Peierls instability for the Holstein model with rational density

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ABSTRACT. *We consider the static Holstein model, describing a chain of Fermions interacting with a classical phonon field, when the interaction is weak and the density is a rational number. We show that the energy of the system, as a function of the phonon field, has two stationary points, defined up to a lattice translation, which are local minima in the space of fields periodic with period equal to the inverse of the density.*

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1. Introduction

1.1. The *Holstein model* [P,H] was introduced to represent the interaction of electrons with optical phonons in a crystal. In the original model the phonons are represented in terms of quantum oscillators but the difficulty of understanding such a fully quantum model has led to a modification of it, called *static Holstein model* (or *adiabatic Holstein model*), in which the phonons are classical oscillators. This corresponds to neglect the vibrational kinetic energy of the phonons, an approximation which can be justified in physical models as the atom mass is much larger than the electron mass.

The Hamiltonian of the model, if we neglect all internal degrees of freedom (the spin, for example, which play no role at zero external magnetic field) is given by

$$
\begin{aligned}
H &\equiv H_L^{\rm el} + \frac{1}{2}\sum_{x\in\Lambda}\varphi_x^2 \\
&= \sum_{x,y\in\Lambda} t_{xy}\,\psi_x^+\psi_y^- - \mu\sum_{x\in\Lambda}\psi_x^+\psi_x^- - \lambda\sum_{x\in\Lambda}\varphi_x\psi_x^+\psi_x^- + \frac{1}{2}\sum_{x\in\Lambda}\varphi_x^2 \;,
\end{aligned}
\tag{1.1}
$$

where x,y are points on the one-dimensional lattice Λ with unit spacing, length L and periodic boundary conditions; we shall identify Λ with $\{x\in\mathbb{Z}:\ -[L/2]\le x\le[(L-1)/2]\}$. Moreover the matrix t_{xy} is defined as $t_{xy}=\delta_{x,y}-(1/2)[\delta_{x,y+1}+\delta_{x,y-1}]$, where $\delta_{x,y}$ is the Kronecker delta, μ is the chemical potential and λ is the interaction strength. The fields $\psi_x^\pm$ are creation $(+)$ and annihilation $(-)$ fermionic fields, satisfying periodic boundary conditions: $\psi_x^\pm=\psi_{x+L}^\pm$. We define also $\psi_{\mathbf{x}}^\pm=e^{tH}\psi_x^\pm e^{-Ht}$, with $\mathbf{x}=(x,t)$, $-\beta/2\le t\le\beta/2$ for some $\beta>0$; on t antiperiodic boundary conditions are imposed. The potential φ_x is a real function representing the classical phonon field.

At finite L, the fermionic Fock space is finite dimensional, hence there is a minimum eigenvalue $E_L^{\rm el}(\varphi,\mu)$ of the operator $H_L^{\rm el}$, for each given phonon field φ and each value of μ; let $\rho_L(\varphi,\mu)$ be the corresponding fermionic density. The aim is to minimize the functional

$$
F_L(\varphi,\mu) = E_L^{\rm el}(\varphi,\mu) + \frac{1}{2}\sum_{x\in\Lambda}\varphi_x^2 \;, \tag{1.2}
$$

subject to the condition

$$
\rho_L(\varphi,\mu) = \rho_L \;, \tag{1.3}
$$

where ρ_L is a fixed value of the density, converging for $L\to\infty$, say to ρ.

The model (1.1) can be considered as an approximation of a *more realistic* continuous model containing also the interaction with a fixed external periodic potential of period one. Then the discreteness is not a pure mathematical artifice, but it has a precise physical interpretation: the properties of the two models are expected to be the same, and we think that this could be easily proven along the lines of the present paper.

1.2. It is generally believed that, as a consequence of Peierls instability argument, [P,F], there is a field $\varphi^{(0)}$, uniquely defined up to a spatial translation, which minimizes (1.2), (1.3), in the limit $L\to\infty$, and is a function of the form $\bar\varphi(2\pi\rho x)$, where $\bar\varphi(u)$ is a 2π-periodic function. This is physically interpreted by saying that one-dimensional metals are unstable at low temperature, in the sense that they can lower their energy through a periodic distortion of the "physical lattice" with period $1/\rho$ (in the continuous version of the model, since $1/\rho$ is not an integer in general): such a distortion is called a *charge density wave*, as the physical lattice and the electronic charge density form a new periodic structure with period bigger than the original lattice period.

The argument in [P,F] is quite simple: the periodic potential $\bar\varphi(2\pi\rho x)$ opens a gap in the electronic dispersion relation in correspondence of the Fermi momentum, and a trivial computation using degenerate perturbation theory shows that the elastic energy increase is less than the fermionic energy decrease. However, see [LRA], in this argument one does not take into account the effects due to the discreteness of the lattice, in particular the fact that the momenta are conserved modulo 2π (*Umklapp*). Neglecting the discreteness of the lattice one loses the difference between commensurate or incommensurate charge density wave (*i.e.* rational or irrational ρ) in the infinite volume limit, whose properties are supposed to be different, especially concerning the conductivity [F,LRA].

Note also that, even if the argument in [P,F] is perturbative, Peierls instability is expected to arise also for large interaction strength, [AL].

An exact result, [KL,LM], makes rigorous the theory of Peierls instability for the model (1.1) in the case $\rho = \rho_L = 1/2$ (*half filled band case*), for any value of λ. In fact, in this case it has been proved that there is a global minimum of $F_L(\varphi)$ of the form $\varepsilon(\lambda)(-1)^x$, where $\varepsilon(\lambda)$ is a suitable function of λ. This means that the periodicity of the ground state phonon field is 2 (recall that in our units 1 is just the lattice spacing): this phenomenon is called *dimerization*. The proof heavily relies on symmetry properties which hold only in the half filled case.

In [AAR,BM] Peierls instability for the Holstein model is proven assuming λ *large enough*: in that case the Fermions are almost classical particles and the quantum effects are treated as perturbations. The results hold for the commensurate or incommensurate case; in particular in the incommensurate case the function $\bar\varphi(u)$, related to the minimizing field through the relation $\varphi_x = \bar\varphi(2\pi\rho x)$, has infinite many discontinuities. On the contrary, in the small λ case, according to numerical results, $\bar\varphi(u)$ has been conjectured to be an analytic function of its argument, both for the commensurate and incommensurate cases, [AAR].

In this paper we study the case of small λ and *any* rational density, for which there are, to our knowledge, no results in the literature besides the simulations in [AAR]. Analytical results in the small λ case can be found for a related model, the *Falikov-Kimball model*, described by a Hamiltonian of the form (1.1), in which the continuous φ_x is replaced by a discrete function taking only the values 0 or 1; see [FGM].

Let $\rho = P/Q$, with P, Q relative prime integers, and let $L = L_i \equiv iQ$, $i \in \mathbb{N}$; we shall prove that, if λ is small enough, there are two stationary points $\varphi^{(\pm,i)}$ of $F_{L_i}(\varphi,\mu)$, defined up to a lattice translation, satisfying (1.3) with $\rho_L = \rho$. These stationary points are periodic functions on L_i of period Q (the smallest multiple of $1/\rho$ which is an integer, hence a multiple of the unit lattice spacing), converging for $i \to \infty$. Moreover, if we restrict $F_{L_i}(\varphi,\mu)$ to functions such that $\varphi_x = \varphi_{x+Q}$, $\varphi^{(\pm,i)}$ are local minima in the norm $||\varphi|| = \sup_{x\in L} |\varphi_x|$, uniformly in i.

The presence of the lattice has the effect that we need the smaller λ the bigger Q is, see (1.16). In particular we are not able to draw conclusions about the incommensurate case neither we know if this is a technical limitation or there is some physical reason behind it, so that we can not draw any conclusions about the analyticity conjecture in [AAR].

1.3. Let $\mathbf{h}_{xy} = t_{xy} - \lambda\varphi_x\delta_{xy}$ be the one-particle Hamiltonian and $e_1(\varphi) \le e_2(\varphi) \le \ldots \le e_L(\varphi)$ its eigenvalues. We have

$$E_L^{\rm el}(\varphi,\mu) = \sum_{n: e_n(\varphi)\le\mu} [e_n(\varphi) - \mu] = \mathrm{Tr}([\mathbf{h} - \mu]\mathbf{P}_\mu) \;, \tag{1.4}$$

where $\mathbf{P}_\mu$ is the projector on the subspace spanned by the eigenvectors of $\mathbf{h}$ with eigenvalues $\le \mu$. As it is well known (see, for example [BM]), $E_L^{\rm el}(\varphi,\mu)$ is a differentiable function of φ and, since $\mathrm{Tr}(\mathbf{h}\partial\mathbf{P}_\mu/\partial\varphi_x) = 0$,

$$\frac{\partial}{\partial\varphi_x} E_L^{\rm el}(\varphi,\mu) = \mathrm{Tr}\Big(\Big[\frac{\partial\mathbf{h}}{\partial\varphi_x}\Big]\mathbf{P}_\mu\Big) = -\lambda\rho_x(\varphi,\mu) \;, \tag{1.5}$$

where $\rho_x(\varphi,\mu) = (\mathbf{P}_\mu)_{xx}$ is the density of the electrons in the point x.

Let us now suppose that μ is not equal to any eigenvalue of $\mathbf{h}$. In this case, given $\tilde\varphi$, also $\rho_L(\varphi,\mu)$ is differentiable in a neighborhood small enough of $\tilde\varphi$ (so small that $e_n(\varphi) - \mu$

stays different from zero, for any n, see again [BM]) and $\partial\rho_L(\varphi,\mu)/\partial\varphi_x = 0$. Hence, a local minimum of (1.2) satisfying (1.3) must satisfy the conditions

$$\begin{aligned} \varphi_x &= \lambda\rho_x(\varphi,\mu)\ ,\\ \rho_L &= \frac{1}{L}\sum_x \rho_x(\varphi,\mu)\ , \end{aligned} \tag{1.6}$$

$$M_{xy} \equiv \delta_{xy} - \lambda\frac{\partial}{\partial\varphi_x}\rho_y(\varphi,\mu) \quad \text{is positive definite .} \tag{1.7}$$

Note that, given $\tilde\varphi$, the previous condition on μ can be in general satisfied only if $||\varphi-\tilde\varphi||$ is of order $1/L$, so that a solution of (1.6) defines in general a local minimum only in a neighborhood of size $1/L$. It follows that the only solutions which are interesting in the limit $L\to\infty$ are those associated with a gap of $\mathbf{h}$ around μ, whose size is independent of L.

1.4. If φ is a solution of (1.6), it must satisfy the condition $\hat\varphi_0 = L^{-1}\sum_x\varphi_x = \lambda\rho_L$. On the other hand, if we define $\chi_x = \varphi_x - \hat\varphi_0$, we can see immediately that $\rho_L(\varphi,\mu) = \rho_L(\chi,\mu+\lambda\hat\varphi_0)$. It follows that we can restrict our search of local minima of (1.2) to fields φ with zero mean, satisfying the conditions

$$\begin{aligned} \varphi_x &= \lambda(\rho_x(\varphi,\mu)-\rho_L)\ ,\\ \rho_L &= \frac{1}{L}\sum_x \rho_x(\varphi,\mu)\ , \end{aligned} \tag{1.8}$$

and condition (1.7).

Of course, if the field φ_x satisfies (1.8), the same is true for the translated field φ_{x+n}, for any integer n. On the other hand, one expects that the solutions of (1.8) are even with respect to some point of Λ; hence we can eliminate the trivial source of non-uniqueness described above by imposing the further condition $\varphi_x = \varphi_{-x}$. We shall then consider only fields of the form

$$\varphi_x = \sum_{n=-[L/2]}^{[(L-1)/2]} \hat\varphi'_n e^{\frac{i2n\pi x}{L}}\ , \qquad \hat\varphi'_{-n} = \hat\varphi'_n \in \mathbb{R}\ , \qquad \hat\varphi_0 = 0\ . \tag{1.9}$$

As we said in §1.2, we want to consider the case of rational density, $\rho = P/Q$, P and Q relatively prime, and we want to look for solutions such that $\varphi_x = \varphi_{x+Q}$. Hence, we shall look for solutions of (1.8) with $L = L_i = iQ$, $\rho_L = \rho$ and

$$\varphi_x = \sum_{n=-[Q/2]}^{[(Q-1)/2]} \hat\varphi_n e^{i2\pi\rho n x}\ , \qquad \hat\varphi_n = \hat\varphi_{-n} \in \mathbb{R}\ , \qquad \hat\varphi_0 = 0\ . \tag{1.10}$$

Note that the condition on L allows to rewrite in a trivial way the field φ_x of (1.10) in the general form (1.9), by putting $\hat\varphi'_n = 0$ for all n such that $(2n\pi)/L \neq 2\pi\rho m, \forall m$, and by relabeling the other Fourier coefficients.

The conditions (1.8) can be easily expressed in terms of the variables $\hat\varphi_n$; if we define $\hat\rho_n$ so that

$$\rho_x(\varphi,\mu) = \sum_{n=-[Q/2]}^{[(Q-1)/2]} \hat\rho_n(\varphi,\mu) e^{i2n\pi\rho x}\ , \tag{1.11}$$

we get

$$\hat\varphi_n = \lambda\hat\rho_n(\varphi,\mu)\ , \qquad n\neq 0\ , \quad n = -[Q/2],\dots,[(Q-1)/2]\ , \tag{1.12}$$

$$\hat\rho_0(\varphi,\mu) = \rho_L\ . \tag{1.13}$$

Also the minimum condition (1.7) can be expressed in terms of the Fourier coefficients; we get that the $L\times L$ matrix

$$\bar M_{nm} \equiv \delta_{nm} - \lambda\frac{\partial}{\partial\hat\varphi'_n}\hat\rho'_m(\varphi,\mu) \tag{1.14}$$

has to be positive definite, if the field φ satisfies (1.12) and (1.13) and $\hat\rho'_m(\varphi,\mu)$ is defined analogously to $\hat\varphi'_m$ in (1.9). Hence, if we restrict the space of phonon fields to those of the form (1.10), we have to show that the $Q\times Q$ matrix

$$\tilde M_{nm} \equiv \delta_{nm} - \lambda \frac{\partial}{\partial\hat\varphi_n}\hat\rho_m(\varphi,\mu) \tag{1.15}$$

has to be positive definite, if the field φ satisfies (1.12) and (1.13).

1.5. REMARK. It is easy to show (by using the expansion described in §3, for example) that $\bar M_{nm}$ can be different from zero only if $2\pi(n-m)/L$ is of the form $2\pi\rho k$ for some k. However, we are not able to get good bounds on all matrix elements $\bar M_{nm}$ not belonging (up to a relabeling of indices) also to the matrix (1.15); therefore, in studying the minimum condition, we restrict ourselves to the fields of the form (1.10).

1.6. THEOREM. *Let $\rho = P/Q$, with P,Q relative prime integers, $L = L_i \equiv iQ$. Then, for any positive integer N, there exist positive constants ε, $\tilde\varepsilon$, c and K, independent of i, ρ and N, such that, if*

$$0 \le \frac{4\pi v_0}{\log(\tilde\varepsilon v_0\, L)} \le \lambda^2 \le \varepsilon\, \frac{v_0^2(1+\log v_0^{-1})^{-1}}{K^N N!\log(c\,Q/v_0^4)}\;, \tag{1.16}$$

where

$$v_0 = \sin(\pi\rho)\;, \tag{1.17}$$

there exist two solutions $\varphi^{(\pm)}$ of (1.8), with $L = L_i$, $1-\mu = \cos(\pi\rho)$ and $\rho_L = \rho$, of the form (1.10). The matrices $\tilde M$ corresponding to these solutions, defined as in (1.15), are positive definite.

Moreover, the Fourier coefficients $\hat\varphi_n^{(\pm)}$ verify, for $|n|>1$, the bound

$$|\hat\varphi_n^{(\pm)}| \le \left(\frac{\lambda^2}{v_0|n|}\right)^N |\hat\varphi_1^{(\pm)}|\;. \tag{1.18}$$

Finally, $\lambda\hat\varphi_1^{(\pm)}$ is of the form

$$\lambda\hat\varphi_1^{(\pm)} = \pm v_0^2 \exp\Big\{-\frac{2\pi v_0 + \beta^{(\pm)}(\lambda,L)}{\lambda^2}\Big\}\;, \tag{1.19}$$

with

$$|\beta^{(\pm)}(\lambda,L)| \le C\lambda^2\left(1+\log\frac{1}{v_0}\right)\;, \tag{1.20}$$

where C is a suitable constant.

The one-particle Hamiltonian **h** *corresponding to this solution has a gap of order $|\lambda\hat\varphi_1|$ around μ, uniformly on i.*

1.7. The above theorem proves that there are two stationary points of the ground state energy in correspondence of a periodic function with period equal to the inverse of the density, if the coupling is small enough and the density is rational, and that these stationary points are local minima at least in the space of periodic functions with that period. The energies associated to such minima are different so that the ground state energy is not degenerate.

The theorem is proved by writing $\rho_x(\varphi,\mu)$ as an expansion convergent for small λ and solving the set of equations (1.12) by a contraction method. As a byproduct we prove that the $\hat\varphi_n$ are fast decaying, (see (1.18)), so that φ_x is really well approximated by its first harmonics (this remark is important as the number of harmonics could be very large).

The results are uniform in the volume, so they are interesting from a physical point of view (a solution defined only for $|\lambda| \le O(1/L)$ should be outside any reasonable physical value for λ). The case in [KL] for the half filled case is contained in Theorem 1.6, but in

[KL] it is also proved that the solution is a global minimum. On the other hand this case is quite special (see Remarks 2.5 in §2).

Finally the lower bound in (1.16) is a large volume condition: this is not a technical condition as, if the number of Fermions is odd, there is Peierls instability only for L large enough. The upper bound for λ in (1.16) requires λ to decrease as Q increases: in particular irrational density are forbidden. This requirement is due to the discreteness of the lattice and to Umklapp phenomena. Note that the dependence of the maximum λ allowed on Q is not very strong as it is a logarithmic one.

The case of irrational densities (possible only in the infinite volume limit), excluded by our theorem, is physically interesting, but the existence of Peierls instability in this case is proven only for large λ, [AL,BM]. In [BGM] $\rho_x(\varphi,\mu)$ is shown to be well defined for small λ not only in the rational density case, (in which the proof is almost trivial), but also in the irrational case: in fact the small divisor problem due to the irrationality of the density can be controlled thanks to a Diophantine condition. However to solve the set of equations (1.12) we use a contraction method which is not trivially adaptable in the latter case (see Remarks 2.5 in §2). The same kind of problem arises in proving the positive definiteness of $\bar{M}_{nm}$ in the rational case (and this is the reason why we are able to prove that the stationary points are local minima only in the space of periodic functions with prefixed period). As we said above, we do not know if such problems are only technical or there is some physical reason for this to happen.

2. Solution of the self-consistence equation

2.1. Let $\rho = P/Q$, with P, Q relatively prime integers such that $0 < P < Q$, and $L = L_i \equiv iQ$; we have to look for a solution of (1.12) and (1.13), which is well defined for $|\lambda| \le \varepsilon_0$, with ε_0 *independent* of i (otherwise our solution is meaningless from a physical point of view). As discussed in §1.3, this means that our solution has to be looked for in a class of functions for which the one-particle Hamiltonian $\mathbf{h}$ has a gap around μ of width independent of L. This class of functions is described by the following lemma, to be proved in §5.4.

2.2. LEMMA. *Let φ_x be a field of the form (1.10), $L = L_i$, $1 - \mu = \cos(\pi\rho)$, $|\lambda\hat\varphi_1| > 0$ and $|\lambda\hat\varphi_n| \le a|\lambda\hat\varphi_1|/|n|^N$ for some positive constants a and N. Then there exists $\varepsilon_0 > 0$, independent of i and ρ, such that, if $|\lambda\hat\varphi_1| \le \varepsilon_0 v_0^4/Q$, with $v_0 = \sin(\pi\rho)$, the one-particle Hamiltonian $\mathbf{h}$ has a gap of width $\ge |\lambda\hat\varphi_1|/2$ around μ. Moreover, $\hat\rho_n(\varphi, \mu)$ is a continuous function of λ, which converges to a continuous function of λ as $i \to \infty$, and $\hat\rho_0(\varphi, \mu) = \rho$.*

2.3. We can write the self-consistence equation (1.12) as

$$\hat\varphi_n = -\lambda^2 c_n(\sigma)\hat\varphi_n + \lambda\tilde\rho_n(\sigma, \Phi) , \quad \sigma \equiv \lambda\hat\varphi_1 , \quad \Phi \equiv \{\lambda\hat\varphi_n\}_{|n|>1} , \tag{2.1}$$

where $c_n(\sigma)$ depends on φ only through σ. We write $\hat\rho_n$ as a perturbative expansion in λ (different from the power expansion in λ); this expansion is described in §3. If $|n| > 1$, we are here defining $-\lambda c_n(\sigma)\hat\varphi_n$ the contribution to $\hat\rho_n$ proportional to $\hat\varphi_n$ of order 1 in the expansion, while $-\sigma c_1(\sigma)$ is the contribution to $\hat\rho_1$ proportional to σ of order ≤ 1 in the expansion (explicit expressions for $c_n(\sigma)$ and $c_1(\sigma)$ will be given in (4.16) and (4.39) respectively); $\tilde\rho_n$ takes into account all the remaining terms of first order plus all terms of order higher than 1.

The equation (2.1) has of course the trivial solution $\hat\varphi_n = 0, \forall n$, but it is easy to see that this is not a local minimum, by using the expansion for $\hat\rho_n$ of §3. Therefore we shall look for solutions such that $\sigma \ne 0$, so that we can rewrite (2.1) as

$$(1 + \lambda^2 c_1(\sigma)) = \frac{\lambda^2\tilde\rho_1(\sigma, \Phi)}{\sigma} , \tag{2.2}$$

$$\Phi_n \equiv \lambda\hat\varphi_n = \frac{\lambda^2\tilde\rho_n(\sigma, \Phi)}{(1 + \lambda^2 c_n(\sigma))} , \qquad |n| > 1 . \tag{2.3}$$

Note that the equation for $n = -1$ does not appear simply because $\rho_{-1} = \rho_1$, as a consequence of the condition $\hat\varphi_n = \hat\varphi_{-n} \in \mathbb{R}$, see (1.10).

2.4. We prove Theorem 1.6 in three steps as follows.

• We first study the self-consistence equation (2.3), considering σ as a variable belonging to the interval

$$J = (\, -\exp(-\pi\, v_0/\lambda^2) \, , \, \exp(-\pi\, v_0/\lambda^2) \,) . \tag{2.4}$$

We find a solution, that we denote $\Phi(\sigma)$, if λ is small enough.

• We then prove that, if L is large enough, the equation (in λ)

$$1 + \lambda^2 c_1(\sigma) = \frac{\lambda^2\tilde\rho_1(\sigma, \Phi(\sigma))}{\sigma} \tag{2.5}$$

has two solutions $\sigma^{(\pm)} \in J$, of the form (1.19). Therefore $(\sigma^{(\pm)}(\lambda)/\lambda,\ \Phi(\sigma^{(\pm)}(\lambda))/\ \lambda)$ turn out to be solutions of (1.12), which verify, thanks to Lemma 2.2, (1.13) with $L = L_i$.

• We finally prove that the Hessian matrices (1.15) corresponding to these two solutions are positive definite.

2.5. REMARKS. The coefficient $\hat\varphi_1$ has a privileged role with respect to the other coefficients. In fact, as we shall see in §5, the properties of the system when only $\hat\varphi_1$ is different from 0 are very close to the properties of the case in which all the coefficients are non vanishing.

This suggests that the "important" equation is (2.2), so explaining the strategy outlined above.

The previous remark also implies that $1+\lambda^2 c_1(\sigma) \simeq 0$. It follows that $1+\lambda^2 c_n(\sigma) \simeq 0$, for all n such that $2\pi\rho n \simeq 2\pi\rho \pmod{2\pi}$. Since $\min_{|n|>1} |2\pi\rho n - 2\pi\rho| = 2\pi/Q$, we can expect that our bounds will not be uniform in Q. This is the reason why Theorem 1.6 can not be extended to irrational density; at most one can hope that a Diophantine condition on ρ is needed, but we have only been able to prove that the Q dependence can be substituted with a dependence on the Diophantine constants in some of the bounds described below.

Note also that, if $Q=2$, the only equation to discuss is just the equation (2.2) with $\Phi = 0$ and the r.h.s. equal to zero; its solution is well known in this case, see [KL,LM] for example. If $Q=3$, again (2.2) is the only equation to discuss, but the r.h.s. is different from zero; however it is easy to prove that the solution has essentially the same properties as in the case $Q=2$. Hence, in the following we shall consider only the case $Q \geq 4$. The following lemma, furnishing a bound on the constants $c_n(\sigma)$ and their derivatives, is proven in §4.9.

2.6. LEMMA. *There exists a constant C, independent of i and ρ, such that, if $|n| \geq 2$,*

$$|c_n(\sigma)| \leq \frac{C}{v_0}\left(1 + \log\frac{1}{v_0}\right)\log Q\ , \tag{2.6}$$

$$\left|\frac{\partial c_n(\sigma)}{\partial\sigma}\right| \leq \frac{C}{v_0|\sigma|}\ , \tag{2.7}$$

2.7. Fixed $L = L_i$, Φ is a finite sequence of $Q-3$ elements, which can be thought as a vector in $\mathbb{R}^{Q-3}$, which is a function of σ. In order to study the equation (2.2) for σ, we shall need a bound on Φ and on the derivative of Φ with respect to σ. Hence we consider the space $\mathcal{F} = \mathcal{C}^1(J, \mathbb{R}^{Q-3})$ of C^1-functions of $\sigma \in J$ with values in $\mathbb{R}^{Q-3}$; the solutions of (2.3) can be seen as fixed points of the operator $\mathbf{T}_\lambda : \mathcal{F} \to \mathcal{F}$, defined by the equation:

$$[\mathbf{T}_\lambda(\Phi)]_n(\sigma) = \frac{\lambda^2 \tilde{\rho}_n(\sigma, \Phi(\sigma))}{(1+\lambda^2 c_n(\sigma))}\ , \tag{2.8}$$

We shall define, for each positive integer N, a norm in $\mathcal{F}$ in the following way:

$$\|\Phi\|_{\mathcal{F}} \equiv \sup_{|n|>1, \sigma\in J} \left\{ |n|^N \left[|\sigma|^{-1}|\Phi_n(\sigma)| + \left|\frac{\partial\Phi_n}{\partial\sigma}(\sigma)\right| \right] \right\}\ . \tag{2.9}$$

We shall also define

$$\mathcal{B} = \{\Phi \in \mathcal{F} : \|\Phi\|_{\mathcal{F}} \leq 1\}\ ; \tag{2.10}$$

$$R(\Phi)_n(\sigma) = \tilde{\rho}_n(\sigma, \Phi(\sigma))\ , \quad |n| \geq 2\ . \tag{2.11}$$

The following two lemmata, to be proved in §5.5 and §5.6, respectively, resume the main properties of $R(\Phi)$.

2.8. LEMMA. *There are two constants $C_1 > 1$ and C_2, independent of i, ρ and N, such that, if $\Phi, \Phi' \in \mathcal{B}$ and*

$$C_1 Q v_0^{-4} |\sigma| [1 + \log(v_0^2/|\sigma|)] \leq 1\ , \tag{2.12}$$

then

$$\|R(\Phi) - R(\Phi')\|_{\mathcal{F}} \leq \frac{C_2 3^N N!}{v_0}\left(1 + \log\frac{1}{v_0}\right)\|\Phi - \Phi'\|_{\mathcal{F}}\ . \tag{2.13}$$

2.9. LEMMA. *There is $C > 1$, such that, if*

$$C Q v_0^{-3} |\sigma|^{1/2} [1 + \log(v_0^2/|\sigma|)] \leq 1\ , \tag{2.14}$$

then

$$\|R(0)\|_{\mathcal{F}} \le \frac{C}{v_0}\left(1+\log\frac{1}{v_0}\right)\sup_{|n|>1}\left\{|n|^N\left(\frac{|\sigma|}{v_0^2}\right)^{\frac{|n|}{10}}\right\} . \tag{2.15}$$

2.10. LEMMA. *There are ε, c, K, independent of i, ρ and N, such that, if $\sigma \in J$ and*

$$\lambda^2 \le \varepsilon\,\frac{v_0^2(1+\log v_0^{-1})^{-1}}{K^N N!\log(c\,Q/v_0^4)} , \tag{2.16}$$

there exists a unique solution $\Phi \in \mathcal{B}$ of (2.3); moreover the solution satisfies the bound

$$\|\Phi\|_{\mathcal{F}} \le \left(\frac{\lambda^2}{v_0}\right)^N . \tag{2.17}$$

2.11. *Proof of Lemma 2.10.* It is easy to see that, if $\sigma \in J$, the conditions on σ of Lemma 2.8 and Lemma 2.9 are satisfied, if

$$\lambda^2 \le \varepsilon_0/\log(cQ/v_0^4) , \tag{2.18}$$

with suitable values of ε_0 and c. Moreover, if $\varepsilon_0 \le \varepsilon_1 v_0(1+\log v_0^{-1})^{-1}$ and ε_1 is chosen small enough, (2.6) and (2.18) imply that $\lambda^2|c_n(\sigma)| \le 1/2$, so that, by using (2.7), (2.8) and Lemma 2.8, we have that, if $\Phi \in \mathcal{B}$,

$$\|\mathbf{T}_\lambda(\Phi)\|_{\mathcal{F}} \le 4\lambda^2\left(1+\lambda^2\frac{C}{v_0}\right)\left[\|R(0)\|_{\mathcal{F}} + \frac{C_2}{v_0}\left(1+\log\frac{1}{v_0}\right)3^N N!\|\Phi\|_{\mathcal{F}}\right] . \tag{2.19}$$

Therefore, by (2.15) and (2.4), there exist constants C_3 and C_4, such that, if $\varepsilon_1 \le \varepsilon v_0(C_4^N N!)^{-1}$ and ε is small enough,

$$\|\mathbf{T}_\lambda(\Phi)\|_{\mathcal{F}} \le \frac{C_3\lambda^2}{v_0^2}\left(1+\log\frac{1}{v_0}\right)\left[3^N N! + \sup_{|n|>1}|n|^N\exp\left(-\frac{\pi v_0|n|}{10\lambda^2}\right)\right] \le 1 . \tag{2.20}$$

Moreover, by (2.13), if $\Phi, \Phi' \in \mathcal{B}$ and similar conditions on λ are satisfied, we have

$$\|\mathbf{T}_\lambda(\Phi) - \mathbf{T}_\lambda(\Phi')\|_{\mathcal{F}} \le \frac{C_5^N N!\lambda^2}{v_0^2}\left(1+\log\frac{1}{v_0}\right)\|\Phi-\Phi'\|_{\mathcal{F}} \le \frac{1}{2}\|\Phi-\Phi'\|_{\mathcal{F}} . \tag{2.21}$$

The bounds (2.20) and (2.21) imply that $\mathcal{B}$ is invariant under the action of $\mathbf{T}_\lambda$ and that $\mathbf{T}_\lambda$ is a contraction on $\mathcal{B}$. Hence, by the contraction mapping principle, there is a unique fixed point $\bar{\Phi}$ of $\mathbf{T}_\lambda$ in $\mathcal{B}$, which can be obtained as the limit of the sequence $\Phi^{(k)}$ defined through the recurrence equation $\Phi^{(k+1)} = \mathbf{T}_\lambda(\Phi^{(k)})$, with any initial condition $\Phi^{(0)} \in \mathcal{B}$. If we choose $\Phi^{(0)} = 0$, we get, by (2.21),

$$\|\bar{\Phi}\|_{\mathcal{F}} \le \sum_{i=1}^{\infty}\|\Phi^{(i)} - \Phi^{(i-1)}\|_{\mathcal{F}} \le \sum_{i=1}^{\infty}\frac{1}{2^{i-1}}\|\Phi^{(1)}\|_{\mathcal{F}} \le \|\Phi^{(1)}\|_{\mathcal{F}} . \tag{2.22}$$

On the other hand, by (2.15),

$$\|\Phi^{(1)}\|_{\mathcal{F}} = \|\mathbf{T}_\lambda(0)\|_{\mathcal{F}} \le \frac{C_6^N N!\lambda^2}{v_0^2}\left(1+\log\frac{1}{v_0}\right)\left(\frac{\lambda^2}{v_0}\right)^N , \tag{2.23}$$

which immediately implies the bound (2.17), if $\varepsilon_1 \le \varepsilon v_0(C_6^N N!)^{-1}$, with ε small enough. ■

2.12. Let us now consider the equation (2.5). We want to prove that it has two solutions of the form (1.19), if $\sigma \in J$ and L_i is large enough. In order to achieve this result, we need some detailed properties of the function $c_1(\sigma)$, which are described in the following Lemma 2.13, to be proved in §4.10. We need also the bounds on $\tilde\rho_1(\sigma,\Phi(\sigma))$ and its derivative with respect to σ, contained in Lemma 2.14, to be proved in §5.7.

2.13. LEMMA. *There is a constant C, such that, if*

$$\frac{v_0}{L_i|\sigma|} \le \tilde\varepsilon \le \frac{1}{8\pi} , \qquad \frac{|\sigma|}{v_0^2} \le 1 , \tag{2.24}$$

then

$$-c_1(\sigma) = \frac{1}{2\pi v_0}\left[\log\frac{v_0^2}{|\sigma|} + r_1(\sigma)\right] , \tag{2.25}$$

with

$$\begin{aligned} |r_1(\sigma)| &\le C\left(1+\log\frac{1}{v_0}\right) , \\ \left|\frac{\partial r_1(\sigma)}{\partial\sigma}\right| &\le C\left(\frac{1}{v_0^2}+\frac{\tilde\varepsilon}{|\sigma|}\right) . \end{aligned} \tag{2.26}$$

2.14. LEMMA. *If $\sigma\in J$, λ satisfies the inequality (2.16), with ε small enough, $\Phi(\sigma)$ is the solution of the equation (2.3) described in Lemma 2.10 and*

$$r_2(\sigma) \equiv \frac{2\pi v_0\tilde\rho_1(\sigma,\Phi(\sigma))}{\sigma} , \tag{2.27}$$

then there is a constant C, such that

$$\begin{aligned} |r_2(\sigma)| &\le C\left[\left(\frac{|\sigma|}{v_0^2}\right)^{1/4} + \left(\frac{\lambda^2}{v_0}\right)^N\right] , \\ \left|\frac{\partial r_2(\sigma)}{\partial\sigma}\right| &\le \frac{C}{|\sigma|}\left[\left(\frac{|\sigma|}{v_0^2}\right)^{1/4} + \left(\frac{\lambda^2}{v_0}\right)^N\right] . \end{aligned} \tag{2.28}$$

2.15. LEMMA. *There exist positive constants ε, $\tilde\varepsilon$, c and K, independent of i, ρ and N, such that, if λ satisfies the inequalities (1.16), there are two solutions $\sigma^{(\pm)}(\lambda)\in J$ of equation (2.5) of the form (1.19).*

2.16. *Proof of Lemma 2.15.* By using the definitions of $r_1(\sigma)$ and $r_2(\sigma)$ given in (2.25) and (2.27), we can write the equation (2.5) in the form

$$F(\sigma) \equiv \log\frac{v_0^2}{|\sigma|} - \frac{2\pi v_0}{\lambda^2} + r(\sigma) = 0 , \tag{2.29}$$

where $r(\sigma) = r_1(\sigma) + r_2(\sigma)$.

Let us now suppose that λ satisfies the inequalities (2.16) and that σ belongs to the interval

$$\tilde J = \left(v_0^2 e^{-4\pi v_0/\lambda^2} , \ v_0^2 e^{-\pi v_0/\lambda^2}\right) \subset J . \tag{2.30}$$

If L_i is large enough and the constant ε in (2.16) is chosen small enough, the conditions (2.24) of Lemma 2.13 are satisfied, for $\sigma\in\tilde J$, and

$$\frac{4\pi v_0}{\lambda^2} \le \log(\tilde\varepsilon v_0 L_i) . \tag{2.31}$$

Moreover, if $\tilde\varepsilon$ and ε (hence $|\sigma| v_0^{-2}$) are small enough,

$$\left|\frac{\partial r(\sigma)}{\partial\sigma}\right| \le \frac{1}{2}\left|\frac{\partial}{\partial\sigma}\log\frac{v_0^2}{\sigma}\right| ; \tag{2.32}$$

hence $F(\sigma)$ is a monotone decreasing function of σ in $\tilde{J}$. If we define

$$\sigma^* = v_0^2 e^{-2\pi v_0/\lambda^2} , \qquad M = \sup_{\sigma \in \tilde{J}} |r(\sigma)| , \tag{2.33}$$

we have that $F(\sigma^* \exp(-2M)) > 0$ and $F(\sigma^* \exp(2M)) < 0$. Moreover, the interval $(\sigma^* \exp(-2M)), \sigma^* \exp(2M)))$ is contained in $\tilde{J}$, if ε is small enough, since the bounds (2.26) and (2.28) imply that $M \le C(1+\log v_0^{-1})$. Hence there is a unique solution $\sigma^{(+)}(\lambda)$ of (2.29) in $\tilde{J}$, which can be written as

$$\sigma^{(+)}(\lambda) = v_0^2 e^{-\frac{2\pi v_0 + \beta^{(+)}(\lambda)}{\lambda^2}} , \tag{2.34}$$

with $|\beta^{(+)}(\lambda)| \le C\lambda^2(1+\log v_0^{-1})$.

In the same manner, we can show that there is solution $\sigma^{(-)}(\lambda)$ in the interval

$$(-v_0^2 e^{-\frac{\pi v_0}{\lambda^2}} , -v_0^2 e^{-\frac{4\pi v_0}{\lambda^2}}) \subset J , \tag{2.35}$$

with the same properties. ■

2.17. Lemma. *The constants ε, $\tilde{\varepsilon}$, c and K, appearing in (1.16), can be chosen so that the Hessian matrix (1.15) is positive definite.*

2.18. The proof of Lemma 2.17 is in §5.8. This completes the proof of Theorem 1.6.

3. Graph formalism

3.1. In this section we shall describe the expansion of $\rho_x(\varphi,\mu)$, used to get the results of this paper.

Let us consider the operators $\psi^\pm_{\mathbf{x}} = e^{tH}\psi^\pm_x e^{-Ht}$, with $\mathbf{x}=(x,t)$, $-\beta/2\le t\le \beta/2$ for some $\beta>0$; on t antiperiodic boundary conditions are imposed. As explained, for example, in [BGM], there is a simple (well known) relation between $\rho_x(\varphi,\mu)$ and the *two-point Schwinger function*, defined by

$$S^{L,\beta}(\mathbf{x};\mathbf{y}) = \frac{\mathrm{Tr}\left[\exp(-\beta H)\left(\theta(x_0>y_0)\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}} - \theta(x_0<y_0)\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\right)\right]}{\mathrm{Tr}\left[\exp(-\beta H)\right]}\;, \tag{3.1}$$

given by

$$\rho_x = -\lim_{\beta\to\infty}\lim_{\tau\to 0^-}\frac{1}{L}S^{L,\beta}(x,\tau;x,0)\;. \tag{3.2}$$

In [BGM], which we shall refer to for more details, it is also explained that the two-point Schwinger function can be written as

$$S^{L,\beta}(\mathbf{x};\mathbf{y}) = \lim_{M\to\infty}\frac{\int P(d\psi)\,e^{\mathcal{V}(\psi)}\,\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}}{\int P(d\psi)\,e^{\mathcal{V}(\psi)}}\;, \tag{3.3}$$

where $\psi^\pm_{\mathbf{x}}$ are now anticommuting Grassmanian variables and $P(d\psi)$ is a *Grassmanian Gaussian measure*, formally defined by

$$P(d\psi) = \Big\{\prod_{\mathbf{k}\in\mathcal{D}_{L,\beta}}(L\beta\hat g(\mathbf{k}))\Big\}\exp\left\{-\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}(L\beta\hat g(\mathbf{k}))^{-1}\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}}\right\}d\psi^- d\psi^+\;, \tag{3.4}$$

$\mathbf{k}=(k,k_0)$, $\mathcal{D}_{L,\beta}\equiv\mathcal{D}_L\times\mathcal{D}_\beta$, $\mathcal{D}_L\equiv\{k=2\pi n/L, n\in\mathbb{Z}, -[L/2]\le n\le[(L-1)/2]\}$, $\mathcal{D}_\beta\equiv\{k_0=2(n+1/2)\pi/\beta, n\in\mathbb{Z}, -M\le n\le M-1\}$, in the limit $M\to\infty$,

$$\hat g(\mathbf{k}) = \frac{1}{-ik_0+\cos p_F-\cos k} \tag{3.5}$$

is the *propagator* or the *covariance of the measure*, $p_F=\pi\rho$ is the *Fermi momentum*, defined so that $\cos p_F = 1-\mu$, and

$$\mathcal{V}(\psi) = \sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2}dx_0\left[\lambda\varphi_x\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\right]\;. \tag{3.6}$$

If we insert (1.10) in the r.h.s. of (3.6), we get

$$\mathcal{V}(\psi) = \sum_{n=-[Q/2]}^{[(Q-1)/2]}\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\lambda\hat\varphi_n\,\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}+2n\mathbf{p}_F}\;, \tag{3.7}$$

where $\mathbf{p}_F=(p_F,0)$ and $k+2np_F$ is of course defined modulo 2π.

3.2. Note that $\hat g(\mathbf{k})^{-1}$ is small for $\mathbf{k}\simeq\pm\mathbf{p}_F$. Hence there is no hope to treat perturbatively the terms with $n=\pm1$ and $\mathbf{k}$ near $\mp\mathbf{p}_F$, but we can at most expect that the interacting measure is a perturbation of the measure (whose covariance is not singular at $\mathbf{k}=\pm\mathbf{p}_F$)

$$\bar P_\lambda(d\psi)\equiv\frac{1}{\mathcal{N}}P(d\psi)\exp\left\{\lambda\hat\varphi_1\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\frac{1}{L}\sum_{k\in I_-}\left[\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}+2\mathbf{p}_F}+\psi^+_{\mathbf{k}+2\mathbf{p}_F}\psi^-_{\mathbf{k}}\right]\right\}\;, \tag{3.8}$$

where $\mathcal{N}$ is a normalization constant and I_- is a small interval centered in $-p_F$, so small that $I_- \cap I_+ = \emptyset$, if $I_+ \equiv \{k = \bar{k} + 2p_F, \bar{k} \in I_-\}$.

This remark suggests to apply a multiscale expansion to the integral (3.3), in order to treat in a different way the momenta near $\pm p_F$ and the others. This procedure was applied in [BGM] to study systems of electrons in presence of a potential of the form $\bar{\varphi}(2px)$, with $\bar{\varphi}$ 2π-periodic, p/π an irrational Diophantine number and $p_F = mp$, m arbitrary integer. In [BGM] the aim was mainly to get the best possible results about the dependence of the two-point Schwinger function on λ and we found it useful to realize the multiscale expansion by dividing the momenta near p_F into a number of "slices" of order $|\log \lambda|$.

This expansion could be applied also the case $p_F = p$ with $\rho = p_F/\pi$ rational, without no important difference, and we could get immediately Lemma 2.2. However, in this paper we prefer to use a simpler expansion into only two scales; this expansion gives weaker results about the dependence on λ, but it is sufficient in order to prove Lemma 2.2 and it is more suitable for studying the equation (1.12).

3.3. Let us introduce a smooth positive function $f_0(k')$ on the one dimensional torus $\mathbb{T}^1$, such that

$$f_0(k') = \begin{cases} 1 , & \text{if} \quad \|k'\|_{\mathbb{T}^1} \le t_0/2 , \\ 0 , & \text{if} \quad \|k'\|_{\mathbb{T}^1} \ge t_0 , \end{cases} \tag{3.9}$$

where

$$t_0 = \min\{p_F/2, (\pi - p_F)/2\} . \tag{3.10}$$

and the norm $\|k'\|_{\mathbb{T}^1}$ on $\mathbb{T}^1$ is defined so that $\|k'\|_{\mathbb{T}^1} = |k'|$, if $k' \in [-\pi, \pi)$. Then we write

$$\hat{f}_1(k) = 1 - f_0(k + p_F) - f_0(k - p_F) , \qquad \hat{f}_0(k) = 1 - \hat{f}_1(k) , \tag{3.11}$$

so that (3.5) becomes

$$\hat{g}(\mathbf{k}) = \hat{g}^{(1)}(\mathbf{k}) + \hat{g}^{(0)}(\mathbf{k}) = \sum_{h=0,1} \frac{\hat{f}_h(k)}{-ik_0 + \cos p_F - \cos k} , \tag{3.12}$$

and, for $h = 0$, we define

$$\hat{g}^{(0)}(\mathbf{k}) = \sum_{\omega=\pm 1} \hat{g}^{(0)}_\omega(\mathbf{k}) , \tag{3.13}$$

where, if $\mathbf{k}' = \mathbf{k} - \omega \mathbf{p}_F$,

$$\hat{g}^{(0)}_\omega(\mathbf{k}' + \omega\mathbf{p}_F) \equiv \tilde{g}^{(0)}_\omega(\mathbf{k}') = \frac{f_0(k')}{-ik_0 + \cos p_F - \cos(k' + \omega p_F)} , \tag{3.14}$$

with $\mathbf{k}' = (k', k_0)$; we set also $\tilde{g}^{(1)}_{1,1}(\mathbf{k}') \equiv \hat{g}^{(1)}(\mathbf{k})$, with $k = k' + p_F$, and $f_1(k') = \hat{f}_1(k)$, in order to simplify the notations in the following sections.

3.4. We can associate with the decomposition (3.12) of $\hat{g}(\mathbf{k})$ a decomposition of the Grassmanian Gaussian measure $P(d\psi)$ into a product of two independent Grassmanian Gaussian measures:

$$P(d\psi) = P(d\psi^{(1)}) P(d\psi^{(0)}) , \tag{3.15}$$

if $P(d\psi^{(i)})$ is defined as in (3.4), with $\hat{g}^{(i)}(\mathbf{k})$ in place of $\hat{g}(\mathbf{k})$.

If we insert (3.15) in (3.3) and we perform the integration over the field $\psi^{(1)}$, it is easy to show (see [BGM], §4) that

$$S^{L,\beta}(\mathbf{x};\mathbf{y}) = g^{(1)}(\mathbf{x};\mathbf{y}) + K^{(0)}_{\phi,\phi}(\mathbf{x};\mathbf{y}) + S^{(0)}(\mathbf{x};\mathbf{y}) , \tag{3.16}$$

where $g^{(1)}(\mathbf{x};\mathbf{y}) = (L\beta)^{-1} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \hat{g}^{(1)}(\mathbf{k}) \exp[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]$ and

$$\begin{aligned} S^{(0)}(\mathbf{x};\mathbf{y}) = & \frac{\partial^2}{\partial\phi^+_\mathbf{x} \partial\phi^-_\mathbf{y}} \frac{1}{\mathcal{N}_0} \int P(d\psi^{(0)}) \\ & e^{\int d\mathbf{x} \left(\phi^+_\mathbf{x} \psi^{(0)-}_\mathbf{x} + \psi^{(0)+}_\mathbf{x} \phi^-_\mathbf{x}\right)} e^{\mathcal{V}^{(0)}(\psi^{(0)}) + W^{(0)}(\psi^{(0)},\phi)} \Big|_{\phi^+=\phi^-=0} ; \end{aligned} \tag{3.17}$$

in (3.17) $\int d\mathbf{x}$ is a shortcut for $\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2} dx_0$, $\mathcal{N}_0=\int P(d\psi^{(0)})\exp[\mathcal{V}^{(0)}(\psi^{(0)})]$, $\{\phi^\pm_{\mathbf{x}}\}$ are Grassmanian variables anticommuting with $\{\psi^\pm_{\mathbf{x}}\}$ and $\mathcal{V}^{(0)}(\psi^{(0)})$, the *effective potential on the small momenta scale*, can be easily represented as a series in λ, as well as the function $W^{(0)}(\psi^{(0)},\phi)$ and the function $K^{(0)}_{\phi,\phi}(\mathbf{x};\mathbf{y})$ appearing in (3.16).

A precise description of these series in terms of Feynman graphs can be found in [BGM]; see in particular the equations (4.3), (4.4) and (4.6) in [BGM]. Here we want only to stress that the involved graphs are chains formed by vertices, associated with the Fourier components of the potential $\hat\varphi_n$, connected through lines, associated to the propagators $g^{(1)}$; hence by using the bound (3.27) below, it is easy to prove that these series are convergent, uniformly in L and β. On the contrary, the series obtained by integrating the field $\psi^{(0)}$ do not have this property, for the reason explained before, and we have to look for a different expansion, based on the idea outlined in §3.2.

We can associate with the decomposition (3.13) of $\hat g^{(0)}(\mathbf{k})$ a decomposition $\psi^{(0)}_{\mathbf{k}}=\psi^{(0)}_{\mathbf{k},+}+\psi^{(0)}_{\mathbf{k},-}$ of the field $\psi^{(0)}_{\mathbf{k}}$. The support properties of $f_0(k')$, see (3.9), and the definition (3.14) imply that the field $\psi^{(0)}_{\mathbf{k},\omega}$ has support on the set $\{\mathbf{k}=\mathbf{k}'+\omega\mathbf{p}_F : f_0(k')\neq 0\}$ and that the supports of $\psi^{(0)}_{\mathbf{k},+}$ and $\psi^{(0)}_{\mathbf{k},-}$ are disjoint.

The idea is to modify the *free measure* $P(d\psi^{(0)})$ by multiplying it by the terms present in $\mathcal{V}^{(0)}(\psi^{(0)})$, (see [BGM], §3), which couple the variables $\psi^{(0)}_{\mathbf{k},-}$ and $\psi^{(0)}_{\mathbf{k}+2\mathbf{p}_F,+}$; then we expand the integral by using the new measure as the free measure. The new graphs differ from the previous ones for two respects; first of all they are not singular anymore at $\mathbf{k}=\pm\mathbf{p}_F$, but they are bounded by $C|\lambda\hat\varphi_1|^{-1}$, see below; moreover the two propagators exiting and entering in the same vertex can not have both the momentum equal to $\pm\mathbf{p}_F$. As we shall see, these two properties are sufficient to control the expansions.

The two properties of the new free measure described above are realized also if we only extract from $\mathcal{V}^{(0)}(\psi^{(0)})$ the first order terms coupling $\psi^{(0)}_{\mathbf{k},-}$ and $\psi^{(0)}_{\mathbf{k}+2\mathbf{p}_F,+}$. It is easy to see that these terms are equal to $\lambda\hat\varphi_1 F^{(0)}_\sigma(\psi^{(0)})$, with

$$F^{(0)}_\sigma(\psi^{(0)})=\sum_{\omega=\pm1}\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\psi^{(0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(0)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega}\ . \tag{3.18}$$

Hence we define

$$\tilde P(d\psi^{(0)})=\frac{1}{\mathcal{N}}P(d\psi^{(0)})e^{\lambda\hat\varphi_1 F^{(0)}_\sigma(\psi^{(0)})}\ , \tag{3.19}$$

where $\mathcal{N}$ is a suitable constant, and

$$\tilde{\mathcal{V}}^{(0)}(\psi^{(0)})=\mathcal{V}^{(0)}(\psi^{(0)})-\lambda\hat\varphi_1 F^{(0)}_\sigma(\psi^{(0)})\ . \tag{3.20}$$

By proceeding as in [BGM], §3, one can show that the Grassmanian integration $\tilde P(d\psi^{(0)})$ has propagator

$$g^{(0)}(\mathbf{x};\mathbf{y})=\sum_{\omega,\omega'=\pm1}e^{-i(\omega x-\omega' y)p_F}\,g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})\ , \tag{3.21}$$

with

$$g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})=\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\tilde g^{(0)}_{\omega,\omega'}(\mathbf{k}')\ , \tag{3.22}$$

$$\begin{aligned}\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')&=\frac{[-ik_0-F_1(k')-\omega F_2(k')]\,f_0(k')}{[-ik_0-F_1(k')]^2-[F_2^2(k')+\sigma_0^2(k')]}\ ,\\ \tilde g^{(0)}_{\omega,-\omega}(\mathbf{k}')&=\frac{[\sigma_0(k')]\,f_0(k')}{[-ik_0-F_1(k')]^2-[F_2^2(k')+\sigma_0^2(k')]}\ ,\end{aligned} \tag{3.23}$$

$\sigma_0(k')=\sigma f_0(k')$, if $\sigma=\lambda\hat\varphi_1\neq 0$, and (see (1.17) for the definition of v_0)

$$F_1(k')=(\cos k'-1)\cos p_F\ ,\qquad F_2(k')=v_0\,\sin k'\ . \tag{3.24}$$

3.5. Remark. Note that, if $|k'| \le t_0$, $2(1-\cos k')/|\sin k'| = 2|\tan(k'/2)| \le 2\tan(t_0/2) \le |\tan p_F|$; hence

$$|F_1(k')| \le \frac{1}{2}|F_2(k')| \;, \qquad \text{for } |k'| \le t_0 \;. \tag{3.25}$$

It immediately follows that

$$\begin{aligned} |\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')| &\le C \frac{\sqrt{k_0^2 + (v_0 k')^2}}{k_0^2 + (v_0 k')^2 + \sigma^2} f_0(k') \;, \\ |\tilde g^{(0)}_{\omega,-\omega}(\mathbf{k}')| &\le C \frac{|\sigma|}{k_0^2 + (v_0 k')^2 + \sigma^2} f_0(k') \;. \end{aligned} \tag{3.26}$$

where C denotes a suitable constant. From now on, for simplifying the notation, the symbol C will be used everywhere to denote a generic constant, that we do not need to better specify.

It is also easy to prove that

$$|\tilde g^{(1)}_{1,1}(\mathbf{k}')| \le C \frac{1 - f_0(k')}{\sqrt{|k_0|^2 + (v_0 k')^2}} \;, \tag{3.27}$$

3.6. We now insert (3.19) and (3.20) in (3.17) and represent the result of the integration in terms of Feynman graphs, by using $\tilde P(d\psi^{(0)})$ as the free measure and $\tilde{\mathcal V}^{(0)}(\psi^{(0)})$ as the effective potential; then we apply (3.2). By proceeding as in [BGM], it is easy to show that we get an expansion for ρ_x, which can be described in the following way.

3.7. A graph ϑ of order $q \ge 1$ is a chain of $q+1$ lines $\ell_1, \ldots, \ell_{q+1}$ connecting a set of q ordered points (*vertices*) $v_1, \ldots, v_q$, so that ℓ_i enters v_i and ℓ_{i+1} exits from v_i, $i \le q$; the lines ℓ_1 and ℓ_{q+1} are the *external lines* of the graph and both have a free extreme, while the others are the *internal lines*; we shall denote $\mathrm{int}(\vartheta)$ the set of all internal lines. We say that $v_i < v_j$ if v_i precedes v_j and we denote v'_j the vertex immediately following v_j, if $j < q$. We denote also by ℓ_v the line entering the vertex v, so that $\ell_i \equiv \ell_{v_i}$, $1 \le i \le q$. We say that a line ℓ emerges from a vertex v if ℓ either enters v ($\ell = \ell_v$) or exits from v ($\ell = \ell_{v'}$). By a slight abuse of notation, if $v = v_q$, we still denote by $\ell_{v'_q}$ the line ℓ_{q+1} exiting from v_q even if there is no vertex v_{q+1}.

We shall say that ϑ is a *labeled graph* of order $q \ge 1$, if ϑ is a graph of order q, to which the following *labels* are associated:
- a label n_v for each vertex,
- a *frequency* (or *scale*) label h_ℓ for each (internal or external) line, with the constraint that, if $n_v = \pm 1$ for some v, then $h_{\ell_v} = h_{\ell_{v'}} = 0$ is not possible,
- for each line ℓ, two labels ω^1_ℓ, ω^2_ℓ, such that $\omega^1_\ell = \omega^2_\ell = 1$ if $h_\ell = 1$,
- a momentum $k_{\ell_1} = k = k' + \omega_1 p_F$ for the first line,
- a momentum

$$k_{\ell_v} = k' + \sum_{\tilde v < v} \Big[2 n_{\tilde v} p_F + (\omega^2_{\ell_{\tilde v}} - \omega^1_{\ell_{\tilde v'}}) p_F \Big] \tag{3.28}$$

for each internal line,
- a momentum

$$k_{\ell_{q+1}} = k' + \sum_{v \in \vartheta} \Big[2 n_{\tilde v} p_F + (\omega^2_{\ell_{\tilde v}} - \omega^1_{\ell_{\tilde v'}}) p_F \Big] \tag{3.29}$$

for the last line.

If $\tilde g^{(h_\ell)}_{\omega^1_\ell, \omega^2_\ell}(\mathbf{k}'_\ell)$ denotes the propagator associated with the line ℓ, we will use the shorthand $\tilde g_\ell = \tilde g^{(h_\ell)}_{\omega^1_\ell, \omega^2_\ell}(\mathbf{k}'_\ell)$.

Let us call $\mathcal{T}_{n,q}$ the set of the labeled graphs of order q and such that

$$\sum_{v\in\vartheta} 2n_v p_F + \sum_{\ell\in\vartheta}\left(\omega_\ell^2-\omega_\ell^1\right)p_F = 2np_F \mod 2\pi \ . \tag{3.30}$$

Then, if $\hat\rho_n(\varphi,\mu)$ is defined as in (1.11), we have

$$\begin{aligned}
\hat\rho_n(\varphi,\mu) &= \lim_{\beta\to\infty}\left[-\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\sum_{\omega=\pm1}\delta_{n,\omega}\tilde g_{-\omega,\omega}(\mathbf{k}')\ +\ \sum_{q=1}^\infty \rho_n^q(\sigma,\Phi)\right]\ ,\\
\rho_n^q(\sigma,\Phi) &= \sum_{\vartheta\in\mathcal{T}_{n,q}} \mathrm{Val}(\vartheta)\ ,
\end{aligned}\tag{3.31}$$

where

$$\mathrm{Val}(\vartheta) = -\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\Big(\prod_{i=1}^{q+1}\tilde g_{\ell_i}\Big)\Big(\prod_{v\in\vartheta}\lambda\hat\varphi_{n_v}\Big)\ . \tag{3.32}$$

Hence, the function $\tilde\rho_n(\sigma,\Phi)$ defined in (2.1) can be written as

$$\begin{aligned}
\tilde\rho_n(\sigma,\Phi) &= \lim_{\beta\to\infty}\sum_{q=1}^\infty \tilde\rho_n^q(\sigma,\Phi)\\
\tilde\rho_n^q(\sigma,\Phi) &= \rho_n^q(\sigma,\Phi)\ ,\quad \text{if } q\ge 2\\
\tilde\rho_n^1(\sigma,\Phi) &= \sum_{\vartheta\in\mathcal{T}_{n,1}}(1-\delta_{n,n_v})\mathrm{Val}(\vartheta)\ .
\end{aligned}\tag{3.33}$$

after substituting in the r.h.s. of (3.32) $\lambda\hat\varphi_{n_v}$ either with Φ_{n_v}, if $|n_v|>1$, or with σ, if $|n_v|=1$.

4. First order graphs

4.1. In this section we study the first order contributions to the density, *i.e.* the terms corresponding to graphs with only one vertex in the perturbative expansion (3.31), calculated in the limit $\beta \to \infty$. For these graphs we have, if $L = L_i = iQ$,

$$\lim_{\beta\to\infty} \mathrm{Val}(\vartheta) = \\ -\lambda\hat\varphi_m \frac{1}{L} \sum_{k'\in\mathcal{D}'_L} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi}\, \tilde g^{(h)}_{\omega_1,\omega'_1}(\mathbf{k}')\, \tilde g^{(h')}_{\omega_2,\omega'_2}(\mathbf{k}' + (2m+\omega'_1-\omega_2)\mathbf{p}_F)\ , \tag{4.1}$$

where $\mathcal{D}'_L$ is the set of possible values of the variable k' introduced before (3.14) as the difference between the "space momentum" k and $\pm p_F$. Since $p_F = \pi\rho = \pi P/Q = (2\pi/L)(iP/2)$, we have

$$\mathcal{D}'_L = \{k' = \frac{2\pi}{L}(n+\delta/2), n\in\mathbb{Z}, -[L/2] \le n \le [(L-1)/2]\}\ , \tag{4.2}$$

where $\delta = 1$, if iP (the number of particles) is odd, while $\delta = 0$, if iP is even.

Note that the value of δ will be in general not relevant, except in the proof of Lemma 2.13 in §4.10, the only place were there is a non trivial dependence on the volume.

Note also that, if the graph value (4.1) contributes to $\hat\rho_n(\varphi,\mu)$, then

$$2mp_F = 2np_F + (\omega_1 - \omega'_1 + \omega_2 - \omega'_2)p_F \bmod 2\pi\ . \tag{4.3}$$

The r.h.s of (4.1) can be easily bounded, by using (3.27) and (3.26) and the remark that $\lim_{p_F\to 0} t_0/v_0 = 1/2$. If $h = h' = 1$, one gets, for any integer r,

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left| \tilde g^{(1)}_{1,1}(\mathbf{k}')\, \tilde g^{(1)}_{1,1}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \\ C\int_{-\pi}^{\pi} dk' \int_{-\infty}^{\infty} dk_0 \frac{[1-f_0(k')]}{\sqrt{k_0^2+(v_0k')^2}} \frac{[1-f_0(k'+2rp_F)]}{\sqrt{k_0^2+v_0^2(k'+2rp_F)^2}} \le \\ C\int_{t_0}^{2\pi} dk' \int_0^{\infty} \frac{dk_0}{k_0^2+(v_0k')^2} \le \frac{C}{v_0}\left(1+\log\frac{1}{v_0}\right)\ . \tag{4.4}$$

If $h = 0$, $h' = 1$, for any ω, ω', one gets

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left| \tilde g^{(0)}_{\omega,\omega'}(\mathbf{k}')\, \tilde g^{(1)}_{1,1}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \\ C\int_0^{t_0} \frac{dk'}{2\pi} \int_0^{\infty} \frac{dk_0}{\sqrt{k_0^2+(v_0k')^2}\sqrt{k_0^2+v_0^4}} \le \frac{C}{v_0}\ . \tag{4.5}$$

The bound (4.5) can be improved for $\sigma \to 0$, if $\omega' = -\omega$; we have

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left| \tilde g^{(0)}_{\omega,-\omega}(\mathbf{k}')\, \tilde g^{(1)}_{1,1}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \\ C|\sigma| \int_0^{t_0} \frac{dk'}{2\pi} \int_0^{\infty} \frac{dk_0}{[k_0^2+(v_0k')^2+\sigma^2]\sqrt{k_0^2+v_0^4}} \le \frac{C|\sigma|}{v_0^3}\left(1+\log\frac{v_0^2}{|\sigma|}\right)\ . \tag{4.6}$$

Let us now consider the case $h = h' = 0$; for any ω_i, ω'_i, we get

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left| \tilde g^{(0)}_{\omega_1,\omega'_1}(\mathbf{k}')\, \tilde g^{(0)}_{\omega_2,\omega'_2}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \\ C\int_{-\pi}^{\pi} dk' \int_{-\infty}^{\infty} dk_0 \frac{f_0(k')}{\sqrt{k_0^2+(v_0k')^2+\sigma^2}} \frac{f_0(k'+2rp_F)}{\sqrt{k_0^2+v_0^2(k'+2rp_F)^2+\sigma^2}} \le \\ C\int_0^{t_0} dk' \int_0^{\infty} \frac{dk_0}{k_0^2+(v_0k')^2+\sigma^2} \le \frac{C}{v_0}\left(1+\log\frac{v_0^2}{|\sigma|}\right)\ . \tag{4.7}$$

If $h = h' = 0$ and $\omega_1 \neq \omega_1'$ or $\omega_2 \neq \omega_2'$, the last bound can be improved; in fact we get

$$\begin{aligned} &\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left|\tilde{g}^{(0)}_{\omega_1,\omega_1'}(\mathbf{k}')\,\tilde{g}^{(0)}_{\omega_2,\omega_2'}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \\ &C|\sigma|\int_0^{t_0}dk'\int_0^{\infty}\frac{dk_0}{[k_0^2+(v_0k')^2+\sigma^2]^{3/2}} \le \frac{C}{v_0}\,. \end{aligned} \tag{4.8}$$

The previous bound can be further improved, if we suppose also that $r \neq 0$, by taking into account that, in this case, $\max\{|k'|, |k'+2rp_F|\} \ge \pi/Q$. Let us suppose, for example, that $\omega_2 = -\omega_2'$; we have

$$\begin{aligned} &\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left|\tilde{g}^{(0)}_{\omega_1,\omega_1'}(\mathbf{k}')\,\tilde{g}^{(0)}_{\omega_2,-\omega_2}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \\ &C|\sigma|\int_{-\pi}^{\pi}dk'\int_0^{\infty}dk_0\frac{f_0(k')}{\sqrt{k_0^2+(v_0k')^2+\sigma^2}}\frac{f_0(k'+2rp_F)}{[k_0^2+v_0^2(k'+2rp_F)^2+\sigma^2]} \le \\ &\frac{C|\sigma|Q}{v_0}\int_0^{t_0}dk'\int_0^{\infty}\frac{dk_0}{k_0^2+(v_0k')^2+\sigma^2} \le \frac{C|\sigma|Q}{v_0^2}\left(1+\log\frac{v_0^2}{|\sigma|}\right)\,. \end{aligned} \tag{4.9}$$

In the following we shall need also the bounds of the expression obtained substituting in the r.h.s. of (4.1) one of the two propagators with its derivative with respect to σ. By proceeding as before, one can easily prove that, for any ω and any integer r,

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left|\frac{\partial\tilde{g}^{(0)}_{\omega,\omega}(\mathbf{k}')}{\partial\sigma}\,\tilde{g}^{(1)}_{1,1}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \frac{C}{v_0^3}\,, \tag{4.10}$$

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left|\frac{\partial\tilde{g}^{(0)}_{\omega,-\omega}(\mathbf{k}')}{\partial\sigma}\,\tilde{g}^{(1)}_{1,1}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \frac{C}{v_0^3}\left(1+\log\frac{v_0^2}{|\sigma|}\right)\,; \tag{4.11}$$

that, for any ω_i, ω_i' and any integer r,

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left|\frac{\partial\tilde{g}^{(0)}_{\omega_1,\omega_1'}(\mathbf{k}')}{\partial\sigma}\,\tilde{g}^{(0)}_{\omega_2,\omega_2'}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \frac{C}{v_0|\sigma|}\,, \tag{4.12}$$

and finally that, for any $\omega_1, \omega_1', \omega$ and any integer $r \neq 0$,

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left|\frac{\partial\tilde{g}^{(0)}_{\omega_1,\omega_1'}(\mathbf{k}')}{\partial\sigma}\,\tilde{g}^{(0)}_{\omega,-\omega}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \frac{CQ}{v_0^2}\,, \tag{4.13}$$

$$\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left|\tilde{g}^{(0)}_{\omega_1,\omega_1'}(\mathbf{k}')\,\frac{\partial\tilde{g}^{(0)}_{\omega,-\omega}}{\partial\sigma}(\mathbf{k}'+2r\mathbf{p}_F)\right| \le \frac{CQ}{v_0^2}\,. \tag{4.14}$$

4.2. Remark. All the previous bounds are valid also if we exchange in the l.h.s. $\mathbf{k}'$ with $\mathbf{k}'+2r\mathbf{p}_F$; this immediately follows from the observation that the variable k' is defined modulo 2π.

4.3. We shall now consider the graphs contributing to the constants $c_n(\sigma)$ introduced in (2.1), in order to prove Lemma 2.6. We can write

$$-\lambda\hat{\varphi}_n\,c_n(\sigma) = \sum_{\vartheta\in\mathcal{T}_{n,1}}\delta_{n,n_v}\mathrm{Val}(\vartheta)\,. \tag{4.15}$$

The equations (4.15), (4.1) and (4.3) imply, if $|n|>2$,

$$
\begin{aligned}
c_n(\sigma) =&\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\Big\{\tilde g^{(1)}_{1,1}(\mathbf{k}')\,\tilde g^{(1)}_{1,1}(\mathbf{k}'+2n\mathbf{p}_F)\\
+&\sum_{\omega=\pm1}\Big[\tilde g^{(1)}_{1,1}(\mathbf{k}')\,\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}'+2n\mathbf{p}_F+(1-\omega)\mathbf{p}_F)\\
&+\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')\,\tilde g^{(1)}_{1,1}(\mathbf{k}'+2n\mathbf{p}_F-(1-\omega)\mathbf{p}_F)+\tilde g^{(0)}_{\omega,-\omega}(\mathbf{k}')\,\tilde g^{(0)}_{-\omega,\omega}(\mathbf{k}'+2n\mathbf{p}_F)\\
&+\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')\,\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}'+2n\mathbf{p}_F)+\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')\,\tilde g^{(0)}_{-\omega,-\omega}(\mathbf{k}'+(2n+2\omega)\mathbf{p}_F)\Big]\Big\}\;.
\end{aligned}
\tag{4.16}
$$

By using the bounds (4.4), (4.5) and (4.8), we see that the first four terms in the r.h.s. of (4.16) are bounded by $(C/v_0)(1+\log v_0^{-1})$. However, the remaining terms, *i.e.* those with $h=h'=0$ and $\omega_1-\omega_1'=\omega_2-\omega_2'=0$, need a more careful analysis; these terms will be denoted as

$$
a_n\equiv\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\,\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')\,\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}'+2n\mathbf{p}_F)\;,\tag{4.17}
$$

when $\omega_1=\omega_2$, and

$$
b_{n,\omega}=\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\,\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')\,\tilde g^{(0)}_{-\omega,-\omega}(\mathbf{k}'+(2n+2\omega)\mathbf{p}_F)\;.\tag{4.18}
$$

when $\omega_1=-\omega_2$. The following two Lemmata 4.5 and 4.7 show that the dimensional bounds which would follow from (4.7) in fact can be improved.

4.4. Remark. Note that a_n is a ω-independent quantity, so that we can set $\omega=1$ in (4.17); this property easily follows from the observation that $g^{(0)}_{\omega,\omega}(k',k_0)=g^{(0)}_{-\omega,-\omega}(-k',k_0)$, see (3.23). It is also easy to prove that $b_{n,1}=b_{-n,-1}$.

4.5. Lemma. *Let $|n|\ge2$ and let a_n be defined as in (4.17); then $|a_n|<C/v_0$.*

4.6. *Proof of Lemma 4.5.* By Remark 4.4, it is enough to study the case $\omega=1$ in (4.17). Define

$$
\begin{aligned}
\bar g^{(0)}_{\omega,\omega}(\mathbf{k}')&=\frac{f_0(k')}{-ik_0+\mathcal{F}(\omega k')}\;,\\
\mathcal{F}(k')&\equiv\operatorname{sign}(k')\sqrt{F_2^2(k')+\sigma_0^2(k')}-F_1(k')\;,
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
\bar a_n&\equiv\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\bar g^{(0)}_{1,1}(\mathbf{k}')\bar g^{(0)}_{1,1}(\mathbf{k}'+2n\mathbf{p}_F)\\
&=\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}f_0(k')\,f_0(k'+2np_F)\,\mathcal{A}_n(k')\;.
\end{aligned}
\tag{4.20}
$$

where $\mathcal{A}_n(k')$ is obtained by explicitly performing the integral on k_0. It is easy to see that, defining

$$
s(k')=\operatorname{sign}\Big(\mathcal{F}(k')\Big)\;,\tag{4.21}
$$

if $s(k')=s(k'+2np_F)$, one has $\mathcal{A}_n(k')=0$, while, if $s(k')=-s(k'+2np_F)$, one has

$$
\mathcal{A}_n(k')=s(k')\Big[\mathcal{F}(k'+2np_F)-\mathcal{F}(k')\Big]^{-1}\;.\tag{4.22}
$$

Note that, by (3.25), $s(k')=\operatorname{sign}(k')$, if $|k'|\le t_0$, *i.e.* on the support of $f_0(k')$; hence we have

$$
\bar a_n=-\frac{1}{L}\sum_{k'\in\mathcal{D}'_L\cap\mathcal{D}'_*}\frac{f_0(k')\,f_0(k'+2np_F)}{|\mathcal{F}(k')|+|\mathcal{F}(k'+2np_F)|}\;,\tag{4.23}
$$

where

$$\mathcal{D}'_* = \{k' \in [-t_0, t_0] \ : \ \mathrm{sign}\,(k') = -\mathrm{sign}\,(k' + 2np_F)\} \ . \tag{4.24}$$

We want to show that

$$\max\{|\mathcal{F}(k')|, |\mathcal{F}(k' + 2np_F)|\} \geq \frac{c_2}{2}\|2np_F\|_{\mathbb{T}_1} \equiv \Delta_1 \ , \tag{4.25}$$

if $k' \in \mathcal{D}'_*$, $k' + 2np_F \in [-t_0, t_0]$ and $c_2 = (\sqrt{2}/\pi)v_0$.

If $|\mathcal{F}(k')| \geq \Delta_1$, (4.25) is immediately verified. Let us suppose now that $|\mathcal{F}(k')| < \Delta_1$; then, by using (3.25), we get

$$|\mathcal{F}(k')| \geq |F_2(k')| - |F_1(k')| \geq \frac{1}{2}|F_2(k')| > c_2|k'| \ , \tag{4.26}$$

so that

$$|k'| < \frac{\Delta_1}{c_2} = \frac{1}{2}\|2np_F\|_{\mathbb{T}_1} \ , \tag{4.27}$$

implying

$$\|k' + 2np_F\|_{\mathbb{T}_1} \geq \Big|\|2np_F\|_{\mathbb{T}_1} - |k'|\Big| \geq \frac{1}{2}\|2np_F\|_{\mathbb{T}_1} \ . \tag{4.28}$$

Moreover, since $\|k' + 2np_F\|_{\mathbb{T}_1} \leq t_0$, then

$$|\mathcal{F}(k' + 2np_F)| > c_2\|k' + 2np_F\|_{\mathbb{T}_1} \ ; \tag{4.29}$$

hence, by using (4.28) and (4.29), we get

$$|\mathcal{F}(k' + 2np_F)| \geq \frac{c_2}{2}\|2np_F\|_{\mathbb{T}_1} = \Delta_1 \ , \tag{4.30}$$

which implies (4.25) also when $|\mathcal{F}(k')| < \Delta_1$.

Inserting (4.25) into (4.23) leads to

$$|\bar{a}_n| \leq \int_{\mathcal{D}'_*} \frac{dk'}{\pi c_2 \|2np_F\|_{\mathbb{T}_1}} \leq \frac{\sqrt{2}}{v_0} \ , \tag{4.31}$$

as the size of the set $\mathcal{D}'_*$ is bounded by $2\|2np_F\|_{\mathbb{T}_1}$.

In order to complete the proof of Lemma 4.5, we note that

$$\begin{aligned} a_n - \bar{a}_n = \frac{1}{L}\sum_{k' \in \mathcal{D}'_L} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \Big\{ &\big[\tilde{g}^{(0)}_{1,1}(\mathbf{k}') - \bar{g}^{(0)}_{1,1}(\mathbf{k}')\big]\tilde{g}^{(0)}_{1,1}(\mathbf{k}' + 2n\mathbf{p}_F) \\ &+ \bar{g}^{(0)}_{1,1}(\mathbf{k}')\big[\tilde{g}^{(0)}_{1,1}(\mathbf{k}' + 2n\mathbf{p}_F) - \bar{g}^{(0)}_{1,1}(\mathbf{k}' + 2n\mathbf{p}_F)\big]\Big\} \ . \end{aligned} \tag{4.32}$$

Moreover, by (3.23) and (4.19),

$$|\tilde{g}^{(0)}_{1,1}(\mathbf{k}') - \bar{g}^{(0)}_{1,1}(\mathbf{k}')| = \frac{\sqrt{F_2(k')^2 + \sigma_0^2(k')} - |F_2(k')|}{\big|[-ik_0 - F_1(k')]^2 - [F_2^2(k') + \sigma_0^2(k')]\big|}|f_0(k')| \ , \tag{4.33}$$

so that, by using also (3.25), we get

$$|a_n - \bar{a}_n| \leq C\int_0^{t_0} dk' \int_0^{\infty} dk_0 \frac{\sigma^2}{\sqrt{(v_0k')^2 + \sigma^2}\,[k_0^2 + (v_0k')^2 + \sigma^2]^{3/2}} \leq \frac{C}{v_0} \ . \tag{4.34}$$

The bounds (4.31) and (4.34) imply Lemma 4.5. ∎

4.7. LEMMA. *Let $|n| \geq 2$, and let $b_{n,\omega}$ be defined as in (4.18); then* $|b_{n,\omega}| \leq (C\log Q)/v_0$.

4.8. *Proof of Lemma 4.7.* Let us define $\bar{b}_{n,\omega}$ as

$$\bar{b}_{n,\omega} = \frac{1}{L}\sum_{k' \in \mathcal{D}'_L} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi}\, \bar{g}^{(0)}_{\omega,\omega}(\mathbf{k}')\, \bar{g}^{(0)}_{-\omega,-\omega}(\kappa'_\omega, k_0) \ , \tag{4.35}$$

where $\kappa'_\omega = k' + (2n+2\omega)p_F$ and $\bar g^{(0)}_{\omega,\omega}(\mathbf{k}')$ is defined in (4.19), and set $b_{n,\omega} = \bar b_{n,\omega} + \mathcal{I}'$. By dimensional bounds analogous to those which led to (4.34), it is easy to prove that $|\mathcal{I}'| \le C/v_0$. Moreover, by proceeding as in §4.6, we see that

$$\bar b_{n,\omega} = -\frac{1}{L}\sum_{k'\in\mathcal{D}'_L\cap\mathcal{D}'_\omega} \frac{f_0(k')\,f_0(\kappa'_\omega)}{|\mathcal{F}(\omega k')| + |\mathcal{F}(-\omega\kappa'_\omega)|}\;, \tag{4.36}$$

where $\mathcal{D}'_\omega = \{k' \in [-t_0, t_0] \;:\; \mathrm{sign}\,(\omega k') = \mathrm{sign}\,(\omega\kappa'_\omega)\}$.

By using the bound (4.26), we have

$$|\mathcal{F}(\omega k')| + |\mathcal{F}(-\omega\kappa'_\omega)| \ge c_2(|k'| + \|\kappa'_\omega\|_{\mathbb{T}_1}) \tag{4.37}$$

Moreover, since $|n|\ge 2$, $\|2np_F \pm 2p_F\|_{\mathbb{T}_1} \ge \frac{2\pi}{Q}$; hence

$$|\bar b_{n,\omega}| \le \int_{\mathcal{D}'_\omega} \frac{dk'}{2\pi c_2(|k'| + \|\kappa'_\omega\|_{\mathbb{T}_1})} \le \frac{C\log Q}{v_0}\;. \tag{4.38}$$

This completes the proof of Lemma 4.7. ∎

4.9. *Proof of Lemma 2.6.* The bound (2.6) immediately follows from the remark after (4.16), Lemma 4.5 and Lemma 4.7. The bound (2.8) is easily proven from (4.16) by using the bounds (4.10)÷(4.12). ∎

4.10. *Proof of Lemma 2.13.* The definition of $c_1(\sigma)$ in §2.3, (3.31), (4.1) and (4.3) imply that

$$\begin{aligned} c_1(\sigma) &= \frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\Big\{\frac{\tilde g^{(0)}_{-1,1}(\mathbf{k}')}{\sigma} + \tilde g^{(1)}_{1,1}(\mathbf{k}')\,\tilde g^{(1)}_{1,1}(\mathbf{k}'+2\mathbf{p}_F) \\ &\quad + \sum_{\omega=\pm1}\Big[\tilde g^{(1)}_{1,1}(\mathbf{k}')\,\tilde g^{(0)}_{\omega,\omega}(\mathbf{k}'+(3-\omega)\mathbf{p}_F) + \tilde g^{(0)}_{\omega,\omega}(\mathbf{k}')\,\tilde g^{(1)}_{1,1}(\mathbf{k}'+(1+\omega)\mathbf{p}_F)\Big]\Big\} \\ &\equiv -F(\sigma, L) + \tilde c_1(\sigma)\;, \end{aligned} \tag{4.39}$$

where $-F(\sigma,L)$ denotes the first term in the r.h.s. of (4.39), while $\tilde c_1(\sigma)$ is the sum of the other ones. It turns out that $-F(\sigma,L)$ is the leading term for $\sigma\to0$; moreover it is the only term whose dependence on L is not trivial, hence we decide to indicate it explicitly.

By using the definition (3.23) and by performing the integration over k_0, we get

$$F(\sigma,L) = \frac{1}{2L}\sum_{k'\in\mathcal{D}'_L}\frac{f_0(k')^2}{\sqrt{v_0^2\sin^2 k' + \sigma^2 f_0(k')^2}}\;. \tag{4.40}$$

The definition (4.2) of $\mathcal{D}'_L$ implies that, for any finite volume $L\equiv L_i$, the r.h.s. is singular for $\sigma\to0$, only if $\delta=0$, that is only if the number of Fermions is even; in that case, in fact, $k'=0$ belongs to the set $\mathcal{D}'_L$. It follows that, if $\delta=1$, the equation (2.5) has no solution for λ^2 very small, how small depending on L; this is the main source of the lower bound on λ of Theorem 1.6. [Equivalently, for fixed λ verifying the inequality to the right in (1.16), which is uniform in L, L has to be large enough so that also the inequality to the left in (1.16) can be fulfilled.] We separate the term with $k'=0$, if it is present, by writing

$$F(\sigma,L) = \frac{1-\delta}{2L\sigma} + F_0(\sigma,L)\;. \tag{4.41}$$

It is easy to see that

$$F_0(\sigma,L) = F_1(\sigma,L) + d_1(\sigma,L) + d_2(\sigma,L)\;, \tag{4.42}$$

where

$$F_1(\sigma, L) = \sum_{n\neq 0: |2\pi L^{-1}(n+\delta/2)|\le t_0/2} \frac{1}{\sqrt{(2\pi v_0)^2 (n+\delta/2)^2 + (\sigma L)^2}} \;, \tag{4.43}$$

$$d_1(\sigma, L) = \frac{1}{2L} \sum_{\substack{k'\in \mathcal{D}'_L \\ |k'|\ge t_0/2}} \frac{f_0(k')^2}{\sqrt{v_0^2 \sin^2 k' + \sigma^2 f_0(k')^2}} \;, \tag{4.44}$$

$$d_2(\sigma, L) = \frac{1}{2L} \sum_{\substack{k'\in \mathcal{D}'_L \\ 0\neq |k'|\le t_0/2}} \left[\frac{1}{\sqrt{v_0^2 \sin^2 k' + \sigma^2}} - \frac{1}{\sqrt{(v_0 k')^2 + \sigma^2}} \right] . \tag{4.45}$$

Note that the sum in the r.h.s. of (4.43) is empty, if $[t_0 L/(4\pi) + \delta/2] < 1$; in that case the equation (2.5) may have a solution, for λ small enough, only if $\delta = 0$. Hence we shall suppose that:

$$t_0 L \ge 4\pi \;, \tag{4.46}$$

a condition which is certainly verified, if the conditions (2.24) are satisfied, since

$$\frac{4}{\pi} \le v_0/t_0 \le 2 \;. \tag{4.47}$$

.

By using (4.47) and supposing that

$$\frac{|\sigma|}{v_0^2} \le 1 \;, \tag{4.48}$$

it is easy to show that

$$\sum_{i=1}^{2} |d_i(\sigma)| \le \frac{C}{v_0} \;, \qquad \sum_{i=1}^{2} \left| \frac{\partial d_i(\sigma)}{\partial \sigma} \right| \le \frac{C}{v_0^3} \;. \tag{4.49}$$

By substituting the sum in the r.h.s. of (4.43) with an integral, we can write

$$F_1(\sigma, L) = F_2(\sigma, L) + d_3(\sigma, L) \;, \tag{4.50}$$

where

$$F_2(\sigma, L) = \int_{1-\delta/2}^{t_0 L/(4\pi)} \frac{dx}{\sqrt{(2\pi v_0 x)^2 + (\sigma L)^2}} \;. \tag{4.51}$$

It is easy to see that, if the condition (4.48) is verified, together with

$$\frac{v_0}{L|\sigma|} \le \tilde\varepsilon \le 1 \;, \tag{4.52}$$

then

$$|d_3(\sigma)| \le \frac{C}{v_0} \;, \qquad \left| \frac{\partial d_3(\sigma)}{\partial \sigma} \right| \le \frac{C\tilde\varepsilon}{v_0 |\sigma|} \;. \tag{4.53}$$

The integral defining $F_2(\sigma, L)$ can be explicitly calculated; we get

$$F_2(\sigma, L) = \frac{1}{2\pi v_0} \log \frac{\frac{v_0 t_0}{2|\sigma|} + \sqrt{\left(\frac{v_0 t_0}{2\sigma}\right)^2 + 1}}{\frac{2\pi v_0}{L|\sigma|}\left(1 - \frac{\delta}{2}\right) + \sqrt{\left(\frac{2\pi v_0}{L|\sigma|}\right)^2 \left(1 - \frac{\delta}{2}\right)^2 + 1}} \;. \tag{4.54}$$

If we write

$$F_2(\sigma, L) = \frac{1}{2\pi v_0} \log \frac{v_0^2}{|\sigma|} + d_4(\sigma, L) \;, \tag{4.55}$$

it is easy to prove, using (4.47), (4.48) and (4.52), that

$$|d_4(\sigma)| \le \frac{C}{v_0} , \qquad \left|\frac{\partial d_4(\sigma)}{\partial\sigma}\right| \le \frac{C}{v_0}\left(1+\frac{\tilde\varepsilon}{|\sigma|}\right) . \tag{4.56}$$

Finally, the function $\tilde c_1(\sigma)$ introduced in (4.39) and its derivative can be bounded, by using (4.4), (4.5) and (4.10), as

$$|\tilde c_1(\sigma)| \le \frac{C}{v_0}\left(1+\log\frac{1}{v_0}\right) , \qquad \left|\frac{\partial \tilde c_1(\sigma)}{\partial\sigma}\right| \le \frac{C}{v_0^3} . \tag{4.57}$$

It is now sufficient to define

$$r_1(\sigma) = 2\pi v_0 \left[\sum_{i=1}^{4} d_i(\sigma, L) + \frac{1-\delta}{2L\sigma} - \tilde c_1(\sigma)\right] , \tag{4.58}$$

to complete the proof of Lemma 2.13. ■

5. Bounds on the density perturbative expansion

5.1. In this section we give some bounds about the perturbative expansion (3.33) of the function $\tilde{\rho}_n(\sigma,\Phi)$, introduced in (2.1), and we prove Lemmata 2.2, 2.8, 2.9 and 2.16.

Given $\Phi \in \mathcal{F}$, let us define

$$R(\Phi)_n^{(q)}(\sigma) = \tilde{\rho}_n^q(\sigma,\Phi(\sigma)) \ , \quad |n|>0, \quad q>0 \ . \tag{5.1}$$

Moreover, if $\mathcal{J}$ is the space of the C^1-functions of $\sigma \in J$ with values in $\mathbb{R}$, and $r(\sigma) \in \mathcal{J}$, we shall define, in agreement with (2.9),

$$\|r\|_{\mathcal{J}} \equiv \sup_{\sigma \in J} \left[|\sigma|^{-1}|r(\sigma)| + \left| \frac{\partial r}{\partial \sigma}(\sigma) \right| \right] \ . \tag{5.2}$$

5.2. Lemma. *If $\Phi \in \mathcal{B}$ and $|\sigma| \le v_0^2$, then, for any $n \ne 0$ and $q>0$,*

$$\begin{aligned} &\|R(\Phi)_n^{(q)}\|_{\mathcal{J}} \le \\ &Dv_0 \left(1+\log\frac{1}{v_0}\right) \left(\frac{C}{v_0^2}\right)^q q^2 \left(\frac{3q}{|n|}\right)^N \left[1+(1-\delta_{q,1})\log\frac{v_0^2}{|\sigma|}\right] (|\sigma|Q)^{[q/2]} \ , \end{aligned} \tag{5.3}$$

where C and D are suitable constants.

5.3. *Proof of Lemma 5.2.* In order to bound $\tilde{\rho}_n^q(\sigma,\Phi)$, we shall use the expansion in (3.33). Let $\vartheta \in \mathcal{T}_{n,q}$ be one of the graphs contributing to $\tilde{\rho}_n^q(\sigma,\Phi)$ and v one of its vertices. If $|n_v| \ne 1$, one has (see §3 for notations)

$$\|k'_{\ell_v} - k'_{\ell_{v'}}\|_{\mathbb{T}^1} = \|2n_v p + (\omega^2_{\ell_v} - \omega^1_{\ell_{v'}})p\|_{\mathbb{T}^1} \ge \frac{2\pi}{Q} \ , \tag{5.4}$$

so that

$$\max\{\|k'_{\ell_v}\|_{\mathbb{T}^1}, \|k'_{\ell_{v'}}\|_{\mathbb{T}^1}\} \ge \frac{\pi}{Q} \ . \tag{5.5}$$

Then there is a constant C_2 such that, $\forall v \in \vartheta$, $|n_v| \ne 1$, if $|\sigma| \le 1$,

$$\left|\tilde{g}_{\ell_v}\tilde{g}_{\ell_{v'}}\right| \le \frac{C_2 Q}{v_0^4|\sigma|} \ , \tag{5.6}$$

$$\min\left\{|\tilde{g}_{\ell_v}|, |\tilde{g}_{\ell_{v'}}|\right\} \le \frac{C_2 Q}{v_0^2} \ , \tag{5.7}$$

by (5.5), (3.27) and (3.26), since $v_0 \le 1$.

Note that (5.6) and (5.7) still hold for $|n_v|=1$, as, in such a case, $h_{\ell_v} = h_{\ell_{v'}} = 0$ is not allowed (see §3.7) and the support properties imply that both propagators are bounded by C/v_0^2.

Note also that, thanks to (3.30),

$$|n| \le q+1+\sum_v |n_v| \le 3\sum_v |n_v| \qquad \Rightarrow \qquad \exists v^* \ : \ |n_{v^*}| \ge \frac{|n|}{3q} \ . \tag{5.8}$$

Let us now suppose that $q=2\bar{q}$, with $\bar{q} \ge 1$. It is easy to see that, in this case, it is possible to couple $2\bar{q}$ among the $2\bar{q}+1$ propagators appearing in the expression of $\mathrm{Val}(\vartheta)$, see (3.32), in $\bar{q}$ pairs $\left\{\tilde{g}_{\ell_v}, \tilde{g}_{\ell_{v'}}\right\}$ with $v \ne v^*$; let $\tilde{g}_{\ell^{(1)}}$, $\ell^{(1)} = \ell_{v^*}$, the propagator left alone after this coupling operation. We select in an arbitrary way one of the $\bar{q}$ couples and we use the bound (5.7) for one of the propagators belonging to it; let $\tilde{g}_{\ell^{(2)}}$ the other propagator of the selected couple. The propagators of all the other couples will be bounded by (5.6). We get

$$|\mathrm{Val}(\vartheta)| \le (C_2Q)^{\bar{q}} \frac{|\sigma|^{-\bar{q}+1}}{v_0^{4\bar{q}-2}} \Big(\prod_{i=1}^{2\bar{q}} |\Phi_{n_{v_i}}|\Big) \frac{1}{L\beta} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} |\tilde{g}_{\ell^{(1)}}\tilde{g}_{\ell^{(2)}}| \ . \tag{5.9}$$

Let us now suppose that $|\sigma| \le v_0^2$; then we can use the bounds (4.4)–(4.8), valid also for finite β, to prove that, for any choice of $\tilde g_{\ell(1)}$ and $\tilde g_{\ell(2)}$,

$$\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} |\tilde g_{\ell(1)}\tilde g_{\ell(2)}| \le \frac{D_2}{v_0}\left(1+\log\frac{1}{v_0}\right)\left(1+\log\frac{v_0^2}{|\sigma|}\right) . \tag{5.10}$$

Hence, if $\Phi\in\mathcal{B}$, by using (2.10), (5.8) and (5.10), we get

$$\begin{aligned}\sum_{\vartheta\in\mathcal{T}_{n,2\bar q}} |\mathrm{Val}(\vartheta)| \le \frac{D_2}{v_0^{2q-1}}\left(1+\log\frac{1}{v_0}\right)&\\ D_1^q q C_2^{\bar q}|\sigma|(|\sigma|Q)^{\bar q}\left(1+\log\frac{v_0^2}{|\sigma|}\right)\left(\frac{3q}{|n|}\right)^N&,\end{aligned} \tag{5.11}$$

where $D_1^q q$ takes into account the fact that there are 5 possible choices for the $\omega_\ell^1,\omega_\ell^2,h_\ell$ labels for each line, and q possible choice for the vertex v^*; then the bound (5.3) is proved for even q.

The case $q=2\bar q+1$, with $\bar q\ge 1$, can be treated in a similar way. We note that it is always possible to couple $2\bar q$ among the $2\bar q+2$ propagators appearing in the expression of $\mathrm{Val}(\vartheta)$ in $\bar q$ pairs $\{\tilde g_{\ell_v},\tilde g_{\ell_{v'}}\}$ with $v\ne v^*$; let $\tilde g_{\ell(1)}$ and $\tilde g_{\ell(2)}$ be the propagators left alone after this coupling operation. Then we use (5.6) for all the couples and the bound (5.10) for the two remaining propagators. We get a bound similar to (5.9), with $|\sigma|^{-\bar q}$ in place of $|\sigma|^{-\bar q+1}$, but the final bound is the same as before.

We still have to consider the case $q=1$. We could of course get again the previous bound with $\bar q=0$, but there is now an improvement, which will play an important role. The improvement follows from the observation that, if $q=1$, the graphs contributing to $\tilde\rho_n^1$ have only one vertex with Fourier index $n_{v_1}\ne n$, so that at least one of the two propagators must have different ω-indices. By using the bound (4.8), this implies that the bound (5.10) can be improved by erasing the factor $[1+\log(v_0^2/|\sigma|)]$.

In order to complete the proof of (5.3), we have to bound also $\partial\tilde\rho_n^q(\sigma,\Phi(\sigma))/\,\partial\sigma$. We can proceed as before, by noticing that $\partial\mathrm{Val}(\theta)/\partial\sigma$ can be written as the sum of $2q+1$ terms, each term differing from $\mathrm{Val}(\theta)$ only because there is the derivative acting on a single propagator or a single vertex function. If the derivative acts on one of the coupled propagators, one can use the bounds (5.6) and (5.7) modified so that the r.h.s. is multiplied by $|\sigma|^{-1}$; if the derivative acts on a vertex function, since $\Phi\in\mathcal{B}$, one can use the bound $|\partial\Phi_n(\sigma)/\partial\sigma|\le|n|^{-N}$; if the derivative acts on one of the propagators left alone after the coupling operation, one can use the bound, following from (4.10)÷(4.12), if $|\sigma|\le v_0^2$,

$$\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\left|\frac{\partial\tilde g_{\ell(1)}}{\partial\sigma}\tilde g_{\ell(2)}\right| \le \frac{D_3}{v_0|\sigma|} . \tag{5.12}$$

We get, for any $q>0$,

$$\begin{aligned}\left|\frac{\partial R(\Phi)_n^{(q)}}{\partial\sigma}(\sigma)\right| \le \sum_{\vartheta\in\mathcal{T}_{n,2\bar q}}\left|\frac{\partial\mathrm{Val}(\vartheta)}{\partial\sigma}\right| \le&\\ (2q+1)\frac{D_3}{v_0^{2q-1}}D_1^q q C_2^{\bar q}(|\sigma|Q)^{\bar q}\left(\frac{3q}{|n|}\right)^N&,\end{aligned} \tag{5.13}$$

with $\bar q=[q/2]$. This complete the proof of (5.3). ■

5.4. *Proof of Lemma 2.2.* The bound (5.3) immediately implies that $\tilde\rho_n^q(\sigma,\Phi)$ is summable over q, for $\sigma Q/v_0^4$ small enough, uniformly in i, ρ and β. On the other end, it is easy to see that the bound is valid also if we substitute in the expression (3.32) of $\mathrm{Val}(\vartheta)$ the sum over k_0 with the integral on the real axis and that $\lim_{\beta\to\infty}\sum_{q=1}^\infty\tilde\rho_n^q(\sigma,\Phi)$ is obtained from

$\sum_{q=1}^{\infty}\tilde\rho_n^q(\sigma,\Phi)$ by doing this substitution. The claim of the Lemma about the continuous dependence on λ of $\hat\rho_n(\varphi,\mu)$ is an easy consequence of this remark and (3.31).

In a similar way, one can see that $\lim_{i\to\infty}\hat\rho_n(\varphi,\mu)$ is obtained by substituting in the expression of $\mathrm{Val}(\vartheta)$ the sum over k' with an integral over the interval $[-\pi,\pi]$ and that this limit is also continuous in λ near 0.

The other claims of the Lemma about the density and the gap of $\mathbf{h}$ can be proved as in [BGM], §4.5 and §4.6. In [BGM] a more complicated expansion was used (involving a further decomposition of the field $\psi^{(0)}$), but the proof of these two points can be even more simply reached by using the expansion of this paper and bounds of the graphs similar to (5.11). We shall not give here the details, but we only remark that the main point in the proof is the remark that the propagators (3.23) are analytic in k_0 in the strip $|\mathrm{Im}\,k_0|\le|\sigma|/2$. ■

5.5. *Proof of Lemma 2.8.* By the remark in §5.4, $\lim_{\beta\to\infty}$ can be exchanged with the sum over q in (3.31) and (3.33). In the following, for simplicity, we shall use the notation $\tilde\rho_n^q(\sigma,\Phi)$ to identify $\lim_{\beta\to\infty}\tilde\rho_n^q(\sigma,\Phi)$.

Let us suppose that $\Phi,\Phi'\in\mathcal{B}$ and $q\ge 2$; then, by (3.31)

$$\tilde\rho_n^q(\sigma,\Phi')-\tilde\rho_n^q(\sigma,\Phi)=\sum_{\vartheta\in\mathcal{T}_{n,q}(\Phi,\Phi')}\mathrm{Val}(\vartheta)\;, \tag{5.14}$$

where $\mathcal{T}_{n,q}(\Phi,\Phi')$ is a set of labeled graphs whose definition differs from the definition of $\mathcal{T}_{n,q}$, see §3.7, only because there is a new label $\alpha_v\in\{0,1,2\}$ for each vertex; moreover

$$\mathrm{Val}(\vartheta)=-\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\Big(\prod_{i=1}^{q+1}\tilde g_{\ell_i}\Big)\Big(\prod_{v\in\vartheta}F_{n_v}^{\alpha_v}\Big)\;, \tag{5.15}$$

$$F_{n_v}^{\alpha_v}=\begin{cases}\sigma & \text{if } \alpha_v=0\;,\\ \Phi_{n_v} & \text{if } \alpha_v=1\;,\\ \Phi'_{n_v}-\Phi_{n_v} & \text{if } \alpha_v=2\;,\end{cases} \tag{5.16}$$

and there is the constraint that at least one vertex has label $\alpha_v=2$.

By proceeding as in the proof of Lemma 5.2 and using the definitions (2.9) and (5.1), we get, if $q\ge2$,

$$\begin{aligned}&\|R(\Phi')^{(q)}-R(\Phi)^{(q)}\|_{\mathcal{F}}\le\\ &D\,v_0\left(1+\log\frac{1}{v_0}\right)\left(\frac{C}{v_0^2}\right)^q\|\Phi'-\Phi\|_{\mathcal{F}}\,q^2(3q)^N\left(1+\log\frac{v_0^2}{|\sigma|}\right)(|\sigma|Q)^{[q/2]}\;.\end{aligned} \tag{5.17}$$

Hence, if $Qv_0^{-4}|\sigma|[1+\log(v_0^2/|\sigma|)]\le1$ and $|\sigma|Qv_0^{-4}\le1/(2C^2)$, we have

$$\sum_{q=2}^{\infty}\|R(\Phi')^{(q)}-R(\Phi)^{(q)}\|_{\mathcal{F}}\le\frac{C_1}{v_0}\left(1+\log\frac{1}{v_0}\right)3^N N!\,\|\Phi'-\Phi\|_{\mathcal{F}}\;. \tag{5.18}$$

with a suitable constant C_1.

In order to complete the proof of the lemma, we have to estimate $\|R(\Phi')^{(1)}-R(\Phi)^{(1)}\|_{\mathcal{F}}$. The bound (5.17), with $q=1$, is still valid, but it is not sufficient; however there is the improvement with respect to (5.17) due to the fact that, if ϑ is a graph contributing to $\tilde\rho_n^1(\sigma,\Phi')-\tilde\rho_n^1(\sigma,\Phi)$, the only vertex belonging to ϑ has a Fourier index $n_{v_1}\ne n$. As in the proof of Lemma 5.2, this remark allows to eliminate the factor $[1+\log(v_0^2/|\sigma|)]$ in the bound (5.17) for any value of n. The previous remark implies that

$$\|R(\Phi')^{(1)}-R(\Phi)^{(1)}\|_{\mathcal{F}}\le\frac{C}{v_0}\left(1+\log\frac{1}{v_0}\right)\|\Phi'-\Phi\|_{\mathcal{F}}\;. \tag{5.19}$$

This bound and (5.18) immediately imply Lemma 2.8. ∎

5.6. *Proof of Lemma 2.9.* The graph expansion of $\tilde\rho^q_n(\sigma,0)$ has the property that, given a graph $\vartheta\in\mathcal{T}_{n,q}$ with $\mathrm{Val}(\vartheta)\neq 0$, each vertex of ϑ has Fourier index $n_v=\pm1$. This implies that, for any $v\in\vartheta$ (see §3.7), $h_{\ell_v}=h_{\ell_{v'}}=0$ is not allowed, so that the number of non diagonal propagators is less or equal of $\bar q+1$, if $\bar q=[q/2]$. Hence (3.30) implies that

$$|n|\le q+\bar q+1\,. \tag{5.20}$$

We can bound $\mathrm{Val}(\vartheta)\neq 0$ as in §5.3, by choosing in an arbitrary way the vertex v^* (since we do not need now to extract the factor $|n|^{-N}$), and we get

$$\sum_{\theta\in\mathcal{T}_{n,q}}|\mathrm{Val}(\vartheta)|\le\frac{D}{v_0^{2q-1}}\left(1+\log\frac{1}{v_0}\right)5^{q+1}C^{\bar q}|\sigma|(|\sigma|Q)^{\bar q}\left(1+\log\frac{v_0^2}{|\sigma|}\right)\,. \tag{5.21}$$

It is easy to see that, if $q\ge 2$, $\bar q\ge|n|/5$; hence, if $Qv_0^{-3}|\sigma|^{1/2}[1+\log(v_0^2/|\sigma|)]$ is small enough,

$$\sum_{q=2}^{\infty}||R(0)^{(q)}_n||_{\mathcal{J}}\le\frac{C}{v_0}\left(1+\log\frac{1}{v_0}\right)\left(\frac{|\sigma|}{v_0^2}\right)^{\frac{|n|}{10}}\,. \tag{5.22}$$

In order to complete the proof of Lemma 2.9, we have to improve the bound (5.21) in the case $q=1$. Note that $\tilde\rho^1_n(\sigma,0)$ is different from 0 only if $|n|=2$ (only one propagator may have frequency label $h=0$) and it is given, if $n=2$ (the case $n=-2$ is similar), by

$$\sigma\frac{1}{L}\sum_{k'\in\mathcal{D}'_L}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left[\tilde g^{(1)}_{1,1}(\mathbf{k}')\,\tilde g^{(0)}_{-1,1}(\mathbf{k}'+4\mathbf{p}_F)+\tilde g^{(0)}_{-1,1}(\mathbf{k}')\,\tilde g^{(1)}_{1,1}(\mathbf{k}'+2\mathbf{p}_F)\right]\,. \tag{5.23}$$

Hence, by using (4.6) and (4.11), we get

$$||R(0)^{(1)}_2||_{\mathcal{J}}\le\frac{C}{v_0^3}|\sigma|\left(1+\log\frac{v_0^2}{|\sigma|}\right)\le\frac{C}{v_0}\left(\frac{|\sigma|}{v_0^2}\right)^{2/10}\,. \tag{5.24}$$

This bound and the bound (5.22) imply (2.15). ∎

5.7. *Proof of Lemma 2.14.* By Lemma 5.2, if $Qv_0^{-3}|\sigma|^{1/2}[1+\log(v_0^2/|\sigma|)]$ is small enough, which is certainly true if condition (2.16) is satisfied, with ε small enough, we have

$$\sum_{q\ge2}|\tilde\rho^q_1(\sigma,\Phi)|\le|\sigma|\,\frac{C^N N!}{v_0}\left(1+\log\frac{1}{v_0}\right)\left(\frac{|\sigma|}{v_0^2}\right)^{1/2}\,. \tag{5.25}$$

Moreover, since the graphs contributing to $\tilde\rho^1_1(\sigma,\Phi)$ have only one vertex with index $n_v=\pm2,\pm3$, at least one of its two propagators has frequency index $h=0$ and different ω-indices. It follows, by (4.6), (4.8) and Lemma 2.10, that

$$|\tilde\rho^1_1(\sigma,\Phi)|\le\frac{C}{v_0}(|\Phi_2(\sigma)|+|\Phi_3(\sigma)|)\le\frac{C}{v_0}|\sigma|\left(\frac{\lambda^2}{v_0}\right)^N\,. \tag{5.26}$$

Hence, if $|\sigma|^{1/4}C^N N!(1+\log v_0^{-1})\le v_0^{1/2}$, which is certainly true if condition (2.16) is satisfied, with ε small enough, and $r_2(\sigma)$ is defined as in (2.27), we get

$$\left|\tilde\rho^1_1(\sigma)\right|\le C\frac{|\sigma|}{v_0}\left[\left(\frac{|\sigma|}{v_0^2}\right)^{1/4}+\left(\frac{\lambda^2}{v_0}\right)^N\right]\,, \tag{5.27}$$

which implies the bound in the first line of (2.28).

Let us now consider the derivative of $r_2(\sigma)$. By Lemma 5.2, if $|\sigma|^{1/2}[1+\log(v_0^2/|\sigma|)]\ Qv_0^{-3}$ is small enough, we have

$$\sum_{q\geq 2}\left|\frac{\partial\tilde\rho_1^q(\sigma,\Phi(\sigma))}{\partial\sigma}\right|\leq\frac{C^N N!}{v_0}\left(1+\log\frac{1}{v_0}\right)\left(\frac{|\sigma|}{v_0^2}\right)^{1/2}\ . \tag{5.28}$$

Moreover, by Lemma 2.10 and the remark preceding (5.26),

$$\left|\frac{\partial\tilde\rho_1^1(\sigma,\Phi(\sigma))}{\partial\sigma}\right|\leq\frac{C}{v_0}\|\Phi\|_{\mathcal F}\leq\frac{C}{v_0}\left(\frac{\lambda^2}{v_0}\right)^N\ . \tag{5.29}$$

It follows that, if $|\sigma|^{1/4}C^N N!(1+\log v_0^{-1})\leq v_0^{1/2}$, which is certainly true if condition (2.16) is satisfied, with ε small enough,

$$\left|\frac{\partial r_2}{\partial\sigma}(\sigma)\right|\leq\frac{2\pi v_0}{|\sigma|}||R(\Phi)_1||_{\mathcal J}\leq\frac{C}{|\sigma|}\left[\left(\frac{|\sigma|}{v_0^2}\right)^{1/4}+\left(\frac{\lambda^2}{v_0}\right)^N\right]\ , \tag{5.30}$$

which immediately implies the bound in the second line of (2.28). ■

5.8. *Proof of Lemma 2.17.* The Hessian matrix $\tilde M$, defined in (1.15), is a real matrix; hence, we have to show that

$$\sum_{n,m=-[Q/2]}^{[(Q-1)/2]} x_n\tilde M_{nm}x_m>0\ , \tag{5.31}$$

for any $\{x_n\}_{n=-[Q/2]}^{[(Q-1)/2]}\in\mathbb R^{Q-3}$. This will be done by writing

$$\begin{aligned}\sum_{n,m=-[Q/2]}^{[(Q-1)/2]} x_n\tilde M_{nm}x_m\geq\\ \sum_{n=-[Q/2]}^{[(Q-1)/2]} x_n^2\left[\tilde M_{nn}-\frac12\sum_{\substack{m=-[Q/2]\\ m\neq n}}^{[(Q-1)/2]}\left(|\tilde M_{nm}|+|\tilde M_{mn}|\right)\right]\ ,\end{aligned} \tag{5.32}$$

and showing that the right hand side of the above equation is strictly positive.

Let us find first a lower bound for $\tilde M_{nn}$. If $|n|\neq 1$, by (1.15), (1.12) and (2.1) we have

$$\tilde M_{nn}=1+\lambda^2c_n(\sigma)-\lambda^2\frac{\partial\tilde\rho_n}{\partial\Phi_n}\ , \tag{5.33}$$

where $1+\lambda^2c_n(\sigma)\geq 1/2$, see §2.11, and $\partial\tilde\rho_n/\partial\Phi_n$ obeys to the same bound of $\partial\tilde\rho_n/\partial\sigma$, see §5.3, up to the factor $|n|^{-N}$: simply note that the derivatives can act only on the vertex functions (and not on the propagators), and $|\partial\Phi_n/\partial\sigma|\leq|n|^{-N}$ has to be replaced with $|\partial\Phi_n/\partial\Phi_n|\leq 1$. Then, analogously to (5.13), we obtain, for any $q>0$,

$$\left|\lambda^2\frac{\partial\tilde\rho_n^q}{\partial\Phi_n}\right|\leq\lambda^2Dv_0\left(\frac{C}{v_0^2}\right)^q q^2(3q)^N(|\sigma|Q)^{[q/2]}\ , \tag{5.34}$$

so that, if $|\sigma|Qv_0^{-4}$ is small enough, we have

$$\left|\lambda^2\frac{\partial\tilde\rho_n}{\partial\Phi_n}\right|\leq C\lambda^2v_0^{-1}3^N N!\ . \tag{5.35}$$

It follows that

$$\tilde M_{nn}\geq\frac13\ ,\qquad |n|\neq 1\ , \tag{5.36}$$

for λ satisfying (1.16), with ε small enough and $K \geq 3$.

In the case $n = 1$ (the case $n = -1$ is discussed in the same way) we have

$$\tilde{M}_{11} = 1 + \lambda^2 c_1(\sigma) + \lambda\sigma \frac{\partial c_1(\sigma)}{\partial \hat{\varphi}_1} - \lambda \frac{\partial \tilde{\rho}_1}{\partial \hat{\varphi}_1} \ . \tag{5.37}$$

Note that our definitions of $c_1(\sigma)$ and $\tilde{\rho}_n(\sigma, \Phi)$ do not distinguish the dependence on $\hat{\varphi}_1$ and $\hat{\varphi}_{-1}$, which are equal in the fixed points we are studying (see discussion in §1.4). However, in the definition of $\tilde{M}$, $\hat{\varphi}_1$ and $\hat{\varphi}_{-1}$ have to be treated as independent variables. By taking into account this remark and by using Lemmata 2.13 and 2.14, with $\tilde{\varepsilon}$ and $|\sigma| v_0^{-2}$ small enough, we get

$$\begin{aligned}
\lambda\sigma \frac{\partial c_1(\sigma)}{\partial \hat{\varphi}_1} &= \frac{1}{2}\lambda^2 \sigma \frac{\partial c_1(\sigma)}{\partial \sigma} \geq \frac{1}{6\pi} \frac{\lambda^2}{v_0} \ , \\
\left|1 + \lambda^2 c_1(\sigma)\right| &= \left|\frac{\lambda^2 \tilde{\rho}_1}{\sigma}\right| \leq \frac{C\lambda^2}{v_0} \left[\left(\frac{|\sigma|}{v_0^2}\right)^{1/4} + \left(\frac{\lambda^2}{v_0}\right)^N \right] \ , \\
\left|\lambda \frac{\partial \tilde{\rho}_1}{\partial \hat{\varphi}_1}\right| &\leq \left|\lambda^2 \frac{\partial \tilde{\rho}_1}{\partial \sigma}\right| \leq \frac{C\lambda^2}{v_0} \left[\left(\frac{|\sigma|}{v_0^2}\right)^{1/4} + \left(\frac{\lambda^2}{v_0}\right)^N \right] \ ,
\end{aligned} \tag{5.38}$$

so that

$$|\tilde{M}_{nn}| \geq \frac{1}{8\pi} \frac{\lambda^2}{v_0} \ , \qquad n = \pm 1 \ , \tag{5.39}$$

under the hypotheses of Theorem 1.6.

The non diagonal terms ($n \neq m$) are of the form

$$\tilde{M}_{nm} = \lambda^2 \Phi_m \frac{\partial c_m(\sigma)}{\partial \sigma} \frac{\delta_{n,1} + \delta_{n,-1}}{2} - \lambda \frac{\partial \tilde{\rho}_m}{\partial \hat{\varphi}_n} \ . \tag{5.40}$$

By using (2.7) and (2.17), the first term in the r.h.s. of (5.40), where $m \neq n$ implies $|m| > 1$, can be bounded as

$$\left|\lambda^2 \Phi_m \frac{\partial c_m(\sigma)}{\partial \sigma} \frac{\delta_{n,1} + \delta_{n,-1}}{2}\right| \leq C \left(\frac{\lambda^2}{v_0}\right)^{N+1} \ . \tag{5.41}$$

Moreover, by proceeding again as in §5.3, we obtain

$$\left|\lambda \frac{\partial \tilde{\rho}_m^q}{\partial \hat{\varphi}_n}\right| \leq \lambda^2 D v_0 \left(\frac{C}{v_0^2}\right)^q q^2 (3q)^N (|\sigma| Q)^{[q/2]} \ , \tag{5.42}$$

so that

$$\sum_{n=-[Q/2];n\neq m}^{[(Q-1)/2]} \sum_{q=2}^{\infty} \left|\lambda \frac{\partial \tilde{\rho}_m^q}{\partial \hat{\varphi}_n}\right| \leq C\lambda^2 3^N N! \frac{|\sigma| Q^2}{v_0^5} \ . \tag{5.43}$$

The contributions with $q = 1$ need an improved bound. Let us first suppose that $|n| > 1$; in this case the derivative can act only on the vertex function of the graphs contributing to $\tilde{\rho}_m^1$. Then, if the derivative is different from 0, the vertex function is equal to Φ_n and, since $m - n \neq 0$, at least one of the two propagators must have different ω indices; this follows from (4.3), which also implies that the integer which multiplies $\mathbf{p}_F$ in the value (4.1) of the graph is different from 0. Hence, by using (4.6), (4.9) and the fact that $|n - m| \leq 2$, we get

$$\sum_{n=-[Q/2];n\neq m}^{[(Q-1)/2]} \left|\lambda^2 \frac{\partial \tilde{\rho}_m^1}{\partial \Phi_n}\right| \leq C\lambda^2 \frac{|\sigma| Q}{v_0^3} \left(1 + \log \frac{v_0^2}{|\sigma|}\right) \ . \tag{5.44}$$

The case $q = 1$ and $|n| = 1$ can be treated in a similar way; the main difference is that the derivative can act also on the propagators of the graphs contributing to $\tilde{\rho}_m^1$, but it is still true that the integer which multiplies $\mathbf{p}_F$ in the value (4.1) of each graph is different from 0, an essential point in the previous bound, since it allowed to use the improved bound (4.9) in place of (4.8). By using again (4.6) and (4.9), as well as the improved bounds (with respect to (4.12)) (4.13) and (4.14), we get again the bound (5.44).

The r.h.s. of (5.41), (5.43) and (5.44) can be made arbitrarily small with respect to λ^2/v_0, by suitably choosing the constants in (1.16); hence Lemma 2.17 is proved. ■

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