On Moments of Negative Eigenvalues for the Pauli Operator

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Dedicated to the Memory of Professor Ruilin Long

Abstract. This paper concerns the three-dimensional Pauli operator

$$\mathbb{P} = \big(\sigma\cdot(\mathbf{p}-\mathbf{A}(x))\big)^2 + V(x)$$

with a non-homogeneous magnetic field $\mathbf{B} = \operatorname{curl}\mathbf{A}$. The following Lieb-Thirring type inequality for the moment of negative eigenvalues is established:

$$\sum_{\lambda_j<0} |\lambda_j| \le C_1 \int_{\mathbb{R}^3} |V(x)|_-^{5/2} dx + C_2 \int_{\mathbb{R}^3} [b_p(x)]^{3/2} |V(x)|_- dx$$

where $p > 3/2$ and $b_p(x)$ is the L^p average of $|\mathbf{B}|$ over certain cube centered at x with a side length scaling like $|\mathbf{B}|^{-1/2}$. We also show that, if $\mathbf{B}$ has a constant direction,

$$\sum_{\lambda_j<0} |\lambda_j|^\gamma \le C_{1,\gamma} \int_{\mathbb{R}^3} |V(x)|_-^{\gamma+3/2} dx + C_{2,\gamma} \int_{\mathbb{R}^3} b_p(x) |V(x)|_-^{\gamma+1/2} dx$$

where $\gamma > 1/2$ and $p > 1$.

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1. Introduction

Consider the Pauli operator

$$\mathbb{P} = \mathbb{P}_0 + V(x) = \big(\sigma \cdot (\mathbf{p} - \mathbf{A}(x))\big)^2 + V(x), \tag{1.1}$$

acting on $L^2(\mathbb{R}^3, \mathbb{C}^2)$. Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denotes the vector of Pauli matrices, $\mathbf{p} = -i\nabla$, and $\mathbf{A}(x) = (A_1(x), A_2(x), A_3(x)) \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ is a vector potential. We shall use $\mathbf{B} = \operatorname{curl} \mathbf{A}$ to denote the magnetic field generated by the potential $\mathbf{A}$. The goal of this paper is to establish the Lieb-Thirring type estimates for the moments of negative eigenvalues of $\mathbb{P}$:

$$\mathcal{M}_\gamma = \sum_{\lambda_j < 0} |\lambda_j|^\gamma, \qquad \gamma > 0. \tag{1.2}$$

We remark that in quantum mechanics $\mathbb{P}$ is used to describe the motion of a charged spin-1/2 particle in an electromagnetic field. Lieb-Thirring type estimates are an important tool in the study of stability of matter and in semi-classical analysis [13,10,11,12,2,4,5].

In the case of Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^n$, the classical Lieb-Thirring inequality [13] states that

$$\mathcal{M}_\gamma \le L_{\gamma,n} \int_{\mathbb{R}^n} |V(x)|_-^{\gamma+n/2} dx \tag{1.3}$$

where $\gamma > 0$, $n \ge 2$, and $|V|_- = \max(-V, 0)$ denotes the negative part of V (in the case $\gamma = 0$ and $n \ge 3$, (1.3) is the well-known Rosenblum-Lieb-Cwickel estimate [15]). For the Pauli operator $\mathbb{P}$ in $\mathbb{R}^3$ with a constant field, it was shown in [11] that

$$\mathcal{M}_\gamma \le C_{1,\gamma} \int_{\mathbb{R}^3} |V(x)|_-^{\gamma+3/2} dx + C_{2,\gamma} |\mathbf{B}| \int_{\mathbb{R}^3} |V(x)|_-^{\gamma+1/2} dx \tag{1.4}$$

for $\gamma > 1/2$. Subsequently, L. Erdös [3] initiated the study of Lieb-Thirring estimates for Pauli operators with non-homogeneous fields. He observed that the direct extension of (1.4) to the case of the non-constant fields:

$$\mathcal{M}_\gamma \le C_{1,\gamma} \int_{\mathbb{R}^3} |V(x)|_-^{\gamma+3/2} dx + C_{2,\gamma} \int_{\mathbb{R}^3} |\mathbf{B}(x)|\, |V(x)|_-^{\gamma+1/2} dx \tag{1.5}$$

as well as its consequence (by Hölder's inequality)

$$\mathcal{M}_\gamma \le \widetilde{C}_{1,\gamma} \int_{\mathbb{R}^3} |V(x)|_-^{\gamma+3/2} dx + \widetilde{C}_{2,\gamma} \int_{\mathbb{R}^3} |\mathbf{B}(x)|^{3/2} |V(x)|_-^\gamma \, dx \tag{1.6}$$

is false without substantial regularity conditions on $\mathbf{B}$. Moreover he conjectured that it would be necessary to replace $|\mathbf{B}|$ by some screened version of $|\mathbf{B}|$.

Motivated by Erdös' observation, A. Sobolev [19,20] obtained estimates in the forms of (1.5) and (1.6), but with $|\mathbf{B}|$ replaced by a so-called effective (scalar) magnetic field $b(x)$:

$$\sum_{\lambda_j<0} |\lambda_j| \le C_1 \int_{\mathbb{R}^3} |V(x)|_-^{5/2} dx + C_2 \int_{\mathbb{R}^3} b(x)^{3/2} |V(x)|_- dx. \tag{1.7}$$

Roughly speaking, $b(x)$ is a slow varying function which dominates $|\mathbf{B}(x)|$ pointwise. We remark that the two-dimensional Pauli operator as well as the three-dimensional case with a constant direction field was also investigated in [3,19,20].

Recently, L. Bugliaro, C. Fefferman, J. Fröhlich, G. Graf, and J. Stubbe [2] established (1.7) with a $b(x)$ whose energy is comparable to that of $|\mathbf{B}|$: $\|b\|_{L^2} \approx \|\mathbf{B}\|_{L^2}$. This in particular implies the following estimate of E. Lieb, M. Loss, and J. Solovej [10]:

$$\mathcal{M}_1 \le C_1 \int_{\mathbb{R}^3} |V(x)|_-^{5/2} dx + C_2 \left(\int_{\mathbb{R}^3} |\mathbf{B}(x)|^2 dx \right)^{3/4} \left(\int_{\mathbb{R}^3} |V(x)|_-^4 \, dx \right)^{1/4}. \tag{1.8}$$

In this paper we further extend the results in [3,19,20,2]. The main novelty of our work is that the effective field $b(x)$ we found is much simple and more natural than the previous ones in [20,2]. Indeed, $b(x) = b_p(x)$ is defined to be the L^p average of $|\mathbf{B}|$ over a suitable cube centered at x with a side length scaling like $|\mathbf{B}|^{-1/2}$. We believe that this choice of the effective field is optimal.

To state our main results, we first define the basic length scale:

$$\ell_p(x) = \sup \left\{ \ell > 0 : \ \ell^2 \left(\frac{1}{\ell^3} \int_{Q(x,\ell)} |\mathbf{B}(y)|^p dy \right)^{1/p} \le 1 \right\} \tag{1.9}$$

where $Q(x,\ell)$ denotes the cube centered at x with side length ℓ.

It is easy to see that if $|\mathbf{B}| \in L^p_{loc}(\mathbb{R}^3)$ for some $p > 3/2$, then $0 < \ell_p(x) < \infty$ unless $|\mathbf{B}| \equiv 0$. Our effective field is now given by

$$b_p(x) \equiv \frac{1}{\{\ell_p(x)\}^2} = \left\{ \frac{1}{\ell_p^3(x)} \int_{Q(x,\ell_p(x))} |\mathbf{B}(y)|^p dy \right\}^{1/p}, \tag{1.10}$$

i.e., $b_p(x)$ is the L^p average of $|\mathbf{B}|$ over the cube centered at x with side length $\ell_p(x)$. In particular, we have $\|b_p\|_{L^q} \leq C\|\mathbf{B}\|_{L^q}$ for any $q \geq p$.

Proposition 1.1. *Let $p > 3/2$. Then, for any $q \geq p$, there exists a constant $C_{p,q} > 0$ such that*

$$\int_{\mathbb{R}^3} |b_p(x)|^q dx \leq C_{p,q} \int_{\mathbb{R}^3} |\mathbf{B}(x)|^q dx. \tag{1.11}$$

The following are the main results of the paper.

Theorem 1.1. *Let $p > 3/2$ and $\gamma \geq 1$. Then there exist constants $C_1(\gamma,p) > 0$ and $C_2(\gamma,p) > 0$ such that*

$$\sum_{\lambda_j<0} |\lambda_j|^\gamma \leq C_1(\gamma,p) \int_{\mathbb{R}^3} |V(x)|_-^{\gamma+3/2} dx + C_2(\gamma,p) \int_{\mathbb{R}^3} [b_p(x)]^{3/2} |V(x)|_-^{\gamma} dx.$$

We have a stronger estimate for magnetic fields with constant directions.

Theorem 1.2. *Let $p > 1$ and $\gamma > 1/2$. Suppose that $\mathbf{B}$ has a constant direction. Then there exist constants $C_3(\gamma,p) > 0$ and $C_4(\gamma,p) > 0$ such that*

$$\sum_{\lambda_j<0} |\lambda_j|^\gamma \leq C_3(\gamma,p) \int_{\mathbb{R}^3} |V(x)|_-^{\gamma+3/2} dx + C_4(\gamma,p) \int_{\mathbb{R}^3} b_p(x) |V(x)|_-^{\gamma+1/2} dx.$$

A few remarks are in order.

Remark 1.1. The definition of $b_p(x)$ is motivated by the auxiliary function $m(x,|\mathbf{B}|)$, which is instrumental in the study of eigenvalue problems for the magnetic Schrödinger operator $-(\mathbf{p}-\mathbf{A}(x))^2+V(x)$ with certain degenerate potentials [16,17]. We should point out that, although the definition (1.9) is not rotation invariant, the basic length scales $\ell_p(x)$, hence $b_p(x)$, are equivalent in different coordinates systems. One can certainly use balls centered at x instead of cubes in (1.9). We also remark that, if the components of $\mathbf{B}$ are polynomials, then

$$b_p(x) \approx \sum_{|\alpha|\le k} |\nabla^\alpha \mathbf{B}(x)|^{\frac{2}{|\alpha|+2}} \tag{1.12}$$

where k is the degree of $\mathbf{B}$.

Remark 1.2. Suppose that $\mathbf{B}=\mathbf{B}_1+\mathbf{B}_2$ where $|\mathbf{B}_1|\in L^\infty(\mathbb{R}^3)$ and $|\mathbf{B}_2|\in L^p(\mathbb{R}^3)$ for some $p>3/2$. Let $\ell=\ell_p(x)$, then

$$\ell^2\left(\frac{1}{\ell^3}\int_{Q(x,\ell)}|\mathbf{B}(y)|^p dy\right)^{1/p}=1.$$

It follows that

$$\ell^2\|\mathbf{B}_1\|_{L^\infty}+\ell^{2-3/p}\|\mathbf{B}_2\|_{L^p}\ge 1.$$

A simple computation yields that

$$b_p(x)=\frac{1}{\ell^2}\le C\{\|\mathbf{B}_1\|_{L^\infty}+\|\mathbf{B}_2\|_{L^p}^{\frac{2p}{2p-3}}\}. \tag{1.13}$$

This, together with Theorem 1.1 ($p=2$), gives the estimate in [3, Theorem 2.4, p.635]. Same argument also shows that Theorem 2.1 in [3, p.632] follows from Theorem 1.2.

Remark 1.3. To compare our results with that in [2], we note that the effective field in [2] is defined by $\widetilde{b}(x)=1/[r(x)]^2$ where

$$\frac{1}{r(x)}=\int_{\mathbb{R}^3}\varphi\left(\frac{y-x}{r(x)}\right)|\mathbf{B}(y)|^2 dy \tag{1.14}$$

and $\varphi(z)=(1+\frac{1}{2}z^2)^{-2}$. Let $r=r(x)$. It is easy to see that

$$\int_{Q(x,r)}|\mathbf{B}(y)|^2 dy\le\frac{C}{r}.$$

Thus, if δ is small, we have

$$(\delta r)^2 \left\{ \frac{1}{(\delta r)^3} \int_{Q(x,\delta r)} |\mathbf{B}(y)|^2 dy \right\}^{1/2} \leq C\delta^{1/2} < 1.$$

By definition, $\ell_2(x) \geq \delta r$. Hence,

$$b_2(x) = \frac{1}{\{\ell_2(x)\}^2} \leq \frac{1}{(\delta r)^2} = \frac{\widetilde{b}(x)}{\delta^2}.$$

Thus Theorem 1.1 extends the results in [2], with an effective field much easier to compute. One may deduce the estimates in [20] from Theorems 1.1 and 1.2 with $p = \infty$ by a similar argument.

Remark 1.4. In Theorems 1.1 and 1.2, we have implicitly assumed that $\mathbb{P}_0 + V$ is a self-adjoint operator. In fact, if the right hand side of the estimate in Theorem 1.1 (or 1.2) is finite, then $\mathbb{P}_0 + V$ has a unique self-adjoint realization on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ associated with its quadratic form.

Indeed, under the assumption $\mathbf{A} \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, we may define $\mathbb{P}_0$ as the self-adjoint operator associated with the closed quadratic form $\|\mathbb{D}\psi\|^2_{L^2}$ where $\mathbb{D} = \sigma \cdot (\mathbf{p} - \mathbf{A})$, i.e., $\mathbb{P}_0 = \mathbb{D}^* \mathbb{D}$. Let $V_j = V\chi_{\{|V| \leq j\}}$. Since V_j is bounded, the operator $\mathbb{P}_0 - \frac{1}{\varepsilon}|V_j|$ is self-adjoint with a form core Domain$(\mathbb{D})$ for any $\varepsilon \in (0, 1)$. We now apply the variation principle and Theorem 1.1 to $\mathbb{P}_0 - \frac{1}{\varepsilon}|V_j|$. We obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |\sigma \cdot (\mathbf{p} - \mathbf{A})\psi|^2 dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^3} |V_j|\, |\psi|^2 dx \\ \geq \lambda_1(\varepsilon) \int_{\mathbb{R}^3} |\psi|^2 dx \geq -C(\delta, \varepsilon, V, \mathbf{B}) \int_{\mathbb{R}^3} |\psi|^2 dx \end{aligned}$$

for any $\psi \in$ Domain$(\mathbb{D})$. Thus,

$$\int_{\mathbb{R}^3} |V_j|\, |\psi|^2 dx \leq \varepsilon \int_{\mathbb{R}^3} |\sigma \cdot (\mathbf{p} - \mathbf{A})\psi|^2 dx + \varepsilon\, C(\delta, \varepsilon, V, \mathbf{B}) \int_{\mathbb{R}^3} |\psi|^2 dx.$$

Let $j \to \infty$. By Fatou's Lemma, we have

$$\int_{\mathbb{R}^3} |V|\, |\psi|^2 dx \leq \varepsilon \int_{\mathbb{R}^3} |\sigma \cdot (\mathbf{p} - \mathbf{A})\psi|^2 dx + \varepsilon\, C(\delta, \varepsilon, V, \mathbf{B}) \int_{\mathbb{R}^3} |\psi|^2 dx$$

for $\psi \in \text{Domain}(\mathbb{D})$. It then follows from the KLMN Theorem [14, p.167] that $\mathbb{P}_0 + V$ can be extended to the unique self-adjoint operator associated with its quadratic form. Furthermore, Domain($\mathbb{D}$) is a form core for $\mathbb{P}_0 + V$.

To prove Theorem 1.1, we will compare the Pauli operator $\mathbb{P}_0$ with the magnetic Schrödinger operator $\mathbb{H}_0 = (\mathbf{p} - \mathbf{A})^2$ in an appropriate scale, as in [2,8,19,20]. This is possible since $\mathbb{P}_0 = \mathbb{H}_0 - \sigma \cdot \mathbf{B}$. To this end, our first step is to localize the operators to cubes Q over which the L^p average of $|\mathbf{B}|$ is small compare to $\ell(Q)^{-2}$ (the localization error in the kinetic energy), by a Calderón–Zygmund decomposition. More precisely, we divide $\mathbb{R}^3$ into a grid of disjoint cubes $\{Q_j\}$ where each Q_j is a maximal dyadic cube such that

$$\left\{ \int_{12Q_j} |\mathbf{B}(x)|^p dx \right\}^{1/p} \leq \frac{\varepsilon}{[\ell(Q_j)]^{2-\frac{3}{p}}}. \tag{1.15}$$

Here $\ell(Q_j) = \ell_j$ is the side length of Q_j, and $12Q_j$ denotes the cube which has the same center as Q_j and side length $12\ell(Q_j)$. It can be proved that $\ell_j \approx \ell_k$ if $4Q_j \cap 4Q_k \neq \emptyset$. Using this property, we construct a partition of unity for $\mathbb{R}^3$: $\sum_j \phi_j^2(x) \equiv 1$, with $\phi_j \in C_0^\infty(2Q_j, \mathbb{R})$.

Next we apply the Birman-Schwinger principle which reduces the problem to the estimate of singular values of $|V|^{1/2}(\mathbb{P}_0 + \lambda)^{-1/2}$ for $\lambda > 0$. We then use the resolvent identity to compare $\phi_j(\mathbb{P}_0 + \lambda)^{-1}$ with $(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}\phi_j$ where $\Phi(x) = \sum_j \frac{1}{\ell_j^2} \phi_j^2(x) \approx b_p(x)$. We will show that, if ε in (1.15) is sufficiently small, then

$$\mathbb{H}_0 \leq C\left\{\mathbb{P}_0 + \Phi\right\} \tag{1.16}$$

(Theorem 3.1). With (1.16) at our disposal, we are able to estimate the contribution from $(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}\phi_j$. This leads to the first term in Theorem 1.1. Finally, to deal with the error term $\phi_j(\mathbb{P}_0 + \lambda)^{-1} - (\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}\phi_j$ which can be written as

$$(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}\big\{\big(\Phi + \frac{1}{\ell_j^2}\big)\phi_j + [\mathbb{P}_0, \phi_j]\big\}(\mathbb{P}_0 + \lambda)^{-1},$$

we use the resolvent identity again to compare $\psi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}$ with $(\mathbb{H}_0 + \frac{1}{\ell_j^2})^{-1}\psi_j$ where ψ_j is a bump function such that $\psi_j\phi_j \equiv \phi_j$. The desired result follows from certain regularity estimates for the operator $\psi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}$ (see Lemma 4.2).

A similar approach is used in the case of constant direction fields. Assuming $\mathbf{B} = (0, 0, B(x_1, x_2))$ without the loss of generality, we construct a partition of unity for $\mathbb{R}^2$ associated with B. The inequality

$$\|(-\partial_{x_3}^2 + \lambda)^{1/2}(\mathbb{P}_0 + \lambda)^{-1/2}\|_{L^2 \to L^2} \leq 1, \qquad \lambda > 0 \tag{1.17}$$

is used to exploit the fact that $\mathbb{P}_0$ commutes with ∂_{x_3}.

The paper is organized as follows. In Section 2 we collect some basic facts that will be used concerning the norm in the Neumann-Schatten classes. Section 2 also contains the proof of Proposition 1.1. The partition of unity will be constructed in Section 3. Section 4 is devoted to the proof of Theorem 1.1. Finally we study the case of constant direction fields in Section 5 where Theorem 1.2 is proved.

Throughout this paper we will use $\|\psi\|_{L^p}$ to denote the L^p norm of the function ψ. $\|T\|_{L^p \to L^q}$ will denote the operator norm, while $\|T\|_p$ is reserved for the norm in the Neumann-Schatten classes (see (2.1)). Finally we will use C to denote constants, which are not necessarily the same at each occurrence, which may depend on p.

2. Some Preliminaries

Most materials in this section on the Neumann-Schatten classes can be found in [18]. We include them here for the reader's convenience. The proof of Proposition 1.1 will be given at the end of the section.

Let T be a compact operator on $L^2(\mathbb{R}^3, \mathbb{C}^2)$. We will use $n(s, T)$ to denote the number of singular values $\{s_n(T)\}$ (counting multiplicity) of T greater than s where $s > 0$. For

$p \geq 1$, let

$$\|T\|_p^p \equiv \sum_n |s_n(T)|^p = p \int_0^\infty s^{p-1} n(s,T) ds. \tag{2.1}$$

The functional $\|T\|_p$ defines a norm on the Neumann-Schatten class $\mathcal{S}_p$ which consists of compact operators T with $\|T\|_p < \infty$.

The following facts will be useful to us.

$$n(s^2, T^*T) = n(s,T), \tag{2.2}$$

$$\|T\|_p^2 = \|T^*T\|_{p/2}, \tag{2.3}$$

$$n(s,T) \leq \frac{\|T\|_p^p}{s^p}. \tag{2.4}$$

We will also need

$$n(s_1 + s_2, T_1 + T_2) \leq n(s_1, T_1) + n(s_2 + T_2), \tag{2.5}$$

$$\|T_1 T_2\|_p \leq \|T_1\|_p \|T_2\|_{L^2 \to L^2}, \tag{2.6}$$

$$\|T_1 T_2\|_p \leq \|T_1\|_r \|T_2\|_s \qquad \text{if} \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s}, \quad r,\ s \geq 1 \tag{2.7}$$

where $\|T\|_{L^2 \to L^2}$ denotes the operator norm of T.

The proof of the following lemma may be found in [18].

Lemma 2.1. *If $f,\ g \in L^p(\mathbb{R}^3)$ with $2 \leq p < \infty$. Then*

$$\|f(x) g(-i\nabla)\|_p \leq C_p \|f\|_{L^p} \|g\|_{L^p}.$$

It follows from Lemma 2.1 that, if $p \geq 2$, $\alpha > \frac{3}{2p}$, and $\lambda > 0$,

$$\|V(-\Delta + \lambda)^{-\alpha}\|_p \leq C_{p,\alpha} \lambda^{\frac{3}{2p} - \alpha} \|V\|_{L^p}. \tag{2.8}$$

Also, by Lemma 2.1, if $\alpha > 1/2$ and $0 < \lambda_1 \leq \lambda_2$,

$$\|V(-\partial_{x_3}^2 + \lambda_1)^{-1/2} (-\Delta + \lambda_2)^{-\alpha}\|_2 \leq C_\alpha \lambda_1^{-\frac{1}{4}} \lambda_2^{-\alpha + \frac{1}{2}} \|V\|_{L^2}. \tag{2.9}$$

Thus, by the diamagnetic inequality [9],

$$|\big(\mathbb{H}_0+\lambda\big)^{-\alpha}\psi| \le (-\Delta+\lambda)^{-\alpha}|\psi|, \tag{2.10}$$

where $\mathbb{H}_0 = (\mathbf{p}-\mathbf{A}(x))^2$, and Theorem 2.13 in [18, p.36], we have

$$\|V(\mathbb{H}_0+\lambda)^{-\alpha}\|_2 \le C_\alpha \lambda^{\frac{3}{4}-\alpha}\|V\|_{L^2} \qquad \text{for } \alpha > 3/4, \tag{2.11}$$

$$\|V(\mathbb{H}_0+\lambda)^{-\alpha}\|_4 \le C_\alpha \lambda^{\frac{3}{8}-\alpha}\|V\|_{L^4} \qquad \text{for } \alpha > 3/8, \tag{2.12}$$

and for $\alpha > 1/2$ and $0 < \lambda_1 \le \lambda_2$,

$$\|V(-\partial_{x_3}^2+\lambda_1)^{-1/2}(\mathbb{H}_0+\lambda_2)^{-\alpha}\|_2 \le C_\alpha \lambda_1^{-\frac{1}{4}}\lambda_2^{-\alpha+\frac{1}{2}}\|V\|_{L^2}. \tag{2.13}$$

The following two lemmas concern the operator norm of $V(-\Delta+\lambda)^{-\alpha}$ on $L^2(\mathbb{R}^3,\mathbb{C}^2)$.

Lemma 2.2. *Let $\alpha > 0$ and $\lambda > 0$. Then*

$$\|V(-\Delta+\lambda)^{-\alpha}\|_{L^2\to L^2} \le C_\alpha \|V\|_{L^{\frac{3}{2\alpha}}} \qquad \text{if} \quad \alpha < 3/4, \tag{2.14}$$

$$\|V(-\Delta+\lambda)^{-\alpha}\|_{L^2\to L^2} \le C_\alpha \lambda^{\frac{3}{4}-\alpha}\|V\|_{L^2} \qquad \text{if} \quad \alpha > 3/4. \tag{2.15}$$

Proof. Note that, if $\alpha < 3/4$,

$$\|V(-\Delta+\lambda)^{-\alpha}\psi\|_{L^2} \le \|V\|_{L^{p_1}}\|(-\Delta+\lambda)^{-\alpha}\psi\|_{L^{p_2}} \le C_\alpha \|V\|_{L^{p_1}}\|\psi\|_{L^2}$$

where $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{2}$, $\frac{1}{p_2}=\frac{1}{2}-\frac{2\alpha}{3}$, and we have used Hölder's inequality and the well-known fractional integral estimate [21]. (2.14) is proved since $p_1 = 3/(2\alpha)$.

Now suppose $\alpha > 3/4$. Using the Fourier transform, we have

$$\begin{aligned}\|(-\Delta+\lambda)^{-\alpha}\psi\|_{L^\infty} &\le C\int_{\mathbb{R}^3}(|\xi|^2+\lambda)^{-\alpha}|\hat{\psi}(\xi)|d\xi \\ &\le C\|\psi\|_{L^2}\left\{\int_{\mathbb{R}^3}(|\xi|^2+\lambda)^{-2\alpha}d\xi\right\}^{1/2} \le C_\alpha\lambda^{\frac{3}{4}-\alpha}\|\psi\|_{L^2}.\end{aligned} \tag{2.16}$$

It follows that

$$\|V(-\Delta+\lambda)^{-\alpha}\psi\|_{L^2} \le \|V\|_{L^2}\|(-\Delta+\lambda)^{-\alpha}\psi\|_{L^\infty} \le C_\alpha\lambda^{\frac{3}{4}-\alpha}\|V\|_{L^2}\|\psi\|_{L^2}.$$

Lemma 2.3. *Let $0 < \alpha < 1/2$ and $\lambda > 0$. Suppose that V depends only on the first two variables. Then*

$$\|V(-\Delta+\lambda)^{-\alpha}\|_{L^2\to L^2} \le C_\alpha \|V\|_{L^{1/\alpha}(\mathbb{R}^2)}.$$

Proof. By the fractional integral estimates [21], if $\psi \in C^\infty(\mathbb{R}^3, \mathbb{C}^2)$ and decays at ∞,

$$\|\psi\|_{L^q(\mathbb{R}^2)} \le C_\alpha \|(-\Delta_2+\lambda)^\alpha \psi\|_{L^2(\mathbb{R}^2)}$$

where $\frac{1}{q} = \frac{1}{2} - \alpha$ and $\Delta_2 = \partial_{x_1}^2 + \partial_{x_2}^2$ denotes the Laplacian in $\mathbb{R}^2$. It then follows from Hölder's inequality that

$$\begin{aligned}\|V\psi\|_{L^2(\mathbb{R}^2)} &\le \|V\|_{L^p(\mathbb{R}^2)} \|\psi\|_{L^q(\mathbb{R}^2)} \\ &\le C_\alpha \|V\|_{L^p(\mathbb{R}^2)} \|(-\Delta_2+\lambda)^\alpha \psi\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Hence

$$\begin{aligned}\int_{\mathbb{R}^3} |V|^2 |\psi|^2 dx &\le C_\alpha^2 \|V\|_{L^p(\mathbb{R}^2)}^2 \int_{\mathbb{R}^3} |(-\Delta_2+\lambda)^\alpha \psi|^2 dx \\ &\le C_\alpha^2 \|V\|_{L^p(\mathbb{R}^2)}^2 \int_{\mathbb{R}^3} |(-\Delta+\lambda)^\alpha \psi|^2 dx\end{aligned}$$

where the second inequality can be verified through the Fourier transform. Thus

$$\|V\psi\|_{L^2} \le C_\alpha \|V\|_{L^p(\mathbb{R}^2)} \|(-\Delta+\lambda)^\alpha \psi\|_{L^2}.$$

The lemma now follows easily since $\frac{1}{p} = \frac{1}{2} - \frac{1}{q} = \alpha$.

We end this section with the

Proof of Proposition 1.1.

It follows from (1.10) that $b_p(x) \le \{M(|\mathbf{B}|^p)(x)\}^{1/p}$ where M is the Hardy-Littlewood maximal operator. This gives the desired estimate in the case $q > p$ since M is bounded on L^s for $s > 1$ [21].

For the case $q = p$, we appeal to an argument in [2]. Let $\widetilde{b}_p(x) = 1/[r_p(x)]^2$ where $r_p(x)$ is defined by

$$\frac{1}{[r_p(x)]^{2p-3}} = \int_{\mathbb{R}^3} \varphi\left(\frac{x-y}{r_p(x)}\right) |\mathbf{B}(y)|^p dy \tag{2.17}$$

and $\varphi(z) = [1 + \frac{1}{2}|z|^2]^{-2}$. Since $b_p(x) \le C\,\widetilde{b}_p(x)$ (see Remark 1.4 for $p = 2$), it suffices to show that

$$\int_{\mathbb{R}^3} |\widetilde{b}_p(x)|^p dx \le C \int_{\mathbb{R}^3} |\mathbf{B}(x)|^p dx. \tag{2.18}$$

Using (2.17) and Fubini's Theorem, one may reduce (2.18) to

$$\int_{\mathbb{R}^3} \varphi\left(\frac{x-y}{r_p(x)}\right) \frac{dx}{[r_p(x)]^3} \le C < \infty. \tag{2.19}$$

The proof of (2.19) relies on the following estimate

$$r_p(y) g_-\left(\frac{|x-y|}{r_p(y)}\right) \le r_p(x) \le r_p(y) g_+\left(\frac{|x-y|}{r_p(y)}\right) \tag{2.20}$$

where $g_+(s)$ and $g_-(s)$ are solutions of the equation

$$s^2 = 2(t^{p+\frac{1}{2}} - 1)(1 - t^{\frac{3}{2}-p}) \tag{2.21}$$

on $(1, \infty)$ and $(0, 1)$ respectively. Since $p > 3/2$, it is easy to see that $g_+(s)$, $g_-(s)$ are well defined. We omit the proof of (2.20), which is similar to that of Lemma 2 in [2, p.569].

It is not hard to see that $g_+(s) \approx s^{\frac{4}{2p+1}}$ if $s \ge 1/2$, and $g_-(s) \ge c_0 > 0$ if $s < 1/2$. This, together with (2.20), implies that

$$\begin{aligned} r_p(x) &\le C\,[r_p(y)]^{1-\frac{4}{2p+1}} |x-y|^{\frac{4}{2p+1}} && \text{if} \quad |x-y| \ge \frac{1}{2} r_p(y), \\ r_p(x) &\ge c\, r_p(y) && \text{if} \quad |x-y| < \frac{1}{2} r_p(y). \end{aligned}$$

It then follows that

$$\begin{aligned} &\int_{\mathbb{R}^3} \varphi\left(\frac{x-y}{r_p(x)}\right) \frac{dx}{[r_p(x)]^3} \le \int_{\mathbb{R}^3} \frac{r_p(x)\,dx}{\{[r_p(x)]^2 + \frac{1}{2}|x-y|^2\}^2} \\ &\le 4 \int_{|x-y| \ge \frac{1}{2} r_p(y)} \frac{r_p(x)\,dx}{|x-y|^4} + \int_{|x-y| < \frac{1}{2} r_p(y)} \frac{dx}{[r_p(x)]^3} \\ &\le C\,\big[r_p(y)\big]^{\frac{2p-3}{2p+1}} \int_{|x-y| \ge \frac{1}{2} r_p(y)} \frac{dx}{|x-y|^{3+\frac{2p-3}{2p+1}}} + \frac{C}{[r_p(y)]^3} \int_{|x-y| < \frac{1}{2} r_p(y)} dx \\ &\le C. \end{aligned}$$

The proof is finished.

3. A Partition of Unity

Throughout this section we fix $p > 3/2$. We will assume that $|\mathbf{B}| \in L^p_{loc}(\mathbb{R}^3)$ and $|\mathbf{B}| \not\equiv 0$.

Let $\varepsilon \in (0,1)$ be a small constant to be determined later. Let $\mathcal{A}$ be the set of all dyadic cubes in $\mathbb{R}^3$ such that

$$\left(\int_{12Q} |\mathbf{B}(x)|^p \, dx \right)^{1/p} \leq \frac{\varepsilon}{[\ell(Q)]^{2-\frac{3}{p}}} \tag{3.1}$$

where $\ell(Q)$ denotes the side length of Q. We say that Q is a maximal element of $\mathcal{A}$ if $Q \in \mathcal{A}$ and Q is not properly contained in any other cube in $\mathcal{A}$.

Let $\mathcal{B}$ denote the set of all maximal elements of $\mathcal{A}$. Clearly, by definition, the interiors of the cubes in $\mathcal{B}$ are disjoint.

Lemma 3.1. *Let $\mathcal{B} = \{Q_j, \ j = 1, 2, \dots\}$. Then,*

$$\mathbb{R}^3 = \bigcup_j Q_j.$$

Proof. It suffices to show that any point in $\mathbb{R}^3$ belongs to some cube Q_j in $\mathcal{B}$. To this end, fix $x \in \mathbb{R}^3$, let $\{Q_{\alpha_k}\}_{k=-\infty}^{\infty}$ be an increasing sequence of dyadic cubes such that $x \in Q_{\alpha_k}$ for all k, $\ell(Q_{\alpha_k}) \to 0$ as $k \to -\infty$, and $\ell(Q_{\alpha_k}) \to \infty$ as $k \to \infty$.

Note that, as $\ell(Q) \to 0$, the l.h.s. of (3.1) goes to 0 while the r.h.s. goes to ∞. This implies that $Q_{\alpha_k} \in \mathcal{A}$ if $\ell(Q_{\alpha_k})$ is small. Also, as $\ell(Q_{\alpha_k}) \to \infty$, the l.h.s. of (3.1) goes to $\|\mathbf{B}\|_{L^p} > 0$ and the r.h.s. goes to 0. Thus, $Q_{\alpha_k} \notin \mathcal{A}$ if $\ell(Q_{\alpha_k})$ is large. Hence there must be a maximal element Q_j in $\mathcal{A}$ (which may not be in $\{Q_{\alpha_k}\}$) such that $x \in Q_j$.

Lemma 3.2. *Let Q_j, $Q_k \in \mathcal{B}$. Suppose that $4Q_j \cap 4Q_k \neq \emptyset$. Then*

$$\frac{1}{2}\ell(Q_k) \leq \ell(Q_j) \leq 2\ell(Q_k).$$

Proof. We adapt an argument found in [8].

Suppose $\ell(Q_j) > 2\ell(Q_k)$. Since $\ell(Q_j)$ and $\ell(Q_k)$ are powers of 2, we have $\ell(Q_j) \geq 4\ell(Q_k)$.

Let Q_k^+ be the dyadic parent of Q_k, i.e., Q_k can be obtained by bisecting Q_k^+. Since Q_k is a maximal element in $\mathcal{A}$, $Q_k^+ \notin \mathcal{A}$. It follows that

$$\left(\int_{12Q_k^+} |\mathbf{B}|^p dx\right)^{1/p} > \varepsilon \left[\frac{1}{\ell(Q_k^+)}\right]^{2-\frac{3}{p}} = \varepsilon \left[\frac{1}{2\ell(Q_k)}\right]^{2-\frac{3}{p}} \geq \varepsilon \left[\frac{2}{\ell(Q_j)}\right]^{2-\frac{3}{p}} \tag{3.2}$$

where we have used $p > 3/2$ in the last inequality.

We claim that

$$12Q_k^+ \subset 12Q_j. \tag{3.3}$$

This, together with (3.2), would imply that

$$\left(\int_{12Q_j} |\mathbf{B}|^p dx\right)^{1/p} \geq \left(\int_{12Q_k^+} |\mathbf{B}|^p dx\right)^{1/p} > \varepsilon \left[\frac{2}{\ell(Q_j)}\right]^{2-\frac{3}{p}}$$

which contradicts with the assumption that $Q_j \in \mathcal{A}$.

To see (3.3), let x_j and x_k be the centers of Q_j and Q_k respectively. Since $4Q_j \cap 4Q_k \neq \emptyset$, there exists z such that

$$|z - x_j|_* \leq 2\ell(Q_j) \quad \text{and} \quad |z - x_k|_* \leq 2\ell(Q_k)$$

where $|\cdot|_*$ is the norm in $\mathbb{R}^3$ defined by

$$|x|_* = |(a, b, c)|_* = \max\{|a|, |b|, |c|\}.$$

Thus,

$$|x_j - x_k|_* \leq 2\ell(Q_j) + 2\ell(Q_k).$$

Now suppose $y \in 12Q_k^+$. Then $|y - x_k|_* \le |y - x_k^+|_* + |x_k^+ - x_k|_* < 13\ell(Q_k)$ where $x_k^+ \in Q_k$ is the center of Q_k^+. It follows that

$$|y - x_j|_* \le |y - x_k|_* + |x_k - x_j|_* < 15\ell(Q_k) + 2\ell(Q_j) < 6\ell(Q_j).$$

Hence, $y \in 12Q_j$. (3.3) is then proved.

Remark 3.1. It follows easily from Lemma 3.2 that $\{4Q_j : Q_j \in \mathcal{B}\}$ has the finite intersection property: $\sum_j \chi_{4Q_j} \le C$ where C is an absolute constant.

We are now in a position to construct the partition of unity associated with the field strength $|\mathbf{B}|$.

Lemma 3.3. *There exists a sequence of functions* $\{\phi_j\}$ *such that*

(i) $\phi_j \in C_0^\infty(2Q_j)$ *and* $0 \le \phi_j \le 1$,

(ii) $|\nabla^\alpha \phi_j(x)| \le c_\alpha / \ell_j^{|\alpha|}$ *where* $\ell_j = \ell(Q_j)$,

(iii) $\sum_j \phi_j^2 \equiv 1$ *in* $\mathbb{R}^3$.

Proof. Choose $\eta \in C_0^\infty(Q(0,2))$ such that $\eta \equiv 1$ in $Q(0,1)$ and $0 \le \eta \le 1$. Let $\mathcal{B} = \{Q_j\}_{j=1}^\infty$ and $Q_j = Q(x_j, \ell_j)$. We define

$$\eta_j(x) = \eta\left(\frac{x - x_j}{\ell_j}\right).$$

and $\varphi(x) = \sum_j \eta_j^2(x)$. Clearly, by Lemma 3.1 and Remark 3.1, $1 \le \varphi \le C$ where C is an absolute constant, and $\varphi \in C^\infty(\mathbb{R}^3)$.

Finally let $\phi_j(x) = \eta_j(x)/\sqrt{\varphi(x)}$. It is easy to check that ϕ_j satisfies (i)-(iii). We omit the details.

Recall that $\ell_j = \ell(Q_j)$ and Q_j is a maximal cube. Define

$$\Phi = \sum_j \frac{1}{\ell_j^2}\,\phi_j^2. \tag{3.4}$$

We will show in Section 4 that $\Phi(x) \approx b_p(x)$ (see Lemma 4.5).

Theorem 3.1. *There exist constants $C > 0$ and $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, we have*

$$\mathbb{H}_0 \le C\{\mathbb{P}_0 + \Phi\}.$$

Proof. We first show that, if ε in (3.1) is small, then

$$\int_{\mathbb{R}^3} |\,|\mathbf{B}|^{1/2}\phi_j\psi|^2 dx \le C \int_{\mathbb{R}^3} |\sigma\cdot(\mathbf{p}-\mathbf{A})(\phi_j\psi)|^2 dx \tag{3.5}$$

for any $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$.

To this end, we note that, by Hölder's inequality

$$\left\{\int_{2Q_j} |\mathbf{B}|^{3/2} dx\right\}^{2/3} \le |2Q_j|^{\frac{2}{3}-\frac{1}{p}} \left\{\int_{2Q_j} |\mathbf{B}|^p dx\right\}^{1/p} \le 4\varepsilon. \tag{3.6}$$

It then follows from the Sobolev embedding and the inequality $|\nabla|\psi|\,| \le |(\mathbf{p}-\mathbf{A})\psi|$ that

$$\begin{aligned}
\left\{\int_{\mathbb{R}^3} |\,|\mathbf{B}|^{1/2}\phi_j\psi|^2 dx\right\}^{1/2} &\le \left\{\int_{2Q_j} |\mathbf{B}|^{3/2} dx\right\}^{1/3} \left\{\int_{\mathbb{R}^3} |\phi_j\psi|^6 dx\right\}^{1/6} \\
&\le C\varepsilon^{1/2}\left\{\int_{\mathbb{R}^3} |\nabla|\phi_j\psi|\,|^2 dx\right\}^{1/2} \\
&\le C\varepsilon^{1/2}\left\{\int_{\mathbb{R}^3} |(\mathbf{p}-\mathbf{A})(\phi_j\psi)|^2 dx\right\}^{1/2} \\
&\le C\varepsilon^{1/2}\left\{\int_{\mathbb{R}^3} |\sigma\cdot(\mathbf{p}-\mathbf{A})(\phi_j\psi)|^2 dx\right\}^{1/2} + C\varepsilon^{1/2}\left\{\int_{\mathbb{R}^3} |\,|\mathbf{B}|^{1/2}\phi_j\psi|^2 dx\right\}^{1/2}
\end{aligned}$$

where we have used the fact that $\mathbb{P}_0 = \mathbb{H}_0 - \sigma\cdot\mathbf{B}$. This gives (3.5) if $C\varepsilon^{1/2} < 1/2$.

Next we use Lemma 3.3 and (3.5) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} |(\mathbf{p}-\mathbf{A})\psi|^2 dx = \sum_j \int_{\mathbb{R}^3} |\phi_j(\mathbf{p}-\mathbf{A})\psi|^2 dx \\
&\le 2\sum_j \int_{\mathbb{R}^3} |(\mathbf{p}-\mathbf{A})(\phi_j\psi)|^2 dx + 2\sum_j \int_{\mathbb{R}^3} |\nabla\phi_j|^2|\psi|^2 dx \\
&\le 2\sum_j \int_{\mathbb{R}^3} |\sigma\cdot(\mathbf{p}-\mathbf{A})(\phi_j\psi)|^2 dx + 2\sum_j \int_{\mathbb{R}^3} |\,|\mathbf{B}|^{1/2}\phi_j\psi|^2 dx + 2\sum_j \int_{\mathbb{R}^3} |\nabla\phi_j|^2|\psi|^2 dx \\
&\le C\sum_j \int_{\mathbb{R}^3} |\sigma\cdot(\mathbf{p}-\mathbf{A})(\phi_j\psi)|^2 dx + 2\sum_j \int_{\mathbb{R}^3} |\nabla\phi_j|^2|\psi|^2 dx \\
&\le C\int_{\mathbb{R}^3} |\sigma\cdot(\mathbf{p}-\mathbf{A})\psi|^2 dx + C\sum_j \frac{1}{\ell_j^2}\int_{2Q_j} |\psi|^2 dx.
\end{aligned}$$

Clearly, the theorem follows if we have

$$\sum_j \frac{1}{\ell_j^2}\,\chi_{2Q_j} \le C\,\Phi. \tag{3.7}$$

We claim that (3.7) is an easy consequence of Lemma 3.2. Indeed, if $x \in Q_k$, the l.h.s. of (3.7) is bounded by

$$\sum_{2Q_j\cap Q_k\neq\emptyset} \frac{1}{\ell_j^2} \le \frac{C}{\ell_k^2}\cdot \#\{Q_j :\; 2Q_j\cap Q_k \neq \emptyset\} \le \frac{C}{\ell_k^2} \le C\,\Phi(x).$$

The proof is now complete.

It follows easily from Theorem 3.1 that, if $0<\lambda_1\le\lambda_2$,

$$\|(\mathbb{H}_0+\lambda_1)^{1/2}(\mathbb{P}_0+\Phi+\lambda_2)^{-1/2}\|_{L^2\to L^2} \le C. \tag{3.8}$$

The following lemma will be used in the next section.

Lemma 3.4. *There exists a constant $C>0$ such that, for $\lambda>0$,*

$$\|(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\phi_j(\mathbb{P}_0+\lambda)^{1/2}\|_{L^2\to L^2} \le C.$$

Proof. By duality, it suffices to show that

$$\|(\mathbb{P}_0+\lambda)^{1/2}\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\|_{L^2\to L^2} \le C. \tag{3.9}$$

To this end, we note that, for any $\psi\in C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} |(\mathbb{P}_0+\lambda)^{1/2}\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\psi|^2dx\\
&=\int_{\mathbb{R}^3}|\sigma\cdot(\mathbf{p}-\mathbf{A})\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\psi|^2dx+\lambda\int_{\mathbb{R}^3}|\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\psi|^2dx\\
&\le 4\int_{\mathbb{R}^3}|\psi|^2+2\int_{\mathbb{R}^3}|\nabla\phi_j|^2|(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\psi|^2dx\\
&\le C\int_{\mathbb{R}^3}|\psi|^2dx
\end{aligned}$$

where we have used $|\nabla\phi_j| \le C/\ell_j$ and

$$\|\sigma\cdot(\mathbf{p}-\mathbf{A})(\mathbb{P}_0+\Phi+\lambda)^{-1/2}\|_{L^2\to L^2} \le 1, \tag{3.10}$$

$$\|(\mathbb{P}_0+\Phi+\lambda)^{-1/2}\|_{L^2\to L^2} \le \frac{1}{\sqrt{\lambda}}. \tag{3.11}$$

The proof is finished.

4. The Proof of Theorem 1.1

For $\lambda > 0$, we denote by $N(\lambda, \mathbb{P}_0+V)$ the number of eigenvalues (counting multiplicity) of $\mathbb{P}_0+V$ smaller than $-\lambda$. Since $V \ge -\lambda - |V+\lambda|_-$, we have

$$N(2\lambda, \mathbb{P}_0+V) \le N(\lambda, \mathbb{P}_0 - |V+\lambda|_-).$$

It then follows from the Birman-Schwinger principle and (2.2) that

$$N(2\lambda, \mathbb{P}_0+V) \le n(1, Y(\mathbb{P}_0+\lambda)^{-1}Y) = n(1, Y(\mathbb{P}_0+\lambda)^{-1/2}) \tag{4.1}$$

where $Y = |V+\lambda|_-^{1/2}$.

As in [19,20], our approach will be based on the following resolvent identity:

$$\varphi H_1^{-1} = H_2^{-1}\varphi + H_2^{-1}\big\{(H_2-H_1)\varphi + [H_1,\varphi]\big\}H_1^{-1} \tag{4.2}$$

where $\varphi \in C_0^\infty(\mathbb{R}^3,\mathbb{R})$, H_1, H_2 are two operators, and $[H_1,\varphi] = H_1\varphi - \varphi H_1$ denotes the commutator between H_1 and φ.

Using (4.2), we may write

$$\begin{aligned}
Y(\mathbb{P}_0+\lambda)^{-1/2} &= \sum_j Y\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1}\phi_j(\mathbb{P}_0+\lambda)^{1/2} \\
&\quad + \sum_j Y\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1}(\Phi+\frac{1}{\ell_j^2})\phi_j(\mathbb{P}_0+\lambda)^{-1/2} \\
&\quad + \sum_j Y\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{P}_0,\phi_j](\mathbb{P}_0+\lambda)^{-1/2} \\
&= I_1 + I_2 + I_3
\end{aligned}$$

where Φ is the function defined by (3.4). Thus, by (2.5),

$$n(1, Y(\mathbb{P}_0+\lambda)^{-1/2}) \le n(1/3, I_1) + n(1/3, I_2) + n(1/3, I_3). \tag{4.3}$$

We begin with the estimate of $n(1/3, I_1)$.

Lemma 4.1. *There exists a constant $C>0$ such that, for any $\lambda>0$,*

$$n(1/3, I_1) \le \frac{C}{\sqrt{\lambda}} \int_{\mathbb{R}^3} |Y|^4 dx.$$

Proof. First observe that, by (2.4),

$$n(1/3, I_1) \le C \| \sum_j Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1} \phi_j (\mathbb{P}_0+\lambda)^{1/2} \|_4^4. \tag{4.4}$$

Let

$$I_{1j} = Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1} \phi_j (\mathbb{P}_0+\lambda)^{1/2}. \tag{4.5}$$

It follows from (2.6) and Lemma 3.4 that

$$\begin{aligned}
\|I_{1j}\|_4 &\le \|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\|_4 \|(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\phi_j(\mathbb{P}_0+\lambda)^{1/2}\|_{L^2\to L^2} \\
&\le C\|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\|_4 \\
&\le C\|Y\phi_j(\mathbb{H}_0+\lambda)^{-1/2}\|_4 \|(\mathbb{H}_0+\lambda)^{1/2}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\|_{L^2\to L^2} \\
&\le C\|Y\phi_j(\mathbb{H}_0+\lambda)^{-1/2}\|_4
\end{aligned}$$

where we also used (3.8) in the last inequality. We now apply (2.12) with $\alpha = 1/2$ to obtain

$$\|I_{1j}\|_4 \le C\lambda^{-1/8} \|Y\phi_j\|_{L^4}. \tag{4.6}$$

Next we note that, by (2.3),

$$\|\sum_j I_{1j}\|_4^4 = \|\sum_j\sum_k\sum_m\sum_n I_{1j}I_{1k}^* I_{1m} I_{1n}^*\|_1 \le \sum_j\sum_k\sum_m\sum_n \|I_{1j}I_{1k}^* I_{1m} I_{1n}^*\|_1.$$

Since ϕ_j is supported in $2Q_j$ and $\phi_j(\mathbb{P}_0+\lambda)^{1/2}(\mathbb{P}_0+\lambda)^{1/2}\phi_k = \phi_j(\mathbb{P}_0+\lambda)\phi_k = 0$ unless $2Q_j \cap 2Q_k \neq \emptyset$, it is not very hard to see that

$$I_{1j}I_{1k}^* I_{1m} I_{1n}^* = 0$$

unless

$$2Q_j \cap 2Q_k \neq \emptyset, \quad 2Q_k \cap 2Q_m \neq \emptyset, \quad \text{and} \quad 2Q_m \cap 2Q_n \neq \emptyset. \tag{4.7}$$

On the other hand, by Lemma 3.2, if we fix any one of indices j, k, m, n, the number of remaining indices which satisfy (4.7) is bounded by an absolute constant. Thus, by (2.7),

$$\begin{aligned} n(1/3, I_1) &\leq C\|\sum_j I_{1j}\|_4^4 \leq C\sum \|I_{1j}\|_4\|I_{1k}\|_4\|I_{1m}\|_4\|I_{1n}\|_4 \\ &\leq C\sum\left\{\|I_{1j}\|_4^4 + \|I_{1k}\|_4^4 + \|I_{1m}\|_4^4 + \|I_{1n}\|_4^4\right\} \\ &\leq C\sum_j \|I_{1j}\|_4^4 \end{aligned}$$

where the second and third sums are over all (j, k, m, n) satisfying (4.7). The lemma now follows from (4.6).

Using $|Y| \leq |V|^{1/2}\chi_{\{x\in\mathbb{R}^3 : V(x)<-\lambda\}}$, we may deduce easily from Lemma 4.1 that

$$\int_0^\infty n(1/3, I_1)\, d\lambda \leq C\int_{\mathbb{R}^3} |V(x)|^{5/2} dx. \tag{4.8}$$

Let $\psi_j \in C_0^\infty(3Q_j, \mathbb{R})$ such that $0 \leq \psi_j \leq 1$, $\phi_j\psi_j \equiv \phi_j$ and $|\nabla^\alpha \psi_j| \leq C_\alpha/\ell_j^{|\alpha|}$.

We will need the following lemma in the estimates of $n(1/3, I_2)$ and $n(1/3, I_3)$.

Lemma 4.2. *There exist constants $C > 0$ and $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$,*

$$\|\,[\mathbb{P}_0, \psi_j](\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\|_{L^2\to L^2} \leq C, \tag{4.9}$$

$$\|\psi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\|_{L^2\to L^\infty} \leq C\ell_j^{\frac{1}{2}}, \tag{4.10}$$

$$\|(\mathbb{H}_0 + \frac{1}{\ell_j^2})^{-\frac{1}{4}+\delta}(\sigma\cdot\mathbf{B})\psi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\|_{L^2\to L^2} \leq C_\delta\, \ell_j^{\frac{1}{2}-2\delta} \tag{4.11}$$

where $0 < \delta < \min\big(1/4, (p-(3/2))/2\big)$.

Proof. To show (4.9), we note that

$$[\mathbb{P}_0, \psi_j] = \big[\mathbb{D}^*, [\mathbb{D}, \psi_j]\big] + [\mathbb{D}^*, \psi_j]\mathbb{D} + [\mathbb{D}, \psi_j]\mathbb{D}^* \tag{4.12}$$

where $\mathbb{P}_0 = \mathbb{D}^*\mathbb{D}$ and $\mathbb{D} = \sigma \cdot (\mathbf{p} - \mathbf{A})$ ($\mathbb{P}_0$ is defined as the operator associated with the quadratic form $\|\mathbb{D}\psi\|_{L^2}^2$. We do not need that $\mathbb{D}$ is self-adjoint). Since

$$|\,[\mathbb{D}^*, \psi_j]\psi| + |\,[\mathbb{D}, \psi_j]\psi| \le 2|\nabla\psi_j|\,|\psi| \le \frac{C}{\ell_j}|\psi|, \tag{4.13}$$

$$|\,\big[\mathbb{D}^*, [\mathbb{D}, \psi_j]\big]\psi| \le |\nabla^2\psi_j|\,|\psi| \le \frac{C}{\ell_j^2}|\psi|, \tag{4.14}$$

we have

$$\|\,[\mathbb{P}_0, \psi_j](\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\psi\|_{L^2} \le C\|\psi\|_{L^2}. \tag{4.15}$$

where we also used (3.10)–(3.11), $\mathbb{D}^* = \mathbb{D}$ on the domain of $\mathbb{D}$, and $\|(\mathbb{P}_0+\Phi+\lambda)^{-1}\|_{L^2\to L^2} \le \lambda^{-1}$.

To prove (4.10), we use the resolvent identity (4.2) and $\mathbb{P}_0 = \mathbb{H}_0 - \sigma\cdot\mathbf{B}$ to write

$$\begin{aligned} &\psi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1} = (\mathbb{H}_0 + \frac{1}{\ell_j^2})^{-1}\psi_j \\ &\qquad + (\mathbb{H}_0 + \frac{1}{\ell_j^2})^{-1}\big\{(\sigma\cdot\mathbf{B} - \Phi)\psi_j + [\mathbb{P}_0, \psi_j]\big\}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}. \end{aligned} \tag{4.16}$$

Hence, by the diamagnetic inequality (2.10) and (2.16), we get

$$\begin{aligned} &|\psi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\psi| \\ &\le (-\Delta + \frac{1}{\ell_j^2})^{-1}|\psi_j\psi| + (-\Delta + \frac{1}{\ell_j^2})^{-1}|\mathbf{B}|\psi_j|(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\psi| \\ &\qquad + (-\Delta + \frac{1}{\ell_j^2})^{-1}\big|(-\Phi\psi_j + [\mathbb{P}_0, \psi_j])(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\psi\big| \\ &\le C\ell_j^{\frac{1}{2}}\|\psi\|_{L^2} + C\ell_j^{2\delta}\|(-\Delta + \frac{1}{\ell_j^2})^{\delta-\frac{1}{4}}|\mathbf{B}|\psi_j|(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\psi|\,\|_{L^2} \end{aligned} \tag{4.17}$$

where $0 < \delta < \min\big(1/4, (p-(3/2))/2\big)$ and we also used (4.15) in the second inequality.

To deal with the second term in r.h.s. of (4.17), we apply (2.14) to obtain

$$
\begin{aligned}
&\|(-\Delta+\frac{1}{\ell_j^2})^{\delta-\frac14}|\mathbf{B}|\psi_j|(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi|\,\|_{L^2}\\
&\le \|(-\Delta+\frac{1}{\ell_j^2})^{\delta-\frac14}|\mathbf{B}|^{-\delta+\frac14}\chi_{3Q_j}\|_{L^2\to L^2}\|\,|\mathbf{B}|^{\delta+\frac34}\psi_j|(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi|\,\|_{L^2}\\
&\le C\|\,|\mathbf{B}|^{-\delta+\frac14}\|_{L^{\frac{6}{1-4\delta}}(3Q_j)}\|\,|\mathbf{B}|^{\delta+\frac34}\|_{L^2(3Q_j)}\|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi\|_{L^\infty}\\
&\le C\ell_j^{2-2\delta-\frac{3}{p}}\left\{\int_{3Q_j}|\mathbf{B}|^p\,dx\right\}^{1/p}\|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi\|_{L^\infty}\\
&\le C\ell_j^{-2\delta}\varepsilon\|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi\|_{L^\infty}
\end{aligned}
\tag{4.18}
$$

where we also used Hölder's inequality, $(3/2)+2\delta<p$ and (3.1). This, together with (4.17), implies that

$$
\|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi\|_{L^\infty}\le C\ell_j^{1/2}\|\psi\|_{L^2}+C\varepsilon\|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi\|_{L^\infty}.
$$

(4.10) then follows by choosing ε_0 small so that $C\varepsilon_0<1/2$, provided that, for any $\psi\in C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)$, we know

$$
\|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi\|_{L^\infty}<\infty. \tag{4.19}
$$

(4.19) can be shown by a bootstrap argument. Indeed, by (4.9) and (4.16)–(4.17),

$$
|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi|\le C\ell_j^{1/2}\|\psi\|_{L^2}+(-\Delta+\frac{1}{\ell_j^2})^{-1}|\mathbf{B}|\,|\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi|.
$$

Using the well-known fractional integral estimates for $(-\Delta+\frac{1}{\ell_j^2})^{-1}$ and Hölder's inequality, we see that, if the function $\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\psi$ is in L^{p_1}, then it is in L^{p_2} where $(1/p_2)=(1/p_1)+(1/p)-(2/3)$. Starting with L^2, since $p>3/2$, we may deduce that the function is in L^q for any $q<\infty$. It then follows that it is in L^∞ by the same argument.

Finally (4.11) follows from (4.18) and (4.10).

We now are ready to estimate $n(1/3,I_2)$ and $n(1/3,I_3)$.

Lemma 4.3. *There exists a constant $C > 0$ such that*

$$\int_0^\infty n(1/3, I_2)\, d\lambda \le C \int_{\mathbb{R}^3} |V(x)|^{5/2} dx + C \int_{\mathbb{R}^3} \Phi(x)^{3/2} |V(x)|\, dx.$$

Proof. We first note that, since $\|(\mathbb{P}_0 + \lambda)^{-1/2}\|_{L^2 \to L^2} \le 1/\sqrt{\lambda}$,

$$\begin{aligned}
n(1/3, I_2) &= n\big(1/3, \sum_j Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j(\mathbb{P}_0 + \lambda)^{-1/2}\big) \\
&\le n\big(\sqrt{\lambda}/3, \sum_j Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j\big) \\
&\le n\big(\sqrt{\lambda}/6, \sum_j Y\phi_j\{(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1} - (\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\}(\Phi + \frac{1}{\ell_j^2})\phi_j\big) \\
&\qquad + n\big(\sqrt{\lambda}/6, \sum_j Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j\big) \\
&= K_1 + K_2.
\end{aligned}$$

Using

$$(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1} - (\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1} = -\lambda(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}$$

and (2.4), we have

$$\begin{aligned}
K_1 &= n\big(\sqrt{\lambda}/6, \sum_j Y\phi_j\lambda(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j\big) \\
&\le C\lambda^2 \|\sum_j Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j\|_4^4 \\
&\le C\lambda^2 \sum_j \|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j\|_4^4 \\
&\le C\lambda^2 \sum_j \|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}\phi_j\|_4^4 \cdot \frac{1}{\ell_j^8}
\end{aligned}$$

where we have used the finite intersection property of the supports of ϕ_j as in the proof of Lemma 4.1.

We now use (2.6) and the (L^2, L^2) bound of $(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-\alpha}$ $(\alpha = 1, 1/2)$ to obtain

$$
\begin{aligned}
&\|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}\phi_j\|_4 \\
&\qquad \leq \|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\|_4 \cdot \frac{1}{\lambda + \frac{1}{\ell_j^2}} \\
&\qquad \leq \|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1/2}\|_4 \cdot \frac{\ell_j}{\lambda + \frac{1}{\ell_j^2}} \\
&\qquad \leq \|Y\phi_j(\mathbb{H}_0 + \frac{1}{\ell_j^2})^{-1/2}\|_4 \|(\mathbb{H}_0 + \frac{1}{\ell_j^2})^{1/2}(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1/2}\|_{L^2 \to L^2} \cdot \frac{\ell_j}{\lambda + \frac{1}{\ell_j^2}} \\
&\qquad \leq C\|Y\phi_j(\mathbb{H}_0 + \frac{1}{\ell_j^2})^{-1/2}\|_4 \cdot \frac{\ell_j}{\lambda + \frac{1}{\ell_j^2}} \\
&\qquad \leq C\|Y\phi_j\|_{L^4} \cdot \frac{\ell_j^2}{\lambda^{5/8}}
\end{aligned}
$$

where we also used (3.8) and (2.12) in the last two inequality. It then follows that

$$
K_1 \leq \frac{C}{\sqrt{\lambda}} \sum_j \int_{\mathbb{R}^3} |Y\phi_j|^4 dx \leq \frac{C}{\sqrt{\lambda}} \int_{\{x \in \mathbb{R}^3 : V(x) < -\lambda\}} |V|^2 dx. \tag{4.20}
$$

It remains to estimate K_2.

Since $Y \leq |V|^{1/2}$, we have

$$
K_2 \leq n\big(\sqrt{\lambda}/6, \sum_j |V|^{1/2}\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j\big).
$$

It follows that

$$
\begin{aligned}
\int_0^\infty K_2\, d\lambda &\leq C\|\sum_j |V|^{1/2}\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}(\Phi + \frac{1}{\ell_j^2})\phi_j\|_2^2 \\
&\leq C\sum_j \|\, |V|^{1/2}\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2})^{-1}\phi_j\|_2^2 \cdot \frac{1}{\ell_j^4}.
\end{aligned} \tag{4.21}
$$

Using $\phi_j\psi_j \equiv \phi_j$, (4.16), and (4.9), we have

$$
\begin{aligned}
&\| |V|^{1/2}\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\phi_j\|_2 \\
&\le \| |V|^{1/2}\phi_j(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-1}\phi_j\|_2 + \| |V|^{1/2}\phi_j(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-1}(\sigma\cdot\mathbf{B})\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\phi_j\|_2 \\
&+ \| |V|^{1/2}\phi_j(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-1}(-\Phi\psi_j+[\mathbb{P}_0,\psi_j])(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\phi_j\|_2 \\
&\le C\| |V|^{1/2}\phi_j(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-1}\|_2 \\
&\quad + \| |V|^{1/2}\phi_j(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-\frac{3}{4}-\delta}\|_2 \|(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-\frac{1}{4}+\delta}(\sigma\cdot\mathbf{B})\psi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2})^{-1}\phi_j\|_{L^2\to L^2} \\
&\le C\| |V|^{1/2}\phi_j(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-1}\|_2 + C\ell_j^{\frac{1}{2}-2\delta}\| |V|^{1/2}\phi_j(\mathbb{H}_0+\frac{1}{\ell_j^2})^{-\frac{3}{4}-\delta}\|_2 \\
&\le C\ell_j^{\frac{1}{2}}\| |V|^{1/2}\phi_j\|_{L^2}
\end{aligned}
$$

where we also used (4.11) in the third and (2.11) in the last inequality.

Inserting the estimate above into the r.h.s. of (4.21), we obtain

$$\int_0^\infty K_2\,d\lambda \le C\sum_j \frac{1}{\ell_j^3}\int_{\mathbb{R}^3}|V|\,\phi_j^2\,dx \le C\int_{\mathbb{R}^3}\Phi(x)^{3/2}|V(x)|dx.$$

This, together with (4.20), gives the desired estimate.

Lemma 4.4. *There exists a constant $C>0$ such that*

$$\int_0^\infty n(1/3, I_3)\,d\lambda \le C\int_{\mathbb{R}^3}|V(x)|^{5/2}dx + \int_{\mathbb{R}^3}\Phi(x)^{3/2}|V(x)|dx.$$

Proof. It follows from (2.5) and (4.12) that

$$
\begin{aligned}
n(1/3, I_3) &= n\big(1/3, \sum_j Y\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{P}_0,\phi_j](\mathbb{P}_0+\lambda)^{-1/2}\big) \\
&\le n\big(1/6, \sum_j Y\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{D}^*,[\mathbb{D},\phi_j]](\mathbb{P}_0+\lambda)^{-1/2}\big) \qquad (4.22)\\
&\quad + n\big(1/6, 2\sum_j Y\phi_j(\mathbb{P}_0+\Phi+\frac{1}{\ell_j^2}+\lambda)^{-1}\left\{[\mathbb{D}^*,\phi_j]\mathbb{D}+[\mathbb{D},\phi_j]\mathbb{D}^*\right\}(\mathbb{P}_0+\lambda)^{-1/2}\big)
\end{aligned}
$$

where $\mathbb{D} = \sigma \cdot (\mathbf{p} - \mathbf{A})$.

Since $|[\mathbb{D}^*, [\mathbb{D}, \phi_j]]\psi| \le |\nabla^2 \phi_j|\,|\psi|$, the first term on the r.h.s. of (4.22) can be treated exactly as $n(1/3, I_2)$. With $\|\mathbb{D}(\mathbb{P}_0 + \lambda)^{-1/2}\|_{L^2 \to L^2} \le 1$ and $\|\mathbb{D}^*(\mathbb{P}_0 + \lambda)^{-1/2}\|_{L^2 \to L^2} \le 1$, we may use the same argument as in the proof of Lemma 4.1 to bound the second term by

$$
\begin{aligned}
C\|\sum_j Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}[\mathbb{D}, \phi_j]\|_4^4 &+ C\|\sum_j Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}[\mathbb{D}^*, \phi_j]\|_4^4 \\
&\le C \sum_j \|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1}\|_4^4 \cdot \frac{1}{\ell_j^4} \\
&\le C \sum_j \|Y\phi_j(\mathbb{P}_0 + \Phi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\|_4^4 \\
&\le C \sum_j \|Y\phi_j(\mathbb{H}_0 + \lambda)^{-1/2}\|_4^4 \\
&\le \frac{C}{\sqrt{\lambda}} \int_{\{x \in \mathbb{R}^3 : V(x) < -\lambda\}} |V|^2 dx.
\end{aligned}
$$

The lemma then follows by integration.

We need one more lemma before we carry out the proof of Theorem 1.1.

Lemma 4.5. *There exist two constants $C_1 > 0$, $C_2 > 0$ depending on ε and p, such that, for every $x \in \mathbb{R}^3$,*

$$C_1 \Phi(x) \le b_p(x) \le C_2 \Phi(x).$$

Proof. Suppose $x \in Q_j$ for some j. Then $\Phi(x) \approx 1/\ell_j^2$.

Let $\delta \in (0, 1)$ and $\ell_j = \ell(Q_j)$. Since $Q(x, \delta\ell_j) \subset 12Q_j$, we have

$$
\begin{aligned}
(\delta\ell_j)^2 \left(\frac{1}{(\delta\ell_j)^3} \int_{Q(x,\delta\ell_j)} |\mathbf{B}(y)|^p dy \right)^{1/p} &= (\delta\ell_j)^{2-\frac{3}{p}} \left(\int_{Q(x,\delta\ell_j)} |\mathbf{B}(y)|^p dy \right)^{1/p} \\
&\le (\delta\ell_j)^{2-\frac{3}{p}} \left(\int_{12Q_j} |\mathbf{B}(y)|^p dy \right)^{1/p} \\
&\le (\delta\ell_j)^{2-\frac{3}{p}} \cdot \frac{\varepsilon}{\ell_j^{2-\frac{3}{p}}} = \delta^{2-\frac{3}{p}} \varepsilon < 1.
\end{aligned}
$$

Hence, by the definition of $\ell_p(x)$ (see (1.9)), we have $\ell_p(x) \geq \ell_j$. It follows that

$$b_p(x) = \frac{1}{\{\ell_p(x)\}^2} \leq \frac{1}{\ell_j^2} \leq C_2 \Phi(x).$$

To show $b_p(x) \geq C_1 \Phi(x)$, we note that, if N is large, $12Q_j^+ \subset Q(x, N\ell_j)$ where Q_j^+ is the dyadic parent of Q_j. Hence

$$\begin{aligned}
(N\ell_j)^2 \left(\frac{1}{(N\ell_j)^3} \int_{Q(x,N\ell_j)} |\mathbf{B}(y)|^p dy \right)^{1/p} &= (N\ell_j)^{2-\frac{3}{p}} \left(\int_{Q(x,N\ell_j)} |\mathbf{B}(y)|^p dy \right)^{1/p} \\
&\geq (N\ell_j)^{2-\frac{3}{p}} \left(\int_{12Q_j^+} |\mathbf{B}(y)|^p dy \right)^{1/p} \\
&> (N\ell_j)^{2-\frac{3}{p}} \cdot \frac{\varepsilon}{(2\ell_j)^{2-\frac{3}{p}}} = \varepsilon \left(\frac{N}{2} \right)^{2-\frac{3}{p}} > 1
\end{aligned}$$

if N is large enough. Again, by definition, this implies that $\ell_p(x) \leq N\ell_j$. Therefore

$$b_p(x) = \frac{1}{\{\ell_p(x)\}^2} \geq \frac{1}{N^2 \ell_j^2} \geq C_1 \Phi(x).$$

Finally we are in a position to give the

Proof of Theorem 1.1.

It suffices to prove the theorem for $\gamma = 1$ (see [20]). We may also assume, without the loss of generality, that $V \leq 0$ a.e..

By means of (4.1) and (4.3), we have

$$\begin{aligned}
\mathcal{M}_1 = \sum_{\lambda_j < 0} |\lambda_j| &= 2 \int_0^\infty N(2\lambda, \mathbb{P}_0 + V)\, d\lambda \\
&\leq 2 \int_0^\infty n(1, Y(\mathbb{P}_0 + \lambda)^{-1/2})\, d\lambda \\
&\leq 2 \sum_{j=1}^{3} \int_0^\infty n(1/3, I_j)\, d\lambda.
\end{aligned}$$

The theorem now follows from (4.8) and Lemmas 4.3–4.5.

5. Fields with Constant Directions

In this section we study the special case where the magnetic field $\mathbf{B}$ has a constant direction. The goal is to prove Theorem 1.2 stated in the Introduction.

Without the loss of generality, we may assume that $\mathbf{A} = (A_1(x_1, x_2), A_2(x_1, x_2), 0)$ and $\mathbf{B} = (0, 0, B)$ where $B = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$. With this assumption, we have the following identity:

$$\int_{\mathbb{R}^3} |\sigma \cdot (\mathbf{p} - \mathbf{A})\psi|^2 dx = \int_{\mathbb{R}^3} |[\sigma_1(p_1 - A_1) + \sigma_2(p_2 - A_2)]\psi|^2 dx + \int_{\mathbb{R}^3} |\frac{\partial \psi}{\partial x_3}|^2 dx.$$

This in particular implies that, for $\lambda > 0$,

$$\|(-\partial_{x_3}^2 + \lambda)^{1/2}(\mathbb{P} + \lambda)^{-1/2}\|_{L^2 \to L^2} \leq 1. \tag{5.1}$$

We shall use $x' = (x_1, x_2)$, $y' = (y_1, y_2)$ to denote points in $\mathbb{R}^2$.

Our approach to the case of constant direction fields is similar to that in Section 4 for arbitrary fields.

We begin by observing that, if $\mathbf{B} = (0, 0, B(x'))$, the basic length scale $\ell_p(x)$ defined by (1.9) is reduced to

$$\ell_p(x') = \sup \left\{ \ell > 0 : \quad \ell^2 \left\{ \frac{1}{\ell^2} \int_{S(x',\ell)} |B(y')|^p dy' \right\}^{1/p} \leq 1 \right\} \tag{5.2}$$

where $S(x', \ell)$ denotes the square in $\mathbb{R}^2$ centered at x' with side length ℓ.

We will assume that $B \not\equiv 0$ and $B \in L^p_{loc}(\mathbb{R}^2)$ for some $p > 1$. With this we have $0 < \ell_p(x') < \infty$ for any $x' \in \mathbb{R}^2$.

We now sketch the construction of the partition of unity for $\mathbb{R}^2$ associated with B, which is parallel to that in Section 3.

First we write

$$\mathbb{R}^2 = \bigcup_j S_j \tag{5.3}$$

where $\{S_j\}_{j=1}^{\infty}$ are maximal elements in the set of all dyadic squares S in $\mathbb{R}^2$ such that

$$\left\{\int_{12S} |B(x')|^p dx'\right\}^{1/p} \leq \frac{\varepsilon}{[\ell(S)]^{2-\frac{2}{p}}} \tag{5.4}$$

where $\varepsilon \in (0,1)$ is a constant to be determined later.

Next we use the same argument as in the proof of Lemma 3.2 to show that,

$$\frac{1}{2}\ell(S_k) \leq \ell(S_j) \leq 2\ell(S_k), \qquad \text{if} \quad 4S_j \cap 4S_k \neq \emptyset. \tag{5.5}$$

It follows from (5.5) that there exists a sequence of functions $\{\varphi_j\}$ such that

$$\varphi_j \in C_0^\infty(2S_j, \mathbb{R}) \quad \text{and} \quad 0 \leq \varphi_j \leq 1, \tag{5.6}$$

$$|\nabla^\alpha \varphi_j| \leq C_\alpha / \ell_j^{|\alpha|} \quad \text{where} \quad \ell_j = \ell(S_j), \tag{5.7}$$

$$\sum_j \varphi_j^2 \equiv 1 \quad \text{in} \ \mathbb{R}^2. \tag{5.8}$$

Let

$$\Psi = \sum_j \frac{1}{\ell_j^2}\, \varphi_j^2. \tag{5.9}$$

Theorem 5.1. *There exist constants $C > 0$ and $\varepsilon_0 > 0$ such that, if ε in (5.4) is less than ε_0, then*

$$\mathbb{H}_0 \leq C\left\{\mathbb{P}_0 + \Psi\right\}.$$

Proof. We will only show that, if ε in (5.4) is small, then

$$\int_{\mathbb{R}^3} |\,|B|^{1/2}\varphi_j\psi|^2 dx \leq C\int_{\mathbb{R}^3} |\sigma\cdot(\mathbf{p}-\mathbf{A})(\varphi_j\psi)|^2 dx \tag{5.10}$$

for any $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$. The rest of the proof is exactly the same as that of Theorem 3.1.

To show (5.10), we need the following Sobolev inequality

$$\left(\int_S |f|^q dx'\right)^{1/q} \leq C_q |S|^{1/q}\left(\int_S |\nabla_2 f|^2 dx'\right)^{1/2} \tag{5.11}$$

for $f \in C_0^\infty(S,\mathbb{R})$ where S is a square, $\nabla_2 = (\partial_{x_1}, \partial_{x_2})$, and $1 < q < \infty$.

Let $\frac{1}{p} + \frac{1}{q} = 1$. By (5.11) and the inequality $|\nabla|\psi|\,| \le |(\mathbf{p} - \mathbf{A})\psi|$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |\,|B|^{1/2}\varphi_j\psi|^2 dx &= \int_{\mathbb{R}} dx_3 \int_{\mathbb{R}^2} |B|\,|\varphi_j\psi|^2 dx' \\
&\le \int_{\mathbb{R}} dx_3 \left\{\int_{2S_j} |B|^p dx'\right\}^{1/p} \left\{\int_{2S_j} |\varphi_j\psi|^{2q} dx'\right\}^{1/q} \\
&\le C\int_{\mathbb{R}} dx_3 \left\{\int_{2S_j} |B|^p dx'\right\}^{1/p} \left\{\int_{2S_j} |\nabla_2|\varphi_j\psi|\,|^2 dx'\right\} \cdot \ell_j^{2/q} \\
&\le C\varepsilon \int_{\mathbb{R}^3} |\nabla|\varphi_j\psi|\,|^2 dx \\
&\le C\varepsilon \int_{\mathbb{R}^3} |(\mathbf{p} - \mathbf{A})(\varphi_j\psi)|^2 dx \\
&\le C\varepsilon \int_{\mathbb{R}^3} |\sigma\cdot(\mathbf{p} - \mathbf{A})(\varphi_j\psi)|^2 dx + C\varepsilon \int_{\mathbb{R}^3} |\,|B|^{1/2}\varphi_j\psi|^2 dx
\end{aligned}$$

where we also used (5.4) in the third inequality. (5.10) then follows by choosing ε small so that $C\varepsilon < 1/2$.

Note that Theorem 5.1 implies that

$$\|(\mathbb{H}_0 + \lambda)^{1/2}(\mathbb{P}_0 + \Psi + \lambda)^{-1/2}\|_{L^2\to L^2} \le C. \tag{5.12}$$

Also, same arguments as in the proof of Lemma 3.4 give

$$\|(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\varphi_j(\mathbb{P}_0 + \lambda)^{1/2}\|_{L^2\to L^2} \le C. \tag{5.13}$$

To prove Theorem 1.2, we will show that

$$N(2\lambda, \mathbb{P}_0 + V) \le \frac{C}{\sqrt{\lambda}} \int_{\{x\in\mathbb{R}^3 : V(x) < -\lambda\}} |V(x)|^2 dx + \frac{C}{\sqrt{\lambda}} \int_{\{x\in\mathbb{R}^3 : V(x) < -\lambda\}} b_p(x)|V(x)| dx.$$

To this end, we use the resolvent identity (4.2) to write

$$\begin{aligned}
Y(\mathbb{P}_0 + \lambda)^{-1/2} &= \sum_j Y\varphi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}\varphi_j(\mathbb{P}_0 + \lambda)^{1/2} \\
&\quad + \sum_j Y\varphi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Psi + \frac{1}{\ell_j^2})\varphi_j(\mathbb{P}_0 + \lambda)^{-1/2} \\
&\quad + \sum_j Y\varphi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}[\mathbb{P}_0, \varphi_j](\mathbb{P}_0 + \lambda)^{-1/2} \\
&= J_1 + J_2 + J_3
\end{aligned}$$

where $Y = |V+\lambda|_-^{1/2}$. As before,

$$N(2\lambda, \mathbb{P}_0+V) \le n(1, Y(\mathbb{P}_0+\lambda)^{-1/2}) \le n(1/3, J_1) + n(1/3, J_2) + n(1/3, J_3). \tag{5.14}$$

Lemma 5.1. *There exists a constant $C>0$ such that, for any $\lambda>0$,*

$$n(1/3, J_1) \le \frac{C}{\sqrt{\lambda}} \int_{\{x\in\mathbb{R}^3 : V(x) < -\lambda\}} |V(x)|^2 dx.$$

The proof of Lemma 5.1, which uses (5.12)–(5.13) and (2.12), is similar to that of Lemma 4.1. We leave it to the reader.

Let $\xi_j \in C_0^\infty(3S_j, \mathbb{R})$ such that $0 \le \xi_j \le 1$, $\xi_j\varphi_j \equiv \varphi_j$, and $|\nabla^\alpha \xi_j| \le C_\alpha/\ell_j^{|\alpha|}$.

The following lemma is needed to estimate $n(1/3, J_2)$ and $n(1/3, J_3)$.

Lemma 5.2. *Let $0 < \delta \le \frac{1}{2}(1-\frac{1}{p})$. There exists a constant $C_\delta > 0$ such that, for any $\lambda > 0$,*

$$\|(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}\|_{L^2\to L^2} \le C_\delta\, \ell_j^{1-2\delta}.$$

Proof. Note that, by (5.12),

$$\begin{aligned}
&\|(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}\|_{L^2\to L^2} \\
&\quad\le \|(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\|_{L^2\to L^2}\cdot \ell_j \\
&\quad\le \|(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\|_{L^2\to L^2} \\
&\qquad\quad \cdot \|(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{1/2}(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1/2}\|_{L^2\to L^2}\cdot \ell_j \\
&\quad\le C\ell_j \|(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}}\|_{L^2\to L^2} \\
&\quad\le C\ell_j^{1+2\delta} \|(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{H}_0 + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}\|_{L^2\to L^2}.
\end{aligned}$$

Thus, by the diamagnetic inequality (2.10) and duality, it suffices to show that

$$\|\,|B|^{1/2}\chi_{3S_j}(-\Delta + \frac{1}{\ell_j^2} + \lambda)^{-\frac{1}{2}+\delta}\|_{L^2\to L^2} \le C_\delta\, \ell_j^{-2\delta}. \tag{5.15}$$

We claim that (5.15) follows from Lemma 2.3. Indeed, by Lemma 2.3, the l.h.s. of (5.15) is bounded by

$$C_\delta \, \| \, |B|^{1/2} \|_{L^{\frac{2}{1-2\delta}}(3S_j)} \le C_\delta \ell_j^{-2\delta}$$

where we used $p \ge 1/(1-2\delta)$, Hölder's inequality, and (5.4) in the inequality.

The proof is complete.

We are now ready to deal with $n(1/3, J_2)$ and $n(1/3, J_3)$.

Lemma 5.3. *There exists a constant $C > 0$ such that*

$$n(1/3, J_2) \le \frac{C}{\sqrt{\lambda}} \int_{\{x \in \mathbb{R}^3 : V(x) < -\lambda\}} \Psi(x) |V(x)| dx.$$

Proof. First we note that, by (2.4), (2.6), and (5.1),

$$\begin{aligned}
n(1/3, J_2) &= n\big(1/3, \sum_j Y\varphi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Psi + \frac{1}{\ell_j^2})\varphi_j(\mathbb{P}_0 + \lambda)^{-1/2}\big) \\
&\le C \| \sum_j Y\varphi_j(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Psi + \frac{1}{\ell_j^2})\varphi_j(\mathbb{P}_0 + \lambda)^{-1/2}\|_2^2 \\
&\le C \| \sum_j Y\varphi_j(-\partial_{x_3}^2 + \lambda)^{-1/2}(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Psi + \frac{1}{\ell_j^2})\varphi_j\|_2^2 \\
&\qquad \cdot \|(-\partial_{x_3}^2 + \lambda)^{1/2}(\mathbb{P}_0 + \lambda)^{-1/2}\|_{L^2 \to L^2}^2 \\
&\le C \| \sum_j Y\varphi_j(-\partial_{x_3}^2 + \lambda)^{-1/2}(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Psi + \frac{1}{\ell_j^2})\varphi_j\|_2^2 \\
&\le C \sum_j \|Y\varphi_j(-\partial_{x_3}^2 + \lambda)^{-1/2}(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}(\Psi + \frac{1}{\ell_j^2})\varphi_j\|_2^2 \\
&\le C \sum_j \|Y\varphi_j(-\partial_{x_3}^2 + \lambda)^{-1/2}(\mathbb{P}_0 + \Psi + \frac{1}{\ell_j^2} + \lambda)^{-1}\varphi_j\|_2^2 \cdot \frac{1}{\ell_j^4}
\end{aligned}$$

where we used the finite intersection property of suppφ_j in the fourth inequality. The fact that $\mathbb{P}_0 + \Psi$ commutes with ∂_{x_3} is essential in the estimate above.

Next we use the resolvent identity (4.2) to obtain

$$\begin{aligned}
&\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\varphi_j\|_2\\
&=\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}\xi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\varphi_j\|_2\\
&\le\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-1}\varphi_j\|_2\\
&\quad+\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-1}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\varphi_j\|_2\\
&\quad+\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-1}\left\{-\Psi\xi_j+[\mathbb{P}_0,\xi_j]\right\}(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\varphi_j\|_2\\
&=K_1+K_2+K_3.
\end{aligned}$$

By (2.13),

$$K_1\le C\lambda^{-1/4}(\frac{1}{\ell_j^2}+\lambda)^{-1/2}\|Y\varphi_j\|_{L^2}\le C\lambda^{-1/4}\ell_j\|Y\varphi_j\|_{L^2}. \tag{5.16}$$

To estimate K_2, we use (2.6), Lemma 5.2, and (2.13) to obtain

$$\begin{aligned}
K_2&\le\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-\frac12-\delta}\|_2\\
&\qquad\cdot\|(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-\frac12+\delta}(\sigma\cdot\mathbf{B})\xi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\|_{L^2\to L^2}\\
&\le C\,\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-\frac12-\delta}\|_2\cdot\ell_j^{1-2\delta}\\
&\le C\,\lambda^{-1/4}(\frac{1}{\ell_j^2}+\lambda)^{-\delta}\|Y\varphi_j\|_{L^2}\cdot\ell_j^{1-2\delta}\\
&\le C\,\lambda^{-1/4}\ell_j\|Y\varphi_j\|_{L^2}
\end{aligned} \tag{5.17}$$

where $\delta=\frac12(1-\frac1p)$.

To bound K_3, we observe that

$$\|\left\{-\Psi\xi_j+[\mathbb{P}_0,\xi_j]\right\}(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\|_{L^2\to L^2}\le C$$

using the same argument as in the proof of (4.9). It follows that

$$\begin{aligned}
K_3&\le\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-1}\|_2\\
&\qquad\cdot\|\left\{-\Psi\xi_j+[\mathbb{P}_0,\xi_j]\right\}(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\|_{L^2\to L^2}\\
&\le C\lambda^{-1/4}\ell_j\|Y\varphi_j\|_{L^2}.
\end{aligned} \tag{5.18}$$

Finally, we add (5.16), (5.17) and (5.18) to conclude that

$$\|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\varphi_j\|_2 \leq K_1+K_2+K_3 \leq C\lambda^{-1/4}\ell_j\|Y\varphi_j\|_{L^2}.$$

It follows that

$$\begin{aligned} n(1/3,J_2) &\leq C\sum_j \|Y\varphi_j(-\partial_{x_3}^2+\lambda)^{-1/2}(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\varphi_j\|_2^2\cdot\frac{1}{\ell_j^4} \\ &\leq \frac{C}{\sqrt{\lambda}}\sum_j\frac{1}{\ell_j^2}\int_{\mathbb{R}^3}|Y\varphi_j|^2dx \\ &\leq \frac{C}{\sqrt{\lambda}}\int_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}\Psi(x)|V(x)|dx \end{aligned}$$

since $|Y|\leq|V|^{1/2}\chi_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}$. The proof is finished.

Lemma 5.4. *There exists a constant $C>0$ such that*

$$n(1/3,J_3)\leq\frac{C}{\sqrt{\lambda}}\int_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}|V(x)|^2dx+\frac{C}{\sqrt{\lambda}}\int_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}\Psi(x)|V(x)|dx.$$

Proof. Using $[\mathbb{P}_0,\varphi_j]=\big[\mathbb{D}^*,[\mathbb{D},\varphi_j]\big]+[\mathbb{D},\varphi_j]\mathbb{D}^*+[\mathbb{D}^*,\varphi_j]\mathbb{D}$, we have

$$\begin{aligned} J_3 &= \sum_j Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{P}_0,\varphi_j](\mathbb{P}_0+\lambda)^{-1/2} \\ &= \sum_j Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\big[\mathbb{D}^*,[\mathbb{D},\varphi_j]\big](\mathbb{P}_0+\lambda)^{-1/2} \\ &\quad+\sum_j Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\left\{[\mathbb{D},\varphi_j]\mathbb{D}^*+[\mathbb{D}^*,\varphi_j]\mathbb{D}\right\}(\mathbb{P}_0+\lambda)^{-1/2} \\ &= J_{31}+J_{32}. \end{aligned}$$

Thus,

$$n(1/3,J_3)\leq n(1/6,J_{31})+n(1/6,J_{32}).$$

Since $|[\mathbb{D}^*,[\mathbb{D},\varphi_j]]\psi|\leq|\nabla^2\varphi_j|\,|\psi|$, we may use the same argument as in the proof of Lemma 5.3 to show that

$$n(1/6,J_{31})\leq\frac{C}{\sqrt{\lambda}}\int_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}\Psi(x)|V(x)|dx.$$

On the other hand, using $\|\mathbb{D}(\mathbb{P}_0+\lambda)^{-1/2}\|_{L^2\to L^2}\le 1$, $\|\mathbb{D}^*(\mathbb{P}_0+\lambda)^{-1/2}\|_{L^2\to L^2}\le 1$, and (5.12), we may bound $n(1/6, J_{32})$ by

$$
\begin{aligned}
&C\|\sum_j Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}\left\{[\mathbb{D}^*,\varphi_j]\mathbb{D}+[\mathbb{D},\varphi_j]\mathbb{D}^*\right\}(\mathbb{P}_0+\lambda)^{-1/2}\|_4^4\\
&\le C\|\sum_j Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{D}^*,\varphi_j]\|_4^4+C\|\sum_j Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{D},\varphi_j]\|_4^4\\
&\le C\sum_j \|Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{D}^*,\varphi_j]\|_4^4+C\sum_j \|Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1}[\mathbb{D},\varphi_j]\|_4^4\\
&\le C\sum_j \|Y\varphi_j(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\|_4^4\cdot\frac{1}{(\frac{1}{\ell_j^2}+\lambda)^2}\cdot\frac{1}{\ell_j^4}\\
&\le C\sum_j \|Y\varphi_j(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\|_4^4\|(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{1/2}(\mathbb{P}_0+\Psi+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\|_{L^2\to L^2}^4\\
&\le C\cdot\sum_j \|Y\varphi_j(\mathbb{H}_0+\frac{1}{\ell_j^2}+\lambda)^{-1/2}\|_4^4\\
&\le \frac{C}{\sqrt{\lambda}}\int_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}|V(x)|^2dx
\end{aligned}
$$

where (2.12) is used in the last inequality.

The proof is complete.

Finally we are in a position to give the

Proof of Theorem 1.2.

It follows from (5.14), Lemmas 5.1, 5.3, and 5.4 that

$$
\begin{aligned}
N(2\lambda,\mathbb{P}_0+V)&\le n(1/3,J_1)+n(1/3,J_2)+n(1/3,J_3)\\
&\le \frac{C}{\sqrt{\lambda}}\int_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}|V(x)|^2dx+\frac{C}{\sqrt{\lambda}}\int_{\{x\in\mathbb{R}^3:V(x)<-\lambda\}}\Psi(x)|V(x)|dx.
\end{aligned}
$$

Thus, if $\gamma>1/2$,

$$
\begin{aligned}
\mathcal{M}_\gamma=\sum_{\lambda_j<0}|\lambda|^\gamma&=\gamma\int_0^\infty\lambda^{\gamma-1}N(\lambda,\mathbb{P}_0+V)\,d\lambda\\
&=\gamma 2^\gamma\int_0^\infty\lambda^{\gamma-1}N(2\lambda,\mathbb{P}_0+V)\,d\lambda\\
&\le C_\gamma\int_{\mathbb{R}^3}|V(x)|_-^{\gamma+3/2}dx+C_\gamma\int_0^\infty\Psi(x)|V(x)|_-^{\gamma+1/2}dx.
\end{aligned}
$$

Noting that $\Psi(x) \approx b_p(x)$ for $p > 1$ by an argument similar to that in the proof of Lemma 4.5, we are done.

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