On the existence of additional solutions of the equations in the $(1/2,0)\oplus(0,1/2)$ representation space*

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We analyze dispersion relations of the equations recently proposed by Ahluwalia for describing neutrino. Equations for type-II spinors are deduced on the basis of the Wigner rules for left- and right- 2-spinors and the Ryder-Burgard relation. It is shown that equations contain acausal solutions which are similar to those of the Dirac-like second-order equation. The latter is obtained in a similar way, provided

that we do not apply to any constraints in the calculation process.

Recently, Ahluwalia proposed a new wave equation for describing self/anti-self charge conjugate states $\lambda^{S,A}(p^\mu)$ of any spin [1]:

$$\mathcal{D}\lambda(p^\mu) = \begin{pmatrix} -\mathbb{1} & \zeta_\lambda \exp(\mathbf{J}\cdot\boldsymbol{\varphi})\,\Theta_{[j]}\,\Xi_{[j]}\exp(\mathbf{J}\cdot\boldsymbol{\varphi})\\ \zeta_\lambda\exp(-\mathbf{J}\cdot\boldsymbol{\varphi})\,\Xi^{-1}_{[j]}\,\Theta_{[j]}\exp(-\mathbf{J}\cdot\boldsymbol{\varphi}) & -\mathbb{1}\end{pmatrix}\lambda(p^\mu) = 0\,. \tag{1}$$

Analogous equations for $\rho^{S,A}(p^\mu)$ bispinors have been derived in ref. [2d]. In the $j=1/2$ case spin matrices $\mathbf{J}$ are chosen to be the Pauli matrices $\boldsymbol{\sigma}/2$; in the $j=1$ case, the Barut-Muzinich-Williams matrices; $\boldsymbol{\varphi}$ are the parameters of the Lorentz boost. The notation coincides with that of refs. [1,2]. $\Theta_{[j]}$ is the Wigner's operator defined as $(\Theta_{[j]})_{\sigma,\sigma'} = (-1)^{j+\sigma}\delta_{\sigma',-\sigma}$ with σ and σ' as eigenvalues of $\mathbf{J}$. $\Xi_{[j]}$ is a $(2j+1)\times(2j+1)$ matrix which connects the "$2j+1$ spinor" and its complex conjugate such that $[\phi_L^\sigma(\mathring{p}^\mu)]^* = \Xi_{[j]}\phi^\sigma_L(\mathring{p}^\mu)$.

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And ζ_λ are phase factors which are fixed by conditions of self/anti-self charge conjugacy imposed on the type-II spinors λ.

While formally the $j = 1/2$ equation "may be put in the form $(\Gamma^{\mu\nu} p_\mu p_\nu + m\Gamma^\mu p_\mu - 2m^2 \mathbb{1})\lambda(p^\mu) = 0$ ... it turns out that $\Gamma^{\mu\nu}$ and Γ^μ do not transform as Poincaré tensors." Other forms of neutrino equations have

been presented in refs. [3,2,4] and gauge interactions have been introduced there.[1] These constructs give alternative insights in neutrino dynamics, which is different from that based on the common-used Weyl massless equations. Indications that neutrino may not be a Dirac particle and may have different dynamical features have appeared

in analyses of the present experimental situation [5]. Earlier considerations of this problem can be found in refs. [6–9].

Both the equations (1) and the equations of ref. [2,4,10] have been obtained by using different forms of the Ryder-Burgard relation [11,12,1,2,4,10] which connects zero-momentum $(0, j)$ left- and $(j, 0)$ right- spinors, and the Wigner rules for their transformations to the frame with the momentum $\mathbf{p}$. The Dirac equation may also be obtained in such a way [1, footnote # 1]. The detailed discussion of this techniques can be found in [13]. It was claimed in ref. [1] that $\lambda^S(p^\mu)$ spinors answer for "*positive* energy solutions, ...[meanwhile], $\lambda^A(p^\mu)$ are the *negative* energy solutions". We, in fact, used this interpretation in [2]. Let us now check by straightforward calculations, which dispersion relations has the equation (1) in the case of $j = 1/2$? Rewriting it to the form (31) of

ref. [1] yields the equation of the second order in p_0 and the matrix in the left side has the dimension four. So, one should have eight solutions. The system for analytical calculation MATEMATICA 2.2 enables us to found that the determinant of the matrix $\mathcal{D}$ is

equal to

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[1]The question of equivalence of these equations still deserves further elaboration and this paper presents a certain part of this analysis.

$$\mathrm{Det}\,[\mathcal{D}] = \left(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2\right)^2 + 4\mathrm{mp}_0)^2 16m^4(p_0+m)^4 \quad . \tag{2}$$

As a result of equating the determinant to zero we deduce that, indeed, the equation (1) has eight solutions in total with

$$p_0 = \pm\sqrt{\mathbf{p}^{\,2} + m^2} \quad , \tag{3}$$

each two times; and with the acausal dispersion relations:

$$p_0 = -2m \mp \sqrt{\mathbf{p}^{\,2} + m^2} \quad , \tag{4}$$

each two times.

We come across the same situation when deriving the Dirac equation by the Ryder-Burgard-Ahluwalia technique provided that we do *not* apply to the constraint $p_0^2 - \mathbf{p}^{\,2} = m^2$ from the beginning. Indeed, one has

$$\Lambda_{_R}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\Lambda_{_L}^{-1}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu) = \frac{p_0^2 + 2mp_0 + \mathbf{p}^2 + m^2 + 2(p_0+m)(\boldsymbol{\sigma}\cdot\mathbf{p})}{2m(p_0+m)} \quad , \tag{5a}$$

$$\Lambda_{_L}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu)\Lambda_{_R}^{-1}(p^\mu \leftarrow \overset{\circ}{p}{}^\mu) = \frac{p_0^2 + 2mp_0 + \mathbf{p}^2 + m^2 - 2(p_0+m)(\boldsymbol{\sigma}\cdot\mathbf{p})}{2m(p_0+m)} \quad . \tag{5b}$$

Thus, the second-order momentum-representation "Dirac" equation can be written:

$$\frac{1}{2m(p_0+m)}\left[(\gamma^\mu p_\mu \mp m)\gamma^0 + 2m\right](\gamma^\nu p_\nu \mp m)\Psi_\pm(p^\mu) = 0 \quad , \tag{6}$$

or

$$\frac{1}{2m(p_0+m)}(\gamma^\nu p_\nu \mp m)\left[\gamma^0(\gamma^\mu p_\mu \mp m) + 2m\right]\Psi_\pm(p^\mu) = 0 \quad . \tag{7}$$

The corresponding coordinate-representation of these equations ($m \neq 0$ and $p_0 \neq -m$) is

$$\left[(i\gamma^\mu\partial_\mu - m)\gamma^0 + 2\wp_{u,v}m\right](i\gamma^\nu\partial_\nu - m)\Psi(x^\mu) = 0 \quad , \tag{8}$$

or

$$(i\gamma^\mu\partial_\mu - m)\left[\gamma^0(i\gamma^\mu\partial_\mu - m) + 2\wp_{u,v}m\right]\Psi(x^\mu) = 0 \quad , \tag{9}$$

where $\wp_{u,v} = \pm 1$ depending on what solutions, with either positive or negative energies, are considered.

Can the equation (1) be put in a more convenient form? The eight-component form, we proposed recently [2d,Eqs.(17,18)], does not have acausal solutions. In the process of its deriving we assumed certain relations[2] between $\lambda^{S,A}(p^\mu)$ and $\rho^{S,A}(p^\mu)$. In the present article we are not going to apply them. Following the procedure of deriving the equations (6,7) one can arrive at a rather complicated equation:

$$\frac{1}{4m(p_0+m)}\left\{(\gamma^\mu p_\mu + m\gamma^0)\left[\mathcal{S}(\gamma^\nu p_\nu + m\gamma^0) - 2m\gamma^0\right] + \right. \\ \left. + \left[(\gamma^\mu p_\mu + m\gamma^0)\mathcal{S} - 2m\gamma^0\right](\gamma^\nu p_\nu + m\gamma^0)\right\}\lambda^{S,A}(p^\mu) = 0 \quad , \tag{10}$$

where

But, as mentioned in [1], one may consider that ϕ, which presents in the generalized Ryder-Burgard relation (see Eq. (27) of ref. [1] or Eq. (38) in [2c]), is the azimuthal angle *associated with* $\mathbf{p}$, the 3-momentum of a particle. In this case one can find commutation relations between $\hat{p} \equiv \gamma^\mu p_\mu$, matrices γ^5, γ^0 and $\mathcal{S}$.

$$[\hat{p}, \mathcal{S}]_- = 0 \quad , \quad \left[\gamma^0, \mathcal{S}\right]_- = 0 \quad , \quad \left[\gamma^5, \mathcal{S}\right]_+ = 0 \quad , \tag{12}$$

and

$$\mathcal{S}\lambda^{S,A}(p^\mu) = \lambda^{S,A}(p^\mu) \quad , \tag{13}$$

because in this case

$$\Lambda^*_{L,R} = \Xi \Lambda_{L,R} \Xi^{-1} \quad . \tag{14}$$

We finally arrive at

$$\left[\hat{p}^2 - m^2\right] 1\!\!1_{4\times 4}\ \lambda^{S,A}(p^\mu) = 0 \quad , \tag{15}$$

[2]See, *e.g.*, formulas (48) of ref. [1].

i.e., at the Klein-Gordon equation for each component of $\lambda^{S,A}(p^\mu)$. Why

did acausal solutions fall out? It appears bispinors $\lambda^A(p^\mu) \equiv -\gamma^5\lambda^S(p^\mu)$ can satisfy the positive-energy equation ($\zeta_\lambda = i$) and bispinors $\lambda^S \equiv -\gamma^5\lambda^A(p^\mu)$, the negative-energy one ($\zeta_\lambda = -i$), but dispersion relations will be acausal, Eq. (4), in this unusual case.[3] So, assuming that in the equations (10) one should take $\zeta_\lambda = i$ for describing λ^S and $\zeta_\lambda = -i$, for λ^A

we, in fact, implicitly impose mass-shell constraints. The same situation is for the equations (6,7), $u(p^\mu)$ and $v(p^\mu) \equiv \gamma^5 u(p^\mu)$ can satisfy both the positive- and the negative-energy equations, but the dispersion relations could be unusual.

¿From a mathematical viewpoint the origin of appearance of these solutions seems to be related with the properties of the Lorentz transformation operators with respect to herimitian conjugation operation, see [14, p.404] for discussion. One should further note that the problem of acausal solutions have interrelations with a mathematical possible situation when operators of the continuous Lorentz transformations are combined with other transformations of the Poincaré group to give $\Lambda_{_R} = -\Lambda_{_L}^{-1}$. Thus, the question, whether these solutions would have some physical significance, should be solved on the basis of the rigorous analysis of the general structure of the Poincaré transformation group and of the experimental situation in neutrino physics.

Finally, let us mention that another second-order equation in the $(1/2,0)\oplus(0,1/2)$ representation space has been investigated in [15] and relations with the problem of the lepton mass spectrum have been revealed (see also [7,16]).

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[3] In the process of the proof one should take into account commutation relations (12) and hence that $\mathcal{S}^{+}\gamma^5\lambda^S(p^\mu) = \lambda^A(p^\mu)$ and $\mathcal{S}^{-}\gamma^5\lambda^A(p^\mu) = \lambda^S(p^\mu)$.

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