$$Z_{s,q-5}(\mathbf{u}) = \kappa_s Z_p^{sp(2)}(\mathbf{u}) \tag{5.29}$$

and from the second relation (5.5) we also infer $h_{s,q}(\mathbf{u}) = \kappa_s h_p^{sp(2)}(\mathbf{u})$ up to a Casimir of $Q^{sp(2)}$, i.e.,

$$h_{s,q}(\mathbf{u}) = \kappa_s h_p^{sp(2)}(\mathbf{u}) + \chi_s h_0^{sp(2)}(\mathbf{u}) + \eta_s h_1^{sp(2)}(\mathbf{u}) \ . \tag{5.30}$$

Again, we can exclude the appearing of any term proportional to either $\int dx\ u$ or $\int\ dx(-u^2+8v)$ in $h_{s,q}(\mathbf{u})$, so $\chi_s = \eta_s = 0$, while $\kappa_s \neq 0$ to avoid a contradiction with the hypothesis $h_{s,q} \neq 0$. On the grounds of Eq.s(5.24),(5.28-30) statements ii), iii) and iv) are proved.
To complete the proof of the Lemma, we only have to show that the case $q > 6$, $q$ odd cannot occur. Assume by absurd that the first nonzero term $h_{s,q}$ in the expansion (5.4) appears for $q$ odd; then Eq.s(5.22-23) still hold, and their analysis leads again to the conclusion (5.26). The coefficients $\xi_{s,i}$ are characterized by Eq.(5.27) where the exponent $q-5$ is now even, while $2i-3$ is obviously odd. In conclusion, $\xi_{s,i} = 0$ for each $i$ and $B_{s,q-5}(\mathbf{u}) = 0$, which also implies $Z_{s,q-5} = Q^{sp(2)} dh_{s,q} = 0$. The Casimir $h_{s,q}$ is necessarily a linear combination of $h_0^{sp(2)}$, $h_1^{sp(2)}$; the usual analysis gives $h_{s,q} = 0$, which contradicts the initial assumption. $\Diamond$

Writing $q = 2p+2$, we can express the statements of the previous Lemma 5.2 in the following way:

$$h_s^\epsilon(\mathbf{u}) = \gamma_s + \kappa_s \epsilon^{2p+2} h_p^{sp(2)}(\mathbf{u}) + O(\epsilon^{2p+3}) \qquad (p \in \{0,1,2,...\}\ , \kappa_s \neq 0)\ , \tag{5.31}$$

$$Z_s^\epsilon(\mathbf{u}) = \kappa_s \epsilon^{2p-3} Z_p^{sp(2)}(\mathbf{u}) + O(\epsilon^{2p-2})\ , \tag{5.32}$$

$$B_s^\epsilon(\mathbf{u}) = \kappa_s \epsilon^{2p-3} B_p^{sp(2)}(\mathbf{u}) + O(\epsilon^{2p-2})\ . \tag{5.33}$$

(At a first sight, Eq.(5.33) is granted by Lemma 5.1 only for $q \geq 6$, i.e., $p \geq 2$; however, if $p = 0$ or $1$, (5.33) is again satisfied in a trivial way, for in these cases it is $B_p^{sp(2)}(\mathbf{u}) = 0$ and the exponent $2p-2$ is $\leq 0$, while the operator $B_s^\epsilon(\mathbf{u})$ has an expansion in positive powers of $\epsilon$.)

**Lemma 5.3.** Let the $(s+2)$-tuple $(c_{s,-1}, c_{s,0}, c_{s,1}, ...., c_{s,s})$ satisfy the $(\mathbf{R}_s)$ condition (4.24). Then the integer $p$ and the constants $\gamma_s$, $\kappa_s$ appearing above are determined as follows

$$p = s\ , \qquad \gamma_s = 0\ , \qquad \kappa_s = 1\ , \tag{5.34}$$

and the relations (4.28-30), appearing in Prop.4.4, are satisified.

**Proof.** Let us specialize Eq.(5.31) to the point $\mathbf{u} = (2,1)$; on account of Lemma 5.1, we obtain

$$h_s^\epsilon(2,1) = \gamma_s + \frac{2^{2s-2}}{2s+1} \binom{s+\frac{1}{2}}{s+1} \kappa_s \epsilon^{2p+2} + O(\epsilon^{2p+3})\ . \tag{5.35}$$

On the other hand, Eq.s(4.26),(4.21) give $h^\epsilon_s(2,1) = c_{s,-1} + \sum_{j=0}^s c_{s,j} f^\epsilon_j(2,1) = c_{s,-1} + \sum_{j=0}^s c_{s,j} p_j(\epsilon^2)$. From here and from the $(\mathbf{R}_s)$ condition (4.24) we infer

$$h^\epsilon_s(2,1) = \frac{2^{2s-2}}{2s+1} \binom{s+\frac{1}{2}}{s+1} \epsilon^{2s+2} + O(\epsilon^{2s+4}) \ . \tag{5.36}$$

Comparing Eq.s(5.35-36) we obtain the thesis. $\diamondsuit$

On the grounds of the previous results, the proof of Prop.4.4 is now trivial; in fact, the statements (4.26-28) of this Proposition are obtained inserting Eq.(5.34) into Eq.s(5.31-33).

## 6. On the continuous limit of the $V_N$ theories

Let $N$ be an arbitrary integer, and $\mathcal{A}$ be the phase space of the $V_N$ system in $N$ fields $\mathbf{a} = (a_1, a_2, ..., a_N)$. The Lax operator for this theory is given by Eq.(1.3). Also, let us consider the $sp(N)$ KdV theory in $N$ fields $\mathbf{u} = (u_1, ..., u_N)$, and denote with $\mathcal{U}$ its phase space. Using the identity

$$\partial^m f = \sum_{h=0}^m \binom{m}{h} f_{(m-h)} \partial^h \tag{6.1}$$

(where $f_{(m-h)}$ denotes the $(m-h)^{\text{th}}$ derivative of the function $f$) we obtain for the Lax operator (1.6) the more explicit representation

$$L^{sp(N)}(\mathbf{u}) = \partial^{2N} + \frac{1}{2} \sum_{j=0}^{2N-1} \sum_{l=1}^{[N-j/2]} (1 + \delta_{2l,2N-j}) \binom{2N-2l}{j} u_{l(2N-2l-j)} \partial^j \ . \tag{6.2}$$

In order to connect the continuous limit of the $V_N$ system with the $sp(N)$ KdV, we search for a transformation

$$\Psi^\epsilon := \mathcal{U} \to \mathcal{A} \qquad \mathbf{u} = (u_1, ..., u_N) \mapsto \mathbf{a} = (a_1, ..., a_N) = \Psi^\epsilon(u_1, ..., u_N) \tag{6.3}$$

such that, setting $L^\epsilon(\mathbf{u}) := L^\epsilon(\mathbf{a})\big|_{\mathbf{a}=\Psi^\epsilon(\mathbf{u})}$, one gets

$$L^\epsilon(\mathbf{u}) = \text{const.} + \epsilon^{2N} L^{sp(N)}(\mathbf{u}) + O(\epsilon^{2N+1}) \ . \tag{6.4}$$

The results obtained for $N = 1, 2$ suggest the following *ansatz* :

$$\Psi^\epsilon(\mathbf{u}) = \mathbf{a} \ , \qquad a_p := C_p + \sum_{l=1}^p \epsilon^{2l} \Big( \sum_{q=0}^{2p-2l} C_{p,l,q} u_{l[q\epsilon]} \Big) \qquad (p = 1, 2, ..., N) \ . \tag{6.5}$$

Here $C_p$ and $C_{p,l,q}$ ($1 \le p \le N$, $1 \le l \le p$, $0 \le q \le 2p-2l$) are unknown constants, that we will often denote collectively with the symbol C. (We take the occasion to repeat that the subscript [...] denotes the shift, while (...) refers to $x$-derivatives. So, formulas (6.2) and (6.5) contain, respectively, the derivative of order $2N-2l-j$ of the function $u_l$ and the shifted function $u_{l[q\epsilon]}(x)=u_l(x+q\epsilon)$). The map $\Psi^\epsilon$ defined by Eq.(6.5) is invertible if the coefficients $C_{p,p,0}$ are nonzero for each $p$; the inverse map sends $\mathbf{a}$ into $\mathbf{u}$, where

$$u_l=\frac{1}{\epsilon^{2l}}\big(D_l+\sum_{p=1}^{l}\sum_{q=0}^{2l-2p}D_{l,p,q}a_{p[q\epsilon]}\big)\ , \tag{6.6}$$

the constants $D_l$ and $D_{l,p,q}$ being uniquely determined by the constants C.
Let us show that the ansatz (6.5) reduces the problem of the continuous limit to the problem of solving a linear system with numerical coefficients. In fact, we have:

**Lemma 6.1.** The condition (6.4) on the continuous limit of the $V_N$ Lax operator is fulfilled by the map $\Psi^\epsilon$ if and only if
i) the coefficients $C_p$ $(1\le p\le N)$ satisfy a linear system of $N$ equations;
ii) for each $l$ between 1 and $N$, the $(N+1-l)^2$ coefficients $C_{p,l,q}$ $(l\le p\le N,$ $0\le q\le 2p-2l)$ satisfy a linear system with the same number of equations.

**Proof.** We start by computing the $\epsilon$-expansion of the $V_N$ Lax operator, after having expressed the fields $a_p$ by means of Eq. (6.5). For each integer $q$, the shift of $a_p$ by $q\epsilon$ is

$$a_{p[q\epsilon]}=C_p+\sum_{r=0}^{+\infty}\sum_{l=1}^{p}\epsilon^{2l+r}\big(\sum_{h=0}^{2p-2l}\frac{(q+h)^r}{r!}C_{p,l,h}u_{l(r)}\big)\ . \tag{6.7}$$

From here and Eq.(4.11) we obtain, after some manipulations with multiple sums,

$$L^\epsilon(\mathbf{u})=\sum_{p=1}^{N}C_p+\sum_{m=2}^{+\infty}\epsilon^m L_m(\mathbf{u}) \tag{6.8}$$

where, for each $m$, $L_m(\mathbf{u})$ is a differential operator with explicit expression

$$L_m(\mathbf{u})=b_m(\mathrm{C})+\sum_{j=0}^{m-1}\big(\sum_{l=1}^{[\frac{m-j}{2}]}b_{m,j,l}(\mathrm{C})u_{l(m+j-2l)}\big)\partial^j\ . \tag{6.9}$$

In the above formula, we have put

$$b_m(\mathrm{C}):=\begin{cases}0 & \text{for } m \text{ odd}\\ 2\sum_{k=1}^{N}(2k-1)^m C_k & \text{for } m \text{ even}\end{cases} \tag{6.10}$$

$$b_{m,j,l}(\mathrm{C}):=\frac{1}{2(m-2l-j)!j!}\sum_{k=l}^{N}\sum_{q=0}^{2k-2l}\big((-1)^j q^{m-2l-j}+(1-2k+q)^{m-2l-j}\big)C_{k,l,q}\ . \tag{6.11}$$

Our purpose is to pick up the coefficients C so as to fulfill Eq. (6.4), i.e., the conditions

$$L_m(\mathbf{u})=0\quad\text{for}\quad 2\le m\le 2N-1\ ,\qquad L_{2N}(\mathbf{u})=L^{sp(N)}(\mathbf{u})\ . \tag{6.12}$$

Let us recall the symmetry of the operators $L^{sp(N)}(\mathbf{u})$ and $L^{\epsilon}(\mathbf{u})$; the latter also implies all the operators $L_m(\mathbf{u})$ to be automatically symmetric. On the other hand, for two symmetric differential operators $F = \sum_k f_k \partial^k$ and $G = \sum_k g_k \partial^k$ to be equal, it is sufficient that $f_k = g_k$ for each *even* $k$. Due to this remark, Eq.(6.4) holds if and only if the coefficients of all even powers of $\partial$ :
a) are zero in $L_m(\mathbf{u})$, for $2 \le m \le 2N-1$;
b) are equal in $L_{2N}(\mathbf{u})$ and $L^{sp(N)}(\mathbf{u})$.
This leads to the following set of equations:

$$b_m(\mathrm{C}) = \delta_{m,2N} \qquad (2 \le m \le 2N, \quad m \text{ even}) \; ; \tag{6.13}$$

$$b_{m,j,l}(\mathrm{C}) = \delta_{m,2N}(1+\delta_{2l,2N-j}) \binom{2N-2l}{j} \tag{6.14}$$

$$(2 \le m \le 2N, \quad 0 \le j \le m-1, \quad j \text{ even}, \quad 1 \le l \le [\frac{m-j}{2}]) \; .$$

On account of the definition (6.10), the equations (6.13) form a linear system of $N$ equations in $N$ unknowns $C_k$ $(k = 1,...,N)$; so, statement i) of this Lemma is proved. Recalling the definition (6.11) we also see that, for each fixed $l$ between 1 and $N$, the equations (6.14) (with $m$ and $j$ variable) form a linear system in the unknowns $C_{k,l,q}$ $(k = l,...,N,\; q = 0,...,2k-2l)$. It is easily checked that the number of unknowns and the number of equations are both equal to $(N+1-l)^2$ for each $l$, and this proves statement ii) of this Lemma. ◊

On account of the previous Lemma, the $sp(N)$ Lax operator can be attained as a continuous limit of the $V_N$ operator through a field rescaling of the form (6.5), provided that the linear systems mentioned in items i) and ii) have a solution; clearly, the solution exists and is unique if the square matrices associated to these systems have nonzero determinant.

The $N \times N$ matrix associated to the system (6.13) is essentially of a Vandermonde type, its determinant can be proved to be nonzero. As for the matrices of the systems (6.14) (one for each $l$ between 1 and $N$), showing that their determinants are nonzero is a nontrivial task, of which we do not have a proof for arbitrary $N$. Of course, from the treatment given in the previous Sections for the cases of $sp(1)$ and $sp(2)$, we already know that the envolved determinants are nonzero for $N = 1$ and 2; we have checked directly that the determinants are nonzero also for $N = 3, 4$. In particular, by solving the systems (6.13-14) for $N = 3$ we have found the transformation $\Psi^{\epsilon}$ sending $(u_1, u_2, u_3)$ into

$$\begin{cases} a_1 := \frac{15}{4} + \frac{1}{8}\epsilon^2 u_1 \; , \\ a_2 := -\frac{5}{8} - \frac{3}{16}\epsilon^2(u_1 - u_{1[\epsilon]} + u_{1[2\epsilon]}) - \frac{1}{8}\epsilon^4 u_2 \; , \\ a_3 := \frac{3}{40} + \frac{1}{16}\epsilon^2(u_1 - u_{1[\epsilon]} + u_{1[2\epsilon]} - u_{1[3\epsilon]} + u_{1[4\epsilon]}) \\ \qquad -\frac{1}{8}\epsilon^4(2u_2 - 5u_{2[\epsilon]} + 2u_{2[2\epsilon]}) + \epsilon^6 u_3 \; . \end{cases} \tag{6.15}$$

Composing with this transformation the $V_3$ Lax operator (given by Eq.(1.1) with $N = 3$), we find $L^{\epsilon}(\mathbf{u}) = \frac{16}{5} + \epsilon^6 L^{sp(3)}(\mathbf{u}) + O(\epsilon^7)$; the $sp(3)$ Lax operator (see Eq.(6.2)) is given explicitly by

$$L^{sp(3)}(\mathbf{u}) = \partial^6 + u_1\partial^4 + 2u_{1x}\partial^3 + (u_2 + 3u_{1xx})\partial^2 + (2u_{1xxx} + u_{2x})\partial + (u_3 + \frac{1}{2}u_{1xxxx}) \; . \tag{6.16}$$

With convenient choices of the functional spaces $\mathcal{A}$ and $\mathcal{U}$ to which these triplets of fields belong, the map $\Psi^\epsilon$ is one-to-one, with inverse sending $(a_1, a_2, a_3)$ into

$$\begin{cases} u_1 = \frac{1}{\epsilon^2}(-30 + 8a_1) \ , \\ u_2 = \frac{1}{\epsilon^4}(40 - 12a_1 + 12a_{1[\epsilon]} - 12a_{1[2\epsilon]} - 8a_2) \ , \\ u_3 = \frac{1}{\epsilon^6}(-\frac{16}{5} - \frac{7}{2}a_1 + 11a_{1[\epsilon]} - 14a_{1[2\epsilon]} + 11a_{1[3\epsilon]} - \frac{7}{2}a_{1[4\epsilon]} \\ \qquad -2a_2 + 5a_{2[\epsilon]} - 2a_{2[2\epsilon]} + a_3) \ . \end{cases} \tag{6.17}$$

In a similar way, we have found a unique transformation of the form (6.5), with inverse as in Eq.(6.6), relating the $V_4$ and $sp(4)$ Lax operators.

## 7. Comments and acknowledgments

The discussion of the $\epsilon \mapsto 0$ limit for the lattices considered in this paper and our previous one is centered on the algebraic/geometrical structures, and not on the behaviour of the solutions. In [1] the examination of this problem was confined to the continuous limit of the one-soliton solutions for the KM system and to some comments on the rigorous convergence theorem of Schwarz [13] for the $\epsilon \mapsto 0$ limit of the general initial data problem on the lattice. Here, we repeat some of the considerations of [1] and take the occasion for quoting, apart from references [13] [14] already mentioned therein, the paper of Lax, Levermore and Venakides [15]; we acknowledge the anonymous referee for this bibliographical indication.

In [15] the authors analyze a limit process for the KM system, based on the rescaling $a(x) \mapsto u(x) := \frac{1}{4}\epsilon a(x)^2$ which is different from the transformation (2.1) and more similar to the one considered by Brockett and Bloch [14] for the Toda lattice. Under the rescaling of [15], the zero-spacing limit of the KM vector field $X_1^\epsilon$ is the vector field $Z^H(u) := uu_x$, giving rise to the so-called Hopf (inviscid Burgers) evolution equation. The numerical experiment performed by the authors for the Cauchy problem with a particular initial datum shows the existence of a critical time $t_{crit}$, after which the solution $u^\epsilon(t)$ of the lattice evolution equation develops irregular space oscillations, with wavelength proportional to $\epsilon$. So, for $t > t_{crit}$ the zero spacing limit of $u^\epsilon(t)$ does not exist in a strong sense; this is the discrete counterpart of a shock phenomenon related to the limiting equation $u_t = uu_x$, namely the fact that the $x$ derivative of the solution $u(t)$ develops a divergency in a finite time, for certain choices of the initial datum; such a kind of shock also occurs in the limiting continuous system of reference [14].

Let us point out some facts which can explain the difference between this singular behaviour and the regular behaviour outlined in [13] for the zero-spacing limit of the lattice solutions, under a limit process for the KM system similar to the one considered in [1] and yielding the KdV equation.

It is known that the ordinary KdV and all the equations in the same hierarchy possess a global flow in each Sobolev space (see for instance [16] and references therein); from a technical viewpoint, global existence and uniqueness of the KdV flows can be traced back to the existence of conserved Hamiltonian densities, depending quadratically on the higher order $x$-derivative of the field $u$. These conservation laws can be employed to obtain a priori bounds on the Sobolev norms; discrete counterparts

for such estimates are possible for the KM hierarchy, yielding the $\epsilon$-independent a priori bounds which control in [13] the zero-spacing limit of the solutions.
The situation is quite different for the Hopf hierarchy considered in [15]: the conserved functionals have the form $\int dx\ u^n$ $(n = 1, 2, 3, ...)$ and do not control the derivatives of the field. With the rescaling considered therein, the conserved quantities of the KM system give the same functionals in the zero spacing limit, so they do not allow any bound on the derivatives of the interpolated solution; a similar remark can be made for the system discussed in [14].
The KdV-type hierarchies obtained from the limit process in this paper do possess Hamiltonian densities which are quadratic in the higher derivatives of the fields (for instance, the Hamiltonians $h_2^{sp(2)}$ and $h_3^{sp(2)}$ in Table IIb contain the squares of $u_x$ and $u_{xx}$ respectively; quadratic terms in $u_{xxx}, u_{xxxx}$ ... and $v_x$, $v_{xx}$, ... appear if one writes down the higher order Hamiltonians). So, in the situation considered here a regular behaviour *à la Schwarz* (globality of the flow for the limiting KdV's and convergence of the solutions for all times, in the zero-spacing limit of the Volterra systems) seems much more likely than the singular behaviour of the dispersionless equations considered in [14][15].

This paper has been partially supported by Consiglio Nazionale delle Ricerche, G.N.F.M, and by Ministero dell'Università e della Ricerca Scientifica, Project "Metodi Geometrici e Probabilistici in Teorie di Campo".

## REFERENCES


[1] C. Morosi, L. Pizzocchero, *On the continuous limit of integrable lattices I. The Kac-Moerbeke system and KdV theory*, Commun. Math. Phys. **180** (1996), 505–528.

[2] Y.B. Zeng, S. Rauch-Woyciekowski, *Continuous limit for the Kac-van Moerbeke hierarchy and for their restricted flows*, J. Phys. A **28** (1995), 3825–3840.

[3] M. Bruschi, S.V. Manakov, O. Ragnisco, D. Levi, *The nonabelian Toda lattice-discrete analogue of the matrix Schrödinger spectral problem*, J. Math. Phys. **21** (1980), 2749–2753.

[4] I. Merola, O. Ragnisco, T. Gui-Zhang, *A novel hierarchy of integrable lattices*, Inverse Problems **10** (1994), 1315–1334.

[5] V.G. Drinfeld, V.V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, J. Sov. Math. **30** (1985), 1975–2036.

[6] A.W. Knapp, *Lie Groups beyond an Introduction*, Birkhäuser, Basel, 1996.

[7] B.A. Kupershmidt, *Discrete Lax equations and differential-difference calculus*, Astérisque **123** (1985), Soc. Math. de France.

[8] C. Morosi, L. Pizzocchero, *R-matrix theory, formal Casimirs and the periodic Toda lattice*, J. Math. Phys. **37** (1996), 4484–4513.

[9] W. Oevel, O. Ragnisco, *R-matrices and higher Poisson brackets for integrable systems*, Physica A **161**, 181–220 (1989).

[10] L. C. Li, S. Parmentier, *Nonlinear Poisson structures and r-matrices*, Commun. Math. Phys. **125**, 545-563 (1989).

[11] P.A. Damianou, *Multiple Hamiltonian structures for Toda-type systems*, J. Math. Phys. **35** (1994), 5511–5541.

[12] C. Morosi, L. Pizzocchero, *On the biHamiltonian interpretation of the Lax formalism*, Rev. Math. Phys. **7** (1995), 389–430.

[13] M. Schwarz, *Korteweg-de Vries and nonlinear equations related to the Toda lattice*, Adv. in Math. **44** (1982), 132–154.

[14] R.W. Brockett, A. Bloch, *Sorting with the dispersionless limit of the Toda lattice*, Proceedings of the CRM Workshop on Hamiltonian systems, transformation groups and spectral transform methods (J. Harnad, J.E. Marsden Ed.s) (1990), 103–112.

[15] P.D. Lax, C.D. Levermore, S. Venakides, *The generation and propagation of oscillations in dispersive initial value problems and their limiting behavior*, Important developments in soliton theory (A.S. Fokas, V. E. Zakharov Ed.s) (1993), Springer, Berlin, 205–241.

[16] M. Schwarz, *The initial value problem for the sequence of generalized Korteweg-de Vries equations*, Adv. in Math. **54** (1984), 22–56.