Temperature Corrections to Conformal Field Theory

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We consider finite temperature dynamical correlation functions in the interacting delta-function Bose gas. In the low-temperature limit the asymptotic behaviour of correlation functions can be determined from conformal field theory. In the present work we determine the deviations from conformal behaviour at low temperatures.

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I. INTRODUCTION

The calculation of finite temperature correlation functions is a long-standing problem in the theory of integrable models. Apart from the obvious conceptual importance of the problem there are many direct applications of the results to experiments on quasi-1D materials like $KCuF_3$ [1] or CuBenz [2], which are described by integrable models. Finite temperature dynamical correlation functions in the systems have been measured by Neutron scattering and NMR and it is highly desirable to develop a method to calculate them exactly.

Important progress was made during the eighties when it was realized that in gapless models Conformal Field Theory can be used to obtain the low-temperature asymptotics of dynamical correlation functions (see *e.g.* [3]). However, models with a spectral gap as well as higher temperatures in gapless models remain outside the scope of the conformal approach.

In a remarkable further development it became possible to determine the behaviour of *static* correlators at finite temperatures through an ingenious mapping of integrable d-dimensional quantum theories to d+1-dimensional integrable classical statistical models [4]. However, dynamical correlation functions cannot presently be calculated by this approach.

For integrable models with free fermionic spectra powerful methods to calculate finite temperature dynamical correlation functions have been available for some time [5,6]. Very recently the method of [5] was successfully extended to cases corresponding to *interacting* fermions [7,8]. In particular, in [8] a formula describing the exponential decay of correlations in the delta-function Bose gas at finite temperatures was presented. Said formula is implicit in the sense that it is written in terms of solutions of certain nonlinear integral equations. The purpose of the present work is to analyze these integral equations by both analytical and numerical methods and present explicit expressions for the correlation lengths describing the decay of correlations.

The outline of the paper is as follows. In section 1 we review some relevant facts on the delta-function Bose gas. In section 2 we study the special case of impenetrable bosons, in which particularly simple expressions for the correlation lengths are obtained at low temperatures. In section 3 we consider the general case of interacting bosons and we conclude in section 4.

II. REVIEW OF THE δ-FUNCTION BOSE GAS

The δ-function Bose gas is one of the paradigms of exactly solvable strongly correlated many-body problems in one spatial dimension. It describes N bosons interacting *via* a repulsive δ-function potential of strength $c > 0$. The Hamiltonian is

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{N \geq j > k \geq 1} \delta(x_j - x_k). \tag{1}$$

At the special value $c = \infty$ the model describes noninteracting hard-core bosons and essential simplifications occur in the exact solution. We refer to this case as "impenetrable bosons". The second-quantized form of the model is known as the Quantum Nonlinear Schrödinger equation and the Hamiltonian is expressed in terms of the canonical Bose field $\psi(x)$ as

$$H=\int_{-\infty}^{\infty} dx\,\left(\partial_x\psi^\dagger(x)\partial_x\psi(x)+c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)\right). \tag{2}$$

The model is solvable by Bethe Ansatz [9] and in what follows we recall some important ingredients of the exact solution.

- **Ground State and Excitations**

  In momentum space a Pauli principle holds [10] and consequently the zero temperature ground state is given by a filled Fermi sea of negative energy pseudoparticles. At $c=\infty$ this is simply the usual free-fermion ground state. The physics of the model is most conveniently described in terms of a rapidity variable λ, which is related to the momentum k by

$$k(\lambda)=\lambda+\int_{-q}^{q}\theta(\lambda-\mu)\rho_t^0(\mu)\,d\mu\ , \tag{3}$$

  where $\theta(\lambda)=i\ln\left(\frac{ic+\lambda}{ic-\lambda}\right)$. The density $\rho_t^0(\lambda)$ of pseudoparticles in the ground state is described by the integral equation

$$\rho_t^0(\lambda)-\frac{1}{2\pi}\int_{-q}^{q}K(\lambda,\mu)\rho_t^0(\mu)\,d\mu=\frac{1}{2\pi}. \tag{4}$$

  Here q is the rapidity corresponding to the Fermi momentum k_F and

$$K(\lambda,\mu)=\frac{2c}{c^2+(\lambda-\mu)^2}. \tag{5}$$

  Excitations over the ground state can be either particles or holes. The particle energy as a function of the rapidity is given by

$$\varepsilon^0(\lambda)-\frac{1}{2\pi}\int_{-q}^{q}K(\lambda,\mu)\varepsilon^0(\mu)\,d\mu=\lambda^2-h\ , \tag{6}$$

  where $h>0$ is the chemical potential. We note that the excitation energy vanishes on the Fermi surface $\varepsilon^0(\pm q)=0$. The dependence of the integration boundary q on the chemical potential h is determined from the condition

$$\varepsilon^0(\pm q)=0\ . \tag{7}$$

  In the impenetrable case the excitation spectrum is given in terms of free-fermionic particle-hole excitations. The Fermi velocity is defined as usual to be

$$v_F=\left.\frac{\partial\varepsilon(\lambda)}{\partial k(\lambda)}\right|_{\lambda=q}=\frac{\varepsilon^{0\prime}(q)}{k'(q)}=\frac{\varepsilon^{0\prime}(q)}{2\pi\rho_t^0(q)} \tag{8}$$

  where the prime denotes differentiation with respect to λ. The derivative of momentum with respect to rapidity on the Fermi surface is called dressed charge and is related to the density of states on the Fermi surface

$$Z=2\pi\rho_t^0(q). \tag{9}$$

- **Asymptotics of Correlation Functions**

  The asymptotic behaviour of correlation functions at very low temperatures can be determined from the exact finite-size spectrum by conformal field theory techniques [5,11]. The result is found to be

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T\underset{\substack{x\to\infty\\ t\to\infty}}{\longrightarrow}\ \exp\left\{-\frac{2\Delta^+\pi T}{v_F}|x-v_Ft|-\frac{2\Delta^-\pi T}{v_F}|x+v_Ft|\right\}. \tag{10}$$

The conformal dimensions $\Delta^{\pm}$ are related to the dressed charge by

$$2\Delta^{+} = 2\Delta^{-} = \frac{1}{4Z^2}. \tag{11}$$

In the framework of the Luttinger liquid approach these results were obtained by F. D. M. Haldane in [12].

In [8] the following formula describing the exponential decay of correlations at *any* temperature was derived from the determinant approach to quantum correlation functions [5]

$$\langle \psi(0,0)\psi^{\dagger}(x,t)\rangle_T \underset{\substack{x\to\infty \\ t\to\infty}}{\longrightarrow} \exp\left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi\rho_t(\lambda)} |x - v(\lambda)t| \ln\left| \frac{e^{\frac{\varepsilon(\lambda)}{T}} - 1}{e^{\frac{\varepsilon(\lambda)}{T}} + 1} \right| \right\} =: \exp(\chi) \ . \tag{12}$$

The functions $\varepsilon(\lambda)$, $\rho_t(\lambda)$ and $v(\lambda)$ are the finite-temperature equivalents of the dressed energy, pseudoparticle density and Fermi velocity defined above. They are solutions of the following integral equations [13]

$$\begin{aligned} \varepsilon(\lambda) &= \lambda^2 - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} d\mu\ K(\lambda,\mu) \ln\left(1 + e^{-\frac{\varepsilon(\mu)}{T}}\right) \ , \\ \rho_t(\lambda) &= \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu\ K(\lambda,\mu)\ \frac{1}{1 + e^{\frac{\varepsilon(\mu)}{T}}}\ \rho_t(\mu) \ . \end{aligned} \tag{13}$$

The velocity $v(\lambda)$ is given by

$$v(\lambda) = \frac{1}{2\pi\rho_t(\lambda)} \frac{\partial \varepsilon(\lambda)}{\partial \lambda} \ . \tag{14}$$

Note that for $T \to 0$ these equations reduce to (6), (4) and (8) respectively.

III. IMPENETRABLE BOSONS

We first investigate the simpler case of impenetrable bosons. In the limit $c \to \infty$ (12) simplifies to [5]

$$\langle \psi(0,0)\psi^{\dagger}(x,t)\rangle_T \underset{\substack{x\to\infty \\ t\to\infty}}{\longrightarrow} \exp\left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\ |x - 2\lambda t|\ \ln\left| \frac{e^{\frac{\lambda^2 - h}{T}} - 1}{e^{\frac{\lambda^2 - h}{T}} + 1} \right| \right\}, \tag{15}$$

and we are left with performing a single integral. Expanding the exponentially decaying part of the integrand as

$$\ln\left| \frac{e^{\frac{\lambda^2 - h}{T}} - 1}{e^{\frac{\lambda^2 - h}{T}} + 1} \right| = -2\sum_{n=1}^{\infty} \frac{1}{2n-1} \exp\left(- \left| \frac{\lambda^2 - h}{T} \right| (2n-1) \right) \tag{16}$$

we obtain an expansion of (15) in terms of error functions and exponential functions, which in turn yield an *asymptotic* low-temperature series on χ in powers of T. The Fermi velocity is $v_F = 2\sqrt{h}$, and we distinguish two cases.

- Space-like region $x/t > v_F$:

  After some elementary calculations we find the following expansion for χ (as defined in (12))

$$\chi \sim -\frac{4xT}{\pi v_F} \sum_{m=0}^{\infty} \frac{(4m)!}{(2m)!} \left(\frac{T}{4h}\right)^{2m} \left(1 - 2^{-2-2m}\right) \zeta(2m+2) \ , \tag{17}$$

  where $\zeta(x)$ is the Riemann zeta-function. Note that this expansion breaks down as we approach $x/t \to v_F$. The first few terms are

$$\chi = -\frac{xT\pi}{2v_F} \left[1 + \left(\frac{\pi T}{v_F^2}\right)^2 + 14\left(\frac{\pi T}{v_F^2}\right)^4 + 5049\left(\frac{\pi T}{v_F^2}\right)^6 + \mathcal{O}(T^8) \right] \ . \tag{18}$$

The inverse correlation length thus has an asymptotic power series expansion in *odd* powers of temperature for $T \to 0$. We also see that in the space-like regime there is no exponential decay with respect to the time t at low temperatures as the t-dependence only enters in terms such as $\exp(\text{const}/T)$ which are smaller than any power of the temperature T.

- Time-like region $x/t < v_F$:

  In the time-like regime we find that there are no power-law corrections to the conformal result

$$\chi = -\frac{\pi t T}{2} - \frac{2tT}{\pi} \exp\left(-[h - x^2/4t^2]/T\right) + \ldots \tag{19}$$

  Thus, up to exponentially small corrections the conformal result is exact in the time-like regime and there is exponential decay of correlations with respect to t only at small temperatures.

For general finite temperatures one needs to resort to numerical integration of (15). If we define a correlation length $\xi(T, h, \frac{t}{x})$ by

$$\chi = -\Theta(x - v_F t)\frac{x}{\xi(T, h, \frac{t}{x})} - \Theta(v_F t - x)\frac{v_F t}{\xi(T, h, \frac{t}{x})} , \tag{20}$$

where $\Theta(x)$ is the Heaviside theta-function, we can study the dependence of ξ on T and the "direction" t/x by determining χ numerically. From the above low-temperature analysis we already know that at very low temperatures ξ becomes independent of t/x. In Fig 1 we plot the correlation length ξ as a function of temperature for various values of t/x for the special values of chemical potential $h = 1$.

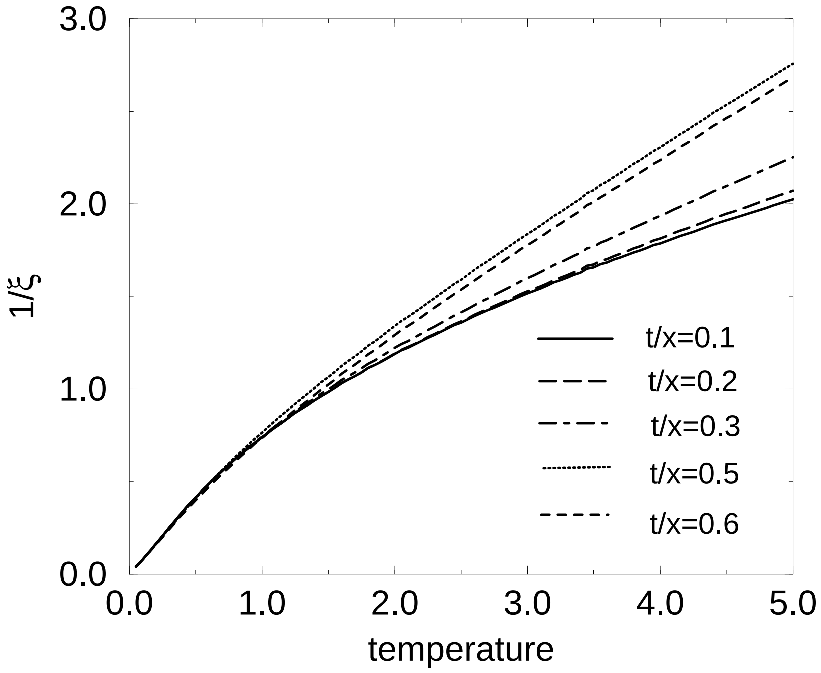

FIG. 1. Inverse correlation length as a function of temperature for impenetrable bosons at $h = 1$.

We clearly see the deviations from linear-T behaviour as temperature increases. The dependence on t/x becomes more pronounced at higher temperatures. In the regions $tv_F \ll x$ and $x \ll v_F t$ the dependence on t/x disappears as expected. We also see that the exponential decay of correlations is fastest in the direction of the "light-cone" $x/t = v_F$.

IV. INTERACTING BOSONS

We now consider the case of finite coupling constant $c < \infty$. In order to derive an analytical low-temperature expansion of the correlation length we first need to perform an (asymptotic) expansion of $\varepsilon(\lambda)$, $v(\lambda)$ and $\rho_t(\lambda)$ in powers of T.

A. Low temperature expansion of integral equations

The expansion of $\varepsilon(\lambda)$ in powers of T is readily established along the lines of [14]. We first note that the function $\varepsilon(\lambda)$ has precisely one zero for $\lambda > 0$ [5]. We denote the corresponding value of the spectral parameter by q_T. As $\varepsilon(\lambda)$ is a symmetric function this then implies that $\varepsilon(\pm q_T) = 0$. From (13) and (6) we find

$$\begin{aligned}
&\varepsilon(\lambda) - \varepsilon^0(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} d\mu\ K(\lambda,\mu)\ \left[\varepsilon(\mu) - \varepsilon^0(\mu)\right] \\
&\quad = -\frac{T}{2\pi}\int_{-\infty}^{\infty} d\mu\ K(\lambda,\mu)\ \ln\left(1 + \exp(-\left|\frac{\varepsilon(\mu)}{T}\right|)\right) - \frac{T}{2\pi}\int_{q}^{q_T} + \int_{-q_T}^{-q} d\mu\ K(\lambda,\mu)\ \ln\frac{1+\exp(-\frac{\varepsilon(\mu)}{T})}{1+\exp(-|\frac{\varepsilon(\mu)}{T}|)}\ . \qquad (21)
\end{aligned}$$

The last two terms in (21) are easily evaluated by expanding the integrand around $\pm q_T$. The leading contribution to the first term in (21) also comes from the regions $\mu = \pm q_T$ and we again expand the integrand around these points and perform the resulting integrals. This gives

$$\begin{aligned}
&\varepsilon(\lambda) - \varepsilon^0(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} d\mu\ K(\lambda,\mu)\ \left[\varepsilon(\mu) - \varepsilon^0(\mu)\right] \\
&\quad = -\frac{\pi}{12\varepsilon'(q)}\left[K(\lambda,q) + K(-\lambda,q)\right]T^2 + \frac{\varepsilon'(q_T)}{4\pi}\left[K(\lambda,q_T) + K(-\lambda,q_T)\right](q_T - q)^2 + \mathcal{O}(T^3)\ . \qquad (22)
\end{aligned}$$

Setting $\lambda = q$ in (21) we find that $q - q_T = \mathcal{O}(T^2)$ so that the last term in (21) does not contribute to $\mathcal{O}(T^2)$. Putting everything together we then have

$$\begin{aligned}
\varepsilon(\lambda) &= \varepsilon^0(\lambda) - \frac{\pi}{12{\varepsilon^0}'(q)}T^2 u(\lambda) + \mathcal{O}(T^3)\ , \\
u(\lambda) &= K(\lambda,q) + K(-\lambda,q) + \frac{1}{2\pi}\int_{-q}^{q} d\mu\ K(\lambda,\mu)\ u(\mu)\ . \qquad (23)
\end{aligned}$$

It immediately follows that

$$q_T = q + \frac{\pi}{12}\frac{u(q)}{[{\varepsilon^0}'(q)]^2}T^2 + \mathcal{O}(T^3)\ . \qquad (24)$$

The finite-temperature corrections to the pseudoparticle density $T^2\Delta\rho(\mu) = \rho_t(\mu) - \rho_t^0(\mu)$ can be determined in a similar way

$$\Delta\rho(\lambda) = \frac{\pi}{12}\frac{\rho_t^0(q)}{[{\varepsilon^0}'(q)]^2}\left\{\left[\frac{u(q)}{2\pi} - \frac{{\varepsilon^0}''(q)}{{\varepsilon^0}'(q)} + \frac{{\rho_t^0}'(q)}{\rho_t^0(q)}\right]u(\lambda) - w(\lambda)\right\} + \mathcal{O}(T)\ , \qquad (25)$$

with

$$w(\lambda) = K'(\lambda,q) + K'(-\lambda,q) + \frac{1}{2\pi}\int_{-q}^{q} d\mu\ K(\lambda,\mu)\ w(\mu)\ , \qquad (26)$$

where $K'(\lambda,q) = dK(\lambda,q)/d\lambda$. Finally, the low-temperature expansion for $v(\lambda)$ is found to be

$$v(\lambda) = \frac{{\varepsilon^0}'(\lambda)}{2\pi\rho_t^0(\lambda)} - \frac{1}{24{\varepsilon^0}'(q)}\frac{u'(\lambda)}{\rho_t^0(\lambda)}\ T^2 - \frac{{\varepsilon^0}'(\lambda)\Delta\rho(\lambda)}{2\pi[\rho_t^0(\lambda)]^2}T^2 + \mathcal{O}(T^3)\ . \qquad (27)$$

B. Asymptotics of the correlator at small temperatures

Having obtained the leading corrections (in temperature) of $\varepsilon(\lambda)$, $\rho_t(\lambda)$ and $v(\lambda)$ we are now in a position to determine the leading temperature correction to the conformal result (10) for general values of the coupling c. In order to do so we need to expand the integral in (12), which is of the form

$$\chi = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\ f(\lambda) \ln\left|\frac{e^{\frac{\varepsilon(\lambda)}{T}}-1}{e^{\frac{\varepsilon(\lambda)}{T}}+1}\right| , \tag{28}$$

where $f(\lambda) = \frac{|x-v(\lambda)t|}{2\pi\rho_t(\lambda)}$. The leading contributions to $I(T)$ come from the vicinity of the points $\pm q_T$, where $\varepsilon(\lambda)$ vanishes. Expanding $f(\lambda)$ in powers of T, using (23), (25) and (27), and integrating around $\pm q_T$ we find

$$\chi = AT + BT^3 + \mathcal{O}(T^4) . \tag{29}$$

We find that A reproduces the "conformal" result (10). There is no correction to order T^2. B is a complicated expression as it reflects the operator content of the theory so that we give an explicit expression only in the case $x \gg v_F t$.

$$\begin{aligned} B = &-\frac{\pi^3 Z' \varepsilon^{0''}(q)}{8Z^2(\varepsilon^{0'}(q))^4} - \frac{\pi^3 Z'^2}{24Z^3(\varepsilon^{0'}(q))^3} + \frac{\pi^3 Z''}{48Z^2(\varepsilon^{0'}(q))^3} - \frac{\pi^3(\varepsilon^{0''}(q))^2}{8Z(\varepsilon^{0'}(q))^5} \\ &+ \frac{\pi^2 u(q)\varepsilon^{0''}(q)}{24Z(\varepsilon^{0'}(q))^4} - \frac{\pi^2 u'(q)}{24Z(\varepsilon^{0'}(q))^3} + \frac{\pi^2 u(q) Z'}{24Z^2(\varepsilon^{0'}(q))^3} + \frac{\pi^3 \varepsilon^{0'''}(q)}{24Z\varepsilon^{0'}(q)} + \frac{\pi^2 \Delta\rho(q)}{Z^2\varepsilon^{0'}(q)} , \end{aligned} \tag{30}$$

where $Z' = 2\pi\rho_t^{0'}(q)$ and $Z'' = 2\pi\rho_t^{0''}(q)$.

At higher temperatures we again resort to a numerical solution of the relevant integral equations and integrals. In Fig. 2 we plot the inverse correlation length (20) as a function of temperature for $c = 1$ and $h = 1$.

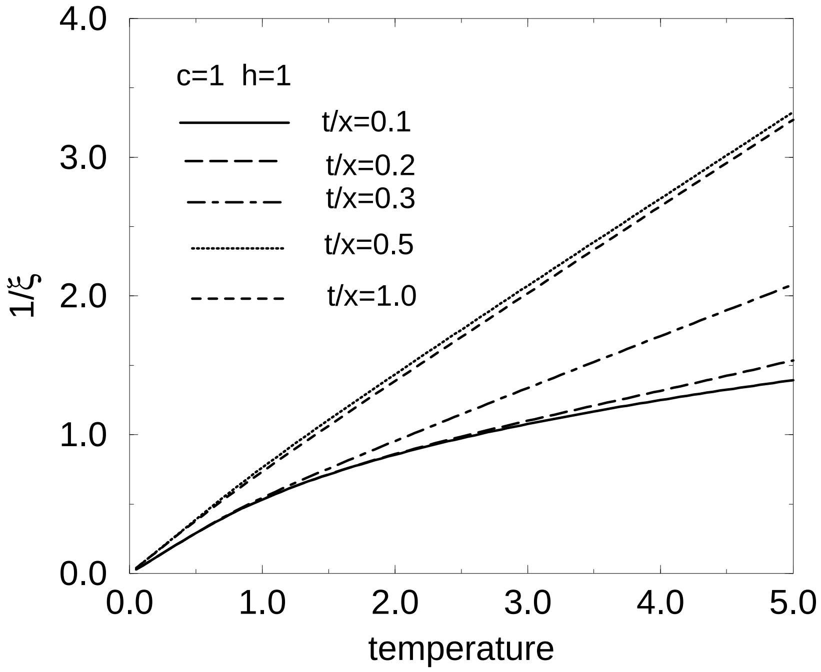

FIG. 2. Inverse correlation length as a function of temperature at $c = 1$, $h = 1$.

We obtain a qualitatively similar picture to the impenetrable case. Once again the exponential decay of correlations is fastest in the direction $x = v_F t$. However, the dependence on t/x is more pronounced than in the impenetrable case. This particularly clear in the space-like regime.

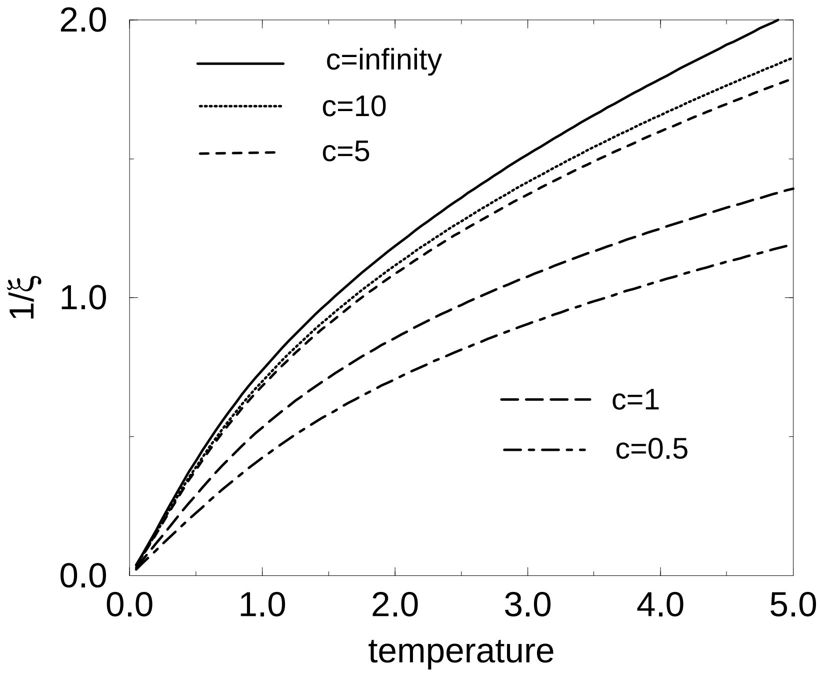

FIG. 3. Inverse correlation length for several values of c as a function of temperature at $h = 1$.

V. CONCLUSIONS

We have studied the rate of exponential decay of finite-temperature dynamical correlation functions of local fields in the δ-function Bose gas. We have explicitly evaluated corrections to conformal behaviour at low temperatures. For impenetrable bosons the results are qualitatively the same, and in addition we are able to get all the terms in an asymptotic series for the low-temperature behaviour of the rate of decay.

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