KHINTCHINE AND PALEY INEQUALITIES FOR $\mathbb{D}$-SYSTEMS
IN SYMMETRIC OPERATOR SPACES

by

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Abstract *We consider Khintchine inequality in symmetric operator spaces in the situation when the sequence of Rademacher functions is replaced by sequence of eigenvectors of some representation of dyadic group $\mathbb{D}$ corresponding to lacunary sequence of characters from $\hat{\mathbb{D}}$. The same approach we apply to study of Paley inequality in non-commutative Hardy spaces and its dyadic generalizations.*

0. Introduction. Let $\mathbb{T}$ be the circle group with Lebesgue measure dt and let $\mathbb{Z}$ be its dual group. As usual we identify the character $n \in \mathbb{Z}$ with the function $e^{int}, t \in \mathbb{T}$. Let $\mathbb{Z}_2$ be the discrete cyclic group of order 2, that is $\mathbb{Z}_2$ is the set $\{0, 1\}$ with the discrete topology and addition modulo 2 and equipped with Haar measure. The *dyadic group* $\mathbb{D}$ is defined to be the compact abelian group formed by taking the Cartesian product of countably many copies of $\mathbb{Z}_2$ so that

$$\mathbb{D} := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots,$$

equipped with the product topology and induced product (again Haar) measure μ. We may identify the dual group $\hat{\mathbb{D}}$ with the set of all finitely non-zero $\{0, 1\}$-valued sequence $\gamma = \{n_k\}$, acting on $x = \{x_n\}_{n=0}^{\infty} \in \mathbb{D}$ as follows

$$\gamma(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}.$$

Let us set

$$U_0 = \{0\}, \quad U_n = \{\{\gamma_k\}_{k=1}^{\infty} \in \hat{\mathbb{D}} : \gamma_n = 1, \gamma_k = 0 \quad \text{for all } k > n\}.$$

and for each $k \in \mathbb{Z}^+$, and $x = \{x_n\}_{n=0}^{\infty} \in \mathbb{D}$, we define the character $\rho_k \in \hat{\mathbb{D}}$ by setting

$$\rho_k(x) = (-1)^{x_k} = \begin{cases} 1 & \text{if } x_k = 0 \\ -1 & \text{if } x_k = 1 \end{cases}.$$

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The sequence $\{\rho_k\}_{k=1}^{\infty}$ generates $\hat{\mathbb{D}}$ i.e. for each $\gamma \in \hat{\mathbb{D}}$ there is unique sequence (n_k) such that

$$\gamma = \prod_{k=0}^{\infty} \rho_k^{n_k})$$

and is called Rademacher sequence, since its image under canonical isomorphism between $L_p(\mathbb{D}, \mu)$ and $L_p([0,1], m)$ is usual Rademacher sequence. Moreover, this isomorphism identifies the whole group $\hat{\mathbb{D}}$ with usual Walsh system $\{w_n\}_{n=1}^{\infty}$ taken in Walsh-Paley numeration, i.e. $\gamma \in \hat{\mathbb{D}}$ corresponds to w_n whenever $n = \sum_{k=0}^{\infty} n_k 2^k$. The next two theorems are the classical ones and have been generalized in many ways. Both of them describe the behavior of certain sequences of characters in functional spaces.

Theorem 0.1 (Khintchine's Inequality) *For all $1 \leq p < \infty$, there exist constant $A_p, B_p > 0$ such that for all sequences $\{a_k\}_{k=1}^{\infty}$ and for all $n = 1, 2...$*

$$A_p \left(\sum_{k=1}^{n} \mid a_k \mid^2 \right)^{\frac{1}{2}} \leq \| \sum_{k=1}^{n} a_k \rho_k \|_{L^p(\mathbb{D},\mu)} \leq B_p \left(\sum_{k=1}^{n} \mid a_k \mid^2 \right)^{\frac{1}{2}}.$$

Theorem 0.2 (Paley Gap Theorem) *There exists a constant $C > 0$ such that for all functions $f = \sum_{n=0}^{\infty} \hat{f}(n) e^{int} \in H^1(\mathbb{T})$ we have*

$$(\sum_{n=0}^{\infty} |\hat{f}(2^n)|^2)^{\frac{1}{2}} \approx \| \sum_{n=0}^{\infty} \hat{f}(2^n) e^{i2^n(\cdot)} \|_{H_1(\mathbb{T},dt)} \leq \|f\|_{H_1(\mathbb{T},dt)}.$$

There is one very natural way to distinguish the characters of compact Abelian group G with Haar measure m among the functions from $L_p(G, m)$. Namely, to consider them as eigenvectors of representation of G at the $L_p(G, m)$ as forward translation group (see [E 1]). This way can be extended to the case of vector-valued spaces $L_p(G, X)$ and $H_p(G, X)$ where again group G is represented as forward translation group. Such generalization obtained in the recent years a great deal of attention. In other words it is just the replacement of the scalars a_k on the elements x_k of Banach space X and studying the behavior of the norms $\| \sum_{k=1}^{n} x_k \rho_k \|_{L^p(\mathbb{D},X)}$ and $\| \sum_{n=0}^{\infty} \hat{x}_k e^{i2^n(\cdot)} \|_{H_1(\mathbb{T},X)}$ respectively for different classes of X. We mention here well-known Maurey's theorem (see [LT 2, 1.d.6]) for the case when X is a Banach lattice and recent series of results [L-P 1,2], [L-PP], [BP], [HP] where many results were obtained for X being either symmetrically normed ideal of compact operators with additional assumptions at its order-topological structure or predual to von Neumann algebra. Important information is also contained in [P 1,2]. We follow to the same idea of study eigenspaces of action groups $\mathbb{D}$ and $\mathbb{T}$ but represented on X rather than X-valued space. This idea is well-known in the theory of non-commutative Hardy spaces, which has been defined to be span of "positive" eigenspaces of non-commutative L_p-spaces. Thus, we represent G as group of authomorphisms of Banach space X and study appearing systems of eigenspaces. We mostly concentrate at the representations of the groups $\mathbb{D}$ and $\mathbb{T}$ on the "non-commutative" $L_p(M,\tau)$-spaces, which usually appear as extentions to some authomorphic action of these groups on von Neumann algebra M. For instance, besides the algebra $L^\infty(\mathbb{D},\mu)$ itself another imporatant example of von Neumann algebras which admit natural representation of $\mathbb{D}$ is approximatively finite-dimensional factor $\mathcal{R}$ of type II_1. The systems of eigenvectors appearing there are called $\mathbb{D}$-systems have strong resemblance with usual Rademacher system and their behavior in symmetric operator spaces associated with underlying von Neumann algebra was recently examined in papers [SF 1,2] (see also [AFS], [FS]). Here we examine whether the "Rademacher" subspaces generated by eigenvectors are isomorphic to l_2 and complemented in enveloping symmetric operator space.

We will present here the direct analogues of Theorems 0.1 and 0.2 for actions of group $\mathbb{D}$ and $\mathbb{T}$ respectively and also introduce non-commutative H_1-spaces connected with actions of the group $\mathbb{D}$ (which are straight generalizations of dyadic H_1-spaces), where some analogue of theorem 0.2 has been also held.

1. Preliminaries. Throughout the article $\mathcal{M}$ is a semifinite von Neumann algebra, τ is a semifinite normal faithful trace on $\mathcal{M}$ with $\tau(\mathbf{1}) = a \leq \infty$ (here $\mathbf{1}$ is the unit of $\mathcal{M}$.) The $*$-algebra of all τ -measurable operators (see [FK]) affiliated with $\mathcal{M}$ is denoted by $K(\mathcal{M},\tau)$. For symmetric function space $E[0,a)$ (see [KPS], [LT 2]) we define the symmetric space of measurable operators $E(\mathcal{M},\tau)$ associated with $E[0,a)$ and the semifinite von Neumann algebra $(\mathcal{M},\tau)$, as follows:

$$E(\mathcal{M},\tau) = \{x \in K(\mathcal{M},\tau) : \mu(x) \in E[0,a)\},\ \|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_{E[0,a)},$$

where $\mu_t(x)$ for fixed $t \in (0,\infty)$ is defined by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(\chi_{[s,\infty)}(\mid x \mid) \leq t\},$$

where $\chi_{[s,\infty)}(\mid x \mid)$ is the spectral projection of $\mid x \mid = (x^*x)^{1/2}$ corresponding to the interval (s,∞). The function $\mu(x) : [0,\infty) \to [0,\infty]$ is called the generalized singular value function (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. If $\mathcal{M} = L_\infty(0,a)$ the function $\mu(x)$ is the usual decreasing rearrangement $\tilde{x}(t)$ (see [KPS]). If $E[0,a)$ is separable, then $(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is a Banach space whose Banach dual $E(\mathcal{M},\tau)^*$ can be identified as follows

$$E(\mathcal{M},\tau)^* = \{x \in K(\mathcal{M},\tau) : \|x\|_{E(M,\tau)^*} = \sup\{\tau(\mid xy \mid) : y \in E(\mathcal{M},\tau), \|y\|_{E(\mathcal{M},\tau)} \leq 1\} < \infty\}$$

and the action of $y \in E(\mathcal{M},\tau)^*$ on x is $\tau(xy^*)$. It is easily can be seen that if $M = L_\infty(0,a)$, $a \leq \infty$, equipped with the trace μ(=Lebesgue integral) generated by Lebesgue measure, then $E(\mathcal{M},\mu)$ coincides with $E(0,a)$. Proofs of assertions above and another information about these spaces can be found in [CS], [DDP 1,2,3], [SC].

Let us now recall some of the elements of the theory of representation of compact Abelian groups. Let X be an arbitrary Banach space, let G be a compact Abelian group with Haar measure dt and dual group $\hat{G}$ and let $\{R_t\}_{t\in G}$ be a strongly continuous group of isometries of X (or even group of invertible maps of X onto itself with $c = \sup_{t\in G} \|R_t\|_{X\to X} < \infty$). For each $\gamma \in \hat{G}$, we define the projection $E_\gamma : X \to X$ by setting

$$E_\gamma x = \int_G \gamma(u) R_{-u} x du,$$

where the integral on the right is a Bochner integral. The range of E_γ is the eigenspace

$$X_\gamma := \{x \in X : R_t x = \gamma(t)x \quad \forall t \in G\}.$$

If $X = L^p(G, dt), 1 \leq p < \infty$, and if $S = \{S_t\}_{t\in G}$ is given by forward translation

$$(S_t f)(x) := f(x+t), \quad x \in G, \quad t \in G,$$

then

$$\begin{aligned} X_\gamma &= \{f \in L^p(G, dt) : (S_t f)(x) = \gamma(t) f(x) \quad \forall t \in G\} \\ &= \{f \in L^p(G, dt) : f(x+t) = \gamma(t) f(x) \quad \forall t \in G\} \\ &= \{f \in L^p(G, dt) : f(t) = \gamma(t) f(0) \quad \forall t \in G\} \end{aligned}$$

and this is just the (one-dimensional) linear subspace spanned by the character γ. In particular, the usual trigonometric system $\{e^{2\pi i n(\cdot)}\}_{n\in\mathbb{Z}}$ appears as a system of eigenvectors arising from the representation of the group $\mathbb{T}$ by forward translation on $L^p(\mathbb{T}, m)$, while the Walsh system appears as a system of eigenvectors arising from the representation of the group $\mathbb{D}$ by forward translation on $L^p(\mathbb{D}, \mu)$.

Recall also that the representations of G on von Neumann algebra $\mathcal{M}$ as automorphisms group is called *action* of G on $\mathcal{M}$. In this case we will always assume that authomorphisms R_t preserves trace for any $t \in G$.

Let (A, Σ_A, μ_A) be an arbitrary probability space. One says that Banach space X has the unconditional property for martingale differences (written "X is an UMD-space" or "$X \in (UMD)$") if for $1 < p < \infty$ there are constants $C(p, X)$ such that for every martingale difference sequence $\{d_i\}_{i=1}^\infty \in L^p(A, X)$ and every sequence $\{\epsilon_i\}_{i=1}^\infty,\ \epsilon_i = \pm 1$

$$\|\sum_{i=1}^\infty \epsilon_i d_i\|_{L^p(A,X)} \leq C(p, X) \|\sum_{i=1}^\infty d_i\|_{L^p(A,X)}. \tag{1.1}$$

The sequence $\{\gamma_{[n]}\}_{n=1}^\infty \subseteq \hat{\mathbb{D}}$ will be called *lacunary* if and only if $\gamma_{[n]} \in U_n$ for all $n \geq 1$. Thus the sequence $\{\rho_k\}_{k=1}^\infty$ delivers the example of lacunary sequence. It is well-known that (1.1) holds if and only if

$$\|\sum_{i=1}^\infty \epsilon_i \gamma_{[i]} x_i \|_{L^p(\mathbb{D},X)} \leq C(p, X) \|\sum_{i=1}^\infty \gamma_{[i]} x_i \|_{L^p(\mathbb{D},X)}. \tag{1.2}$$

holds for any lacunary sequence of characters $\{\gamma_{[i]}\}$ and any $\{x_i\} \subseteq X$. Arbitrary non-commutative L^p-spaces, $1 < p < \infty$ are UMD-spaces (see [BGM 1]); in particular, the L^p-spaces associated with an arbitrary measure space as well as the von Neumann-Schatten classes C_p (non-commutative L^p-spaces associated with von Neumann algebra $B(H)$ of all linear operators on separable Hilbert space H) for $1 < p < \infty$ are UMD-spaces. The next result transferes the "unconditionality" from $L^p(\mathbb{D}, X)$ to X.

Theorem 1.1 ([SF 1,2] and [FS]) *Let X be a (UMD)-space and let R be a strongly continuous representation of $\mathbb{D}$ on X. If $X_n = \operatorname{span}\{X_\gamma : \gamma \in U_n\}, n \geq 0$ then the sequence $\{X_n\}_{n=0}^{\infty}$ is an unconditional decomposition for X.*

We say that $T_A : X \to X$ is the projection along A (with respect to some strongly continuous representation $\{R_t\}_{t\in G}$ of some group G) if T_A is a bounded operator such that $T_A \mid_{X_\gamma} = Id$ for all $\gamma \in A$ and $T_A \mid_{X_\gamma} = 0$ for all $\gamma \notin A$.

By definition, if T_A exists, then it is uniquely determined. The authors of [BGM 1,2] developed a simple and powerful variant of the transference method of Coifmann and Weiss. This method was applied to the projections along A (see Theorem 1.2 below) in [SF 1,2] and [FS].

Let X be an arbitrary Banach space and let $\{R_t\}_{t\in G}$ be a strongly continuous representation of G on X. Fix some $p \in [1, +\infty)$ and consider the Banach space $L^p(G, X)$ with canonical representation G on this space as the group of forward translations. Let A be an arbitrary subset of $\hat{G}$.

Theorem 1.2 ([SF 1,2] and [FS]) *Fix some $A \subseteq \hat{G}$. If the projection P_A along A with respect to the representation G on $L^p(G, X)$ by forward translation exists, then the projection T_A along A with respect to an arbitrary representation $\{R_t\}_{t\in G}$ in X also exists and moreover*

$$\|T_A\|_{X\to X} \leq c^2 \|P_A\|_{L^p(G,X)\to L^p(G,X)},$$

where $c = \sup_{t\in G} \|R_t\|_{X\to X}$.

It has been already mentioned in the Introduction that the sequence $\{\rho_i\}_{i=1}^{\infty}$ generates $\hat{\mathbb{D}}$. Denote J_k the subset of all characters of $\hat{\mathbb{D}}$ which can be represented as product of precisely k different elements from $\{\rho_i\}_{i=1}^{\infty}$ (k-fold product of Rademacher characters) and denote by R_k the orthogonal projection onto span in $L_2(\mathbb{D}, \mu)$ of J_k. The following remarkable result was established in [P].

Theorem 1.3 (Pisier G.) *The projection $R_k \otimes Id_X$ is bounded in $L_2(\mathbb{D}, X)$ if and only if X does not contain l_1^n's uniformly.*

Combining Theorems 1.2 and 1.3 we get the following immediate corollary

Corollary 1.4 *Projection along T_{R_k} is bounded in X whenever X does not contain l_1^n's uniformly, in particular, if $X \in (UMD)$.*

For easy references we will now show that any lacunary set in $\hat{\mathbb{D}}$ is a system of Bernoulli random variables. We need first the following definition.

Definition 1.5 *The set $\{\gamma_n\}_{n=0}^{\infty} \subseteq \hat{\mathbb{D}}$ is said to be algebraically independent if and only if for any sequence of positive integers $\{k_n\}$ and any positive integer N the equality*

$$\prod_{n=0}^{N} \gamma_n^{k_n} = 1$$

implies that

$$\gamma_n^{k_n} = 1, \quad \forall 0 \le n \le N.$$

Proposition 1.6 *Any lacunary subset in $\hat{\mathbb{D}}$ is a system of Bernoulli random variables.*

Proof At firts we show that any such system is algebraically independent. We proceed by induction on N. For the case that $N = 0$ the assertion is obvious. We assume then that the assertion is valid for all $n < N$, suppose that $\{\gamma_{[n]}\}_{n=0}^{\infty}$ is a lacunary set and that

$$\prod_{n=0}^{N} \gamma_{[n]}^{k_n} = 1.$$

If k_N is odd, then let $t = \{t_k\}_{k=0}^{\infty} \in \mathbb{D}$ be given by setting

$$t_N = 1, \quad t_k = 0, \quad k \ne N.$$

We then have

$$\left(\prod_{n=0}^{N} \gamma_{[n]}^{k_n}\right)(t) = \prod_{n=0}^{N} \gamma_{[n]}^{k_n}(t) = \gamma_{[N]}(t) = -1,$$

with the last equality following from the fact that $\{\gamma_{[n]}\}_{n=0}^{\infty}$ is a lacunary set.Thus, k_N is even and so $\gamma_{[N]}^{k_N} = 1$, and the proof now follows from the induction hypothesis. $\square$.
From [PP], Proposition 6.3 it follows that any algebraically independent system in $\hat{\mathbb{D}}$ is a system of independent random variables. In particular, any lacunary sequence in $\hat{\mathbb{D}}$ is a linearly independent sequence of identically distributed $\{1, -1\}$-valued random variables of mean 0, and thus a sequence of Bernoulli random variables. $\square$

2. Khintchine inequality in symmetric operator spaces Throughout this section $\{R_t\}_{t\in\mathbb{D}}$ is a σ-weakly continuous representation of the dyadic group $\mathbb{D}$ as a group of trace preserving $*$-automorphisms of $(\mathcal{M}, \tau)$. The set $M_0 = \{x \in M : R_t(x) = x \text{ for all } t \in \mathbb{D}\}$ is von Neumann subalgebra of $\mathcal{M}$. It can be checked that $\{R_t\}_{t\in\mathbb{D}}$ extends uniquely to a strongly continuous group of isometries $R^E = \{R_t^E\}_{t\in\mathbb{D}}$ on $E(\mathcal{M}, \tau)$ for any $E(\mathcal{M}, \tau)$ associated with separable $E(0, a)$ (see also [DDPS]).

Proposition 2.1 *If $(\mathcal{M}_0, \tau)$ is semifinite algebra with semifinite trace, then $\{x \in E(\mathcal{M}, \tau) : R_t(x) = x$ for all $t \in \mathbb{D}\}$ coincides with $E(\mathcal{M}_0, \tau)$.*

Proof The proof follows immediately from the strong continuity of $\{R_t^E\}_{t\in\mathbb{D}}$ and density of $\mathcal{M}_0 \cap L_1(\mathcal{M}_0,\tau)$ in $E(\mathcal{M}_0,\tau)$ (see [DDP 1-3], [SC]).

Let $\hat{\mathbb{D}}'$ denote the collection of all characters $\gamma \in \hat{\mathbb{D}}$ for which $(E(\mathcal{M},\tau))_\gamma \neq \{0\}$. Any system of normalised elements $\{x_\gamma\}_{\gamma\in\hat{\mathbb{D}}'}$ with $x_\gamma \in (E(\mathcal{M},\tau))_\gamma$ will be called a non-commutative Walsh system corresponding to the representation R^E. The corresponding system $\{x_{\rho_k}\}_{\rho_k\in\hat{\mathbb{D}}'}$ will be called a non-commutative Rademacher system corresponding to the representation R^E. If $\{\gamma_{[n]}\} \subseteq \hat{\mathbb{D}}$ is a lacunary sequence, then the corresponding system $\{x_{\gamma_{[n]}}\}_{\gamma_{[n]}\in\hat{\mathbb{D}}'}$ will be called a lacunary system of normalised eigenvectors corresponding to the representation R^E. We will often omit letter "E" from notation of representation, that shouldn't confuse.

Proposition 2.2 *If $\gamma, \gamma' \in \hat{\mathbb{D}}$, and let $x_\gamma, x_{\gamma'}$ be eigenvectors corresponding to $\gamma, \gamma' \in \hat{\mathbb{D}}$ respectively, then*

(i) *$x_\gamma x_{\gamma'}$ is an eigenvector corresponding to the character $\gamma + \gamma'$.*

(ii) *x_γ^* is an eigenvector corresponding to the character γ.*

(iii) *If $\gamma \neq 0$, then $\tau(x_\gamma) = 0$.*

In particular, if $\mathcal{M}$ is finite von Neumann algebra and $\mathcal{M}_0 = \{\lambda\mathbf{1}\}$, then each x_γ is scalar multiple of a unitary operator.

Proof Let $\gamma, \gamma' \in \hat{\mathbb{D}}$, and let $x_\gamma, x_{\gamma'}$ be eigenvectors corresponding to $\gamma, \gamma' \in \hat{\mathbb{D}}$ respectively. We observe that

$$\begin{aligned} R_t(x_\gamma x_{\gamma'}) &= R_t(x_\gamma)R_t(x_{\gamma'}) \\ &= \gamma(t)\gamma'(t)x_\gamma x_{\gamma'} \\ &= (\gamma+\gamma')(t)x_\gamma x_{\gamma'}, \quad \forall t \in \mathbb{D} \end{aligned}$$

so that $x_\gamma x_{\gamma'}$ is an eigenvector corresponding to the character $\gamma + \gamma'$, which verifies (i). Further,

$$R_t(x_\gamma^*) = R_t(x_\gamma))^* = \gamma(t)x_\gamma^*, \quad \forall t \in \mathbb{D},$$

and so also x_γ^* is an eigenvector corresponding to the character γ, which is the assertion of (ii). Since the representation R is trace-preserving, it follows that

$$\tau(x_\gamma) = \tau(R_t x_\gamma) = \gamma(t)\tau(x_\gamma), \quad \forall t \in \mathbb{D}.$$

Consequently,

$$(1-\gamma(t))\tau(x_\gamma) = 0, \quad \forall t \in \mathbb{D},$$

and if $\gamma \neq 0$, it follows that $\tau(x_\gamma) = 0$, and this is just (iii). Finally, if $\mathcal{M}$ is finite and $\mathcal{M}_0 = \{\lambda\mathbf{1}\}$, then it follows from (i) and (ii) that

$$|x_\gamma| = \lambda_\gamma \mathbf{1},$$

for some scalar $0 \leq \lambda_\gamma$. $\square$

Remark 2.3 Assume that all eigenspaces $(E(\mathcal{M},\tau))_\gamma$ are *one-dimensional*. It easily follows then, that for any finite sequence of characters $\{\gamma_i\}_{i=1}^n$ and any their rearrangement $\{\gamma_{i(j)}\}_{j=1}^n$ there exists unimodular number α (which, probably depends on $\{\gamma_i\}_{i=1}^n$ as well as on $\{\gamma_{i(j)}\}_{j=1}^n$) such that

$$x_{\gamma_1}x_{\gamma_2}...x_{\gamma_n} = \alpha x_{\gamma_{i(1)}}x_{\gamma_{i(2)}}...x_{\gamma_{i(n)}}.$$

It is clear that for commutative algebras (such as $L_\infty(0,1)$) the $\alpha = 1$ for any combinations of characters and their rearrangements. It is not the case for non-commutative algebras. The variety in the choice of this number defines the difference of pure Banach space properties of corresponding Rademacher systems (see below Examples 2.7 and 2.9).

Theorem 2.4 *Let $\{x_{\gamma_{[n]}}\}_{\gamma_{[n]}\in\hat{\mathbb{D}}'} \subseteq L^p(\mathcal{M},\tau)$ be a lacunary system of eigenvectors corresponding to the representation R^p. Then for any $2 \le p < \infty$:*

$$\|\sum_{k=1}^n a_k x_{\gamma_{[k]}}\|_p \approx \max\left\{ \left\| \left(\sum_{k=1}^n |a_k|^2 |x_{\gamma_{[k]}}|^2 \right)^{1/2} \right\|_{L^p(\mathcal{M}_0,\tau)}, \left\| \left(\sum_{k=1}^n |a_k|^2 |x^*_{\gamma_{[k]}}|^2 \right)^{1/2} \right\|_{L^p(\mathcal{M}_0,\tau)} \right\}.$$

Proof Since R^p is a group of isometries, it follows immediately that

$$\|x\|_p = \|R^p_t x\|_{L^p(\mathbb{D},L^p(\mathcal{M},\tau))}. \qquad (i)$$

Consequently, if $\{a_k\}_{k=1}^\infty$ is any finitely non-zero sequence of scalars, we obtain that

$$\begin{aligned}\|\sum_{k=1}^n a_k x_{\gamma_{[k]}}\|_p &= \|R_t\left(\sum_{k=1}^n a_k x_{\gamma_{[k]}}\right)\|_{L^p(\mathbb{D},L^p(\mathcal{M},\tau))} \\ &= \|\sum_{k=1}^n a_k R_t x_{\gamma_{[k]}}\|_{L^p(\mathbb{D},L^p(\mathcal{M},\tau))} \\ &= \|\sum_{k=1}^n a_k x_{\gamma_{[k]}}\gamma_{[k]}(t)\|_{L^p(\mathbb{D},L^p(\mathcal{M},\tau))}\end{aligned}$$

.

Since $\{\gamma_{[n]}\}_{n=0}^\infty$ is a sequence of Bernoulli random variables (see Proposition 1.6), it follows from ([L-PP], [L-P] ??????????????????? duality arguments ???????????, whatever) and from Proposition 2.2 that

$$\|\sum_{k=1}^n a_k x_{\gamma_{[k]}}\gamma_{[k]}(t)\|_{L^p(\mathbb{D},L^p(\mathcal{M},\tau))} \approx$$

$$\max\left\{\left\|\left(\sum_{k=1}^{n}|a_k|^2|x_{\gamma_{[k]}}|^2\right)^{1/2}\right\|_{L^p(\mathcal{M}_0,\tau)},\left\|\left(\sum_{k=1}^{n}|a_k|^2|x^*_{\gamma_{[k]}}|^2\right)^{1/2}\right\|_{L^p(\mathcal{M}_0,\tau)}\right\}.\square$$

$\mathcal{M}_0$. In many important cases the structure of $\mathcal{M}_0$ is very simple. For example, if action of $\mathbb{D}$ on $\mathcal{M}$ is ergodic (see [T], Definition $V.7.3$), then $\mathcal{M}_0 = \{\lambda\mathbf{1}\}$. In this case the Khintchine's inequality in finite algebras gets the most simple form, with strong resemblance with Theorem 0.1.

Theorem 2.5 *Let E be a symmetric Banach function space on $[0,1]$ which is an intermediate space for some couple $(L^{p_1}, L^{p_2}), 1 < p_1 \leq p_2 < \infty$. Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra with $\tau(1) = 1$ and let $\{x_{\gamma_{[n]}}\} \subseteq \mathcal{M}$ be a lacunary sequence of eigenvectors corresponding to the ergodic action of $\mathbb{D}$ on $(\mathcal{M},\tau)$. Then there exists a constant $C_E > 0$ such that for all finitely non-zero scalar sequences $\{a_k\}_{k=0}^{\infty}$,*

$$C_E^{-1}\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2} \leq \|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_{E(\mathcal{M},\tau)} \leq C_E\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}.$$

Proof From the discussion preceding the theorem it follows that if $2 \leq p < \infty$, then there exists a constant $C_p > 0$ such that for all finitely non-zero scalar sequences $\{a_k\}_{k=0}^{\infty}$,

$$C_p^{-1}\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2} \leq \|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_p \leq C_p\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}.$$

Let now q be the conjugate index to p. We have

$$\|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_{L_q(\mathcal{M},\tau)} \geq \frac{\tau\left((\sum_{k=1}^{n}a_k x_{\gamma_{[k]}})(\sum_{k=1}^{n}a_k x_{\gamma_{[k]}})^*\right)}{\|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_{L_p(\mathcal{M},\tau)}} = \frac{\tau\left(\sum_{k=0}^{n}|a_k|^2\mathbf{1}\right)}{\|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_{L_p(\mathcal{M},\tau)}}$$
$$\approx \frac{\sum_{k=0}^{n}|a_k|^2}{(\sum_{k=1}^{n}|a_k|^2)^{1/2}} = \left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}.$$

The remainig inequality follows from the continuity of the embedding of L^2 into $L_q(\mathcal{M},\tau)$ and the fact that

$$\left\|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\right\|_2 = \left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}.$$

We suppose now that E is an intermediate space for some couple $(L^{p_1}, L^{p_2}), 1 < p_1 \leq p_2 < \infty$. Thus $L^{p_2}(\mathcal{M},\tau) = L^{p_1}(\mathcal{M},\tau) \cap L^{p_2}(\mathcal{M},\tau)$ embeds continuously with constant K_2

into $E(\mathcal{M},\tau)$, which in turn embeds continuously into $L^{p_1}(\mathcal{M},\tau) = L^{p_1}(\mathcal{M},\tau)+L^{p_2}(\mathcal{M},\tau)$ with constant K_1. We obtain

$$C_{p_1}^{-1}K_1^{-1}\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2} \le K_1^{-1}\|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_{p_1} \le \|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_{E(\mathcal{M},\tau)}$$

$$\le K_2\|\sum_{k=1}^{n}a_k x_{\gamma_{[k]}}\|_{p_2} \le K_2 C_{p_2}\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}. \quad \square$$

It is now of interest to exhibit some explicit lacunary systems and we turn to this question. We begin with a basic finite-dimensional example. Our exposition based on [SF 1,2], [FS], [AFS].

Example 2.6 We set

$$U(0) = V(0) = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let

$$\mathbb{H} := \mathbb{Z}_2 \times \mathbb{Z}_2$$

and note that $\hat{\mathbb{H}}$ may also be identified with $\mathbb{Z}_2 \times \mathbb{Z}_2$ via the duality given by setting

$$\gamma(t) = (-1)^{\langle t,\gamma\rangle}, \quad t \in \mathbb{H}, \quad \gamma \in \hat{\mathbb{H}}.$$

We define the representation $\{r_t\}_{t\in\mathbb{H}}$ of the group $\mathbb{H}$ on $(M_2(\mathbb{C}), tr)$, where tr denotes normalised trace, by setting

$$r_t(x) = U(t_1)V(t_2)xV(t_2)U(t_1), \quad t = (t_1,t_2) \in \mathbb{H}, \quad x \in= M_2(\mathbb{C}).$$

It is clear that each r_t is a trace preserving $*$-automorphism of $(M_2(\mathbb{C}), tr)$. That $\{r_t\}_{t\in\mathbb{H}}$ is a representation of $\mathbb{H}$ on $(M_2(\mathbb{C}), tr)$ may be verified directly by tedious calculation. The eigenspaces corresponding to the representation $\{r_t\}_{t\in\mathbb{H}}$ may be calculated by observing that

$$E_\gamma x = \int_{\mathbb{H}} \gamma(t) r_{-t} x\, dt = \frac{1}{4}\sum_{t\in\mathbb{H}}(-1)^{\langle t,\gamma\rangle} r_t x, \quad \gamma \in \hat{\mathbb{H}}, x \in M_2(\mathbb{C}).$$

Direct computation using this formula now yields that the eigenspaces $(M_2(\mathbb{C}))_\gamma, \gamma \in \hat{\mathbb{H}}$ are the one-dimensional subspaces generated by the matrices

$$e_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_{(1,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_{(1,1)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Example 2.7 First (semi-commutative) Rademacher system We let $\mathcal{R} = \bigotimes_{k=1}^{\infty}(M_2(\mathbb{C}),\tau)$ be the unique approximatevely finite-dimensional factor of type II_1

with faithful normal finite trace $\tau = \bigotimes_{k=1}^{\infty} tr$. The finite-dimensional factors $\bigotimes_{k=1}^{n}(M_2(\mathbb{C}), tr) \otimes_{k=n+1}^{\infty} \mathbb{C}.1_{(M_2(\mathbb{C}), tr}$ will be denoted by $\mathcal{R}_n$. We let

$$G = \prod_{n=1}^{\infty} (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

and define the representation

$$R_t = \bigotimes_{k=1}^{\infty} r_{t_k}, \quad t = (t_k)_{k=1}^{\infty} \in G$$

of G on $\mathcal{R}$ by setting

$$R_t x = \bigotimes_{k=1}^{\infty} r_{t_k} x_k, \quad t = (t_k)_{k=1}^{\infty} \in G, \quad x = \bigotimes_{k=1}^{\infty} x_k \in \mathcal{R}.$$

It is now easily checked that the restriction of each R_t to $\mathcal{R}_n$ is a trace preserving $*$-automorphism, and we omit the further details needed to show that R in fact defines a strongly continuous representation of G on $(\mathcal{R}, \tau)$. Such an argument may be based on the density of $\bigcup_{k=1}^{\infty} \mathcal{R}_k$ in $\mathcal{R}$.

To compute the eigenspaces, let $\gamma = \{\gamma_k\}_{k=1}^{\infty} \in \hat{G}$ and observe that there exists a natural number N such that $\gamma_k = 0$ for all $k > N$. If $x = \bigotimes_{k=1}^{\infty} x_k \in \mathcal{R}$ then

$$\begin{aligned} E_\gamma(x) &= \int_G \gamma(t) R_{-t} x dt \\ &= (\bigotimes_{k=1}^{N} \int_{\mathbb{H}} \gamma_k(t_k) r_{t_k} x_k dt_k) \otimes (\bigotimes_{k>N} x_k) \\ &= (\bigotimes_{k=1}^{N} E_{\gamma_k}(x_k)) \otimes (\bigotimes_{k>N} x_k). \end{aligned}$$

Consequently, we may write formally

$$E_\gamma = \bigotimes_{k=1}^{\infty} E_{\gamma_k}, \quad \gamma \in \hat{G},$$

and for each $\gamma = (\gamma_1, \gamma_2 \ldots) \in \hat{G}$, it follows that the corresponding eigenspace $((\mathcal{R}, \tau))_\gamma$ is just the one-dimensional subspace spanned by $\bigotimes_{k=1}^{\infty} e_{\gamma_k}$,for each $\gamma \in \hat{G}$.

We now identify

$$G = \prod_{k-1}^{\infty} (\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{D} \times \mathbb{D}$$

with $D = \prod_{k-1}^{\infty} \mathbb{Z}_2$ via the mapping

$$\Big((t_1^{(1)}, t_1^{(2)}), (t_2^{(1)}, t_2^{(2)}), \ldots, (t_n^{(1)}, t_n^{(2)}), \ldots\Big) \to (t_1^{(1)}, t_1^{(2)}, t_2^{(1)}, t_2^{(2)}, \ldots, t_n^{(1)}, t_n^{(2)}, \ldots),$$

with a similar identification of $\hat{G} = \hat{\mathbb{D}} \times \hat{\mathbb{D}}$ with $\hat{\mathbb{D}}$. We let $\{R_t\}_{t \in \mathbb{D}}$ be the induced representation on $(\mathcal{R}, \tau)$ given by

$$R_t x = \bigotimes_{k=1}^{\infty} r_{(t_k, t_{k+1})} x_k, \quad t = (t_k)_{k=1}^{\infty} \in \mathbb{D}, \quad x = \bigotimes_{k=1}^{\infty} x_k \in \mathcal{R}.$$

The eigenspace $((\mathcal{R}, \tau))_\gamma$ is just the one-dimensional subspace spanned by $\bigotimes_{k=0}^{\infty} e_{(\gamma_{2k+1}, \gamma_{2k})}$, for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \hat{D}$. The Rademacher system $\{x_{\rho_n}\}_{n=1}^{\infty} \subseteq (\mathcal{R}, \tau)$ corresponding to the representation R may now be taken to be given by the choice

$$x_{\rho_n} = \begin{cases} (\bigotimes_{k=1}^{m} e_{(0,0)}) \otimes e_{(1,0)} \otimes \bigotimes_{k=m+2}^{\infty} e_{(0,0)} & \text{if } n = 2m+1, , m = 0, 1, 2, \ldots; \\ (\bigotimes_{k=1}^{m} e_{(0,0)}) \otimes e_{(0,1)} \otimes \bigotimes_{k=m+2}^{\infty} e_{(0,0)} & \text{if } n = 2m, , m = 1, 2, \ldots. \end{cases}$$

A further example of a lacunary sequence of eigenvectors, to which Theorem 2.5 applies, is given by setting, for example,

$$y_n = \begin{cases} (\bigotimes_{k=1}^{m} e_{(0,0)}) \otimes e_{(1,0)} \otimes \bigotimes_{k=m+2}^{\infty} e_{(0,0)} & \text{if } n = 2m+1, , m = 0, 1, 2, \ldots; \\ (\bigotimes_{k=1}^{m} e_{(0,0)}) \otimes e_{(1,1)} \otimes \bigotimes_{k=m+2}^{\infty} e_{(0,0)} & \text{if } n = 2m, , m = 1, 2, \ldots. \end{cases}$$

Let us return to the Rademacher system $\{x_{\rho_n}\}_{n=1}^{\infty} \subseteq (\mathcal{R}, \tau)$. We observe first that the commutative subsystem $\{x_{\rho_{2m+1}}\}_{m=0}^{\infty}$ may be identified with the usual Rademacher sequence on the interval $[0,1]$. In fact, there is a trace preserving $*$-isomorphism of the algebra $\bigotimes_{n=1}^{\infty} \mathrm{diag} M_2(\mathbb{C})$ onto $L^\infty[0,1]$ which maps $x_{\rho_{2m+1}}$ to $\rho_{m+1}, m = 0, 1, 2, \ldots$. A similar statement may be made concerning the commutative subsystem $\{x_{\rho_{2m}}\}_{m=1}^{\infty}$. To see this, let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and observe that the $*$-automorphism of $M_2(\mathbb{C})$ given by the map $X \to U^* X U$, $X \in M_2(\mathbb{C})$, where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

maps A onto the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This $*$-automorphism extends in a natural way to a $*$-automorphism of $\bigotimes_{n=1}^{\infty} \mathrm{diag} M_2(\mathbb{C})$ which maps $x_{\rho_{2m}}$ to $x_{\rho_{2m-1}}$, $m = 1, 2, \ldots$.

We now recall the result of Rodin-Semyonov ([LT2] 2.b.4(i)) that if E is any rearrangement invariant Banach function space on $[0,1]$, then the span of the Rademacher sequence $\{r_k\}_{k=0}^{\infty}$ in E is equivalent to the the unit vector basis of l^2 if and only if there exists a constant $K > 0$ such that

$$\|f\|_{E} \leq K \|f\|_{M}, \quad f \in L^{\infty}[0,1],$$

where M denotes the Orlicz function

$$M(t) = (e^{t^2} - 1)/(e - 1)$$

and $\|\cdot\|_{M}$ denotes the norm in the Orlicz space $L_{M}[0,1]$. We shall refer to this condition as the condition of Rodin-Semyonov. We can now state the following form of the Khintchine inequalities for the first Rademacher system $\{x_{\rho_n}\}_{n=1}^{\infty} \subseteq (\mathcal{R}, \tau)$. The result which follows shows that the Khintchine inequalities hold in the non-commutative space $E(\mathcal{R}, \tau)$ if and only if the classical form of the inequalities hold in the commutative space E.

Theorem 2.8 *Let E be a rearrangement-invariant Banach function space on $[0,1]$, and let $\{x_{\rho_n}\}_{n=1}^{\infty} \subseteq (\mathcal{R}, \tau)$ be the first Rademacher system. The following statements are equivalent.*

(i) *There exists a constant $C > 0$ such that for all finitely non-zero scalar sequences $\{a_k\}_{k=0}^{\infty}$,*

$$C^{-1} \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \leq \| \sum_{k=1}^{n} a_k x_{\rho_k} \|_{E(\mathcal{R},\tau)} \leq C \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2}.$$

(ii) *E satisfies the condition of Rodin-Semyonov.*

Proof The implication (i)$\Rightarrow$(ii) follows from the theorem of Rodin-Semyonov and the fact that

$$\| \sum_{k=1}^{n} a_k x_{\rho_{2k}} \|_{E(\mathcal{R},\tau)} = \| \sum_{k=0}^{n} a_k x_{\rho_{2k+1}} \|_{E(\mathcal{R},\tau)} = \| \sum_{k=1}^{n} a_k r_k \|_{E}.$$

(ii)$\Rightarrow$ (i). We assume that E satisfies the condition of Rodin-Semyonov. Again using the equalities

$$\| \sum_{k=1}^{n} a_k x_{\rho_{2k}} \|_{E(\mathcal{R},\tau)} = \| \sum_{k=0}^{n} a_k x_{\rho_{2k+1}} \|_{E(\mathcal{R},\tau)} = \| \sum_{k=1}^{n} a_k r_k \|_{E},$$

and the theorem of Rodin-Semyonov, it follows that

$$\begin{aligned}\|\sum_{k=1}^{n} a_k x_{\rho_k}\|_{E(\mathcal{R},\tau)} &\le \|\sum_{1<2k\le n} a_{2k} x_{\rho_{2k}}\|_{E(\mathcal{R},\tau)} + \|\sum_{1\le 2k+1\le n} a_{2k+1} x_{\rho_{2k+1}}\|_{E(\mathcal{R},\tau)} \\ &\le C\left(\sum_{1<2k\le n} |a_{2k}|^2\right)^{1/2} + \left(\sum_{1\le 2k+1\le n} |a_{2k-1}|^2\right)^{1/2} \\ &\le 2C\left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2}.\end{aligned}$$

Further,

$$\begin{aligned}\left(\sum_{1<2k\le n} |a_k|^2\right)^{1/2} &\le \left(\sum_{k=1}^{n} |a_{2k}|^2\right)^{1/2} + \left(\sum_{1\le 2k+1\le n} |a_{2k-1}|^2\right)^{1/2} \\ &\le C^{-1}\left(\|\sum_{1<2k\le n} a_{2k} x_{\rho_{2k}}\|_{E(\mathcal{R},\tau)} + \|\sum_{1\le 2k+1\le n} a_{2k+1} x_{\rho_{2k+1}}\|_{E(\mathcal{R},\tau)}\right) \\ &.\end{aligned}$$

To complete the proof, it suffices therefore to establish the inequalities

$$\|\sum_{k=1}^{n} a_{2k} x_{\rho_{2k}}\|_{E(\mathcal{R},\tau)}, \|\sum_{k=0}^{n} a_{2k+1} x_{\rho_{2k+1}}\|_{E(\mathcal{R},\tau)} \le \|\sum_{k=1}^{n} a_k x_{\rho_k}\|_{E(\mathcal{R},\tau)}.$$

To this end, it suffices to observe that $\sum_{k=0}^{n} a_{2k+1} x_{\rho_{2k+1}}$ (respectively, $\sum_{k=1}^{n} a_{2k} x_{\rho_{2k}}$) is the image of $\sum_{k=1}^{n} a_k x_{\rho_k}$ under the conditional expectation of $\mathcal{R}$ onto $\bigotimes_{n=1}^{\infty} \mathrm{diag} M_2(\mathbb{C})$ (respectively, $\bigotimes_{n=1}^{\infty} N$, where N is commutative subalgebra of $M_2(\mathbb{C})$ generated by 1 and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). In fact, it is easily checked that if T is conditional expectation of $M_2(\mathbb{C})$ onto $\mathrm{diag} M_2(\mathbb{C})$, then $T(A) = 0$. Similarly, it is not difficult to check that the image of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ under conditional expectation of $M_2(\mathbb{C})$ onto N is also 0.

Example 2.9 Second (anti-commutative) Rademacher system The following example is taken from [AFS]. Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and define the sequence $\{r_k\}_{k=0}^{\infty} \subseteq \mathcal{R}$ by setting

$$r_0 = 1, \quad r_1 = \sigma_1 \otimes \bigotimes_{k=2}^{\infty} 1_{M_2(\mathbb{C})}, \quad r_2 = \sigma_2 \otimes \bigotimes_{k=2}^{\infty} 1_{M_2(\mathbb{C})},$$

and

$$r_n = \begin{cases} (\bigotimes_{k=1}^{m} \sigma_3) \otimes \sigma_1 \otimes (\bigotimes_{k=m+2}^{\infty} 1_{M_2(\mathbb{C})}) & \text{if } n = 2m+1, m = 1, 2, \ldots; \\ (\bigotimes_{k=1}^{m-1} \sigma_3) \otimes \sigma_2 \otimes (\bigotimes_{k=m+1}^{\infty} 1_{M_2(\mathbb{C})}) & \text{if } n = 2m, m = 2, 3, \ldots. \end{cases}$$

It is shown in [AFS] Lemma 1 that $r_n^* = r_n, r_n^2 = 1$, and $r_n r_m = -r_m r_n$ for all $n, m = 1, 2, \ldots, n \neq m$. We will now show that this system arises as a Rademacher sequence corresponding to some representation of $\mathbb{D}$ on $L^p(\mathcal{R}, \tau) 1 \leq p < \infty$.

Following [AFS], we set

$$w_0 = 1, \quad w_\gamma = \prod_{n \in A_\gamma} r_n, \quad 0 =\neq \gamma \in \hat{\mathbb{D}},$$

where

$$A_\gamma = \{n \in \mathbb{N} : \gamma_n = 1\}, \quad \gamma = (\gamma_1, \gamma_2, \ldots) \in \hat{\mathbb{D}}.$$

For each finite subset $\mathcal{F} \subseteq \hat{\mathbb{D}}, t \in \mathbb{D}$ and scalars $\{\alpha_\gamma\}_{\gamma \in \mathcal{F}}$, we set

$$R_t(\sum_{\gamma \in \mathcal{F}} \alpha_\gamma w_\gamma) = \sum_{\gamma \in \mathcal{F}} \alpha_\gamma \gamma(t) w_\gamma.$$

We observe first that $R_0 = Id$, and that for all $s, t \in \mathbb{D}$,

$$\begin{aligned} R_t R_s(\sum_{\gamma \in \mathcal{F}} \alpha_\gamma w_\gamma) &= \sum_{\gamma \in \mathcal{F}} \alpha_\gamma \gamma(t)\gamma(s) w_\gamma \\ &= \sum_{\gamma \in \mathcal{F}} \alpha_\gamma \gamma(t+s) w_\gamma \\ &= R_{t+s}(\sum_{\gamma \in \mathcal{F}} \alpha_\gamma w_\gamma). \end{aligned}$$

Using now [AFS] Lemma 2(i), we obtain

$$
\begin{aligned}
R_t\left(\sum_{\gamma\in\mathcal{F}}\alpha_\gamma w_\gamma \sum_{\mu\in\mathcal{F}'}\beta_\mu w_\mu\right) &= R_t(\sum_{\gamma,\mu}\alpha_\gamma\beta_\mu w_{\gamma+\mu}(-1)^{I(\gamma+\mu)}) \\
&= \sum_{\gamma,\mu}\alpha_\gamma\beta_\mu(\gamma+\mu)(t)w_{\gamma+\mu}(-1)^{I(\gamma+\mu)} \\
&= \sum_{\gamma,\mu}\alpha_\gamma\beta_\mu\gamma(t)\mu(t)w_{\gamma+\mu}(-1)^{I(\gamma+\mu)} \\
&= (\sum_{\gamma\in\mathcal{F}}\alpha_\gamma\gamma(t)w_\gamma)(\sum_{\mu\in\mathcal{F}'}\beta_\mu\mu(t)w_\mu) \\
&= R_t(\sum_{\gamma\in\mathcal{F}}\alpha_\gamma w_\gamma)R_t(\sum_{\mu\in\mathcal{F}'}\beta_\mu w_\mu).
\end{aligned}
$$

Further, from [AFS] Lemma 2(ii), it follows immediately that

$$
\tau\left(R_t(\sum_{\gamma\in\mathcal{F}}\alpha_\gamma w_\gamma)\right) = \tau(\sum_{\gamma\in\mathcal{F}}\alpha_\gamma w_\gamma).
$$

The above calculations show that, for each $t \in \mathbb{D}$,the restriction of the mapping R_t to each finite-dimensional factor $\mathcal{R}_n$ is a trace preserving $*$-automorphism, and consequently is an isometry in the L^p-norm for each $1 \leq p < \infty$. Since the set $\cup_{n=1}^{\infty}\mathcal{R}_n$ is dense in $L^p(\mathcal{R},\tau)$ for every $1 \leq p < \infty$, it follows that $R = \{R_t\}_{t\in\mathbb{D}}$ extends to a group of isometries (which should also be checked to be strongly continuous) on $L^p(\mathcal{R},\tau)$ for every $1 \leq p < \infty$, whose restriction to $(\mathcal{R},\tau)$ is a group of trace-preserving $*$-automorphisms of $(\mathcal{R},\tau)$. It now remains to check that the system $\{w_\gamma\}_{\gamma\in\hat{\mathbb{D}}}$ is in fact a Walsh system corresponding to the representation R on $\mathrm{L}^p(\mathcal{R},\tau), 1 \leq p < \infty$. For each $\gamma \in \hat{\mathbb{D}}$, it follows from the definition of R_t that

$$
R_t w_\gamma = \gamma(t)w_\gamma, \quad \forall t \in \mathbb{D},
$$

and consequently, it follows that $w_\gamma \in (L^p(\mathcal{R},\tau))_\gamma$, for all $\gamma \in \hat{\mathbb{D}}$. Denoting by E_γ the projection onto the eigenspace $(L^p(\mathcal{R},\tau))_\gamma$, observe that it follows [BGM] from the fact that

$$
(L^p(\mathcal{R},\tau))_\gamma \cap (L^p(\mathcal{R},\tau))_\mu = \{0\}, \quad \gamma \neq \mu.
$$

On the other hand, for each $\gamma \in \hat{\mathbb{D}}$,

$$
\begin{aligned}
E_\gamma(L^p(\mathcal{R},\tau)) &= E_\gamma(\overline{\{\sum_{\mu\in\mathcal{F}}\beta_\mu w_\mu)\}} \\
&= \overline{\{E_\gamma(\sum_{\mu\in\mathcal{F}}\beta_\mu w_\mu)\}} \\
&= \overline{\{\sum_{\mu\in\mathcal{F}}\beta_\mu E_\gamma w_\mu\}} = \overline{\{\beta_\gamma w_\gamma\}}.
\end{aligned}
$$

This shows that the eigenspaces correspnding to the representation $\mathcal{R}$ are just the one dimensional eigenspaces spanned by elements of the system $\{w_\gamma\}_{\gamma\in\hat{\mathbb{D}}}$.

Let us note that the real linear span of the anti-commutative Rademacher system $\{r_n\}$ in any $L^p(\mathcal{R},\tau)$ is even isometric to l^2. This follows by observing that, if $x=\sum_{k=1}^N \alpha_n r_n$ is any finite linear combination with real coefficients, then

$$|x|^2=\sum_{n,m}\alpha_n\alpha_m r_n^* r_m=\sum_{n,m}\alpha_n\alpha_m r_n r_m=\sum_{n=1}^N|\alpha_n|^2 1.$$

Suppose that E is *any* rearrangement invariant space on the interval $[0,1]$ and suppose, for the sake of definiteness, that $\|\chi_{[0,1]}\|_{E(\mathcal{R},\tau)}=1$. It follows immediately that

$$\|\sum_{k=1}^N\alpha_n r_n\|_{E(\mathcal{R},\tau)}=\left(\sum_{k=1}^N|\alpha_k|^2\right)^{1/2}=\|\{\alpha_k\}_{k=1}^N\|_2$$

for any real sequence $\{\alpha_k\}_{k=1}^N$. Now suppose that $\{\alpha_k\}_{k=1}^N\subseteq\mathbb{C}$ and observe that

$$\begin{aligned}\|\sum_{k=1}^N\alpha_n r_n\|_{E(\mathcal{R},\tau)}&\le\|\sum_{k=1}^N\Re(\alpha_n)r_n\|_{E(\mathcal{R},\tau)}+\|\sum_{k=1}^N\Im(\alpha_n)r_n\|_{E(\mathcal{R},\tau)}\\&=\|\{\Re(\alpha_k)\}_{k=1}^N\|_2+\|\{\Im(\alpha_k)\}_{k=1}^N\|_2\\&\le 2\|\{\alpha_k\}_{k=1}^N\|_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}2\|\sum_{k=1}^N\alpha_n r_n\|_{E(\mathcal{R},\tau)}&\ge\|\sum_{k=1}^N\alpha_n r_n+\sum_{k=1}^N\overline{\alpha}_n r_n^*\|_{E(\mathcal{R},\tau)}\\&=\|\sum_{k=1}^N\Re(\alpha_n)r_n\|_{E(\mathcal{R},\tau)}=\|\{\Re(\alpha_k)\}_{k=1}^N\|_2.\end{aligned}$$

Similarly,

$$2\|\sum_{k=1}^N\alpha_n r_n\|_{E(\mathcal{R},\tau)}\ge\|\{\Im(\alpha_k)\}_{k=1}^N\|_2.$$

We obtain therefore that the Khintchine inequalities

$$\frac{1}{4}\|\{\alpha_k\}_{k=1}^N\|_2\le\|\sum_{k=1}^N\alpha_n r_n\|_{E(\mathcal{R},\tau)}\le 2\|\{\alpha_k\}_{k=1}^N\|_2$$

are valid for any rearrangement invariant space E on $[0,1]$.

The restriction of ergodicity of the action on the von Neumann algebra is natural. Indeed, the conclusion of Theorem 2.5 fails in the case of the Schatten ideals $\mathcal{C}_p$, even for natural (but non-ergodic !) representations of $\mathbb{D}$, as the following example shows.

Example 2.10 Let $(\mathcal{M}, \tau)$ be $\mathcal{L}(\mathcal{H})$, with $\mathcal{H}$ separable, equipped with the canonical trace τ. We let $\{\varphi_n\}_{n=1}^{\infty}$ be some orthonormal basis in $\mathcal{H}$ and let $e_{nm} = \varphi_n \otimes \varphi_m$, $n, m = 1, 2, \ldots$ be the corresponding system of matrix units. For each $x \in \mathcal{M}$, we set

$$R_t x = u_t x u_t, \quad t \in \mathbb{D}$$

where

$$u_t = \sum_{n=1}^{\infty} \rho_n(t) e_{nn}, \quad t \in \mathbb{D}$$

with convergence in the strong operator topology. It is clear that each $u_t, t \in \mathbb{D}$ is a self-adjoint unitary operator and moreover

$$u_s u_t = u_{s+t}, \quad s, t \in \mathbb{D}.$$

To calculate the eigenspaces, let $x \in \mathcal{M}$, $\gamma \in \mathbb{D}$ and observe that

$$\begin{aligned} e_{mm}(E_\gamma x)e_{nn} &= \int_{\mathbb{D}} e_{mm}\Big(\sum_{k=1}^{\infty} \rho_k(t)e_{kk}\Big)x\Big(\sum_{j=1}^{\infty} \rho_j(t)e_{jj}\Big)e_{nn}\gamma(t)dt \\ &= \int_{\mathbb{D}} \rho_m(t)e_{mm}xe_{nn}\rho_n(t)\gamma(t)dt \\ &= e_{mm}xe_{nn}\int_{\mathbb{D}} (\rho_n + \rho_m + \gamma)(t)dt \\ &= \begin{cases} 0, & \text{if } \rho_n + \rho_m + \gamma \neq 0; \\ e_{mm}xe_{nn}, & \text{if } \rho_n + \rho_m + \gamma = 0. \end{cases} \end{aligned}$$

We obtain that

$$\begin{aligned} E_0(\mathcal{L}(\mathcal{H})) &= \left\{ \sum_{n=0}^{\infty} \lambda_n e_{nn} : \{\lambda_n\} \in l^\infty \right\}, \\ E_\gamma(\mathcal{L}(\mathcal{H})) &= \{\lambda e_{mn} + \mu e_{nm} : \lambda, \mu \in \mathbb{C}\} \end{aligned}$$

if $\gamma = \rho_n + \rho_m$, and $\{0\}$ otherwise. Thus, in contrast to the earlier examples for the finite case, the eigenspace corresponding to the 0 character is far from one-dimensional; moreover, $E_{\rho_n} = 0$ for all $n = 1, 2, \ldots$, so that the only Rademacher system corresponding to the representation R is trivial. Nonetheless, there is an abundance of lacunary systems. We consider, in particular, the lacunary system of characters $\{\gamma_{[n]}\}_{n=1}^{\infty}$ defined by setting

$$(\gamma_{[n]})_k = \begin{cases} 1, & \text{if } k = 2n-1, 2n; \\ 0, & \text{otherwise .} \end{cases}$$

We let $p \geq 2$ and let

$$x_{\gamma_{[n]}} = \alpha_n e_{2n,2n-1} + \beta_n e_{2n-1,2n}, \quad n \geq 1,$$

where

$$|\alpha_n|^p + |\beta|^p = 1, \quad n \geq 1,$$

be a corresponding lacunary system in the Schatten ideal $\mathcal{C}_p$. It is clear that

$$x_{\gamma_{[n]}} x_{\gamma_{[m]}} = 0, \quad n \neq m.$$

Let $\{a_n\}_{n=1}^N$ be any finitely non-zero sequence of scalars. Observe that $\left(\sum_{k=0}^N |a_k x_{\gamma_{[k]}}|^2\right)^{1/2}$ is a diagonal matrix with entries

$$|a_1\beta_1|, |a_1\alpha_1|, |a_2\beta_2|, |a_2\alpha_2|, \ldots, |a_N\beta_N|, |a_N\alpha_N|,$$

while $\left(\sum_{k=0}^N |\overline{a}_k x^*_{\gamma_{[k]}}|^2\right)^{1/2}$ is a diagonal matrix with entries

$$|a_1\alpha_1|, |a_1\beta_1|, |a_2\alpha_2|, |a_2\beta_2| \ldots, |a_N\alpha_N|, |a_N\beta_N|.$$

3. Paley Gap Theorem in non-commutative H_1-spaces Our aim in this section is to establish analogue of the Theorem 0.2 for non-commutative H_1-spaces associated with actions of group $\mathbb{T}$ and $\mathbb{D}$. The first ones are well-known (see , for example [Z], [S])the second ones are being defined here are the non-commutative counterparts of well-known dyadic H_1-spaces (see, for example [SWS] and [G]).

Definition 3.1 *If $R = \{R_t\}_{t\in\mathbb{T}}$ is some σ-weakly continuous action of $\mathbb{T}$ on $(\mathcal{M}, \tau)$, then non-commutative H_1-space (associated with R and $(\mathcal{M}, \tau)$) is defined to be the closure in $L_1(\mathcal{M}, \tau)$-norm of all finite linear sums $\sum x_\gamma$ where $x_\gamma \in (L_1(\mathcal{M}, \tau))_\gamma, \gamma \in \mathbb{Z}, \gamma \geq 0$.*

We suppose that $R = \{R_t\}_{t\in\mathbb{D}}$ is some σ-weakly continuous ergodic action of $\mathbb{D}$ on $\widetilde{\mathcal{M}}$. We will assume also that each eigenspace $\mathcal{M}_\gamma, \gamma \in \hat{\mathbb{D}}$ is one -dimensional (see discussion in Remark 2.3). It should be pointed out, that all results of the present section will hold also without the latter assumption, but our main examples of action of $\mathbb{D}$ on non-commutative algebras exposed in the Examples 2.7 and 2.9 obey this requirement. This assumption allows us to define generalized Walsh system (i.e. the system of eigenvectors) directly, exactly as classical functional system (see [SWS],p.1). We are going at first to discuss this matter in more details.
indent Let us first choose our Rademacher system $\{r_n\}_{n=1}^\infty$ by the requirements that $r_n \in \mathcal{M}_{\rho_n}$ is a self-adjoint unitary operator. We may now *define* the corresponding Walsh system by setting

$$w_0 = 1, \quad w_\gamma := \prod_{n\in A_\gamma} r_n, 0 \neq \gamma \in \hat{\mathbb{D}},$$

where $A_\gamma = \{n \in \mathbb{N} : \gamma(n) = 1\}$. Equivalently,

$$w_\gamma = r_{n_1} r_{n_2} \cdots r_{n_k}, \quad \text{if} \quad \gamma = \rho_{n_1}\rho_{n_2}\cdots\rho_{n_k}.$$

From Proposition 2.2, it follows that w_γ is a unitary operator in $\mathcal{M}_\gamma$ for all $\gamma \in \hat{\mathbb{D}}$. The argument of [AFS], Lemma 2 (vi) shows that the system $\{w_\gamma\}_{\gamma\in\hat{\mathbb{D}}}$ is a minimal and total system in each $L^p(\mathcal{M}, \tau), 1 \le p < \infty$ and [FS] Mat.Zam. shows that the system $\{w_\gamma\}_{\gamma\in\hat{\mathbb{D}}}$ is a Schauder basis in each $L^p(\mathcal{M}, \tau), 1 < p < \infty$.

For each $n = 1, 2, \ldots$, we let

$$G_n := \{\gamma \in \hat{\mathbb{D}} : \gamma(k) = 0, k > n\}, \quad \mathcal{M}_n = \mathrm{clm}\{w_\gamma : \gamma \in G_n\}.$$

Since each r_n is a self-adjoint unitary, it follows now that each $\mathcal{M}_n$ is a finite-dimensional von Neumann subalgebra of $\mathcal{M}$. It is clear that $\mathcal{M}_n \uparrow_n$, and that $\mathcal{M} = \overline{\cup_{n=1}^\infty \mathcal{M}_n}^{w.o.}$ The argument of [AFS], Lemma 3, shows that the projection along G_n exists in any $L^p(\mathcal{M}, \tau), 1 \le p < \infty$, and coincides with the conditional expectation $\mathcal{E}(\cdot \mid \mathcal{M}_n)$ for each $n \in \mathbb{N}$. Observe that $U_{n+1} = G_{n+1}\backslash G_n, n \ge 1$. If $1 < p < \infty$ and if $x \in L^p(\mathcal{M}, \tau)$, then $\{\mathcal{E}(x \mid \mathcal{M}_n)\}_{n\ge1}$ is a martingale in $L^p(\mathcal{M}, \tau)$ and the difference sequence

$$\Delta_n(x) := \mathcal{E}(x \mid \mathcal{M}_{n+1}) - \mathcal{E}(x \mid \mathcal{M}_n) \in X_{n+1}, \quad n \ge 0,$$

where $X_n = \mathrm{clm}\{w_\gamma : \gamma \in U_n\}$. From Theorem 1.1, it follows that the martingale difference sequence $\Delta_n(x)$ is unconditional in each $L^p(\mathcal{M}, \tau), 1 < p < \infty$.

Definition 3.2 *Non-commutative dyadic $H_p(\mathcal{M}, \tau)$ (associated with action $R = \{R_t\}_{t\in\mathbb{D}}$) is the space of all elements $x \in L_1(\mathcal{M}, \tau)$ with*

$$\|x\|_{H_p(\mathcal{M},\tau)} := \|\sum_{n=1}^{\infty} \rho_n(\cdot)\Delta_n(x)\|_{L_p(\mathbb{D}, L_p(\mathcal{M},\tau))} < \infty.$$

Remark 3.3 It is clear, that the definition above can be considered for any von Neumann algebra $\mathcal{M}$ and sequence $\mathcal{M}_n \uparrow_n$ such that $\mathcal{M} = \overline{\cup_{n=1}^\infty \mathcal{M}_n}^{w.o.}$. If $\mathcal{M}$ is $L_\infty(\mathbb{D}, \mu)$ and $\mathbb{D}$ acts there as forward translation group, then the martingale in the question above is exactly classical Walsh-Paley martingale then the assertion about unconditionality of martingale differences is reformulation of another famous Paley's Theorem (Paley 1932). Using [LT 2, 1.d.6] it is immediately verified that $H_p(\mathbb{D}, \mu)$ is usual dyadic H_p-space (see SWS], [G]).

In order to present the perfect non-commutative analogue of Walsh-Paley martingales consider the case that $\mathcal{M}$ is the unique hyperfinite II_1 factor $\mathcal{R}$. In this case, it is not difficult to see that for both Rademacher systems from Examples 2.7 and 2.9 $\mathcal{M}_{2n}$ is just the finite-dimensional factor

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In particular, the preceding arguments justify the following theorem

Theorem 3.4 *If $(\mathcal{R},\tau)$ is the unique hyperfinite II_1 factor, if $1 < p < \infty$ and if $x \in L^p(\mathcal{R},\tau)$, then the martingale difference sequence*

$$\mathcal{E}(x \mid \mathcal{R}_{n+1}) - \mathcal{E}(x \mid \mathcal{R}_n)$$

is unconditional in $L^p(\mathcal{R},\tau)$. In particular

It is worth while observing that Theorem 4.3 above contains as a special case Paley's theorem concerning the unconditionality of Walsh-Paley martingales in $L^p(\mathbb{D})$ for $1 < p < \infty$ (Theorem 4.16 (ii)-in Peter's notes). To see this, we need only identify $L^\infty(\mathbb{D})$ with the commutative von Neumann subalgebra $\mathcal{N}$ of $\mathcal{R}$ generated by the finite-dimensional subalgebras

$$\mathcal{N}_n = \bigotimes_{k=1}^{n} \mathrm{diag} M_2(\mathbb{C}) \otimes \bigotimes_{k=n+1}^{\infty} \mathbb{C}1_{M_2(\mathbb{C})},$$

and check that the restriction to $\mathcal{N}$ of conditional expectation relative to $\mathcal{R}$ is conditional expectation relative to $\mathcal{N}_n$.

We proceed now to the analogues of Theorem 0.2 for non-commutative H_1-spaces. At first we present the most general version of our arguments which are slight refinement of Theorem 1.2. As usual we assume, that G is a compact Abelian group with (countable) dual group $\hat{G}$, and normalised Haar measure m. Let $R = \{R_g\}_{g\in G}$ be a strongly continuous representation of the group G on the Banach space X, and set

$$C_R = \sup_g \|R_g\|$$

. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a generalised "Walsh system", that is, for each $\gamma \in \Gamma$, x_γ is an element of the eigenspace X_γ, where

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We let $\{\Delta_k\}_{k=1}^\infty$ be a partition of the dual group Γ and $\Lambda = \{\gamma_{[k]} : k \in \mathbb{N}\} \subseteq \hat{G}$ be a Sidon set such that $\gamma_{[k]} \in \Delta_k$ for all $k \in \mathbb{N}$.

Theorem 3.5 *Let $G, \hat{G}, \{x_\gamma\}_{\gamma\in\hat{G}}, \{\Delta_k\}$ and Λ be as above.*

(i) Then for every family $\{\alpha_\gamma\}_{\gamma\in\hat{G}}$ of finitely non-zero scalars,

$$\|\sum_{\gamma\in\Lambda} \alpha_\gamma \gamma(\cdot) x_\gamma\|_{L_1(G,X)} \le C_R \|\sum_k \gamma_{[k]}(\cdot)(\sum_{\gamma\in\Delta_k} \alpha_\gamma x_\gamma)\|_{L_1(G,X)}.$$

(ii) Fix some set of coefficients $\{a_\gamma\}$ and assume that there exists the constant M such that

$$\|\sum_{\gamma\in\Lambda} \alpha_\gamma \gamma(\cdot) x_\gamma\|_{L_1(G,X)} \le M \|\sum_{\gamma\in\hat{G}} \alpha_\gamma \gamma(\cdot) x_\gamma\|_{L_1(G,X)}.$$

We have then

$$\left\|\sum_{\gamma\in\Lambda}\alpha_\gamma x_\gamma\right\|_X \le MC_R^2\left\|\sum_{\gamma\in\hat G}\alpha_\gamma x_\gamma\right\|_X .$$

Proof (i) To this end we have that

$$\begin{aligned}
\|\sum_{\gamma\in\Lambda}\alpha_\gamma\gamma(\cdot)x_\gamma\|_{L_1(G,X)} &= \int_G\|\sum_{\gamma\in\Lambda}\alpha_\gamma x_\gamma\gamma(g)\|_X dm(g)\\
&= \int_G\|\sum_{k=0}^{\infty}\left(\int_G\left(\sum_{\gamma\in\Delta_k}\alpha_\gamma x_\gamma\gamma(h)\gamma_{[k]}(-h)\right)dm(h)\right)\gamma_{[k]}(g)\|_X dm(g)\\
&= \int_G\|\int_G\left(\sum_{k=0}^{\infty}\left(\sum_{\gamma\in\Delta_k}\alpha_\gamma x_\gamma\gamma(h)\gamma_{[k]}(-h)\right)dm(h)\right)\gamma_{[k]}(g)\|_X dm(g)\\
&\le \int_G\int_G\|\sum_{k=1}^{\infty}\left(\sum_{\gamma\in\Delta_k}\alpha_\gamma x_\gamma\gamma(h)\right)\gamma_{[k]}(-h)\gamma_{[k]}(g)\|_X dm(h)dm(g)\\
&= \int_G\int_G\|\sum_{k=1}^{\infty}\left(\sum_{\gamma\in\Delta_k}\alpha_\gamma x_\gamma\gamma(h)\right)\gamma_{[k]}(g-h)\|_X dm(h)dm(g)\\
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&= \int_G\int_G\|R_u\left(\sum_{k=1}^{\infty}\left(\sum_{\gamma\in\Delta_k}\alpha_\gamma x_\gamma\right)\gamma_{[k]}(v)\right)\|_X dm(u)dm(v)\\
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&= C_R\|\sum_{k=1}^{\infty}\left(\sum_{\gamma\in\Delta_k}\alpha_\gamma x_\gamma\right)\gamma_{[k]}(\cdot)\|_{L_1(G,X)}
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that establishes (i). In order to prove (ii) we will use a transference argument. Let $\{k_n\}_{n=1}^{\infty}\subseteq L^1(G)$ be a Féjèr kernel, that is

(i) $\|k_n\|_1\le 2,\quad n\ge 1.$

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We set

$$k_n^\Lambda=\sum_{\gamma\in\Lambda}\hat k_n(\gamma)\gamma,\quad n\ge 1,$$

and observe that

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\end{aligned}$$

We obtain that

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as $n \to \infty$.

We now write $x = \sum_{\gamma\in\hat{G}} \alpha_\gamma x_\gamma$, and observe that

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&= C_R \int_G \|k_n^\Lambda *_R R_g x\|_X dm(g) \\
&= C_R \int_G \|k_n^\Lambda *_R \left(\sum_{\gamma\in\hat{G}} \alpha_\gamma x_\gamma \gamma(g)\right) \|_X dm(g) \\
&= C_R \int_G \|k_n *_R \left(\sum_{\gamma\in\Lambda} \alpha_\gamma x_\gamma \gamma(g)\right) \|_X dm(g) \\
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\end{aligned}$$

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