ON DARBOUX THEOREM FOR WEAK SYMPLECTIC MANIFOLDS

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Abstract. *A new tool to study reducibility of a weak symplectic form to a constant one is introduced and used to prove a version of Darboux theorem more general than previous ones. More precisely, at each point of the considered manifold a Banach space is associated to the symplectic form (dual of the phase space with respect to the symplectic form), and it is shown that Darboux theorem holds if such a space is locally constant. The following application is given. Consider a weak symplectic manifold M on which Darboux theorem is assumed to hold (e.g. a symplectic vector space). It is proved that Darboux theorem holds also for any finite codimension symplectic submanifolds of M, and for symplectic manifolds obtained from M by Marsden–Weinstein reduction procedure.*

1. Introduction

It was first pointed out by Marsden [1] that Darboux theorem fails for weak symplectic forms. Afterwards, some generalizations of the theorem to weak symplectic manifolds were proved [2] (see also [3]), but a complete characterization of the situations in which the symplectic form can be locally reduced to a constant one is still lacking. In the present paper a new tool to study reducibility of a weak symplectic form to a constant one is introduced and used to prove a version of Darboux theorem more general than previous ones. A necessary condition for Darboux theorem is also obtained.

The main idea of this paper is (following [4]) to associate, at each point of the manifold, a suitable Banach space to the symplectic form (classifying space); it turns out that Darboux theorem can be proved if the classifying space does not depend on the point of the manifold and a suitable smoothness condition holds. Applying this version of Darboux theorem

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the following results are proven: Let M be a weak symplectic manifold in which Darboux theorem is assumed to hold (e.g. a symplectic vector space) then Darboux theorem holds for each closed symplectic submanifold of M having finite codimension (see theorem 3.1). Then, the case where a finite dimensional Lie group acts symplectically on M is considered: it is proven that Darboux theorem holds for the symplectic manifold obtained from M by Marsden–Weinstein reduction procedure (see corollary 3.5). As far as I know these results are new.

I come to the necessary condition for the reducibility of a weak symplectic form to a constant one. It turns out that the assumptions of the present version of Darboux theorem are not invariant under general coordinate transformations, and so they do not directly provide a necessary condition for the reducibility of the symplectic form to a constant one. Anyway it is possible to exploit the classifying space to give a completely coordinate independent necessary condition for the reducibility of a symplectic form to a constant one. Such a condition is exploited to give a counterexample to Darboux theorem. Actually this is just a variant of Marsden's counterexample, and is given here mainly for the sake of completeness.

2. A necessary and a sufficient condition for Darboux theorem.

2.1 Statements

First, recall that a bilinear form $\Omega : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, $\mathcal{P}$ being a Banach space, is said to be weakly nondegenerate if

$$\{\Omega(X,Y) = 0 \ , \quad \forall Y \in \mathcal{P}\} \Longrightarrow X = 0 \ . \tag{1}$$

It is well known that Ω defines a linear map $\Omega^\flat : \mathcal{P} \to \mathcal{P}^*$ by

$$\langle \Omega^\flat Y, X \rangle := \Omega(Y,X) \ , \tag{2}$$

and that, by (1) such a map is injective; if it is also surjective, then Ω is said to be strongly nondegenerate.

A weak symplectic manifold (M,Ω) is a differentiable manifold M endowed with a closed differentiable 2-form Ω, which at each $x \in M$ is weakly nondegenerate as a bilinear form on T_xM. I will denote by $\Omega_x : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ the value of Ω at $x \in M$, and by J_x the inverse of the map $\Omega_x^\flat$. J_x is usually called the Poisson tensor.

Since Darboux theorem is a local result it is enough to consider the case where M is an open set $\mathcal{U}$ of a Banach space $\mathcal{P}$ which will be assumed to be *reflexive*; I will assume also $0 \in \mathcal{U}$, and all functions will be assumed to be C^∞.

Using Ω_x define on $\mathcal{P}$ the norm

$$\|X\|_{\mathcal{F}_x} := \sup_{\|Y\|_{\mathcal{P}}=1} |\Omega_x(X,Y)| = \left\|\Omega_x^\flat X\right\|_{\mathcal{P}^*} \ , \tag{3}$$

and consider the completion $\mathcal{F}_x$ of $\mathcal{P}$ in such a norm.

Definition *The space $\mathcal{F}_x$ thus obtained will be called "the dual of $\mathcal{P}$ with respect to Ω_x."*

It is clear that Ω_x can be extended to a continuous bilinear form on $\mathcal{P} \times \mathcal{F}_x$.

Theorem 2.1. *Assume that there exists a neighbourhood $\mathcal{W}$ of 0 such that, for all $x \in \mathcal{W}$ the spaces $\mathcal{F}_x$ coincide (i.e. the norms $\|.\|_{\mathcal{F}_x}$ are equivalent), and moreover that the map $x \mapsto \Omega_x$ is differentiable as an application from $\mathcal{W}$ to the continuous bilinear forms on $\mathcal{F} \times \mathcal{P}$, where $\mathcal{F} := \mathcal{F}_x$; then there exists a neighbourhood $\mathcal{V}$ of 0 and a change of coordinates ψ defined on $\mathcal{V}$ which reduces Ω to the constant two form $\Omega_0 := \Omega_x\big|_{x=0}$, namely such that*

$$\psi^* \Omega = \Omega_0 \ .$$

Remark 2.2. *One can conjecture that the differentiability of Ω as an application form $\mathcal{V}$ to the bilinear forms on $\mathcal{P} \times \mathcal{F}$ is automatic. It is easy to construct a counterexample in the case of Ω_x which is C^k ($k < \infty$) dependent on x, however I do not have a counterexample in the C^∞ case.*

Remark 2.3. *The Darboux theorem proved in [3] is contained in the above theorem 2.1. This is easily seen exploiting the lemma at p.10 of [3]. The same holds for the Darboux theorem proved in [2], which is based on ideas quite close to those presented here.*

Theorem 2.1 can be reformulated in terms of Poisson tensor. Indeed the norm $\|X\|_{\mathcal{F}_x}$ can obviously be defined by $\|X\|_{\mathcal{F}_x} := \left\|J_x^{-1} X\right\|_{\mathcal{P}^*}$, and this allows to obtain the following

Corollary 2.4. *Define $\mathcal{F}$ as the completion of $\mathcal{P}$ in the norm $\|X\|_{\mathcal{F}} := \left\|J_0^{-1} X\right\|_{\mathcal{P}^*}$, where $J_0 := J_x\big|_{x=0}$. Assume that there exists a neighbourhood of the origin where J_x extends to a bounded linear operator from $\mathcal{P}^*$ to $\mathcal{F}$ smoothly dependent on x, then there exists a chart about zero in which the symplectic form (and therefore the Poisson tensor) is constant.*

Remark 2.5. *The constancy of the spaces $\mathcal{F}_x$ is a property which is not completely chart invariant; indeed one can construct examples of change of coordinates which change the spaces $\mathcal{F}_x$ everywhere, but not at one single point.*

One can use the spaces $\mathcal{F}_x$ to give a coordinate independent necessary condition for Darboux theorem.

Proposition 2.6. *The existence of chart about 0 in which the symplectic form is constant implies that there exists a neighbourhood of 0 such that for each x in such a neighbourhood there exists an isomorphism between $\mathcal{F}_x$ and $\mathcal{F}_0$ which restricts to an isomorphism of $\mathcal{P}$ with itself.*

2.2 Proofs

The proof of theorem 2.1. will be obtained along the lines of [5] (which follows an idea by Moser). The original part consists of the proof of the smoothness of the vector field generating the transformation we are looking for.

I begin by two simple lemmas. Consider the map $\tilde{\Omega}^\flat_x : \mathcal{F}_x \to \mathcal{P}^*$ defined by $\langle \tilde{\Omega}^\flat_x V; W\rangle := \Omega_x(V,W)$; we have the following

Lemma 2.7. *$\tilde{\Omega}^\flat_x$ is an isomorphism between $\mathcal{F}_x$ and $\mathcal{P}^*$.*

Proof. It is immediate to remark that $\tilde{\Omega}^\flat_x$ is a linear *unitary* operator, so that it is invertible on its range, and its inverse is continuous. I prove now that it is surjective. Denote by $\mathcal{R}$ the range of $\tilde{\Omega}^\flat_x$ (which is close by the above consideration) and assume by contradiction that there exists a covector $\alpha \in \mathcal{P}^*$ such that $\alpha \notin \mathcal{R}$. By the Hahn–Banach theorem there exists a $X \in \mathcal{P}^{**}$ such that

$$X(\alpha) = 1 \tag{4}$$

and

$$X(\beta) = 0\ , \quad \forall \beta \in \mathcal{R}\ . \tag{5}$$

But $\mathcal{P}$ is reflexive, and therefore $X \in \mathcal{P}$, so (5) is equivalent to

$$\Omega_x(Y,X) = 0\ , \forall Y \in \mathcal{F}_x\ ,$$

which, by (1), implies $X = 0$ against (4). □

The above lemma is used to prove

Lemma 2.8. *The map $\Omega^\flat_x : \mathcal{P} \to \mathcal{F}^*_x$, defined by $X \mapsto \Omega_x(X,.)$, is an isomorphism.*

Proof. It is clear that the map is injective; I prove now that it is surjective by constructing its inverse. Consider the map that associates to any $\alpha \in \mathcal{F}^*_x$ the vector $X \in \mathcal{P}^{**}$ defined by

$$X(\beta) := -\alpha(J_x\beta)\ , \forall \beta \in \mathcal{P}^*$$

where $J_x := (\tilde{\Omega}^\flat_x)^{-1}$. But $\mathcal{P}$ is reflexive, so that $X \in \mathcal{P}$. It is easy to see that this map is the inverse of $\Omega^\flat_x$. □

In what follows we will not distinguish between $\Omega^\flat_x$ and $\tilde{\Omega}^\flat_x$.

Remark 2.9. *Lemmas 2.7 and 2.8 hold also for any nondegenerate bilinear form (skew–symmetry plays no role).*

Remark 2.10. *In the hypotheses of theorem 2.1 the map*

$$\begin{aligned} \mathcal{W} &\to L(\mathcal{P},\mathcal{F}^*) \\ x &\mapsto \Omega^\flat_x \end{aligned} \tag{6}$$

is smooth; here $L(\mathcal{P},\mathcal{F}^*)$ is the Banach space of linear operators from $\mathcal{P}$ to $\mathcal{F}^*$. Indeed it is the composition of the map $x \mapsto \Omega_x$ which is smooth as an application from $\mathcal{W}$ to the bilinear forms on $\mathcal{P}\times\mathcal{F}$ by hypothesis, and the linear map $^\flat : \Omega \mapsto \Omega^\flat$, which is analytic.

Proof of theorem 2.1. Denote $\bar{\Omega} := \Omega_0 - \Omega$ and $\Omega^t := \Omega + t\bar{\Omega}$, where $\Omega_0 := \Omega_x|_{x=0}$. By the Poincaré lemma we have

$$\bar{\Omega} = d\alpha\ , \quad \text{with } \alpha_x X := \int_0^1 s\bar{\Omega}_{sx}(x,X)ds\ . \tag{7}$$

We look for a smooth vector field $Y_t : \mathcal{P} \to \mathcal{P}$ such that

$$i_{Y_t}\Omega^t = -\alpha \ . \tag{8}$$

By (7) the one form α_x can be extended to an element of $\mathcal{F}^*$. Consider now

$$(\Omega^t)^\flat_x : \mathcal{P} \to \mathcal{F}^* \ . \tag{9}$$

When $x = 0$ one has $\Omega^t_x|_{x=0} \equiv \Omega_0$, and therefore at such point (9) is an isomorphism. By remark 2.10 $(\Omega^t_x)^\flat$ depends smoothly on x and, for x close to zero, it is an isomorphism. So, one can define

$$Y_t := ((\Omega^t_x)^\flat)^{-1}\alpha \ ,$$

and Y_t is a smooth, time dependent vector field taking values in $\mathcal{P}$.

Then the proof is easily completed following the scheme by Moser. So, let F_t be the flow generated by Y_t, then

$$\frac{d}{dt}F_t^*\Omega_t = F_t^*(\pounds_{Y_t}\Omega_t) + F_t^*\frac{d}{dt}\Omega_t = F_t^*(-d\alpha + \bar{\Omega}) = 0 \ ,$$

so $F_1^*\Omega_1 = \Omega$, and F_1^{-1} is the wanted change of variables. □

Proof of corollary 2.4. By lemma 2.7, J_0 is an isomorphism of $\mathcal{P}^*$ and $\mathcal{F}$, it follows (from the implicit function theorem) that in a neighbourhood of zero also J_x is an isomorphism of $\mathcal{P}^*$ and $\mathcal{F}$, and therefore the spaces $\mathcal{F}_x$ coincide with $\mathcal{F}$, moreover one clearly has that

$$\begin{aligned} \Omega_x : \mathcal{F} \times \mathcal{P} &\to \mathbb{R} \\ (X, Y) &\mapsto \langle J_x^{-1}X, Y\rangle \end{aligned}$$

depends smoothly on x as a bilinear form on $\mathcal{F} \times \mathcal{P}$, therefore theorem 2.1 gives the result. □

Proof of proposition 2.6. Assume that there exists a change of coordinates ψ about 0 such that $\psi^*\Omega = \Omega_0$. Then, for $X, Y \in \mathcal{P}$ one has

$$\Omega_x(\psi_{*x}X, \psi_{*x}Y) = \Omega_0(X, Y) \ ,$$

where we denoted by ψ_{*x} the tangent map to ψ at x. Denote by A_x the inverse of ψ_{*x}, then one has

$$\Omega_x(X, Y) = \Omega_0(A_xX, A_xY) \ ,$$

and therefore

$$\|A_xY\|_{\mathcal{F}_0} = \sup_{Z\neq 0}\frac{|\Omega_0(Z, A_xY)|}{\|Z\|_{\mathcal{P}}} = \sup_{X\neq 0}\frac{|\Omega_0(A_xX, A_xY)|}{\|A_xX\|_{\mathcal{P}}} = \sup_{X\neq 0}\frac{|\Omega_x(X, Y)|}{\|A_xX\|_{\mathcal{P}}} \ , \tag{10}$$

where $X = A_x^{-1}Z$. But, since A_x is an isomorphism, one has

$$\|A_xX\|_{\mathcal{P}} \geq \left\|A_x^{-1}\right\|^{-1}\|X\|_{\mathcal{P}} \ ,$$

from which, substituting in (10) and taking into account the definition of the norm $\mathcal{F}_x$

$$\|A_xY\|_{\mathcal{F}_0} \leq \left\|A_x^{-1}\right\|\|Y\|_{\mathcal{F}_x} \ .$$

It follows that A_x can be extended to a continuous operator from $\mathcal{F}_x$ to $\mathcal{F}_0$, moreover, the same holds for the operator A_x^{-1}, so A_x is the wonted isomorphism. □

3. Applications to symplectic submanifolds

Let M be a weakly symplectic manifold modelled on a reflexive Banach space; assume that at each point of M there exists a chart in which the symplectic form is constant. The following theorem will be proved

Theorem 3.1. *Let $N \subset M$ be a closed symplectic submanifold of M having finite codimension. Then at each point of N there exists a chart in which the symplectic form of N is constant.*

Before giving the proof of theorem 3.1 I give two lemmas on symplectic vector spaces which will be used below.

Let $(\mathcal{P}, \Omega)$ be a weakly symplectic vector space, with $\mathcal{P}$ a reflexive Banach space; as above denote by $\mathcal{F}$ the dual of $\mathcal{P}$ with respect to Ω. Given a closed subspace $V \subset \mathcal{P}$ define $\overline{V}^{\mathcal{F}}$ to be the closure of V in the topology of $\mathcal{F}$, and

$$V^{\perp} := \{X \in \mathcal{P} \ : \ \Omega(X,Y) = 0 \ \forall Y \in V\} \ ,$$
$$V^{\dagger} := \{X \in \mathcal{F} \ : \ \Omega(X,Y) = 0 \ \forall Y \in V\} \ ;$$

Remark that

$$\overline{V^{\perp}}^{\mathcal{F}} = V^{\dagger} \ . \tag{11}$$

If $W \subset \mathcal{F}$ is a closed subspace I will denote

$$W^{\perp} := \{X \in \mathcal{P} \ : \ \Omega(X,Y) = 0 \ \forall Y \in W\} \ ,$$

this is not ambiguous since (for $V \subset \mathcal{P}$) one has

$$V^{\perp} = \left(\overline{V}^{\mathcal{F}}\right)^{\perp} \tag{12}$$

Lemma 3.2. *Let $V \subset \mathcal{P}$ be a closed subspace, then one has*

$$V^{\perp\perp} = V \ , \quad \left(V^{\perp}\right)^{\dagger} = \overline{V}^{\mathcal{F}} \tag{13}$$

Proof. Let $X \notin V$ be a vector of $\mathcal{P}$, then by the Hahn–Banach theorem there exists $\alpha \in \mathcal{P}^*$ such that $\alpha(X) = 1$ and $\alpha|_V = 0$. Consider $J\alpha \in \mathcal{F}$: clearly one has $J\alpha \in V^{\dagger}$ and $\Omega(J\alpha, X) = 1$, so that

$$X \notin \left(V^{\dagger}\right)^{\perp} = V^{\perp\perp}$$

where the last equality follows from (11) and (12). This proves $X \in V \Longrightarrow X \in V^{\perp\perp}$; the inverse inclusion is trivial. To obtain the second of (13) remark that

$$\overline{V}^{\mathcal{F}} = \overline{V^{\perp\perp}}^{\mathcal{F}} = \left(V^{\perp}\right)^{\dagger} \ ,$$

where the second equality follows from (11). ☐

For a previous proof of $V^{\perp\perp} = V$ in weak symplectic spaces see [6].

Lemma 3.3. *Let $V \subset \mathcal{P}$ be closed and symplectic, then*

1. $$\mathcal{P} = V \oplus V^{\perp}$$
2. *Let π be the projection on V, then π extends to a continuous map from $\mathcal{F}$ to $\overline{V}^{\mathcal{F}}$.*
3. *There exists C such that $\forall X \in V$ one has*

$$\|X\|_{\mathcal{F}} \leq C \sup_{Y \in V\setminus\{0\}} \frac{|\Omega(X,Y)|}{\|Y\|} \ .$$

Proof. Clearly one has $V \cap V^{\perp} = \{0\}$ and $\overline{V}^{\mathcal{F}} \cap V^{\dagger} = \{0\}$. Define $\mathcal{P}_1 := V \oplus V^{\perp}$, and assume that it does not coincide with $\mathcal{P}$, then there exists $0 \neq X \in \mathcal{P}\setminus\mathcal{P}_1$. By the Hahn–Banach theorem there exists $\alpha \in \mathcal{P}^*$ such that $\alpha(X) = 1$ and $\alpha\big|_{\mathcal{P}_1} = 0$. Then, by lemma 3.2, one has

$$J\alpha \in V^{\dagger} \cap (V^{\perp})^{\dagger} = V^{\dagger} \cap \overline{V}^{\mathcal{F}} = \{0\} \ .$$

Since $\alpha \neq 0$, this is a contradiction. To prove 2 fix $X \in \mathcal{P}$; we have

$$\begin{aligned} \|\pi X\|_{\mathcal{F}} &= \sup_{Y \in \mathcal{P}\setminus\{0\}} \frac{|\Omega(\pi X, Y)|}{\|Y\|} = \sup_{Y \in \mathcal{P}\setminus\{0\}} \frac{|\Omega(\pi X, \pi Y)|}{\|Y\|} \\ &= \sup_{Y \in \mathcal{P}\setminus\{0\}} \frac{|\Omega(X, \pi Y)|}{\|Y\|} \leq \sup_{Y \in \mathcal{P}\setminus\{0\}} \frac{\|\pi Y\| \, \|X\|_{\mathcal{F}}}{\|Y\|} \leq C \|X\|_{\mathcal{F}} \ , \end{aligned}$$

where the last equality is just boundedness of π. I come to 3. Fix $Y \in V$, we have

$$\begin{aligned} \|Y\|_{\mathcal{F}} = \sup_{X \in \mathcal{P}\setminus\{0\}} \frac{|\Omega(X,Y)|}{\|X\|} &= \sup_{X \in \mathcal{P}\setminus\{0\}} \frac{|\Omega(\pi X, Y)|}{\|X\|} \\ &\leq \sup_{X \in \mathcal{P}\setminus\{0\}} \frac{|\Omega(\pi X, Y)|}{\frac{\|\pi X\|}{C}} \end{aligned}$$

where the last equality is due to the boundedness of π. From this 3 immediately follows. ☐

Proof of theorem 3.1 The result is local, so I consider only the case where $M = \mathcal{U} \subset \mathcal{P}$ is an open subset of $\mathcal{P}$ endowed by a constant symplectic form Ω. Moreover assume $0 \in \mathcal{U} \cap N$; I will construct a chart of N about 0 in which the assumptions of theorem 2.1 are satisfied. Denote by $\mathcal{F}$ the dual of $\mathcal{P}$ with respect to Ω, and $V := T_0 N$; then V is symplectic and therefore $\mathcal{P} = V \oplus V^{\perp}$; denote by $\pi : \mathcal{P} \supset \mathcal{U} \to V$ the projection on V. Clearly there exists a neighbourhood $\mathcal{W} \subset N$ of 0 where

$$\begin{aligned} \varphi : \mathcal{W} &\to V \\ x &\mapsto \varphi(x) := \pi x \end{aligned}$$

is a chart of N. I claim this is a chart in which the assumptions of theorem 2.1 are satisfied.

To start with remark that in this chart the expression Ω^N of the symplectic form of N is given by

$$\begin{aligned} \Omega^N_x &: V\times V\to \mathbb{R} \\ (X,Y) &\mapsto \Omega^N_x(X,Y) := \Omega\left(\varphi^{-1}_{*x}X, \varphi^{-1}_{*x}Y\right) \ ; \end{aligned}$$

I will show that φ_{*x} extends to an isomorphism of $\overline{T_xN}^{\mathcal{F}}$ and $\overline{V}^{\mathcal{F}}$; then the result will easily follow.

First remark that, from the linearity of π one has

$$\begin{aligned} \varphi_{*x} &: T_xN\to V \\ X &\mapsto \varphi_{*x}X=\pi X \ . \end{aligned}$$

I use this fact to construct more explicitly φ^{-1}_{*x}. Fix a basis $v_1(x),...,v_{2n}(x)$ of $(T_xN)^\perp$ smoothly dependent on x, define the one forms $\alpha^x_i := \Omega(v_i(x),.)$, and remark that they extend to elements of $\mathcal{F}^*$ smoothly dependent on x. Then define the linear operator

$$\begin{aligned} A_x &: \mathcal{P}\to V\oplus \mathbb{R}^{2n} \\ X &\mapsto (\pi X;\alpha^x_1X,...,\alpha^x_{2n}X) \ , \end{aligned} \tag{14}$$

clearly $A_x\big|_{T_xN}\equiv(\varphi_{*x},0)$. It is immediate to realize that A_0 is an isomorphism which extends to an isomorphism of $\mathcal{F}$ with $\overline{V}^{\mathcal{F}}\oplus\mathbb{R}^{2n}$. Moreover, A_x depends smoothly on x both as linear bounded operator from $\mathcal{P}$ to $V\oplus\mathbb{R}^{2n}$ and as bounded linear operator from $\mathcal{F}$ to $\overline{V}^{\mathcal{F}}\oplus\mathbb{R}^{2n}$. So there exists a neighbourhood $\mathcal{W}$ of $0\subset\mathcal{U}$ where it is an isomorphism. It follows that

$$\begin{aligned} \varphi^{-1}_{*x} &: V\to T_xN \\ X &\mapsto A^{-1}_x(X,0) \end{aligned}$$

extends to an isomorphism of $\overline{V}^{\mathcal{F}}$ with $\overline{T_xN}^{\mathcal{F}}$ smoothly dependent on x.

Denote now by F_x the dual of V with respect to Ω^N_x. By lemma 3.3 (assertion 3), one has $F_0=\overline{V}^{\mathcal{F}}$. I am going to prove that the same holds for x in a neighbourhood of 0. By the very definition of $\|.\|_{F_x}$ and of Ω^N_x, for $Y\in V$ one has

$$\|Y\|_{F_x}=\sup_{X\in T_xN\setminus\{0\}}\frac{|\Omega(X,\varphi^{-1}_{*x}Y)|}{\|\varphi_{*x}X\|} \ ,$$

but, since φ_{*x} is an isomorphism of T_xN and V there exist constants such that

$$\|Y\|_{F_x}\le C_1\left\|\varphi^{-1}_{*x}Y\right\|_{\mathcal{F}}\le C_2\left\|Y\right\|_{\mathcal{F}} \ ,$$
$$\|Y\|_{F_x}\ge C_3\sup_{X\in T_xN\setminus\{0\}}\frac{|\Omega(X,\varphi^{-1}_{*x}Y)|}{\|X\|}\ge C_4\left\|\varphi^{-1}_{*x}Y\right\|_{\mathcal{F}}\ge C_5\left\|Y\right\|_{\mathcal{F}} \ ,$$

where I used also the fact that φ_{*x} is an isomorphism from $\overline{V}^{\mathcal{F}}$ and $\overline{T_xN}^{\mathcal{F}}$, and lemma 3.3.

Finally one has to prove that Ω^N_x depends smoothly on x as a bilinear form on $V\times F\simeq V\times\overline{V}^{\mathcal{F}}$. But this is a trivial consequence of the smoothness of A_x^{-1} both as a map from $V\oplus\mathbb{R}^{2n}$ to $\mathcal{P}$ and as a map from $\overline{V}^{\mathcal{F}}\oplus\mathbb{R}^{2n}$ to $\mathcal{F}$, and of the definition of Ω^N. $\square$

Remark 3.4. *I think that theorem 3.1 should hold also for ∞-codimensional symplectic submanifolds of M, however, I was not able to prove that $(T_xN)^\perp$ depends smoothly on x, and therefore I did not succeeded in defining an operator of the kind of the operator (14)*

I come to symplectic manifolds obtained by Marsden–Weinstein reduction procedure.

Fix again M as above, and let G be a finite dimensional Lie group. Let $\Phi : M\times G\to M$ be a smooth symplectic action of G on M admitting an Ad^*-equivariant momentum map $I : M\to g^*$. Let $\mu\in g^*$ be a regular value of I, and denote by G_μ the isotropy group of μ, namely

$$G_\mu := \left\{g\in G \ : \ Ad^*_g\mu=\mu\right\} \ .$$

Assume G_μ acts properly and freely on $I^{-1}(\mu)$, and denote

$$M_\mu := \frac{I^{-1}(\mu)}{G_\mu} \ ,$$

then it is well known that M_μ is a symplectic manifold.

Corollary 3.5. *About each point of M_μ there exists a chart in which the symplectic form is constant.*

Proof. Let $z\in M_\mu$, then $z=[x]$ for some $x\in I^{-1}(\mu)$; denote by $G_\mu x$ the orbit of G_μ through x. Let $P\subset M$ be a closed submanifold of M passing through x such that $T_xM\equiv T_xP\oplus T_x(G_\mu x)$. Then it is well known that M_μ is locally isomorphic, as a symplectic manifold, to a neighbourhood of x in $P\cap I^{-1}(\mu)$. But $P\cap I^{-1}(\mu)$ is a finite codimension submanifold of M, and therefore theorem 3.1 gives the result. $\square$

4. A counterexample to Darboux theorem (following Marsden)

Let Q be a Banach manifold modelled on a reflexive Banach space $\mathcal{B}$, then it is well known that the canonical symplectic structure of T^*Q is strong [1]. Suppose that Q is endowed by a metric, then it is possible to pull back the symplectic form of T^*Q to TQ. If the metric of Q is only weakly nondegenerate (i.e. $g_x : T_xQ\times T_xQ\to\mathbb{R}$ is weakly nondegenerate as a bilinear form) then the symplectic form on TQ is only weakly nondegenerate. Marsden idea is to define a symplectic form Ω on TQ using a metric, which for a given $x_0\in Q$ is only weakly nondegenerate, while it is strongly nondegenerate at the points x_n of a sequence having x_0 as an accumulation point. It clearly turns out that $\forall X\in T_{x_0}Q$ the form Ω_X is weakly nondegenerate, and therefore $\mathcal{F}_X$ strictly contains $T_XTQ\simeq\mathcal{B}\times\mathcal{B}$, while for any

$Y \in T_{x_n}Q$ the form Ω_Y is strongly nondegenerate, and therefore $\mathcal{F}_Y \equiv T_Y TQ \simeq \mathcal{B} \times \mathcal{B}$. This violates the necessary condition 2.6, and therefore Darboux theorem fails.

A concrete example of this kind can be obtained choosing Q to be the Sobolev space $H^1(\mathbf{T}^1)$ of L^2 functions on $[0, \pi]$ having L^2 weak derivatives, and fulfilling periodic boundary conditions; the weak metric can be defined by

$$g_u(X, Y) := \int_{\mathbf{T}^1} X(x)Y(x)dx + \|u\|^2_{L^2} \int_{\mathbf{T}^1} X_x(x)Y_x(x)dx \ , \tag{15}$$

which at $u = 0$ is weak (it is just the L^2 pairing) while for $u \neq 0$ it is strong.

A simple calculation shows that, denoting (as usual) by H^{-1} the dual of $H^1(\mathbf{T}^1)$ with respect to the L^2 pairing, the map

$$\Omega^\flat_{(u,\dot u)} : H^1(\mathbf{T}^1) \times H^1(\mathbf{T}^1) \to H^{-1} \times H^{-1}$$

corresponding to the so obtained symplectic form is given by

$$\Omega^\flat_{(u,\dot u)} \begin{pmatrix} V \\ \dot V \end{pmatrix} = \begin{pmatrix} - \|u\|^2_{L^2} \dot V_{xx} + \dot V - \dot u_{xx} 2\langle u, V\rangle_{L^2} + 2\langle V, \dot u_{xx}\rangle u \\ \|u\|^2_{L^2} V_{xx} - V \end{pmatrix} \ , \tag{16}$$

which is an isomorphism for $u \neq 0$.

Remark 4.1. *Choosing $Q = H^s(\mathbf{T}^1)$ $(s > 1)$ endowed by the metric (15) one obtains a symplectic form which is nowhere strong, however exploiting (16) it is easy to show that also in this case Darboux theorem fails. Indeed it is clear that the reason why Darboux theorem is violated is that such a map is a differential operator whose degree change at $u = 0$.*

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