RUC-SYSTEMS IN NON-COMMUTATIVE SYMMETRIC SPACES.

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ABSTRACT. Let $r_i \in \{0,1\}$, $i = 0, \dots$ be a sequence of independent random variables. A biorthogonal system (x_n, x_n^*) in a Banach space X is called a RUC (an abbreviation for random unconditional convergence) if for every z from the closed linear hull of the system the series

$$\sum \epsilon_i(\omega)\alpha_i x_i$$

converges for almost all $\omega \in \Omega$. If the system (x_n) forms (Schauder) basis, then it is called RUC-basis. The aim of the article is to present a general procedure of constructing of RUC-bases in symmetric operator spaces associated with different von Neumann algebras M, in particular, with hyperfinite factor of type II_1.

1. INTRODUCTION

Let (ϵ_n) be a sequence of independent ± 1-valued random variables on probability space (Ω, μ) with $\mu(\epsilon_n = 1) = \mu(\epsilon_n = -1) = \frac{1}{2}$ (for instance, Rademacher functions over $(0,1)$). Let X be a Banach space with biorthogonal system (x_n, x_n^*). Such system is called *RUC-system* if for every $z \in [x_n]$ the series

$$\sum \epsilon_i(\omega)\alpha_i x_i$$

converges for almost all $\omega \in \Omega$. If the system (x_n) forms (Schauder) basis, then it is called *RUC-basis.* It is clear, that any unconditional basis is also RUC basis, but there are examples of RUC-bases which are not unconditional. Such examples are of especial interest in spaces without any unconditional basis; among the latter ones symmetric operator spaces appear to be the most natural object to be studied. The ideals of compact operators that provide the simplest example of spaces of such sort can be described as follows.

Definition 1.1. *If E is a symmetric sequence space, i.e. a Banach space of sequences such that the standard unit vectors (e_n) forms a normalized, 1-symmetric basis of E (see [15]), then we denote by C_E the space of compact operators x on l_2 for which $s(x) \in E$, equipped* *with a norm*

$$\|x\|_{C_E} = \|s(x)\|_E,$$

where $s(x) = \{s_n(x)\}_{n=1}^{\infty}$ *is the sequence of s-numbers of* x*, i.e. the eigenvalues of* $|\, x\,|$ *arranged in the non-increasing order, counting multiplicity.*

Each element of C_E has a matrix representation with respect to an orthonormal basis (e_n) of l_2. Elements e_{ij} of C_E with the following matrix representation

$$e_{ij}(k,l) = \delta_i^k \delta_j^l,\ 1 \le i,j,k,l < \infty,$$

ordered lexicographically form a basis (z_n) in every C_E (see [12]). On the other hand, ideals C_E have no unconditional bases, unless $E = l_2$. G.Pisier proved that if $2 \le p < \infty$ then (x_n) is a RUC-basis in every $C_p (= C_{l_p})$ and moreover in every ideal C_E of type 2. Analogous result was obtained in [18] for non-commutative symmetric spaces $E(L_\infty \otimes B(l_2))$ and the basis $(h_n \otimes e_{kl})$, where (h_n) is Haar system. It also has been established in [1], that Walsh system is a RUC-basis in every L_p-space provided $p \in [2,\infty)$.

The aim of the present article is to show that Pisier's proof can be naturally generalized for the case of general UMD-spaces of type 2 that admit "good" representation of the dyadic group. It allows us, in particular, to present new examples of RUC-bases in operator spaces associated with the hyperfinite factor of type II_1.

2. Preliminaries and formulation of the main result

Let $\mathbb{Z}_2$ be the discrete cyclic group of order 2. The *dyadic group* $\mathbb{D}$ is by definition the product of countably many copies of $\mathbb{Z}_2$ so that

$$\mathbb{D} := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots,$$

equipped with the product topology. Thus the compact Abelian group $\mathbb{D}$ consists of all $\{0,1\}$-valued sequences with coordinatewise addition modulo two. Consider now the Haar measure on each copy of $\mathbb{Z}_2$. The induced product measure on $\mathbb{D}$ is a translation invariant Borel probability measure and therefore it is the Haar measure on $\mathbb{D}$. We may identify the dual group $\hat{\mathbb{D}}$ with the set of all $\{0,1\}$-valued sequences $\{n_k\}$ with only finitely many non-zero elements, acting on $x = \{x_n\}_{n=0}^{\infty} \in \mathbb{D}$ as follows

$$\gamma(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}. \tag{1}$$

For every $k \in \mathbb{Z}^+$, and $x = \{x_n\}_{n=0}^{\infty} \in \mathbb{D}$, we define the character $\rho \in \hat{\mathbb{D}}$ by

$$\rho_k(x) = (-1)^{x_k} = \begin{cases} 1, \text{if } x_k = 0, \\ -1, \text{if } x_k = 1 \end{cases} .$$

It is easy to check that $\rho_k \in \hat{\mathbb{D}}$ for all $k \in \mathbb{Z}^+$. Obviously, $\{\rho_k\}$ is a set of generators of $\hat{\mathbb{D}}$. In fact, (1) yields

$$\gamma(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k} = \prod_{k=0}^{\infty} ((-1)^{x_k n_k} = \prod_{k=0}^{\infty} \rho_k^{n_k}(x),$$

or, equivalently,

$$\gamma = \prod_{k=0}^{\infty} \rho_k^{n_k} \tag{2}$$

The last relation allows to introduce an numeration of the elements of $\hat{\mathbb{D}}$, known as the Walsh-Paley numeration as follows. If the dyadic expansion of $n \in \mathbb{Z}^+$ is

$$n = \sum_{k=0}^{\infty} n_k 2^k, \tag{3}$$

where n_k are equal to either 0 or 1 for all k, we define the character ψ_n according to the following rule

$$\psi_n := \prod_{k=0}^{\infty} \rho_k^{n_k}.$$

It is clear that if $x = \{x_k\} \in \mathbb{D}$ and if n has the dyadic expansion (3), then

$$\psi_n(x) = (-1)^{\langle n,x \rangle}$$

where

$$\langle n, x \rangle = n_0 x_0 + n_1 x_1 + n_2 x_2 \dots \pmod 2.$$

Moreover , it is straightforward that

$$\rho_n = \psi_{2^n}, \quad n = 0, 1, 2, \dots.$$

For the sake of completeness, let us remind that if

$$r(x) = \begin{cases} 1 \text{ if } x \in [0, \frac{1}{2}) \\ -1 \text{ if } x \in [\frac{1}{2}, 1) \end{cases}$$

then the Rademacher system $\{r_n\}_{n=0}^{\infty}$ on $(0,1)$ is by definition

$$r_n(x) := r(2^n x), \quad x \in [0, 1), \quad n \in \mathbb{Z}^+.$$

The *Walsh-Paley* system $\{w_n\}_{n=0}^{\infty}$ in this case is defined as

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k},$$

if the dyadic expansion $n \in \mathbb{Z}^+$ is given by (3). The Walsh system is an orthonormal system in $L^2([0,1], m)$. We will usually identify the system $(w_n)_{n=0}^{\infty}$ with $(\psi_n)_{n=0}^{\infty}$ and the system $(r_n)_{n=0}^{\infty}$ with $(\rho_n)_{n=0}^{\infty}$ respectively.

Let X be an arbitrary Banach space, let G be a compact Abelian group with Haar measure dt and the dual group $\hat{G}$ and let $\{R_t\}_{t\in G}$ be a strongly continuous representation of G on X. We set

$$c_R := \sup_{t\in G} \|R_t\|_{X\to X}.$$

For every $\gamma \in \hat{G}$, consider the projection $E_\gamma : X \to X$, where

$$E_\gamma x = \int_G \gamma(u) R_{-u} x du,$$

and the integral on the right hand side is the Bochner integral. The range of E_γ is called an eigenspace

$$X_\gamma := \{x \in X : R_t x = \gamma(t)x \quad \text{for all } t \in G\}.$$

If $X = L^p(G, dt), 1 \le p < \infty$, and the action $S = \{S_t\}_{t\in G}$ is just the forward shift

$$(S_t f)(x) := f(x+t), \quad x \in G, \quad t \in G,$$

then

$$\begin{aligned} X_\gamma &= \{f \in L^p(G, dt) : (S_t f)(x) = \gamma(t) f(x) \quad \forall t \in G\} \\ &= \{f \in L^p(G, dt) : f(x+t) = \gamma(t) f(x) \quad \text{for all } t \in G\} \\ &= \{f \in L^p(G, dt) : f(t) = \gamma(t) f(0) \quad \text{for all } t \in G\} \end{aligned}$$

and this is nothing else but a (one-dimensional) linear subspace spanned by the character γ. In particular, the standard Walsh system can be considered as a system of eigenvectors of the shift forward action of the group $\mathbb{D}$ on $L^p(\mathbb{D}, \mu)$. The following definition has been introduced in [11].

Definition 2.1. *If eigenspaces X_γ are such that $dim X_\gamma \le 1$ for all $\gamma \in \Gamma$, then the system of non-zero vectors (x_γ), where $x_\gamma \in X_\gamma$, is called a $\mathbb{D}$-system.*

There are many natural examples of orthogonal $\mathbb{D}$-systems in symmetric operator spaces, in particular, the system of matrix identities in C_E can be considered as $\mathbb{D}$-system (see [20], [21], and [11]). The principal result about these systems is the following theorem that has been announced first in [20] and proved in [10].

Theorem 1. *Any $\mathbb{D}$-system in (UMD)-space X endowed with the Walsh-Paley enumeration is a Schauder basis of X.*

The main result of the present article is

Theorem 2. *If X is Banach space of type 2, then any $\mathbb{D}$-system $(x_k) \subseteq X$ is a RUC-system in X.*

3. The proof of Theorem 2

Our analysis is based on certain inequality that involves Gaussian and Rademacher random variables. Denote via (g_k) a sequence of independent Gaussian variables of standard distribution. The following inequalities can be found in [17]:

Theorem 3. *Let X be Banach space, $(x_k) \subseteq X$. Then there exists a uniform constant c such that*

$$\|\sum_{k=1}^{m} x_k r_k\|_{L^2(G,X)} \leq c\|\sum_{k=1}^{m} x_k g_k\|_{L^2(G,X)}.$$

Moreover, if X is of type 2, then there exists a constant K which depends only on the type 2 constant of the space X, such that

$$\|\sum_{k=1}^{m} x_k g_k(t)\|^2_{L_2(\mathbb{D},X)} \leq K^2 \sum_{k=1}^{m} \|x_k\|_X.$$

We also will need the following general fact about Gaussian random variables (see, for instance, Theorem II.2.10 in [14]).

Theorem 4. *Let $(u_k)_{k=1}^n$ be an orthonormal sequence of vectors $u_k = (u_{kl})_{l=1}^n$ in $\mathbf{C}^n$. Then the sequence*

$$\theta_l = \sum_{k=1}^{n} u_{kl} g_k, \ \ l = 1, ..., n$$

is a sequence of standard independent Gaussian random variables.

Proof of Theorem 2.

It is sufficient to prove (see Proposition 1.1. in [1]) that there exists a constant C such that for any finite sequence of scalars (a_γ) one has

$$\|\sum_{\gamma} a_\gamma r_\gamma x_\gamma\|_{L_2(\mathbb{D},X)} \leq C\|\sum_{\gamma} a_\gamma x_\gamma\|_X \tag{4}$$

where we have numerated the Rademacher system using elements of the group $\hat{\mathbb{D}}$. Let us fix arbitrary finite sequences $(a_\gamma)_{\gamma\in\Gamma}$, $(x_\gamma)_{\gamma\in\Gamma}$, where $\Gamma \subseteq \hat{\mathbb{D}}$. Then there exists a positive integer n such that any character $\gamma \in \Gamma$ belongs to a finite subgroup

$$\hat{G} = \coprod_{k=1}^{n} \hat{\mathbb{Z}}_2 \otimes \coprod_{k=n+1}^{\infty} \{0\} \in \hat{\mathbb{D}},$$

which is in turn dual to a finite subgroup $G = \prod_{k=1}^{n} \mathbb{Z}_2$ of group $\mathbb{D}$. We set $N = \mathrm{Card} G$, fix a finite sequence $(g_s)_{s\in G}$, and consider a set of N vectors $(u_\gamma)_{\gamma\in\hat{G}} \subseteq \mathbf{C}^N$, where

$$u_\gamma = (u_{\gamma,s})_{s\in G} = (N^{-\frac{1}{2}}\gamma(s))_{s\in G}.$$

Since

$$u_{\gamma_1} u_{\gamma_2} = N^{-1}\sum_{s\in G}\overline{\gamma}_1(s)\gamma_2(s) = N^{-1}\sum_{s\in G}(\gamma_1-\gamma_2)(s) = \begin{cases} 1, \text{if } \gamma_1 = \gamma_2, \\ 0, \text{if } \gamma_1 \neq \gamma_2 \end{cases}$$

we see that $(u_\gamma)_{\gamma\in\hat{G}}$ is an orthonormal system. Thus, by Theorem 4 the following set of random variables

$$\theta_\gamma := \sum_{s\in G} g_s u_{\gamma,s} = N^{-\frac{1}{2}}\sum_{s\in G} g_s\gamma(s)$$

is a sequence of standard independent Gaussian variables. Applying Theorem 3 twice we get that

$$(5)\qquad \|\sum_\gamma a_\gamma r_\gamma x_\gamma\|_{L_2(\mathbb{D},X)} \le c\|\sum_\gamma a_\gamma \theta_\gamma x_\gamma\|_{L_2(\mathbb{D},X)} =$$

$$cN^{-\frac{1}{2}}\|\sum_\gamma a_\gamma \sum_{s\in G} g_s\gamma(s)x_\gamma\|_{L_2(\mathbb{D},X)} = cN^{-\frac{1}{2}}\|\sum_{s\in G} g_s \sum_\gamma a_\gamma x_\gamma\gamma(t)\|_{L_2(\mathbb{D},X)} \le$$

$$cN^{-\frac{1}{2}}K(\sum_{s\in G}\|\sum_\gamma a_\gamma x_\gamma\gamma(s)\|_X)^{\frac{1}{2}}.$$

Next, using the fact that system (x_γ) is a system of eigenvectors of the representation R_t we see that

(6)

$$\sum_{s\in G}\|\sum_\gamma a_\gamma x_\gamma\gamma(t)\|_X^2 = \sum_{s\in G}\|\sum_\gamma a_\gamma R_t x_\gamma\|_X^2 = \sum_{s\in G}\|R_t(\sum_\gamma a_\gamma x_\gamma)\|_X^2 \le$$

$$\sum_{s\in G} c_{\mathbf{R}}\|\sum_\gamma a_\gamma x_\gamma\|_X^2 = c_{\mathbf{R}} N\|\sum_\gamma a_\gamma x_\gamma\|_X^2.$$

Finally, equalities (5) and (6) imply

$$\|\sum_\gamma a_\gamma r_\gamma x_\gamma\|_{L_2(\mathbb{D},X)} \le cN^{-\frac{1}{2}}\|\sum_{s\in G} g_s \sum_\gamma a_\gamma x_\gamma \gamma(s)\|_{L_2(\mathbb{D},X)} \le$$

$$cKN^{-\frac{1}{2}}(\sum_{s\in G}\|\sum_\gamma a_\gamma x_\gamma \gamma(t)\|_X)^{\frac{1}{2}} \le cKc_{\mathbf{R}}\|\sum_\gamma a_\gamma x_\gamma\|_X.$$

Therefore condition (4) is satisfied for the constant $C = cKc_{\mathbf{R}}$ and the proof is completed.

4. Examples and corollaries

We first need to recall some facts about non-commutative symmetric spaces. Let M be a semifinite von Neumann algebra with C^*-norm $\|\cdot\|_M$, let τ be a semifinite normal faithful trace on M. Denote via $\mathbf{1}$ the unit of M, and via $K(M,\tau)$ $*$-algebra of all τ-measurable operators (see [FK]) affiliated with M. For an arbitrary symmetric function space $E[0,\infty)$ we define the symmetric space of measurable operators $E(M,\tau)$ associated with $E[0,\infty)$ and a semifinite von Neumann algebra (M,τ), as the following space:

$$E(M,\tau) = \{x \in K(M,\tau) : \mu(x) \in E[0,\infty)\},\ \|x\|_{E(M,\tau)} = \|\mu(x)\|_{E[0,\infty)},$$

where $\mu_t(x)$ for given $t \in (0,\infty)$ is defined by

$$\mu_t(x) = \inf\{s \ge 0 : \tau(\chi_{[s,\infty)}(\mid x \mid) \le t\},$$

and $\chi_{[s,\infty)}(\mid x \mid)$ is the spectral projection of $\mid x \mid = (x^*x)^{1/2}$ corresponding to the interval (s,∞). Always in the paper we assume that the space $E[0,\infty)$ is separable. In this case it is shown in [6], [7] and [5] that $(E(M,\tau), \|\cdot\|_{E(M,\tau)})$ is a Banach space. The dual Banach space $E(M,\tau)^*$ can be described in the following way:

$$E(M,\tau)^* = \{x \in K(M,\tau) : \|x\|_{E(M,\tau)^*} =$$

$$\sup\{\tau(\mid xy \mid) : y \in E(M,\tau), \|y\|_{E(M,\tau)} \le 1\} < \infty\}$$

and the value of the functional $x^* \in E(M,\tau)^*$ applied to an element x is equal to $\tau(xx^*)$.

It is not difficult to check that any $\mathbb{D}$-flow on $(\mathcal{M},\tau)$ of ultraweakly continuous $*$-automorphisms of $\mathcal{M}$ preserving the trace τ can be uniquely extended to a strongly continuous group of isometries of an arbitrary symmetric space $E(M,\tau)$, provided E is separable symmetric function space. Standard arguments show that in this situation the choice of $\mathbb{D}$-system is coherent for all $E(M,\tau)$.

The following result is announced in [SF 1].

Corollary 4.1. *If $E(M,\tau)$ is of type 2, then every $\mathbb{D}$-system taken in Walsh-Paley numeration is a RUC-basis of $E(M,\tau)$.*

Proof. The result follows immediately from Theorems 1, 2, standard interpolation arguments and the fact that operator L_p-spaces are UMD for every $p \in (1,\infty)$, $\square$

There are several "model" von Neumann algebras M of each particular type and they are usually of an especial interest. These are the algebra $L_\infty(0,1)$ of all complex bounded measurable functions on $(0,1)$, the algebra $B(l_2)$ of all bounded linear operators on l_2, the algebra $L_\infty(0,1)\bar{\otimes}B(l_2)$ of all weakly measurable $B(l_2)$-valued functions on $(0,1)$ with tensor product trace, the approximately finite-dimensional factor R of type II_1 with (unique) normalized trace and the approximately finite-dimensional factor $R_{01} = R\bar{\otimes}B(l_2)$ (which is just a tensor-product of $B(l_2)$ and R) of type II_∞ with tensor-product trace. Representations of the dyadic group $\mathbb{D}$ on operator L_p-spaces associated with these five algebras and examples of $\mathbb{D}$-systems (so-called non-commutative Walsh bases) can be found in [20], [21] and [10]. The following result demonstrates an application of the corollary 4.1 and can be considered as a generalization of the results about (RUC)-property of the bases (w_n) and (e_{ij}).

Corollary 4.2. *The following systems of operators*

1. $w_j\bar{\otimes}e_{kl}$, *where* $k,l \in \mathbf{N}$, $j \in \mathbf{N}\cup\{0\}$;
2. $\omega_\gamma = \otimes_{k=1}^\infty e_{\gamma_k^1,\gamma_k^2}$, *where* $\gamma = (\gamma_k^1,\gamma_k^2) \in \hat{\mathbb{D}}\otimes\hat{\mathbb{D}}$, *and*

$$e_{0,0} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, e_{1,0} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, e_{0,1} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} and e_{1,1} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix};$$

3. $\omega_\gamma\bar{\otimes}e_{kl}$, *where* $k,l \in \mathbf{N}$, $\gamma \in \hat{\mathbb{D}}\otimes\hat{\mathbb{D}}$,

each of which is endowed with the Walsh-Paley numeration, are RUC-bases in any $E(M,\tau)$-space of type 2 where $M = L_\infty(0,1)\bar{\otimes}B(l_2)$, R, $R\otimes B(l_2)$ respectively.

Finally, we consider applications of our results to RUC-bases in symmetric vector-valued spaces.

Corollary 4.3. *Let X be (UMD) space of type 2, let F be of type 2 symmetric function space over $(0,1)$ with non-trivial Boyd indexes, let $(x_m)_{m=1}^\infty$ be some $\mathbb{D}$-system in X. Then $(w_n\otimes x_m)_{n,m=1}^\infty$ is a RUC basis in $F(X)$. In particular, $\mathbb{D}$-system $(w_n\otimes w_m)_{n,m=1}^\infty$ is a RUC-basis in every space $L_F(L_G)$, where L_F and L_G are reflexive Orlicz spaces of type 2.*

Proof. It follows from the results of [21] that there exists a representation of the group $\mathbb{D}$ on the space $L_p(X)$ such that the system $(w_n \otimes x_m)_{n,m=1}^{\infty}$ can be considered as a $\mathbb{D}$-system with respect to the representation. Since $L_p(X)$ is (UMD) space, the system taken in the Walsh-Paley numeration is a (RUC) basis of the space as well.

Let us now recall the following interpolation theorem, which may be found in [2]: If E_0, E_1 are Banach ideal spaces on some measure space, X is a Banach space and if $\mathcal{F}$ is some interpolation functor, then

$$\mathcal{F}(E_0(X), E_1(X)) = \mathcal{F}(E_0, E_1)(X).)$$

The theorem allows us now to conclude that $(w_n \otimes x_m)_{n,m=1}^{\infty}$ is also certain $\mathbb{D}$-system and a basis in $F(X)$. Finally, due to [4] one has that $F(X)$ is of type 2. An application of Theorem 2 completes the proof. □

Using the last two corollaries one can construct several examples of RUC-bases in the spaces of vector-functions that do not have any unconditional basis. We would like to mention just one of them.

Corollary 4.4. *The system*

$$(w_n \otimes e_{ml})_{n=0;m,l=1}^{\infty}$$

is a RUC basis in every space $F(C_E)$ whenever C_E and F are of type 2.

The results proved in [S], that the system of operators $h_j \bar{\otimes} e_{kl}$ (here (h_j) is the Haar system) is a RUC-basis in any $E(L_\infty(0,1)\bar{\otimes}B(l_2))$ of type 2 and that the system $(h_j \otimes e_{kl})$ is a RUC-basis in every space $F(C_E)$ of type 2 also can be obtained from Corollary 4.1 simply by constructing a special representation of the group $\mathbb{D}$ such that the corresponding $\mathbb{D}$-system coincides with $h_j \bar{\otimes} e_{kl}$.

References

[1] P.Billard, S.Kwapien, A.Pelczynski, and Ch.Samuel. Biorthogonal systems of random unconditional convergence in Banach spaces. *Longhorn Notes the univ. of Texas at Austin* (1985-86), 13-35.

[2] A.V.Buhvalov. Interpolation of linear operators in spaces of vector-valued functions and with mixed norm. *Siberian Math J*, 28, (1987), 37-51 (Russian).

[3] A.V.Buhvalov. Theorems on interpolation of sublinear operators in spaces with mixed norm. *Qualitative and Approximate Methods of Investigating Operator Equations* (Russian), Yaroslavl State Univ. (1984) 90-105 (Russian).

[4] A.V.Buhvalov. Geometrical properties ov Banach spaces of measurable vector-functions. *Dokl.AN SSSR*, 238 (6), (1978), 1279-1282 (Russian).

[5] V.Chilin and F.Sukochev. Weak convergence in symmetric spaces of measurable operators *J.Operator Theory*, 31, (1994), 35-65.

[6] P.G. Dodds, T.K. Dodds and B. de Pagter. Non-commutative Banach function spaces. *Math. Z.*, 201, (1989), 583-597.

[7] P.G. Dodds, T.K. Dodds and B. de Pagter. Fully symmetric operator spaces. *Integr. Equat. Oper. Th.* , 15, (1992), 942-972.

[8] P.G. Dodds, T.K. Dodds and B. de Pagter, and F.Sukochev. Lipschitz continuity of the absolute value in symmetric operator spaces. *Reports of the faculty of Tech.Math. and Informatics of Delft Univ. of Tech.*, Rep.95-34, (1995).

[9] T. Fack and H. Kosaki. Generalized s-numbers of τ-measurable operators. *Pacific J. Math.*, 123, (1986), 269-300.

[10] S. Ferleger. Bases in (UMD)-spaces: applications to the non-commutative symmetric spaces. Preprint.

[11] S.Ferleger and F.Sukochev. On the contractibility to a point of the linear groups of reflexive non-commutative L^p-spaces. Math. Proc. Cambridge Philos. Soc. 119 (1996), 3, 545-560.

[12] S.Kwapien and A.Pelczynski. The main triangle projection in matrix spaces and its applications. *Studia Math.*, 34, (1970), 43-68.

[13] S.Krein, Ju.Petunin and E.Semenov. Interpolation of linear operators. *Translations of Mathematical Monographs, Amer. Math. Soc.*, 54, (1982).

[14] B.Kashin and A.Saakyan. Orthogonal series. *Translations of Mathematical Monographs, Amer. Math. Soc.*, 75, (1989).

[15] J.Lindenstrauss and L.Tzafriri. *Classical Banach Spaces I,* (Springer-Verlag, 1977.)

[16] J.Lindenstrauss and L.Tzafriri. *Classical Banach Spaces II,* (Springer-Verlag, 1979.)

[17] G.Pisier. Types des espaces normes. *C.R.Acad.Sci.Paris*, 286, (1973),1673-1674.

[18] F.Sukochev. (RUC)-bases in $E(L_\infty \bigotimes B(H))$ and $F(C_E)$ *Matem.Zametki*, 56, (1994), 88-104 (Russian).

[19] F.Sukochev and V.Chilin. Non-commutative symmetric spaces on semifinite von Neumann algebras. *Soviet Math.Dokl.*, 42 (1)(1991), 97-101.

[20] F.Sukochev and S.Ferleger. Harmonic analysis in symmetric spaces of measurable operators. (Russian) Dokl. Akad. Nauk 339 (1994), 3, 307-310.

[21] F.Sukochev and S.Ferleger. Harmonic analysis in UMD-spaces: applications to basis theory. (Russian) Mat. Zametki 58 (1995), 6, 890-905, 960.

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