On the spectrum of odd order self adjoint ordinary differential operators on the real line with quasi-periodic coefficients

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1 Introduction

In this paper we prove that an odd order self-adjoint differential operator on the real line with quasi-periodic coefficients has only absolute continuous simple spectrum for large enough energies.

Ordinary differential operators with quasi-periodic coefficients have been studied a lot during the last 20 years. These operators arise naturally from physical considerations. People have mainly been interested in the one dimensional Schrödinger operator with a quasi-periodic potential (see [Be-Te], [Din-Sin], [Eli], [Ji-Si], [Mo], [Mo-Po], [Su] and [So-Sp]); such hamiltonians come up naturally in solid state physics (see [Pa-Fi]).
In this paper, we study ordinary differential operators of higher order with quasi-periodic coefficients. Such operators also come up in physics (for exemple, in the study of Boussinesq's equation).
More precisely, we consider the following differential operator, denoted by L,

of order n on the real line.

$$L = D^n + p_{n-2}(t)D^{n-2} + ... + p_1(t)D + p_0(t) \tag{1}$$

where $D = \frac{1}{i}\frac{d}{dt}$
Let us assume n is odd and L is formally self-adjoint.
One notices that the differential expression (1) does not contain terms of order $n-1$; indeed terms of order $n-1$ may be eliminated by conjugation with a unitary operator (see [BDT] and [D-S]).
For every $j = 0, ..., n-2$, p_j is a quasi-periodic function. That is, we define $T^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ to be the d-dimensional torus and B_r to be the space of functions analytic in a complex neighborhood of T^d of the form $\{x+iy;\ x \in T^d,\ |\ y\ |_\infty < r\}$ (here $r > 0$ is fixed) and which are continuous on the closure of this complex neighborhood. B_r is endowed with the norm

$$\forall \psi \in B_r,\ |\psi|_r = \sup_{|Im(x)|<r} |\psi(x)|.$$

Let $p(t) = P(\omega_1 t, ..., \omega_d t)$ where P is in B_r and $\omega = (\omega_1, ..., \omega_d)$ is a vector in $\mathbb{R}^d$ satisfying the diophantine condition

$$|n.\omega| \geq |n|^{-\tau}, n \in \mathbb{Z}^d \setminus 0 \tag{2}$$

for some $\tau > d-1$ and where $|n| = \sum_{j=1}^d |n_j|$. ω is called the frequency vector. It is a well known fact that the set of frequency vectors satisfying our diophantine condition is of full measure in $\mathbb{R}^d$.
We denote by QP_r the set of quasi-periodic functions p constructed as above starting with functions in B_r. The norm defined on B_r naturally endows QP_r with a norm that we will denote by $|\cdot|_r$. For $p = (p_0, ..., p_{n-2}) \in (QP_r)^{n-1}$, we define $|p|_r = \sup_{j=0,...,n-2} |p_j|_r$.

Under these assumptions, L is self-adjoint in $L^2(\mathbb{R})$ with domain $H^n(\mathbb{R})$. Our main result deals with the spectrum of L. We have

Theorem. 1.1 *For every fixed $r > 0$ and ω satisfying (2), there exists a continuously increasing function f from $[0, +\infty)$ into $[0, +\infty)$ with $f(0) = 0$ such that if $(p_0, ..., p_{n-2}) \in (QP_r)^{n-1}$ the spectrum of L contains the set $I(p) = (-\infty, -f(|p|_r)] \cup [f(|p|_r), \infty)$. Furthermore, in $I(p)$, L has only absolutely continuous spectrum; and this spectrum is simple.*

Remark We easily deduce from this theorem that the spectrum of an even order ordinary differential operator which is the square of an odd order self adjoint ordinary differential operator with quasi-periodic coefficient contains a half line and this part of the spectrum is absolutely continuous.

When $n = 3$, L can be rewritten as

$$L_3 = D^3 + qD + Dq + p = D^3 + p_1 D + p_0. \tag{3}$$

It is well known (see for example [DTT]) that L_3 is the self-adjoint operator of the Lax pair associated with the Boussinesq equation

$$\frac{\partial^2 q}{\partial t^2}(x,t) = 3\frac{\partial^4 q}{\partial x^4}(x,t) - 12\frac{\partial^2 (q^2)}{\partial x^2} \tag{4}$$

with

$$\frac{\partial p}{\partial x}(x,t) = -\frac{1}{3}\frac{\partial q}{\partial t}(x,t).$$

Thus the Boussinesq equation and its hierarchy define isospectral flows for (3).

Inverse Scattering theory for (3) has been applied in ([DTT]) to solve the Cauchy problem for (4) with initial data in $S(I\!\!R)$ (the Schwartz space). To implement the same procedure in order to solve the Cauchy problem for (4) with quasi-periodic data the first step would be to know the spectrum of (3). But our method only gives a part of the spectrum and the description of the spectrum of (3) is still in progress (see [GGK]). Moreover one also needs to know how to reconstruct the potentials from the spectral data. To the best of our knowledge this inverse spectral problem for (3) with quasi-periodic coefficients has not been solved yet. Nevertheless one may hope that the Cauchy problem for (4) with quasi-periodic or almost-periodic initial data can be solved along the same lines as the ones for the Korteweg-De Vries equation by [Eg] and the nonlinear Schrödinger equation by [BdM-Eg].

Hence a still more complete information about the spectrum of L_3 is useful. Let $\hat{p}_0$ and $\hat{p}_1$ be the means of p_0 and p_1 on T^d, i.e.

$$\hat{p}_j = \frac{1}{(2\pi)^d}\int_{T^d} P_j(x)dx,\ j = 1, 2.$$

Set

$$\hat{L} = D^3 + \hat{p}_1 D + \hat{p}_0 \tag{5}$$

Let I be the set in $I\!\!R$ such that , if $Z \in I$ the equation

$$X^3 + \hat{p}_1 X + \hat{p}_0 = Z$$

has only one real simple root. We easily check that $I = (-\infty, a) \cup (b, +\infty)$ for some real numbers a and b.

Setting $I_\epsilon = (-\infty, a-\epsilon) \cup (b+\epsilon, +\infty)$, we then have

Theorem. 1.2 *For every $r > 0$ and $\epsilon > 0$ there exists $\eta > 0$ such that if $|p_0(.) - \hat{p}_0|_r < \eta$ and $|p_1(.) - \hat{p}_1|_r < \eta$, the spectrum of L contains I_ϵ. Furthermore, in I_ϵ, L has only absolutely continuous spectrum; and this spectrum is simple.*

So we see that the simple absolutely continuous spectrum of 3rd order ordinary differential operators with constant coefficients is “robust” under quasi-periodic perturbations. In the case of Schrödinger operators on the line with quasi-periodic potential, we know that the spectrum does not stay as nice as in Theorem 1.2 (see [Pa-Fi]). This may be interpreted as follows: for the free Schrödinger operator, one has two free generalized (i.e polynomially bounded at $\pm\infty$) solutions that propagate in directions opposite to each other; this gives rise to interactions. In our case, as the free operator (i.e the one with constant coefficients) has only one generalized solution (at least in the energy range we consider), there are no interactions and the absolutely continuous spectrum is more stable. This is also the case for any odd order quasi-periodic operator at high energies (see Theorem 1.1). For lower energies, even if the spectrum of the free operator is simple, the situation seems trickier; indeed, our proof of Theorem 1.1 and 1.2 needs also the non polynomially bounded solution of the free operator to be non-interacting ([GGK]).

The proofs of the results of this paper rely heavily on the quasi-periodicity of the coefficients of the equation. Note that almost every non rationaly dependent frequency vector ω will satisfy some diophantine condition. Though the heuristic argument given above sounds rather convencing to us, for general (e.g. random) potentials, it is a classical result that nothing can be said about the perturbation of absolutely continuous spectrum. Indeed it can be proved that any absolutely continuous spectrum can be destroyed by an appropriate Schatten class perturbation for any Schatten class (except the trace class see [Re-Si]).

We will now briefly outline a sketch of the proof of Theorem 1.1. The proof of Theorem 1.2 will go along the same lines.
To construct the spectral measure of L, we will construct the Jost functions. Therefore we will follow the method implemented in the case of second order quasi-periodic ordinary differential operators by[Din-Sin], [Mo-Po] and [Eli].

Let the spectral parameter $\lambda = z^n$. The eigenvalue equation $Lu = z^n u$ can be reduced to a first order system

$$Dv = J_z v + Qv \tag{6}$$

where $v = (v_1, ..., v_n)^t$, $v_1 = u$, ..., $v_n = D^{n-1}u$ and

$$J_z = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \\ z^n & 0 & 0 & ... & 0 \end{pmatrix}, \tag{7}$$

$$Q = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ . & . & ... & . & . \\ 0 & 0 & ... & 0 & 0 \\ q_1 & q_2 & ... & q_{n-1} & 0 \end{pmatrix}, \ q_j = -p_{j-1}. \tag{8}$$

Let $(\beta_r)_{r=1,\dots,n} = \left(e^{2i\pi\frac{r-1}{n}}\right)_{r=1,\dots,n}$ be the roots of unity. Set

$$J = \begin{pmatrix} \beta_1 & 0 & ... & 0 \\ 0 & \beta_2 & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ 0 & 0 & ... & \beta_n \end{pmatrix} \tag{9}$$

and

$$\Lambda_z = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & z & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ 0 & 0 & ... & z^{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & ... & 1 \\ \beta_1 & \beta_2 & ... & \beta_n \\ . & . & ... & . \\ . & . & ... & . \\ \beta_1^{n-1} & \beta_2^{n-1} & ... & \beta_n^{n-1} \end{pmatrix}. \tag{10}$$

Λ_z diagonalizes J_z and (6) is equivalent to

$$X'(t) = (A_1(z) + F_1(t,z))X(t) \tag{11}$$

where

$$A_1(z) = izJ \text{ and } F_1(t,z) = i\Lambda_z^{-1}Q(t)\Lambda_z \tag{12}$$

Note that we have $Tr(A_1(z)) = Tr(F_1(t,z)) = 0$.

For every $z \neq 0$, $A_1(z)$ has simple eigenvalues $(\lambda_l(A_1(z)))_{l=1,..n}$. When z is in a domain D close to the real axis and because n is odd there exists $a > 1$ such that

$$a^{-1}\|A_1(z)\| \leq |Re(\lambda_l(A_1(z)) - \lambda_m(A_1(z)))| \leq a\|A_1(z)\|$$

for every $l \neq m$ and $z \in D$.

The main step of the proof of Theorem 1.1 will be to construct Floquet-type solutions to (11) for $|z|$ sufficiently large. By this, in the context we consider, we mean solutions of the form $Y(t,z)e^{tA(z)}$ where $Y(.,z)$ is a $n \times n$ quasi-periodic invertible matrix and $A(z)$ is a $n \times n$ matrix with only one purely imaginary eigenvalues when z is real. The other eigenvalues are $\frac{n-1}{2}$ couples of complex numbers symmetric with respect to the imaginary axis and with a real part different from zero.
Following [Din-Sin], [Mo-Po] and [Eli], we construct such solutions by using KAM-like procedure. Such a procedure is needed because of the existence

of small divisors. Nevertheless, under our assumptions and in the region of energy under study here, there is only one small divisor . If n is even (at all energies) and if n is odd (in certain energy ranges), there arise more than one small divisor and the situation becomes much more complicated; to our knowledge, the only such case that has been studied is the case of Schrödinger operator with quasi-periodic potential (i.e $n = 2$) (see [Din-Sin], [Mo-Po] and [Eli]).

The paper is organized as follows. In the second section we give the KAM procedure that allows us to obtain the needed Floquet decomposition. The Floquet solutions allow us to construct the Jost functions of the operator L. In the third one, from these Jost functions, we construct the spectral measure of L.

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2 KAM procedure

Our aim is to prove that, under suitable assumptions on A_1 and F_1, the equation (11) has matrix solution of the form

$$Y(t, z)e^{A(z)t}$$

Thus Y is solution to

$$Y' = (A_1 + F_1)Y - YA \tag{13}$$

Let us first describe the construction of Y and A formally. We follow the general procedure given in [Din-Sin].
We first suppose that F_1 is small and then we construct Y_2 such that

$$Y_2' = (A_1 + F_1)Y_2 - Y_2(A_2 + F_2) \tag{14}$$

where A_2 is constant and F_2 smaller than F_1. Therefore we solve the homological equation

$$\tilde{Y}' = [A_1, \tilde{Y}] + F_1 + A_1 - A_2 \tag{15}$$

where we define

$$A_2 = A_1 + \hat{F}_1(0) \text{ with } \hat{F}_1(0) = \frac{1}{(2\pi)^d} \int_{T^d} F_1(x)dx$$

Then Y_2 and F_2 are given by

$$Y_2 = Id + \tilde{Y}, \ F_2 = (Id + \tilde{Y})^{-1}(F_1\tilde{Y} - \tilde{Y}\hat{F}_1(0))$$

A simple computation shows that Y_2 and F_2 satisfy (14) formally. Furthermore, F_2 is smaller than F_1 as $\tilde{Y}$ is small.
This is the first step to start up the iteration. At the j-th step assume that $(A_i)_{i=1,\dots,j-1}$ and $(F_i)_{i=1,\dots,j-1}$ have been constructed as above. We then solve the homological equation

$$\tilde{Y}' = [A_{j-1}, \tilde{Y}] + F_{j-1} + A_{j-1} - A_j \tag{16}$$

where we define

$$A_j = A_{j-1} + \hat{F}_{j-1}(0),\ Y_j = Id + \tilde{Y}$$

and

$$F_j = (Id + \tilde{Y})^{-1}(F_{j-1}\tilde{Y} - \tilde{Y}\hat{F}_{j-1}(0))$$

again F_j is smaller than F_{j-1}.
It is easy to check that Y_j is solution to

$$Y_j' = (A_{j-1} + F_{j-1})Y_j - Y_j(A_j + F_j) \tag{17}$$

Furthermore one easily shows that, for every p, $\prod_{j=1}^{p} Y_j$ satisfies

$$(\prod_{j=2}^{p} Y_j)' = (A_1 + F_1)(\prod_{j=2}^{p} Y_j) - (\prod_{j=2}^{p} Y_j)(A_p + F_p) \tag{18}$$

So if $\lim_{j\to\infty} F_j = 0$ and if we set

$$Y = \prod_{j=2}^{\infty} Y_j \text{ and } A = \lim_{j\to\infty} A_j$$

We obtain formally that Y and A satisfy (13).

We now want to a give rigorous meaning to the formal iteration we have described above. First we give a general lemma that allows us to solve the homological equation (15) for some class of matrix A taking into account the spectral properties of the matrices we are dealing with.
For a matrix A in $M_n(\mathbb{C})$ and $(\lambda_i(A))_{i=,\dots,n}$ its eigenvalues counted with their multiplicity, we define

$$\alpha(A) = \inf_{i\neq j} |Re(\lambda_i(A) - \lambda_j(A))| \tag{19}$$

$$\Gamma = \{A \in M_n(\mathbb{C}), \alpha(A) > 0\} \tag{20}$$

Note that in particular the eigenvalues of a matrix $A \in \Gamma$ are all simple. We prove

Lemma. 2.1 *Let A be in Γ and G be a matrix valued function in B_r for some $r>0$ and satisfying $\hat{G}(0)=0$. There exists a unique Y in $B^r=\cap_{r>s>0}B_s$ analytic with respect to A and G such that*

$$\partial_\omega Y=[A,Y]+G \tag{21}$$

with $\hat{Y}(0)=0$ where $\partial_\omega=\omega.\nabla$
Moreover Y satisfies the estimate

$$|Y|_s\leq\frac{C}{\alpha(A)}\left(1+\frac{\|A\|}{\alpha(A)}\right)^{2n-2}\frac{|G|_r}{(r-s)^{d+\tau}} \tag{22}$$

for every $0<s<r$, where the constant C depends only on n,d and τ.

Proof
There exists U and T such that

$$A=UTU^*$$

where U is a unitary matrix and T lower triangular, that is $T_{ij}=0$ when $i<j$. In particular the $T_{ii}'s$ are the eigenvalues of A. Set, $T_{ii}=\lambda_i(A), i=1,...,n$. Let

$$Y=UZU^* \text{ and } G=U\tilde{G}U^*$$

then (21) is equivalent to

$$\partial_\omega Z=[T,Z]+\tilde{G}$$

with$\hat{Z}(0)=\hat{\tilde{G}}(0)=0$
So for every $m\in\mathbb{Z}^d\setminus 0$ and for $i,j=1,...,n$ the m^{th} Fourier coefficient of Z_{ij} satisfies

$$(im\omega-T_{ii}+T_{jj})\hat{Z}(m)_{ij}=\hat{\tilde{G}}(m)_{ij}+\sum_{k=1}^{i-1}T_{ik}\hat{Z}(m)_{kj}-\sum_{k=j+1}^{n}\hat{Z}(m)_{ik}T_{kj} \tag{23}$$

It follows from (19) that for every $i\neq j$

$$|im\omega-T_{ii}+T_{jj}|\geq\alpha(A) \tag{24}$$

Moreover, since U is unitary, we have for every i and j

$$|T_{ij}|\leq\|A\| \tag{25}$$

In particular, for $i=1$, we get

$$(im\omega-T_{11}+T_{nn})\hat{Z}(m)_{1n}=\hat{\tilde{G}}(m)_{1n} \tag{26}$$

and for $2 \leq p \leq n-1$

$$(im\omega - T_{11} + T_{pp})\hat{Z}(m)_{1p} = \hat{\tilde{G}}(m)_{1p} + \sum_{k=p+1}^{n} T_{kp}\hat{Z}(m)_{1k} \tag{27}$$

For $i = j = 1$ we get

$$im\omega\hat{Z}(m)_{11} = \hat{\tilde{G}}(m)_{11} + \sum_{k=2}^{n} T_{k1}\hat{Z}(m)_{1k} \tag{28}$$

Using (24) and (25) we can uniquely solve (26) and by induction (27). Furthermore we get for every $2 \leq p \leq n$

$$|\hat{Z}(m)_{1p}| \leq \frac{1}{\alpha(A)}\|\hat{\tilde{G}}(m)\|\left(1 + \frac{\|A\|}{\alpha(A)}\right)^{n-p} \tag{29}$$

Then we solve (28) and using $|m.\omega| \geq |m|^{-\tau}$ and (29) we obtain

$$|\hat{Z}(m)_{11}| \leq |m|^{\tau}\|\hat{\tilde{G}}(m)\|\left(1 + \frac{\|A\|}{\alpha(A)}\right)^{n-1} \tag{30}$$

By induction we can then solve uniquely (23) for every $i, j = 1, ..., n$. Moreover we get

$$\|\hat{Z}(m)\| \leq \frac{n|m|^{\tau}}{\alpha(A)}\left(1 + \frac{\|A\|}{\alpha(A)}\right)^{2n-2}\|\hat{G}(m)\| \tag{31}$$

Since $\|\hat{G}(m)\| \leq |G|_r e^{-|m|r}$ we deduce from (31) that the Fourier series

$$\sum_{m\in\mathbb{Z}^d\backslash 0} \hat{Z}(m)e^{im.x}$$

converges in B_s for $0 < s < r$ and we get

$$\begin{aligned}
\left|\sum_{m\in\mathbb{Z}^d\backslash 0} \hat{Z}(m)e^{im.x}\right|_s &\leq \frac{n}{\alpha(A)}\left(1 + \frac{\|A\|}{\alpha(A)}\right)^{2n-2}|G|_r \sum_{m\in\mathbb{Z}^d\backslash 0}|m|^{\tau}e^{-|m|(r-s)} \\
&\leq \frac{n}{\alpha(A)}\left(1 + \frac{\|A\|}{\alpha(A)}\right)^{2n-2} 4^d|G|_r\sum_{l>0} l^{d+\tau-1}e^{-l(r-s)} \\
&\leq \frac{C}{\alpha(A)}\left(1 + \frac{\|A\|}{\alpha(A)}\right)^{2n-2}\frac{|G|_r}{(r-s)^{d+\tau}}
\end{aligned}$$

where C depends on n, d and τ. Define for $x \in T^d$

$$Z(x) = \sum_{m\in\mathbb{Z}^d\backslash 0} \hat{Z}(m)e^{im.x} \text{ and } Y(x) = UZ(x)U^*$$

Then Y is in B^r and solves (21) uniquely. Moreover Y satifies (22).
Furthermore U and T are analytic with respect to A because the eigenvalues of A are simple when A is in Γ. Z is also analytic in A and G as a sum of Fourier series whose coefficients are analytic and which converges locally uniformly with respect to $A \in \Gamma$ and $G \in B_r$.
Then Y is analytic in A and G. ■

Now, in order to use Lemma 2.1 to solve the homological equation (16), we need to control the eigenvalues of A_j, $j \geq 2$. Therefore we use

Lemma. 2.2 *Let A be an $n \times n$ complex matrix in Γ, $(\lambda_i(A))_{i=1,\dots,n}$ its eigenvalues and*

$$C_A = \frac{\alpha(A)}{9\left(1 + 6\frac{\|A\|}{\alpha(A)}\right)^{n-1}} \tag{32}$$

(i) Then for every B in $M_n(\mathbb{C})$ such that $\|B - A\| \leq C_A$ we can define $\lambda_j(B)$, $j = 1, \dots, n$, the eigenvalue branches of B analytic in B.
(ii) For every B and B' in $M_n(\mathbb{C})$ such that $\|B - A\| \leq C_A$ and $\|B' - A\| \leq C_A$ we have for j=1,...,n

$$|\lambda_j(B) - \lambda_j(B')| \leq M_A \|B - B'\| \tag{33}$$

where M_A is a non negative constant which only depends on $\frac{\|A\|}{\alpha(A)}$.

Proof
Let γ_j be the circle of center $\lambda_j(A)$ and radius $\frac{\alpha(A)}{3}$. Then using ([Ka] p 28) we get

$$\begin{aligned}\sup_{1\leq j\leq n}\left(\sup_{z\in\gamma_j}\|(z-A)^{-1}\|\right) &\leq \sup_{1\leq j\leq n}\left(\sup_{z\in\gamma_j}\frac{\|(z-A)^{n-1}\|}{|\det(A-z)|}\right)\\ &\leq \frac{\left(2\|A\| + \frac{\alpha(A)}{3}\right)^{n-1}}{\left(\frac{\alpha(A)}{3}\right)^n} = \frac{1}{3C_A}\end{aligned} \tag{34}$$

where $\det(A - z)$ is the determinant of the matrix $(A - z)$.
Let B such that $\|B - A\| \leq C_A$, then, for z in γ_j and $j = 1, \dots, n$, $(z - B)$ is invertible and its inverse is given by

$$(z - B)^{-1} = (z - A)^{-1} \sum_{m\geq 0} \left[(B - A)(z - A)^{-1}\right]^m$$

In particular, one has

$$\|(z - B)^{-1}\| \leq 2\|(z - A)^{-1}\| \tag{35}$$

Now define

$$P_j(B) = \frac{1}{2i\pi}\int_{\gamma_j} (z - B)^{-1} dz$$

then $(P_j(B))_{j=1,...,n}$ are eigenprojectors for B satisfying $P_j(B) \circ P_i(B) = 0$ for $i \neq j$. This follows from the resolvent formula and Cauchy theorem.
Moreover for $t \in [0,1]$ the projector valued function $t \mapsto P_j(A + t(B - A))$ is continuous in t. So its rank is constant, hence rank$(P_j(B))$ =rank$(P_j(A)) = 1$.
Hence

$$\sum_{j=1}^{n} P_j(B) = Id \text{ and } \sum_{j=1}^{n} BP_j(B) = B$$

Hence $P_j(B)$ is an eigenprojector of B and if we define for $j = 1, ..., n$ $\lambda_j(B) = Tr(BP_j(B))$ we obtain (i).
Now for $\|B - A\| \leq C_A$ and $\|B' - A\| \leq C_A$ we have for j=1,...,n

$$\begin{aligned}
|\lambda_j(B) - \lambda_j(B')| &= |Tr(BP_j(B) - B'P_j(B'))| \\
&\leq |Tr((B - B')P_j(B))| + |Tr(B'(P_j(B) - P_j(B')))| \\
&\leq \|B - B'\| + |Tr(B' \frac{1}{2i\pi} \int_{\gamma_j} (z - B)^{-1}(B - B')(z - B')^{-1} dz)| \\
&\leq \|B - B'\|(1 + 4n \frac{\alpha(A)}{3} \|B'\| \|(z - A)^{-1}\|^2)
\end{aligned}$$

where we have used $|Tr(M)| \leq$rank$(M)\|M\|$ and (35).
Using (34) we get

$$|\lambda_j(B) - \lambda_j(B')| \leq \|B - B'\| M_A$$

where

$$\begin{aligned}
M_A &= 1 + \frac{4n\alpha(A)}{3} \frac{\|A\| + C_A}{(3C_A)^2} \\
&= 1 + 36n \left(1 + 6\frac{\|A\|}{\alpha(A)}\right)^{n-1} \left(1 + \frac{\|A\|}{3\alpha(A)} \left(1 + 6\frac{\|A\|}{\alpha(A)}\right)^{n-1}\right). \qquad (36)
\end{aligned}$$

■

We now state the theorem that summarizes our Floquet decomposition result in a general setting. Let for $a > 1$ and $b > 0$

$$D_{a,b} = \{A \in \Gamma, a^{-1} \leq \frac{\alpha(A)}{\|A\|} \leq a \text{ and } \|A\| \geq b\} \qquad (37)$$

Theorem. 2.1 *For every $r > 0$, $1 > a > 0$ and $b > 0$ there exists $\beta > 0$ sufficiently small such that if $A_1 \in D_{a,b}$ with $Tr(A_1) = 0$, if F_1 is a matrix valued function in B_r with $Tr(\hat{F}_1(0)) = 0$ and $|F_1|_r < \beta$, then there exists a matrix A in Γ and a matrix valued function Y in $B_{\frac{r}{2}}$ which are analytic with respect to A_1 and F_1 respectively in $D_{a,b}$ and $\{F \in B_r, |F|_r < \beta\}$ such that $Y(\omega t)e^{At}$ is a solution to*

$$X'(t) = (A_1 + F_1(\omega t))X(t)$$

Furthermore one has $Tr(A) = 0$.

In order to prove Theorem 2.1, we need the following lemma which implements our KAM induction procedure

Lemma. 2.3 *For every $r > 0$, $a > 1$ and $b > 0$ there exists $1 > \delta > 0$ such that if $A_1 \in D_{a,b}$ with $Tr(A_1) = 0$, if $F_1 \in B_r$ with $Tr(\hat{F}_1(0)) = 0$ and $|F_1|_r < \delta^2$ then for every $j \geq 2$ there exist A_j a $n \times n$ complex matrix, F_j in $B_{\frac{r}{2}}$ and Y_j in $B_{\frac{r}{2}}$ such that*

$$\partial_\omega Y_j(\omega t) = (A_{j-1} + F_{j-1}(\omega t))Y_j(\omega t) - Y_j(\omega t)(A_j + F_j(\omega t)) \tag{38}$$

and $TrA_j = Tr\hat{F}_j(0) = 0$
Furthermore A_j, F_j and Y_j satisfy the following estimates

$$\|A_j - A_{j-1}\| \leq \delta^j, \tag{39}$$

$$|Id - Y_j|_{\frac{r}{2}} \leq \frac{\delta}{4j^2}, \tag{40}$$

$$|F_j|_{\frac{r}{2}} \leq \delta^{j+1}. \tag{41}$$

Finally, for every $j \geq 2$, A_j and F_j are analytic with respect to A_1 in $D_{a,b}$ and F_1 in $\{F \in B_r, |F|_r < \delta^2\}$.

Proof
Let $\alpha_1 = \alpha(A_1)$ and $(r_j)_{j\geq 1}$ define by $r_1 = r$ and for $j \geq 1$

$$(r_j - r_{j+1})^{d+\tau} = D_{A_1}(j+1)^2\delta^j \tag{42}$$

where

$$D_{A_1} = \frac{C}{\alpha_1}\left(1 + \frac{2\|A_1\| + 2C_{A_1}}{\alpha_1}\right)^{2n-2} \tag{43}$$

and C is a constant depending on n, d and τ.
Let $1 > \delta > 0$ be such that

$$M_{A_1} \sum_{j\geq 2} \delta^j \leq \frac{\alpha_1}{4} \tag{44}$$

$$\sum_{j\geq 2} \delta^j \leq C_{A_1} \tag{45}$$

$$\sum_{j\geq 1} (r_j - r_{j+1}) \leq \frac{r}{2} \tag{46}$$

Note that for $A_1 \in D_{a,b}$ we can choose δ depending only on a, b and r because, as a consequence of the definitions of C_{A_1}, M_{A_1} and D_{A_1}, for $A_1 \in D_{a,b}$ we have

$$C_{A_1} \leq C(a,b),\ M_{A_1} \geq M(a,b) \text{ and } D_{A_1} \leq D(a,b)$$

where $C(a,b)$, $M(a,b)$ and $D(a,b)$ are positive constants depending only on a and b.

We will proceed by induction. Assume that for $k=2,...,j$ we have constructed A_k, F_k and Y_k such that (38), (39) hold and such that

$$|Id - Y_k|_{r_k} \leq \frac{\delta}{4k^2}, \tag{47}$$

$$|F_k|_{r_k} \leq \delta^{k+1} \tag{48}$$

In particular (40) and (41) hold up to the rank j.
By (39) and (45) we obtain

$$\|A_j - A_1\| \leq \sum_{k=2}^{j} \delta^k \leq C_{A_1}$$

Hence, by Lemma 2.2, the eigenvalues of A_j are simple and using (44) we get

$$|\lambda_l(A_j) - \lambda_l(A_1)| \leq M_{A_1} \sum_{k=2}^{j} \delta^k \leq \frac{\alpha_1}{4} \tag{49}$$

for every $l = 1, \ldots, n$.
Then

$$|Re(\lambda_l(A_j) - \lambda_m(A_j))| \geq \frac{\alpha_1}{2}$$

for every $l, m = 1, \ldots, n$ such that $l \neq m$.
So we can apply Lemma 2.1. Let $Y \in B$ the unique solution of

$$\partial_\omega Y = [A_j, Y] + F_j - \hat{F}_j(0) \tag{50}$$

with $\hat{y}(0) = 0$. We define

$$Y_{j+1} = Id + Y, \tag{51}$$

$$A_{j+1} = A_j + \hat{F}_j(0), \tag{52}$$

$$F_{j+1} = (Id+Y)^{-1}(F_j Y - Y\hat{F}_j(0)). \tag{53}$$

We then check that Y_{j+1} satisfies (38). Moreover we have $TrA_{j+1} = 0$ and (see [Eli])

$$Tr \int_{T^d} F_j(x)dx = (2\pi)^d Tr \hat{F}_{j+1}(0) = \sum_{k\geq 0} \frac{(-1)^{k+1}}{k+1} Tr \int_{T^d} \partial_\omega Y^{k+1}(x)dx = 0.$$

By Lemma 2.1 and (50) we get

$$|Y|_{r_{j+1}} \leq 2\frac{c}{\alpha_1}\left(1 + \frac{2\|A_j\|}{\alpha_1}\right)^{2n-2} \frac{2|F_j|_{r_j}}{(r_j - r_{j+1})^{d+\tau}}$$

$$\leq \left(1 + \frac{2\|A_1\| + 2C_{A_1}}{\alpha_1}\right)^{2n-2} \frac{4C\delta^{j+1}}{\alpha_1(r_j - r_{j+1})^{d+\tau}}.$$

So

$$|Y|_{r_j+1} \leq \frac{1}{4} D_{A_1} \frac{\delta^{j+1}}{(r_j - r_{j+1})^{d+\tau}}. \tag{54}$$

By (42) we obtain

$$|Y|_{r_{j+1}} \leq \frac{\delta}{4(j+1)^2}$$

and

$$|(Y + Id)^{-1}|_{r_{j+1}} \leq 2. \tag{55}$$

Now using (53), (54) and (55) we get

$$|F|_{r_{j+1}} \leq 4\delta^{j+1}|Y|_{r_{j+1}} \leq D_{A_1} \frac{\delta^{2j+2}}{(r_j - r_{j+1})^{d+\tau}}.$$

Using (42) we obtain

$$|F|_{r_{j+1}} \leq \delta^{j+2}.$$

Finally the analytic dependance of A_j and F_j with respect to A_1 and F_1 is a direct consequence of the construction (51)-(53) and Lemma 2.1. ■

Proof of Theorem 2.1

Let $\beta = \delta^2$ where δ is defined in Lemma 2.3. (39) and (40) allow us to define $A = \lim_{j \to +\infty} A_j$ and $Y = \prod_{j \geq 2} Y_j$ where A_j and Y_j are defined in lemma 2.3. It is easy to check, using (18) and (41) that Y and A satisfy

$$Y' = (A_1 + F_1)Y - YA\ .$$

Then $Y(\omega t)e^{At}$ satisfies

$$X' = (A_1 + F_1)X\ .$$

Furthemore $TrA = 0$ because $TrA_j = 0$ for every $j \geq 1$. The analyticity of Y and A with respect to A_1 and F_1 follows from the uniform convergence of $\lim_{j \to +\infty} A_j$ and $\prod_{j \geq 2} Y_j$ with respect to $(A_1, F_1) \in D_{a,b} \times \{F \in B_r, |f|_r < \beta\}$.
■

We now particularize the choice of A_1 and F_1 in order to obtain Theorem 1.1. For $z \in \mathbb{C}$ let $A_1(z) = izJ$ and $F_1(.,z) = \Lambda_z^{-1}Q(.)\Lambda_z$ where J, Q and Λ are given by (9), (8) and (10). Let $\lambda_j(z) = iz\beta_j$ the eigenvalues of A_1 and

$$\alpha_0 = \inf_{j \neq k} \left(|arg(\lambda_j(1) - \lambda_k(1)) - \frac{\pi}{2}|; |arg(\lambda_j(1) - \lambda_k(1)) + \frac{\pi}{2}|\right) \tag{56}$$

α_0 is a nonnegative constant depending only on n because n is odd.
Let for $c > 0$ and $d > 0$

$$\Delta(c,d) = \{z \in \mathbb{C}, |z| > c \text{ and } |arg(z)| < d\} \tag{57}$$

We have the following theorem

Theorem. 2.2 *Let $r > 0$. There exits a continuous increasing function f from $[0,+\infty)$ to $[0,+\infty)$ with $f(0) = 0$ such that for every $(p_0, ..., p_{n-2}) \in (QP_r)^{n-1}$,*

(i) there exists a $n \times n$ matrix valued function $A(z)$ analytic in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$ and a $n \times n$ invertible matrix valued function $Y(x,z)$ analytic in $T^d_\epsilon \times \Delta(f(|p|_r), \frac{\alpha_0}{2})$, where T^d_ϵ is a complex neighborhood of T^d, such that $Y(\omega t, z)e^{A(z)t}$ is a solution to (6) for every $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$.

(ii) for $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$, we can define the eigenvalues branches of $iA(z)$, $\lambda^\infty_j(z)$, $j = 1, \ldots, n$, that are analytic in $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$. Furthermore we can index these eigenvalues in such a way that:
for $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$ and with $n = 2l + 1$:

$$Im\lambda^\infty_j(z) < 0 \text{ when } j = 1, \ldots, l$$

$$Im\lambda^\infty_j(z) > 0 \text{ when } j = l+2, \ldots, n$$

$$Im\lambda^\infty_{l+1}(z) = 0 \text{ when } z \in \Delta(f(|p|_r), \frac{\alpha_0}{2}) \cap I\!R$$

and

$$Imz \cdot Im\lambda^\infty_{l+1}(z) < 0 \text{ when } z \in \Delta(f(|p|_r), \frac{\alpha_0}{2}) \backslash I\!R$$

(iii) $Y(x,z)$ is bounded with respect to $x \in T^d_\epsilon$ and z in a compact set included in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$.

Proof

By (19), we have

$$\alpha(A_1(1)) = \sin\alpha_0 \tag{58}$$

and

$$|z|\sin(\alpha_0 - arg(z)) \leq \alpha(A_1(z)) \leq |z|\sin(\alpha_0 + arg(z)) \tag{59}$$

Furthermore $\|A_1(z)\| = |z|\|A_1(1)\|$. Hence for every $c > 0$, there exist $a > 0$ and $b > 0$ such that

$$\forall z \in \Delta(c, \frac{\alpha_0}{2}),\ A_1(z) \in D_{a,b}$$

Let δ be as in Theorem 2.1. For fixed $r > 0$ it is easy to check that δ is an increasing function of c, let say $\delta(c)$.

Now, because we will need it in the proof of (ii), let us define

$$\tilde{\delta}(c) = \min\left(\delta(c), \frac{\sin\left(\frac{\alpha_0}{2}\right)}{(2M_{A_1})^{\frac{1}{2}}}\right) \tag{60}$$

Using (36), (58) and (59), we see that M_{A_1} depends only on α_0 and not on c. So $\tilde{\delta}(c)$ is a non decreasing function of c.

Notice that for fixed $r > 0$ and for $z \in \mathbb{C}^*$

$$|F_1(., z)|_r \leq \frac{M}{|z|}|p|_r$$

where M is a constant.
Now let f be a continuous increasing function from $I\!R_+$ to $I\!R_+$ such that $f(x)\tilde{\delta}(f(x)) \geq Mx$. Then, for $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$

$$|F_1(.,z)|_r \leq \tilde{\delta}(f(|p|_r)) \leq \delta(f(|p|_r))$$

So we can apply Theorem 2.1 with $\delta = \tilde{\delta}(f(|p|_r))$ and we obtain the existence of $\tilde{A}(z)$ analytic in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$ and $\tilde{Y}(.,z) \in B_{\frac{r}{2}}$ analytic in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$ such that $\tilde{Y}(\omega t, z)e^{\tilde{A}(z)t}$ is a solution for $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$ to (11).
Now if we define

$$Y(x,z) = \Lambda_z \tilde{Y}(x,z)\Lambda_z^{-1} \text{ and } A(z) = \Lambda_z \tilde{A}(z)\Lambda_z^{-1} \tag{61}$$

$Y(\omega t, z)e^{A(z)t}$ is a solution to (6). This ends the proof of (i).

Using (40) $|\tilde{Y}(.,z)|_{\frac{r}{2}}$ is bounded in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$. Furthermore Λ_z and Λ_z^{-1} are bounded on every compact set of $\mathbb{C}^*$. So we obtain (iii).

By (61) $i\tilde{A}(z)$ and $iA(z)$ (resp. $i\tilde{A}_1(z)$ and $iA_1(z)$ have the same eigenvalues. Then by (39), (45) and Lemma 2.2 we know that we can define the eigenvalue branches of $iA(z)$ (resp. $iA_1(z)$), $\lambda_j^\infty(z)$ (resp. $\lambda_j(z)$) $j = 1, \ldots, n$, that are analytic in $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$. Furthermore, using (49), (58) and (59) we obtain for $j = 1, \ldots, n$

$$|\lambda_j^\infty(z) - \lambda_j(z)| \leq \frac{\sin(\frac{\alpha_0}{2})}{4}|z|$$

Now if we index the eigenvalues of $iA_1(z)$ in such a way that $\lambda_j(z) = -\beta_{j+1}z$ for $j = 1, \ldots, l$, $\lambda_j(z) = -\beta_j z$ for $j = l+2, \ldots, n$ and $\lambda_{l+1}(z) = -z$ we have for every z in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$

$$Im\lambda_j^\infty(z) < 0 \text{ when } j = 1, \ldots, l$$

$$Im\lambda_j^\infty(z) > 0 \text{ when } j = l+2, \ldots, n.$$

Furthermore, for every z in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$ with $|arg(z)| > \frac{\alpha_o}{4}$, we have

$$Imz.Im\lambda_{l+1}^\infty(z) < 0.$$

Hence for z in $\Delta(f(|p|_r), \frac{\alpha_0}{2})$ with $arg(z) > \frac{\alpha_o}{4}$ we have $Im(\lambda_{l+1}^\infty(z)) < 0$ and $Im(\lambda_{l+1}^\infty(\bar{z})) > 0$. Thus there exists $\tilde{z} \in [z, \bar{z}]$ such that $Im(\lambda_{l+1}^\infty(\tilde{z})) = 0$ as $Im(\lambda_{l+1}^\infty(z))$ is continuous.

Let us show that $\tilde{z}$ is real. Notice that this immediately implies that $Imz \cdot Im\lambda_{l+1}^\infty(z) < 0$ for z in $\Delta(f(|p|_r), \frac{\alpha_0}{2}) \backslash I\!R$.

By Theorem 2.1 and its proof we find a non trivial solution ϕ to $(L - \tilde{z}^n)\phi = 0$ that is bounded and has all its derivatives bounded.
Thus $\tilde{z}^n$ belongs to the spectrum of L. Indeed, if χ is in $C_0^\infty(I\!R)$, $0 \leq \chi(.) \leq 1$

such that $\chi = 1$ on $[-1,1]$, $\chi = 0$ outside of $[-2,2]$, then put $\chi_N(x) = \chi(\frac{x}{N})$ where N is positive integer and compute

$$(L - \tilde{z}^n)(\chi_N \phi)(x) = \chi_N (L - \tilde{z}^n)\phi(x) + \sum_{k=1}^{n} \frac{1}{N^k}(D^k \chi)(\frac{x}{N}) P_k(x,D)\phi(x)$$

where $P_k(x,D)$ is a differential polynomial of order at most $2p$ with all its coefficients bounded. So, for $k \geq 1$, $\|(D^k\chi)(\frac{x}{N})P_k(x,D)\phi(x)\|_{L^2} = O(\sqrt{N})$ as $N \to +\infty$. Then as $(L - \tilde{z}^n)\phi(x) = 0$ we have $\lim_{N\to\infty} ||(L-\tilde{z}^n)\chi_N\phi|| = 0$ and $||\chi_N \phi|| \geq C > 0$.
So by the Weyl criterion, $\tilde{z}^n$ is in the spectrum of L and, hence, as L is self-adjoint, $\tilde{z}$ is real. ■

3 The spectral result

In this section, we will prove Theorems 1.1 and 1.2. The proof will follow the lines of the proof of [Eli]. That is, we will use the reduction of the eigenvalues problem for the odd order differential operator (14) to the matrix problem (6) and the Floquet solutions we have constructed in Theorem 2.2 to find a basis of generalized eigenfunctions for the differential operator. This will be done for values of the spectral parameter close to the real axis.
Theorem 2.2 will also allow us to exhibit the elements of this basis that are exponentially decreasing at $+\infty$ and those exponentially decreasing at $-\infty$. Then using the constructions given in [D-S] we will write the Green's function and the spectral measure.

Recall that by Theorem 2.2, for $z \in \Delta(f(|p|_r), \frac{\alpha_0}{2})$ we have a matrix solution to (6) of the form, $X(\omega t, z) = Y(\omega t, z)e^{A(z)t}$. $A(z)$ may be reduced to a diagonal matrix $D(z)$ by conjugation by an invertible matrix $P(z)$, that is

$$A(z)P(z) = P(z)D(z)$$

If we define $\tilde{X}(\omega t, z) = X(\omega t, z)P(z)$ then $\tilde{X}$ is a solution to (6). Moreover

$$\tilde{X}(\omega t, z) = Y(\omega t, z)P(z)e^{D(z)t} = \tilde{Y}(\omega t, z)e^{D(z)t} \tag{62}$$

Setting $\tilde{Y}(\omega t, z) = ((y_{ij}))_{ij}$ and $\tilde{X}(\omega t, z) = ((x_{ij}))_{ij}$ we get

$$x_{kj}(\omega t, z) = y_{kj}(\omega t, z)e^{\lambda_j(z)t}$$

where $(\lambda_j(z))_{j=1,\dots,n}$ are the eigenvalues of $A(z)$.
As Y is invertible, so is $\tilde{X}$; moreover x_{kj} is the $(k-1)^{st}$ derivative of x_{1j}, hence the functions $(x_{1j})_{j=1,\dots,n}$ are linearly independent. For $j = 1,\dots,n$, define $\Phi_j(t,z) = x_{1j}(\omega t, z)$. The $(\Phi_j)_{j=1,\dots,n}$ form a basis of the vector space of solutions of $Lu = z^n u$.

By Theorem 2.2 (ii) we can order the eigenvalues of $A(z)$ as follow:

- $(\lambda_j)_{j=1,\ldots,p}$ have positive real part for $z \in \Delta$
- λ_{p+1} has real part zero for z real and its real part has the sign of minus the imaginary part of z for z non real in Δ.
- $(\lambda_j)_{j=p+2,\ldots,2p+1}$ have negative real part for $z \in \Delta$

(here $\Delta = \Delta(f(|p|_r), \alpha_0)$).
With this ordering we get that the $(\Phi_j)_{j=1,\ldots,n}$ are the Jost solutions of the operator (14). More precisely, for $z \in \Delta$,

- $(\Phi_j)_{j=1,\ldots,p}$ are exponentially decreasing at $-\infty$ and thus in $L^2((-\infty, 0])$,
- Φ_{p+1} is exponentially decreasing at $-\infty$ (resp. $+\infty$) if $Im(z) < 0$ (resp. $Im(z) > 0$),
- $(\Phi_j)_{j=p+2,\ldots,n}$ are exponentially decreasing at $+\infty$ and thus in $L^2([0, +\infty))$.

We will now use these Jost solutions and the construction of the Green's function done in ([D-S], chapter XIII-3) to construct the spectral measure. Note that the spectral parameter is $\lambda = z^n$. So if $Im(\lambda) \neq 0$ but small then z is the n^{th} root of λ close to the real axis. As n is odd $Im(z)$ and $Im(\lambda)$ have the same sign.
Let us briefly recall the results of ([D-S], chapter XIII-3) in our particular case. For λ in the complement of the spectrum of L, let

- $(\phi_i(z,t))_{i=1,\ldots,\gamma}$ be a basis of the vector space solutions of $(L - \lambda)\phi = 0$ that are in $L^2((-\infty, 0))$
- $(\psi_i(z,t))_{i=1,\ldots,\beta}$ be a basis of the vector space solutions of $(L - \lambda)\phi = 0$ that are in $L^2((0, +\infty))$
- $(\psi_i^*(z,t))_{i=1,\ldots,\beta}$ and $(\phi_i^*(z,t))_{i=1,\ldots,\gamma}$ defined as above for the problem $(L^* - \bar{\lambda})\phi = 0$.

Note that $\gamma + \beta = n$.
For f and g functions defined on $I\!R$, define

$$F_t(f,g) = \sum_{j,l=0}^{n-1} F_t^{lj} f^{(l)}(t)\overline{g^{(j)}(t)}$$

where $((F_t^{lj}))$ is the boundary matrix for $L - \lambda$ at the point t (see [D-S] p 1286).
By Green's formula, $F_t(\phi_i, \phi_i^*)$, $F_t(\psi_i, \psi_i^*)$, $F_t(\phi_i, \psi_i^*)$ and $F_t(\psi_i, \phi_i^*)$ are independent of t (see [D-S] p 1324). These are analogues for operator of order n of the Wronskian in the 2^{nd} order case.
Define

- $\xi_i = \phi_i$ for $i = 1, ..., \gamma$ and $\xi_i = \psi_{i-\gamma}$ for $i = \gamma+1, ..., n$,
- $\eta_i^* = \phi_i^*$ for $i = 1, ..., \beta$ and $\eta_i^* = -\psi_{i-\beta}^*$ for $i = \beta+1, ..., n$.

Then the matrix $((F_t(\xi_i, \eta_j^*)))_{ij}$ is independent of t and invertible. Its inverse is denoted by $\Gamma = ((\Gamma_{ij}))_{ij}$. The kernel K of the resolvent $(L-\lambda)^{-1}$ is given by

$$\begin{aligned} K(\lambda, t, s) &= \textstyle\sum_{i=1}^{\beta} \sum_{j=\gamma+1}^{n} \Gamma_{ij} \overline{\eta_i^*(s)} \xi_j(t) \quad \textit{if } s < t, \\ K(\lambda, t, s) &= -\textstyle\sum_{i=\beta+1}^{n} \sum_{j=1}^{\gamma} \Gamma_{ij} \overline{\eta_i^*(s)} \xi_j(t) \quad \textit{if } s > t. \end{aligned} \tag{63}$$

By what we said above, in our case we compute the (ξ_i) and (η_j^*) and get:

- for $Im(\lambda) < 0$, $\gamma = p+1$, $\beta = p$ and

$$\xi_j(z,t) = \phi_j(z,t) = \Phi_j(z,t) \ for\ j = 1, ..., p+1$$

$$\xi_j(z,t) = \psi_{j-p}(z,t) = \Phi_j(z,t) \ for\ j = p+2, ..., n$$

$$\eta_j^*(z,t) = \phi_j^*(z,t) = \Phi_j(\bar{z},t) \ for\ j = 1, ..., p$$

$$\eta_j^*(z,t) = -\psi_{j-p}^*(z,t) = -\Phi_j(\bar{z},t) \ for\ j = p+1, ..., n\ .$$

- for $Im(\lambda) > 0$, $\gamma = p$, $\beta = p+1$ and

$$\xi_j(z,t) = \phi_j(z,t) = \Phi_j(z,t) \ for\ j = 1, ..., p$$

$$\xi_j(z,t) = \psi_{j-p}(z,t) = \Phi_j(z,t) \ for\ j = p+1, ..., n$$

$$\eta_j^*(z,t) = \phi_j^*(z,t) = \Phi_j(\bar{z},t) \ for\ j = 1, ..., p+1$$

$$\eta_j^*(z,t) = -\psi_{j-p}^*(z,t) = -\Phi_j(\bar{z},t) \ for\ j = p+2, ..., n\ .$$

We now have to determine the matrix Γ. Therefore we will use the following lemma

Lemma. 3.1 *(i) For $(j,l) \in \{1, ..., p\}^2 \cup \{p+2, ..., n\}^2$ and $z \in \Delta$ one has*

$$F_t(\Phi_j(z,t), \Phi_l(\bar{z},t)) = 0.$$

(ii) For $j \in \{1, ..., 2p+1\} \setminus \{p+1\}$ and $z \in \Delta$ one has

$$F_t(\Phi_{p+1}(z,t), \Phi_j(\bar{z},t)) = F_t(\Phi_j(z,t), \Phi_{p+1}(\bar{z},t)) = 0.$$

Proof

(i) For $(j,l) \in \{1,...,p\}^2 \cup \{p+2,...,n\}^2$ and $z \in \Delta$, $F_t(\Phi_j(z,t), \Phi_l(\bar{z},t))$ is independent of t. So, as $\Phi_j(z,t)$ and $\Phi_l(\bar{z},t)$ vanish both at either $+\infty$ or $-\infty$, we get

$$F_t(\Phi_j(z,t), \Phi_l(\bar{z},t)) = \lim_{t\to\pm\infty} F_t(\Phi_j(z,t), \Phi_l(\bar{z},t)) = 0.$$

(ii) Let $1 \le j \le p$ then for $Im(z) < 0$, using (i), we know that

$$F_t(\Phi_j(z,t), \Phi_{p+1}(\bar{z},t)) = 0.$$

Since the $\Phi_j(z,t)$ are analytic in z (by Theorem 2.2), we know that $F_t(\Phi_j(x+iy,t), \Phi_{p+1}(x-iy,t))$ is real analytic in (x,y) and vanishes for $y > 0$. So by unique continuation, it vanishes also for $y \ge 0$, i.e. $Im(z) \ge 0$. ■

So the matrix $F = ((F_t(\xi_i, \eta_j^*)))$ takes the following form

$$F = \begin{pmatrix} 0 & 0 & -F^+ \\ 0 & \epsilon F_t(p+1,p+1) & 0 \\ F^- & 0 & 0 \end{pmatrix}$$

where

$$F_t(i,j) = F_t(\Phi_i(z,t), \Phi_j(\bar{z},t))$$

and

$$F^+ = ((F_t(i,j)))_{p+2\le j\le n}^{1\le i\le p},\ F^- = ((F_t(i,j)))_{1\le j\le p}^{p+2\le i\le n},$$

and ϵ is the sign of $Im(\lambda)$.

So

$$\Gamma = \begin{pmatrix} 0 & 0 & \Gamma^- \\ 0 & \epsilon(F_t(p+1,p+1))^{-1} & 0 \\ -\Gamma^+ & 0 & 0 \end{pmatrix},$$

where $\Gamma^\pm = (F^\pm)^{-1}$. Hence by (63) one has

- if $Im(\lambda) < 0$ and $t > s$

$$K(\lambda,t,s) = \sum_{i=1}^{p}\sum_{j=p+2}^{n} \Gamma_{ij}\Phi_j(z,t)\overline{\Phi_i(\bar{z},s)}$$

$$= \sum_{i=1}^{p}\sum_{j=p+2}^{n} \Gamma_{ij}^-\Phi_j(z,t)\overline{\Phi_i(\bar{z},s)},$$

- if $Im(\lambda) < 0$ and $t < s$

$$K(\lambda,t,s) = +\sum_{i=p+1}^{n}\sum_{j=1}^{p+1} \Gamma_{ij}\Phi_j(z,t)\overline{\Phi_i(\bar{z},s)}$$

$$= -\sum_{i=p+2}^{n}\sum_{j=1}^{p} \Gamma_{ij}^+\Phi_j(z,t)\overline{\Phi_i(\bar{z},s)} - \frac{1}{F_t(p+1,p+1)}\Phi_{p+1}(z,t)\overline{\Phi_{p+1}(\bar{z},s)},$$

- if $Im(\lambda) > 0$ and $t > s$

$$K(\lambda,t,s) = \sum_{i=1}^{p+1}\sum_{j=p+1}^{n}\Gamma_{ij}\Phi_j(z,t)\overline{\Phi_i(\bar z,s)}$$

$$= \sum_{i=1}^{p}\sum_{j=p+2}^{n}\Gamma_{ij}^{-}\Phi_j(z,t)\overline{\Phi_i(\bar z,s)} + \frac{1}{F_t(p+1,p+1)}\Phi_{p+1}(z,t)\overline{\Phi_{p+1}(\bar z,s)},$$

- if $Im(\lambda) > 0$ and $t < s$

$$K(\lambda,t,s) = -\sum_{i=p+2}^{n}\sum_{j=1}^{p}\Gamma_{ij}^{+}\Phi_j(z,t)\overline{\Phi_i(\bar z,s)}.$$

So, if $z = x + iy$, for $t > s$

$$\lim_{y\to 0^-}(K(z,t,s) - K(\bar z,t,s)) = \frac{-1}{F_t(\Phi_{p+1}(x,t),\Phi_{p+1}(x,t))}\Phi_{p+1}(x,t)\overline{\Phi_{p+1}(x,s)}$$

and for $t < s$

$$\lim_{y\to 0^-}(K(z,t,s) - K(\bar z,t,s)) = \frac{-1}{F_t(\Phi_{p+1}(x,t),\Phi_{p+1}(x,t))}\Phi_{p+1}(x,t)\overline{\Phi_{p+1}(x,s)}.$$

The Titchmarch-Kodaira theorem (see [D-S] p 1364) tells us that for every energy larger than $f(|p|_r)$, where f is defined in Theorem 2.2, the spectrum of L is simple. That is the matrix valued spectral measure has only one non vanishing term. Moreover this measure has the following density with respect to the Lebesgue measure

$$\mu(x) = \frac{-1}{2i\pi F_t(\Phi_{p+1}(x,t),\Phi_{p+1}(x,t))}$$

This density is real analytic in x, hence the spectral measure is absolutely continuous with respect to the Lebesgue measure. This ends the proof of Theorem 1.1 .

4 Sketch of the proof of Theorem 1.2

The spectral part of the proof of Theorem 1.2 is the same as above. So we will only explain briefly how to obtain the Floquet decomposition for the operator L_3 defined by (3) when $p_0(.) - \hat p_0$ and $p_1(.) - \hat p_1$ are small. As we say in the introduction we consider L_3 as a perturbation of $\hat L$ defined by (5). The spectral equation $Lu = \lambda u$ can be reduced to the following first order system

$$X'(t,\lambda) = i(B(\lambda) + G(\omega t))X(t,\lambda)$$

where

$$B(\lambda) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ \hat{p}_0 - \lambda & \hat{p}_1 & 0 \end{pmatrix} \text{ and } G(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \\ p_0(x) - \hat{p}_0 & p_1(x) - \hat{p}_1 & 0 \end{pmatrix}$$

Let I be the set of $I\!R$ such that, if $\lambda \in I$, $B(\lambda)$ has only one real eigenvalue and this eigenvalue is simple. This means that the polynomial $P_\lambda(X) = X^3 + \hat{p}_1 X + \hat{p}_0 - \lambda$ has only one real root and it is simple. It is easy to check that I has the form $(-\infty, a) \cup (b, +\infty)$ for some real numbers a and b. Set $(X_i(\lambda))_{i=1,2,3}$ the eigenvalues of $B(\lambda)$ with $X_1(\lambda)$ real for $\lambda \in I$. Now, for $\lambda \in I$, $|Im(X_i(\lambda) - X_j(\lambda))| > 0$ because, since $\hat{p}_1$ and $\hat{p}_0$ are real, $X_2 = \overline{X_3}$. So for every $\epsilon > 0$, if λ is in complex neighborhood of $I_\epsilon = (-\infty, -\epsilon) \cup (b+\epsilon, +\infty)$, $iB \in D_{c,d}$ for some $c > 1$ and $d > 0$. We can then applied Theorem 2.1 and obtain, for every $\epsilon > 0$ and $r > 0$, the existence of $\eta > 0$ such that if $|p_0(.) - \hat{p}_0|_r < \eta$ and $|p_1(.) - \hat{p}_1|_r < \eta$, L (restricted to the energy set I_ϵ) admits a Floquet decomposition. The arguments we used in the proof of Theorem 1.1 allow us to obtain Theorem 2.2.
In the case $n \geq 5$, for some operators with constant coefficients, $\alpha(iB)$ will be 0 for some energies λ even if the polynomial $P_\lambda(X)$ admits only one real root and that this root is simple. So the method used here does not work in such a straitforward way. This problem is under study ([GGK]).

References

[BdM-Eg] A. Boutet de Monvel and I. Egorova. On the almost periodicity of solutions of the nonlinear Schrödinger equation with the Cantor type spectrum. Preprint, laboratoire de Physique Mathématique et Géométrie, Université Paris 7, 1995.

[BDT] R. Beals, P. Deift and C. Tomei. Direct and Inverse Scattering on the line. Mathematical surveys and monographs, Number 28, A.M.S. 1988.

[Be-Te] J. Bellissard and D. Testard. Almost periodic Schrödinger operators. In Mathematics and Physics, Lectures on recent results, vol. I, edited by L. Streit, p1-64. World Scientific, 1985.

[DTT] P. Deift, C. Tomei and E. Trubowitz. Inverse Scattering and the Boussinesq equation. CPAM, Vol 35, p 567-628, 1982.

[Din-Sin] E.I. Dinaburg and Y.G. Sinai. The one dimensional Schrödinger equation with quasi-periodic potential. Funkt. Anal. i. Priloz., Vol 9, p 8-21, 1975.

[D-S] N. Dunford and J.T. Schwartz. Linear operators part II. Interscience Publishers, 1963.

[Eg] I.E. Egorova. The Cauchy problem for the KdV equation with almost periodic initial data whose spectrum is nowhere dense. Advances in Soviet Math., Vol 19, p 181-208, 1994.

[Eli] L.H. Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. CMP, Vol 146, p 447-482, 1992.

[GGK] B. Grébert, J.-C. Guillot and F. Klopp. Work in progress.

[Ji-Si] S. Jitomirskaya and B. Simon. Operators with Singular Continuous Spectrum III: Almost Periodic Schrödinger Operators. CMP, Vol 165, p 201-205, 1994.

[Ka] T. Kato. Perturbation Theory for Linear Operators. 2nd edition, Springer Verlag , 1976.

[Mo] J Moser. An example of a Schrödinger equation with almost-periodic potential and nowhere dense spectrum. Comment. Math. Helvetici, vol 56, p 198-215, 1981.

[Mo-Po] J. Moser and J. Pöschel. An Extension of a Result by Dinaburg and Sinai on Quasi-Periodic Potentials. Comment. Math. Helvetici, Vol 59, p 39-85, 1984.

[Pa-Fi] L. Pastur and A. Figotin. Spectra of Random and Almost-Periodic Operators. Springer Verlag, 1992

[Re-Si] M. Reed and B. Simon. Methods of Modern Mathematical Physics. Vol IV. Academic Press, 1979.

[Su] Steve Surace. The Schrödinger Equation with a Quasi-Periodic Potential. Trans. of the Am. Math. Soc., Vol 320, Number1, p 321-370, 1990.

[So-Sp] E. Sorets and T. Spencer. Positive Lyapounov Exponents for Schrödinger Operators with Quasi-Periodic Potentials. CMP, Vol 142, p 543-566, 1991.