Convergence of the natural approximations of piecewise monotone interval maps

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Abstract: We consider piecewise monotone interval mappings which are topologically mixing and satisfy the Markov property. It has previously been shown that the invariant densities of the natural approximations converge exponentially fast in uniform pointwise topology to the invariant density of the given map provided it's derivative is piecewise Lipshitz continuous. In this paper we show that in general one does not obtain exponential convergence in the bounded variation norm. Here we prove that if the derivative of the interval map is Hölder continuous and its variation is well approximable (γ-uniform variation for $\gamma > 0$), then the densities converge exponentially fast in the norm.

1 Introduction

The dynamics of (mixing) expanding piecewise monotone maps has been extensively studied, in particular the spectral properties of the associated transfer operators have been established by notably Hofbauer and Keller [2] [3]. On the Banach space of functions of bounded variations the transfer operator has a simple largest real eigenvalue whose eigenfunction is the density of the invariant measure. The remainder of the spectrum has then strictly smaller radius which in particular implies that the correlation functions decay exponentially fast. For a piecewise affine map which satisfies the Markov property the transfer operator can be written as a matrix, which makes is more practical to find its invariant density [4]. It has been shown in [1] that the densities of affine approximations of an expanding piecewise monotone map indeed converge exponetially fast pointwise uniformly to the actual density of the invariant measure if the map has (piecewise) Lipshitz continuous derivatives. In section 3 we provide an example which shows that under these conditions the convergence cannot in general expected to be exponential in the bounded variation norm. However, our main result, theorems 5 and 6, shows that the approximating densities convergence in the norm exponentially fast if one assumes that the variation of the map's

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derivative can sufficiently well be approximated by variations over partitions (the derivative has uniform variation—see below).

Let T be a piecewise $C^{1+\gamma}$ continuous map (that is its first derivative is γ-Hölder continuous) of the unit interval $[0,1]$ to itself such that its restriction to the atoms of a finite partition $\mathcal{A} = \{(a_{j-1}, a_j) : j = 1, \dots J\}$ are monotone with Hölder continous derivative T', where $0 = a_0 < a_1 < a_2 < \cdots < a_J = 1$. We shall furthermore make the following assumptions:

(i) There exists a constant $\rho < 1$ so that $|T'| \geq \rho^{-1}$ on every atom $A \in \mathcal{A}$.

(ii) At the endpoints a_j of the atoms of $\mathcal{A}$ the map T shall assume one or the other value of the continuous extensions of T to the closures of the adjacent intervals.

(iii) T has the Markov property, that is if $T(A) \cap A' \neq \emptyset$ for some $A, A' \in \mathcal{A}$ then $A' \subset T(A)$.

(iv) We require that T is topologically mixing, that is, there is a positive integer N so that the branches of monotonicity of T^N are onto $[0,1]$. Without loss of generality we shall assume that $N = 1$.

We can now define the transfer operator $\mathcal{L}$ by:

$$\mathcal{L}\phi(x) = \sum_{y \in T^{-1}x} \frac{\phi(y)}{|T'(y)|},$$

for suitable functions $\phi : [0,1] \to \mathbf{C}$. On the space BV of functions with bounded variation $\mathcal{L} : \mathrm{BV} \to \mathrm{BV}$, the spectral radius of $\mathcal{L}$ is equal to 1, and 1 is a simple eigenvalue with a strictly positive eigenfunction h which we assume to be normalised. Moreover the essential spectral radius $\vartheta < 1$, where

$$\vartheta = \limsup_{n \to \infty} \frac{1}{|(T^n)'|^{1/n}}.$$

If we put

$$\tau = \max\{|\lambda| : \lambda \neq 1 \text{ in the spectrum of } \mathcal{L} : \mathrm{BV} \to \mathrm{BV}\},$$

(spectral gap), then the convergence of $\mathcal{L}$ to the eigenspace spanned by h is exponential at rate $\tau < 1$, that is

$$\|\mathcal{L}^n \phi - h\mu(\phi)\| \leq \text{const.} \tau'^n \|\phi\|,$$

for every $\tau' > \tau$, where μ denotes the Lebesgue measure on $[0,1]$ and $\|\phi\| = |\phi|_\infty + \mathrm{var}_{[0,1]}\phi$ is the usual BV-norm.

Definition 1 *The* nth piecewise affine approximation *of T is the transform $T_n : [0,1] \to [0,1]$ such that $T|_A$ is affine for all $A \in \mathcal{A}^n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{A}$ (the n-th join) and matches with T at the endpoints of the atoms A.*

Denote by $\mathcal{L}_n$ the transfer operator for the n-th affine approximation:

$$\mathcal{L}_n\phi(x) = \sum_{y\in T_n^{-1}x} \frac{\phi(y)}{|T_n'(y)|}, \quad \phi \in \mathrm{BV}.$$

Clearly 1 is a simple eigenvalue of $\mathcal{L}_n$ with strictly positive eigenfunction h_n. Assume h_n is normalised.

Theorem 2 *Let T be a piecewise monotone interval map which satisfies the conditions (i)–(iv) (for some $\gamma > 0$). There exists a constant C_1 such that*

$$\mathrm{var}\, h_n \leq C_1, \ \forall n.$$

For every $\tau' > \tau^{1/4}$ there exists a constant C_2 such that

$$|h_n - h|_\infty \leq C_2 \tau'^n, \quad n = 1, 2, 3, \ldots.$$

This theorem is proven in [1] for $\tau' > \tau^{1/6}$. However a simplification of [1] lemma 2.2 yields the Ionescu Tulcea-Marinescu type inequality

$$\|(\mathcal{L}^k - \mathcal{L}_n^k)\phi\|_{\mathrm{BV}} \leq \mathrm{const.}\, |\phi|_\infty + \vartheta^k \mathrm{var}\, \phi, \tag{1}$$

for all $k = 0, 1, 2, \ldots, n$ (instead of $k < \frac{2}{3}n$) which then (in [1] lemma 2.3) implies the improved lower bound $\tau^{1/4}$ for τ'.

We shall prove that the convergence is indeed exponential in the BV-norm if T satisfies the stronger property of having uniform variation. In section 3 we shall then produce an example of a piecewise C^{1+L} map (i.e. the derivative of the map is piecewise Lipshitz continuous) for which $\mathrm{var}\,(h - h_{n_j}) \geq \frac{\mathrm{const.}}{\log n_j}$ for a suitable sequence of integers n_j. That is, for that example the convergence of the densities is very slow in the bounded variation norm.

If a finite partition $\mathcal{P}$ of the unit interval consists of the intervals between the points $0 = p_0 < p_1 < p_2 < \cdots < p_m = 1$, then we put

$$\mathrm{var}\,(\phi, \mathcal{P}) = \sum_{j=1}^{m} |\phi(p_{j-1} - \phi(p_j)|$$

for the variation of the function ϕ (if ϕ is continuous on $[0, 1]$) with respect to the partition $\mathcal{P}$. If ϕ is piecewise continuous we take the variation over the intervals of piecewise continuity. Clearly, $\mathrm{var}\, \phi = \sup_{\mathcal{P}} \mathrm{var}(\phi, \mathcal{P})$, where the supremum is over all partitions $\mathcal{P}$ of the unit interval. (Observe that $\mathrm{var}\, \phi = \mathrm{var}(\phi, \mathcal{P})$, for all partitions $\mathcal{P}$, if ϕ is monotonically increasing or decreasing.) Put

$$\mathrm{Var}\,(\phi, \mathcal{P}) = \left|\mathrm{var}\, \phi - \mathrm{var}\,(\phi, \mathcal{P})\right|,$$

and denote by $\mathrm{diam}\, \mathcal{P} = \max_j (p_j - p_{j-1})$ the *diameter* of the partition $\mathcal{P}$.

Definition 3 *We say a function ϕ on the unit interval has* γ-uniform variation *for some $\gamma > 0$ if there exists a constant U so that*

$$\mathrm{Var}\,(\phi, \mathcal{P}) \leq U(\mathrm{diam}\,\mathcal{P})^\gamma,$$

for all partitions $\mathcal{P}$ of $[0,1]$. We shall put $U_\gamma(\phi)$ for the smallest possible choice of the constant U.

If f and g are functions on the unit interval with finite U_γ values, then it is easily seen that $U_\gamma(f+g) \leq U_\gamma(f) + U_\gamma(g)$ and $U_\gamma(cf) = |c| U_\gamma(f)$ for constants c. For constant functions c we have $U_\gamma(c) = 0$. Moreover $U_\gamma(fg) \leq |g|_\infty U_\gamma(f) + |f|_\infty U_\gamma(g)$. We can therefore define a norm $\|\cdot\|_\gamma$ by

$$\|\phi\|_\gamma = |\phi|_\infty + U_\gamma(\phi),$$

and introduce for some partition $\mathcal{P}$ of the unit interval the function space

$$\mathrm{UV}_\gamma = \{\phi \in C^\gamma(\mathcal{P}) : \|\phi\|_\gamma < \infty\}$$

of functions with γ-uniform variation, where $C^\gamma(\mathcal{P})$ consists of all functions on $[0,1]$ which are γ-Hölder continuous on the atoms $\mathcal{P}$. The space $(\mathrm{UV}_\gamma, \|\cdot\|_\gamma)$ is a Banach space. Clearly $\mathrm{UV}_\gamma \subset \mathrm{UV}_{\gamma'}$ for $\gamma' \leq \gamma$, and moreover $\mathrm{UV}_\gamma \subset \mathrm{BV}$ for all positive γ. (Naturally $U_{\gamma'}(\phi) \leq U_\gamma(\phi)$ if $\gamma' \leq \gamma$.)

Continuous functions in UV_γ are not necessarily γ-Hölder continuous as the example $\phi = x^{\gamma/2}$, $x \in [0,1]$ shows. Since ϕ is increasing $\mathrm{var}\,\phi = \mathrm{var}\,(\phi, \mathcal{P})$ and therefore $\mathrm{Var}\,(\phi, \mathcal{P}) = 0$ for every partition $\mathcal{P}$ of the unit interval. Thus $\phi \in \mathrm{UV}_\gamma$ although ϕ is not γ-Hölder continuous.

Let us denote by S_k the inverse branches of T^k and put A_φ for the range of the inverse branches $\varphi \in S_k$. Note that $A_\varphi \in \mathcal{A}^k$.

Lemma 4 *Let T be a transformation of the unit interval to itself. Assume $|T'| \in \mathrm{UV}_\gamma$ ($\gamma > 0$), then $\mathcal{L}$ maps UV_γ into itself.*

Proof. Clearly $|\mathcal{L}\phi|_\infty \leq |\phi|_\infty$. To estimate the variation we proceed as follows

$$\begin{aligned}
\mathrm{Var}\,(\mathcal{L}\phi, \mathcal{P}) &= |\mathrm{var}\,\mathcal{L}\phi - \mathrm{var}\,(\mathcal{L}\phi, \mathcal{P})| \\
&\leq \sum_{\varphi \in S_1} \left| \mathrm{var}\,\frac{\phi\varphi}{|T'\varphi|} - \mathrm{var}\,\left(\frac{\phi\varphi}{|T'\varphi|}, \mathcal{P}\right) \right| \\
&\leq |S_1|\,\mathrm{Var}\,\left(\frac{\phi}{|T'|}, T^{-1}\mathcal{P}\right).
\end{aligned}$$

This implies that $\|\mathcal{L}\phi\|_\gamma \leq c_1 \|\phi\|_\gamma$ for some constant c_1. □

For the transfer operator and its affine approximations one can now deduce inequality (1) on the space UV_γ for $\gamma > 0$ (an explicit proof can be found by picking one's way through the proof of theorem 6). It follows that the approximating densities h_n in fact lie in UV_γ for all $\gamma > 0$. The following theorem, which is proven in the following section, asserts that convergence of the approximating densities is in fact exponential in the spaces of γ-uniform variation.

Theorem 5 *Let T be a transformation of the unit interval to itself so that $|T'| \in \mathrm{UV}_\gamma$ ($\gamma > 0$). Then there exists a number $\sigma \in (0,1)$ and a constant C_3 so that*

$$\|h - h_n\|_\gamma \leq C_3 \sigma^n,$$

for all $n = 1, 2, \ldots$.

Theorem 6 *Let T be a transformation of the interval which satisfies the conditions (i)–(iv) and let $\mathcal{L}$ be the associated transfer operator acting on the Banach space of functions of bounded variation. If we assume that T' has γ-uniform variation for some positive γ, then there exists a constant C_4 and a $\sigma \in (0,1)$ so that $\|h - h_n\|_{\mathrm{BV}} \leq C_4 \sigma^n \quad \forall n$. In particular $\mathrm{var}\,(h - h_n) \leq C_4 \sigma^n$, for all n.*

The proof of these two theorems is the subject of the following section.

2 Convergence

Lemma 7 *There exists a constant C_5 so that for every interval $A \subset [0,1]$ on which T^k is one-to-one:*
(I) For every partition $\mathcal{P}$ of $[0,1]$ one has

$$\mathrm{Var}_A \left(\frac{1}{|(T^k)'|}, \mathcal{P} \right) \leq C_5 \sup_A \frac{1}{|(T^k)'|} (\mathrm{diam}\, T^k \mathcal{P})^\gamma.$$

(II)

$$\mathrm{var}_A \frac{1}{|(T^k)'|} \leq k C_5 \sup_A \frac{1}{|(T^k)'|}.$$

Proof. Let $\mathcal{Q}$ be a partition of A. Assume $\mathcal{Q}$ is given by the points $y_0 < y_1 < \cdots < y_Q$. As $|(T^k)'|$ is (uniformly in k) γ-Hölder continuous, the variation over the partition $\mathcal{Q}$ is then

$$\begin{aligned} \mathrm{var} \left(\frac{1}{|(T^k)'|}, \mathcal{Q} \right) &= \sum_{j=1}^{Q} \left| \frac{1}{|(T^k)'(y_{j-1})|} - \frac{1}{|(T^k)'(y_j)|} \right| \\ &= \sum_{j=1}^{Q} \frac{1}{|(T^k)'(y_j)|} \left| 1 - \frac{|(T^k)'(y_j)|}{|(T^k)'(y_{j-1})|} \right| \\ &= \sum_{j=1}^{Q} \frac{1 + \mathcal{O}((\mathrm{diam}\, \mathcal{Q})^\gamma)}{|(T^k)'(y_j)|} \left| \log \frac{|(T^k)'(y_j)|}{|(T^k)'(y_{j-1})|} \right|. \end{aligned} \tag{2}$$

Hence, letting diam $\mathcal{Q}$ go to zero, one obtains

$$\begin{aligned}
\operatorname{var} \frac{1}{|(T^k)'|} &\leq \sup_A \left| \frac{1}{(T^k)'} \right| \operatorname{var}_A \log |(T^k)'| \\
&\leq \sup_A \left| \frac{1}{(T^k)'} \right| \sum_{\ell=0}^{k-1} \operatorname{var}_{T^\ell A} \log |T'| \\
&\leq \sup_A \left| \frac{k}{(T^k)'} \right| \operatorname{var} \log |T'|.
\end{aligned}$$

This proves the statement (II) of the lemma.

To prove the inequality (I), let $\mathcal{P}$ be some partition. Equation 2 yields

$$\operatorname{var} \frac{1}{|(T^k)'|} \leq \sum_{P \in \mathcal{P}} \sup_{P \cap A} \left| \frac{1}{(T^k)'} \right| \operatorname{var}_{P \cap A} \log |(T^k)'|$$

and

$$\operatorname{var} \left(\frac{1}{|(T^k)'|}, \mathcal{P} \right) \geq (1 - \mathcal{O}((\operatorname{diam} \mathcal{P})^\gamma)) \sum_{P \in \mathcal{P}} \inf_{P \cap A} \left| \frac{1}{(T^k)'} \right| \operatorname{var}_{P \cap A} \left(\log |(T^k)'|, \mathcal{P} \right).$$

Now, since

$$\inf_{P \cap A} \left| \frac{1}{(T^k)'} \right| \geq (1 - \mathcal{O}((\operatorname{diam} \mathcal{P})^\gamma)) \sup_{P \cap A} \left| \frac{1}{(T^k)'} \right|,$$

for $P \in \mathcal{P}$, we can finally estimate as follows:

$$\begin{aligned}
\operatorname{Var} \left(\frac{1}{|(T^k)'|}, \mathcal{P} \right) &\leq \sum_{P \in \mathcal{P}} \Bigg(\sup_{P \cap A} \left| \frac{1}{(T^k)'} \right| \operatorname{var}_{P \cap A} \log |(T^k)'| \\
&\qquad\qquad - \inf_{P \cap A} \left| \frac{1}{(T^k)'} \right| \operatorname{var}_{P \cap A} \left(\log |(T^k)'|, \mathcal{P} \right) \Bigg) \\
&\quad + c_1 (\operatorname{diam} \mathcal{P})^\gamma \sup_A \left| \frac{1}{(T^k)'} \right| \operatorname{var}_A \left(\log |(T^k)'|, \mathcal{P} \right) \\
&\leq \sup_A \left| \frac{1}{(T^k)'} \right| \left(\operatorname{Var}_A \left(\log |(T^k)'|, \mathcal{P} \right) + c_2 (\operatorname{diam} T^k \mathcal{P})^\gamma \right) \\
&\leq C_5 \sup_A \left| \frac{1}{(T^k)'} \right| (\operatorname{diam} T^k \mathcal{P})^\gamma,
\end{aligned}$$

for some constants c_1, c_2. In the last two inequalities we used that with some constant c_3 (observe that $\log |T'| \in \mathrm{UV}_\gamma$)

$$\operatorname{Var}_A \left(\log |(T^k)'|, \mathcal{P} \right) \leq \sum_{\ell=1}^{k} \operatorname{Var}_{T^k A} \left(\log |T'|, T^\ell \mathcal{P} \right) \leq c_3 (\operatorname{diam} T^k \mathcal{P})^\gamma \frac{1}{1-\rho},$$

since T^k is one-to-one on A and $\operatorname{diam} T^\ell\mathcal{P} \leq \rho^{k-\ell}\operatorname{diam} T^k\mathcal{P}$. Similarly $\operatorname{var}_A \big(\log|(T^k)'|, \mathcal{P}\big) \leq \frac{c_3}{1-\rho}(\operatorname{diam} T^k\mathcal{P})^\gamma$.

Put $C_5 = \max(c_2 + c_3/(1-\rho), \operatorname{var} \log|T'|)$. □

Let S_k denote the inverse branches of T^k. If $\varphi \in S_k$, then we denote its range by A_φ which in fact is an atom in $\mathcal{A}^k$. If $k < n$ then every inverse branch $\varphi \in S_k$ has a corresponding inverse branch $\tilde{\varphi}$ of T_n^k whose range $A_{\tilde{\varphi}} \in \mathcal{A}^k$ coincides with A_φ.

Lemma 8 *For every $\vartheta' > \vartheta$ there exists a constant C_6 so that for all $k < n$:*

$$\left|1 - \frac{|(T_n^k)'\tilde{\varphi}|}{|(T^k)'\varphi|}\right|_\infty \leq nC_6\vartheta'^{n-k}.$$

Proof. Given $\vartheta' > \vartheta$ then for any $\vartheta'' \in (\vartheta, \vartheta')$ we can find c_1 so that $\frac{1}{|(T^k)'|} \leq c_1\vartheta''^k$ for all k. Since T^k is one-to-one on the atoms of $\mathcal{A}^k$ we conclude that $\operatorname{diam}\mathcal{A}^k \leq c_1\vartheta''^k$ for all k. Now let $k < n$ and observe that $\varphi(x) = \tilde{\varphi}(x)$ on the endpoints x of atoms in $\mathcal{A}^{n-k}$. Therefore, by the mean value theorem,

$$\begin{aligned}
\left|1 - \frac{|(T_n^k)'\tilde{\varphi}|}{|(T^k)'\varphi|}\right|_\infty &\leq c_2\left|\log\frac{|(T_n^k)'\tilde{\varphi}|}{|(T^k)'\varphi|}\right|_\infty \\
&\leq c_2 \max_{B\in\mathcal{A}^{n-k}} \big(\operatorname{var}_{T^kB} \log|(T^k)'| + \operatorname{var}_{T_n^kB} \log|(T_n^k)'|\big) \\
&\leq c_2 \max\log|(T^k)'|\, 2\operatorname{diam}\mathcal{A}^{n-k} \\
&\leq (n-k)c_3\vartheta''^{n-k} \\
&\leq nC_6\vartheta'^{n-k},
\end{aligned}$$

for some constants c_2, c_3 and C_6. □

Proof of theorem 5. We have to show that there exists a constant $\sigma < 1$ so that the conditional variation $\operatorname{Var}(h - h_n, \mathcal{P}) \leq c_1\sigma^n(\operatorname{diam}\mathcal{P})^\gamma$ for all partitions $\mathcal{P}$ of $[0,1]$ and some constant c_1. Assume $k < n$ and given an inverse branch $\varphi \in S_k$ let, as above, $\tilde{\varphi}$ be the associated inverse branch of T_n^k which has the same range $A_\varphi = A_{\tilde{\varphi}} \in \mathcal{A}^k$. Let $\mathcal{P}$ be a partition of the unit interval. Then, for any k,

$$\begin{aligned}
\operatorname{Var}(h - h_n, \mathcal{P}) &= \operatorname{Var}(\mathcal{L}^k h - \mathcal{L}_n^k h_n, \mathcal{P}) \\
&= \operatorname{Var}\left(\sum_{\varphi\in S_k}\left(\frac{h\varphi}{|(T^k)'\varphi|} - \frac{h_n\tilde{\varphi}}{|(T_n^k)'\tilde{\varphi}|}\right), \mathcal{P}\right) \\
&\leq \operatorname{Var}\left(\sum_{\varphi\in S_k}\frac{1}{|(T_n^k)'\varphi|}(h\varphi - h_n\tilde{\varphi}), \mathcal{P}\right)
\end{aligned}$$

$$
\begin{aligned}
& +\mathrm{Var}\left(\sum_{\varphi\in S_k} h_n\tilde{\varphi}\left(\frac{1}{|(T_n^k)'\tilde{\varphi}|}-\frac{1}{|(T^k)'\varphi|}\right),\mathcal{P}\right)\\
\le\ & I+II+III+IV,
\end{aligned}
$$

where S_k are the inverse branches of T^k and their ranges are the atoms of $\mathcal{A}^k$. Let us now estimate the four parts separately.

The first term can be bounded as follows:

$$
\begin{aligned}
I &= \sum_{\varphi\in S_k}\left|\frac{1}{(T_n^k)'\varphi}\right|_\infty \mathrm{Var}_{T^kA_\varphi}\left(h\varphi-h_n\tilde{\varphi},\mathcal{P}\right)\\
&\le \vartheta'^k\sum_{\varphi\in S_k}\mathrm{Var}_{A_\varphi}\left(h-h_n\hat{\varphi}_0,T^{-k}\mathcal{P}\right)\\
&\le \vartheta'^k(\mathrm{Var}\,(h,T^{-k}\mathcal{P})+\mathrm{Var}\,(h_n,T_n^{-k}\mathcal{P}))\\
&\le c_2\vartheta'^k\rho^{\gamma k}(\mathrm{diam}\,\mathcal{P})^\gamma,
\end{aligned}
$$

where $c_2\le \mathrm{U}_\gamma(h)+\mathrm{U}_\gamma(h_n)$, $\hat{\varphi}_0=\tilde{\varphi}T^k$ is a homeo from T^kA_φ to itself, and $\vartheta'\in(\vartheta,1)$ (k large enough). We used that $\mathrm{diam}\,T^{-k}\mathcal{P}\le\rho^k\mathrm{diam}\,\mathcal{P}$ and $\mathrm{diam}\,T_n^{-k}\mathcal{P}\le\rho^k\mathrm{diam}\,\mathcal{P}$.

For the second term we get with the help of lemma 7(I):

$$
\begin{aligned}
II &= \sum_{\varphi\in S_k}|h\varphi-h_n\tilde{\varphi}|_\infty\mathrm{Var}_{T^kA_\varphi}\left(\frac{1}{|(T_n^k)'\tilde{\varphi}|},\mathcal{P}\right)\\
&\le c_3\kappa^n\sum_{\varphi\in S_k}\mathrm{Var}_{A_\varphi}\left(\frac{1}{|(T_n^k)'|},T_n^{-k}\mathcal{P}\right)\\
&\le c_3C_5c_4\kappa^n(\mathrm{diam}\,\mathcal{P})^\gamma\sum_{\varphi\in S_k}\frac{1}{|(T_n^k)'\tilde{\varphi}x|}\\
&\le c_3C_5c_4\kappa^n(\mathrm{diam}\,\mathcal{P})^\gamma h_n(x),
\end{aligned}
$$

where we have used that $T^kA_\varphi=[0,1]$ (mixing property) for all inverse branches $\varphi\in S_k$, and $\sup_{A_\varphi}|1/(T_n^k)'|\le c_4/|(T_n^k)'(y)|$ for some constant c_4 and any y which we choose to be equal to $\tilde{\varphi}x$. Hence $II\le c_5\kappa^n(\mathrm{diam}\,\mathcal{P})^\gamma$ for some c_5.

Next we estimate the third term as follows

$$
III=\sum_{\varphi\in S_k}\left|1-\frac{|(T_n^k)'\tilde{\varphi}|}{|(T^k)'\varphi|}\right|_\infty\mathrm{Var}_{T^kA_\varphi}\left(\frac{h_n\tilde{\varphi}}{|(T_n^k)'\tilde{\varphi}|},\mathcal{P}\right),
$$

where the first term in the sum was estimated in lemma 8. We thus obtain with lemma 7(I) (applied to the map T_n instead of T)

$$
III \le nC_6\vartheta'^{n-k}\sum_{\varphi\in S_k}\mathrm{Var}_{T^kA_\varphi}\left(\frac{h_n\tilde{\varphi}}{|(T_n^k)'\tilde{\varphi}|},\mathcal{P}\right)
$$

$$
\begin{aligned}
&\leq \quad nC_6\vartheta'^{n-k}\sum_{\varphi\in S_k}\mathrm{Var}_{A_\varphi}\left(\frac{h_n}{|(T_n^k)'|},T_n^{-k}\mathcal{P}\right)\\
&\leq \quad nC_6\vartheta'^{n-k}\sum_{\varphi\in S_k}\left(\left|\frac{1}{(T_n^k)'}\right|_\infty \mathrm{Var}_{A_\varphi}\left(h_n,T_n^{-k}\mathcal{P}\right)\right.\\
&\qquad\qquad\qquad\qquad \left.+|h_n|_\infty \mathrm{Var}_{A_\varphi}\left(\frac{1}{|(T_n^k)'|},T_n^{-k}\mathcal{P}\right)\right)\\
&\leq \quad nC_6\vartheta'^{n-k}\left(\vartheta'^k\mathrm{Var}\left(h_n,T_n^{-k}\mathcal{P}\right)+|h_n|_\infty c_4C_5|h_n|_\infty(\mathrm{diam}\,\mathcal{P})^\gamma\right)\\
&\leq \quad nc_6\vartheta'^{n-k}(\mathrm{diam}\,\mathcal{P})^\gamma,
\end{aligned}
$$

for some constant c_6.

While in the first three expressions we obtain exponential contraction even if $\gamma = 0$, in the last term the assumption that $|T'|$ has uniform variation is essential even in the bounded variation (BV-norm) case (see theorem 6). We have

$$
IV=\sum_{\varphi\in S_k}\left|\frac{h_n}{(T_n^k)'}\circ\tilde\varphi\right|_\infty \mathrm{Var}_{T^kA_\varphi}\left(1-\frac{|(T_n^k)'\varphi|}{|(T^k)'\tilde\varphi|},\mathcal{P}\right).
$$

We estimate the term with the variation as follows:

$$
\begin{aligned}
\mathrm{Var}_{T^kA_\varphi}\left(1-\frac{|(T_n^k)'\tilde\varphi|}{|(T^k)'\varphi|},\mathcal{P}\right) &= \mathrm{Var}_{A_\varphi}\left(1-\frac{|(T_n^k)'\hat\varphi_0|}{|(T^k)'|},T^{-k}\mathcal{P}\right)\\
&\leq c_7\mathrm{Var}_{A_\varphi}\left(\log\frac{|(T^k)'\hat\varphi_0|}{|(T_n^k)'|},T^{-k}\mathcal{P}\right)\\
&\leq c_7\sum_{\ell=0}^{k-1}\mathrm{Var}_{A_\varphi}\left(\log\frac{|T'T^\ell|}{|T_n'T_n^\ell\hat\varphi_0|},T^{-k}\mathcal{P}\right)\\
&= c_7\sum_{\ell=0}^{k-1}\mathrm{Var}_{T^\ell A_\varphi}\left(\log\frac{|T'|}{|T_n'\hat\varphi_\ell|},T^{\ell-k}\mathcal{P}\right),
\end{aligned}
$$

for some constant c_7, where $\hat\varphi_\ell = T_n^\ell\tilde\varphi T^{k-\ell}$ is a homeomorphism of $T^\ell A_\varphi$ to itself. We estimate the terms inside the sum individually as follows (for some c_8, c_9, c_{10}):

$$
\begin{aligned}
&\sum_{\varphi\in S_k}\left|\frac{h_n}{(T_n^k)'}\circ\tilde\varphi\right|_\infty \mathrm{Var}_{T^\ell A_\varphi}\left(\log\frac{|T'|}{|T_n'\hat\varphi_\ell|},T^{\ell-k}\mathcal{P}\right)\\
&\leq \quad \sum_{\varphi\in S_k}\left|\frac{h_n}{(T_n^k)'}\circ\tilde\varphi\right|_\infty c_8\left(\mathrm{Var}_{T^\ell A_\varphi}\left(\log|T'|,T^{\ell-k}\mathcal{P}\right)\right.\\
&\qquad\qquad\qquad\qquad \left.+\mathrm{Var}_{T^\ell A_\varphi}\left(\log|T_n'|,T_n^{\ell-k}\mathcal{P}\right)\right)\\
&\leq \quad c_8\vartheta'^\ell\sum_{\psi\in S_{k-\ell}}\left(\mathrm{Var}_{A_\psi}\left(\log|T'|,T^{\ell-k}\mathcal{P}\right)+\mathrm{Var}_{A_\psi}\left(\log|T_n'|,T_n^{\ell-k}\mathcal{P}\right)\right)
\end{aligned}
$$

$$\begin{aligned}
&\le c_9 \vartheta'^{\ell} \left(\left(\operatorname{diam} T^{\ell-k}\mathcal{P}\right)^{\gamma} + \left(\operatorname{diam} T_n^{\ell-k}\mathcal{P}\right)^{\gamma} \right) \\
&\le c_{10} \vartheta'^{\ell} \rho^{\gamma(k-\ell)} \left(\operatorname{diam} \mathcal{P}\right)^{\gamma},
\end{aligned}$$

as $\operatorname{diam} T^{\ell-k}\mathcal{P} \le \rho^{k-\ell} \operatorname{diam} \mathcal{P}$ and $\operatorname{diam} T_n^{\ell-k}\mathcal{P} \le \rho^{k-\ell} \operatorname{diam} \mathcal{P}$. Thus we obtain with some c_{11}:

$$IV \le c_{10} \sum_{\ell=0}^{k-1} \vartheta'^{k-\ell} \rho^{\gamma\ell} \left(\operatorname{diam} \mathcal{P}\right)^{\gamma} \le k c_{11} \theta^k \left(\operatorname{diam} \mathcal{P}\right)^{\gamma}.$$

where $\theta = \max(\vartheta', \rho^{\gamma})$, $\theta < 1$.

Finally with a suitable choice for k, say $k = n/2$, we get that

$$\operatorname{Var}(h - h_n, \mathcal{P}) \le I + II + III + IV \le c_{12} \left(\vartheta'^k + \kappa^k + n\vartheta'^{n-k} + k\theta^k \right) \le c_1 \sigma^n,$$

for all n and for some $\sigma < 1$ (for which $\sigma^2 \ge \max(\vartheta', \kappa, \theta)$). □

Proof of theorem 6. We have to show that $\operatorname{var}(h - h_n) \le c_1 \sigma^n$ for some $\sigma < 1$ and some constant c_1. As before S_k denote the inverse branches of T^k, and if $\varphi \in S_k$ then we denote the corresponding inverse branch of T_n^k by $\tilde{\varphi}$ of T_n^k. We proceed as in the proof of theorem 5 and obtain for any $k < n$:

$$\operatorname{var}(h - h_n) = \operatorname{var}(\mathcal{L}^k h - \mathcal{L}_n^k h_n) \le I + II + III + IV.$$

Again we proceed to estimate the terms one by one, where the first three terms are essentially estimated by putting $\gamma = 0$.

The first term:

$$\begin{aligned}
I &= \sum_{\varphi \in S_k} \left| \frac{1}{(T_n^k)'\varphi} \right|_{\infty} \operatorname{var}_{T^k A_\varphi} (h\varphi - h_n \tilde{\varphi}) \\
&\le \vartheta'^k \sum_{\varphi \in S_k} \left(\operatorname{var}_{A_\varphi} h + \operatorname{var}_{A_\varphi} h_n \right) \\
&\le c_2 \vartheta'^k.
\end{aligned}$$

The second term is estimated as (using lemma 7(II)):

$$\begin{aligned}
II &= \sum_{\varphi \in S_k} |h\varphi - h_n\tilde{\varphi}|_{\infty} \operatorname{var}_{T^k A_\varphi} \left(\frac{1}{|(T_n^k)'\tilde{\varphi}|} \right) \\
&\le c_3 \kappa^n \sum_{\varphi \in S_k} \operatorname{var}_{A_\varphi} \left(\frac{1}{|(T_n^k)'|} \right) \\
&\le c_3 C_5 \kappa^n \sum_{\varphi \in S_k} \max_{A_\varphi} \frac{1}{|(T_n^k)'\tilde{\varphi} x|} \\
&\le c_4 \kappa^n.
\end{aligned}$$

Using lemma 8, the third term becomes

$$\begin{aligned}
III &= \sum_{\varphi\in S_k}\left|1-\frac{|(T_n^k)'\tilde\varphi|}{|(T^k)'\varphi|}\right|_\infty \mathrm{var}_{T^kA_\varphi}\frac{h_n\tilde\varphi}{|(T_n^k)'\tilde\varphi|}\\
&\le nC_6\vartheta'^{n-k}\sum_{\varphi\in S_k}\mathrm{var}_{A_\varphi}\frac{h_n}{|(T_n^k)'|}\\
&\le nC_6\vartheta'^{n-k}\sum_{\varphi\in S_k}\left(\left|\frac{1}{(T_n^k)'}\right|_\infty \mathrm{var}_{A_\varphi} h_n+|h_n|_\infty \mathrm{var}_{A_\varphi}\frac{1}{|(T_n^k)'|}\right)\\
&\le nC_6\vartheta'^{n-k}\left(\vartheta'^k\mathrm{var}\, h_n+|h_n|_\infty C_5c_5|h_n|_\infty\right)\\
&\le nc_6\vartheta'^{n-k},
\end{aligned}$$

for some constants c_5, c_6, where we used lemma 7(II).

For the fourth term we have to use that $|T'|$ has uniform variation. As before

$$IV=\sum_{\varphi\in S_k}\left|\frac{h_n}{(T_n^k)'}\circ\tilde\varphi\right|_\infty \mathrm{var}_{T^kA_\varphi}\left(1-\frac{|(T_n^k)'\varphi|}{|(T^k)'\tilde\varphi|}\right),$$

where

$$\begin{aligned}
\mathrm{var}_{T^kA_\varphi}\left(1-\frac{|(T_n^k)'\tilde\varphi|}{|(T^k)'\varphi|}\right) &= \mathrm{var}_{A_\varphi}\left(1-\frac{|(T_n^k)'\hat\varphi_0|}{|(T^k)'|}\right)\\
&\le c_7\,\mathrm{var}_{A_\varphi}\log\frac{|(T^k)'\hat\varphi_0|}{|(T_n^k)'|}\\
&= c_7\sum_{\ell=0}^{k-1}\mathrm{var}_{T^\ell A_\varphi}\log\frac{|T'|}{|T_n'\hat\varphi_\ell|}
\end{aligned}$$

with some c_7, where $\hat\varphi_\ell = T_n^\ell\tilde\varphi T^{k-\ell}$. In order to estimate the variation term inside the sum for a given value of ℓ, consider on $T^\ell A_\varphi$ the partition $\mathcal{Q}_\ell$ induced by $T^\ell\mathcal{A}^n$. Notice that $|T_n'|$ and $|T_n'\hat\varphi_\ell|$ are both constant on the atoms of $\mathcal{Q}_\ell$. In every atom $Q\in\mathcal{Q}_\ell$ there is (by the mean value theorem) a point y_Q so that so that $|T'(y_Q)|=|T_n'|_Q|$. The points $\{y_Q : Q\in\mathcal{Q}_\ell\}$ form a partition of $T^\ell A_\varphi$ which we shall call $\mathcal{R}_\ell$. As $\log|T'|$ has γ-uniform variation, one obtains:

$$\mathrm{var}_{T^\ell A_\varphi}\log\frac{|T'|}{|T_n'\hat\varphi_\ell|}=\mathrm{Var}_{T^\ell A_\varphi}\left(\log|T'|,\mathcal{R}_\ell\right)\le \mathrm{U}_\gamma(\log|T'|)\left(\mathrm{diam}\,\mathcal{R}_\ell\right)^\gamma.$$

Moreover, since

$$\mathrm{diam}\,\mathcal{R}_\ell\le 2\,\mathrm{diam}\,\mathcal{Q}_\ell\le 2\,\mathrm{diam}\,T^\ell\mathcal{A}^n\le 2\,\mathrm{diam}\,\mathcal{A}^{n-\ell}\le 2\rho^{n-\ell},$$

we deduce that

$$\begin{aligned}
\sum_{\varphi\in S_k}\left|\frac{h_n}{(T_n^k)'}\circ\tilde\varphi\right|_\infty \mathrm{var}_{T^\ell A_\varphi}\log\frac{|T'|}{|T_n'\hat\varphi_\ell|} &\le \vartheta'^{k-\ell}\sum_{\psi\in S_{k-\ell}}\mathrm{Var}_{A_\psi}\left(\log|T'|,\mathcal{R}_\ell\right)\\
&\le c_8\vartheta'^{k-\ell}\rho^{\gamma(n-\ell)},
\end{aligned}$$

and, with some c_9:

$$IV \le c_8 \sum_{\ell=0}^{k-1} \vartheta'^{k-\ell} \rho^{\gamma(n-\ell)} \le k c_9 \theta^{n-k},$$

where $\theta \ge \max(\vartheta', \rho^\gamma)$, $\theta < 1$.

Finally with a suitable choice for k, say $k = n/2$, we get that

$$\operatorname{var}(h - h_n, \mathcal{P}) \le I + II + III + IV \le c_{11} \left(\vartheta'^k + \kappa^k + n\vartheta'^{n-k} + k\theta^k \right) \le c_1 \sigma^n,$$

for all n and for some $\sigma < 1$. □

3 Example

In this section we present a map T of the unit interval to itself which is piecewise C^{1+L} and for which var $(h - h_{n_j})$ converges subexponentially fast to zero as the sequence n_j goes to infinity. As before h_n are the densities for the n-th natural affine approximation T_n of T.

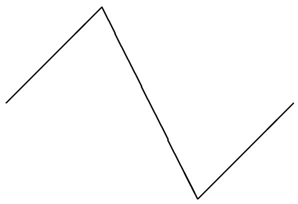
As in the proof of theorem 6 let us consider (for a suitable sequence of n-values)

$$\operatorname{var}(h - h_n) = \operatorname{var}(\mathcal{L}^k h - \mathcal{L}_n^k h_n) \le I + II + III + IV,$$

where as above $I + II + III \le c_1(\vartheta^k + \kappa^n + n\vartheta^{n-k})$ for $k < n$, where the numbers $\vartheta, \kappa < 1$ are as above. We shall now construct T so that the fourth term IV can be estimated from below (polynomially in $\log n_j$). We shall proceed in four steps. We shall construct the map (i); then estimate the conditional variation $\log T'$ from below for a certain partition (ii) and from above for some finer partitions (iii); and finally we shall obtain lower bounds on the variation var $(h - h_n)$ (iv).

(i) We approximate T in the following way in C^1 by a sequence of piecewise C^{1+L}-maps V_j, $j = 1, 2, \ldots$. Let P be some large integer which is to be chosen later and put V_1 for the piecewise linear map $V_1(x) = Px \bmod 1$, for $x \in [0,1]$. Then V_1' is constant equal to P on P subintervals of length $1/P$ that form a partition $\mathcal{A}$ of $[0,1]$. The quantities $\vartheta' = (P - \frac{1}{3P})^{-1}$ and $\vartheta = (P + \frac{1}{3P})^{-1}$ will be a lower respectively upper bound for the contraction rates for the inverse branches of the maps V_j and T. Choose $\beta \ge 2\frac{\log \vartheta}{\log \vartheta'}$ ($\beta > 1$), and define a sequence of integers $n(m) = [\beta^m]$ ($[\cdot]$ denotes integer part). That is $m \sim \log_\beta n(m)$. The sequence of positive numbers $\alpha_m = \vartheta^{n(m)}$, $m = 1, 2, \ldots$, is at least exponentially fast decreasing to zero ($\alpha_m \sim \vartheta^{\beta^m}$). Note that $\vartheta'^{n(m)/2} \le \alpha_{m-1}$.

Suppose we already constructed the maps $V_1, V_2, \ldots, V_{m-1}$, and let us now proceed to find V_m. First, denote by H_j the union of (open) intervals on which V_j' is constant (in fact equal to P). Then H_{m-1} is the disjoint union of (finitely many) intervals, say $I_{m-1,1}, I_{m-1,2}, \ldots$. Each of these intervals $I_{m-1,\ell}$ is divided into $[|I_{m-1,\ell}|\frac{1}{m\alpha_m}]+1$ ($|\cdot|$ denotes the length of the interval) many pieces and on the middle of each of these pieces we replace the constant value P of V'_{m-1} by a *squiggle* (the left and right thirds both have slope 1 while the middle third has slope -2)



Squiggle

of length α_m (the length of a squiggle is the euclidean distance between the left and right endpoints). In this way we find the derivative V'_m of the map V_m. This completes the construction of V_m. Notice that $V_m(x) = V_{m-1}(x)$ for x outside the squiggles which were introduced at the step m (i.e. for $x \in [0,1] \setminus (H_m \setminus H_{m-1})$). For the lebesgue measure (sum of the lengths of subintervals) of the 'linear set' H_m of V_m (i.e. $H_m = \{x \in [0,1] : V'_m(x) = P\}$) we obtain

$$\begin{aligned}
|H_m| &= |H_{m-1}| - \sum_\ell \left(\left[|I_{m-1,\ell}|\frac{1}{m\alpha_m}\right]+1\right)\alpha_m \\
&\geq |H_{m-1}|(1-\frac{1}{m}) \\
&= |H_{m-1}|\frac{m-1}{m} \\
&\geq |H_1|\frac{1}{m}.
\end{aligned}$$

Thus $|H_m| \sim 1/m$. Let us furthermore observe that the number M_m of squiggles of length α_m is

$$M_m \sim |H_m|\frac{1}{m\alpha_m} \sim \frac{\vartheta^{-\beta^m}}{m^2} \sim \frac{\vartheta^{-n}}{m^2}, \tag{3}$$

and the total length of the α_m-squiggles is $|H_{m-1} \setminus H_m| \sim \frac{1}{m^2}$.

It follows from the construction that all the maps V_m are C^{1+L} on the atoms of $\mathcal{A}$ (in fact 2 is a Lipshitz constant for all the derivatives V'_m), and their derivatives satisfy $1/\vartheta \leq V'_m \leq 1/\vartheta'$. By Arzela-Ascoli the maps V_m converge in C^1 to a limit T which is piecewise of class C^{1+L}.

Clearly, T has the Markov property and is topologically mixing (in fact $N=1$). It thus satisfies the conditions (i)–(iv) (with $\rho = \vartheta'$).

If $\mathcal{A}^n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{A}$ denotes the n-th join of $\mathcal{A}$, then $\vartheta'^{-n} \leq |\mathcal{A}^n| \leq \vartheta^{-n}$ (cardinality) and $\vartheta^n \leq |A| \leq \vartheta'^n$ (length) for every atom $A \in \mathcal{A}^n$.

(ii) Let us next estimate $\mathrm{Var}_{T^{k-1}A}(\log T', \mathcal{R}_{k-1})$, for $A \in \mathcal{P}^k$, $k < n$, where $\mathcal{R}_{k-1}$ is a partition of A whose diameter $\leq 2\vartheta'^{n-k}$ and whose cardinality $|\mathcal{R}_{k-1}|$ is bounded from above by $\vartheta^{-(n-k)}$. Since in our case $T^{k-1}A = [0,1]$ for all $A \in \mathcal{A}^k$, we therefore get that (recall that by construction $T' > 1$)

$$\mathrm{Var}_{T^{k-1}A}(\log T', \mathcal{R}_{k-1}) = \mathrm{Var}\,(\log T', \mathcal{R}_{k-1}).$$

If we put $k = [n/2]$ and $n = n(m)$, then $|\mathcal{A}^{n-k}| \leq M_m$ by (3) for large enough m. We can therefore conclude that the total length of those squiggles of lengths $\leq \alpha_m$ which are subsets of single atoms of the partition $\mathcal{R}_{k-1}$ is at least $|H_{m+1}| \sim \frac{1}{m+1}$. Every such squiggle contributes $\frac{4}{3}$-times its length to the variation $\mathrm{Var}\,(\log T', \mathcal{R}_{k-1})$. Hence

$$\mathrm{Var}\,(\log T', \mathcal{R}_{k-1}) \geq \frac{4}{3}|H_{m+1}| \geq \frac{1}{m}. \tag{4}$$

(iii) Now let $\ell < k-1$, and let us find upper estimates for $\mathrm{Var}_{T^\ell A}(\log T', \mathcal{R}_\ell)$ for $A \in \mathcal{A}^k$, where $\mathcal{R}_\ell$ are partitions whose diameters are bounded above by $2\vartheta'^{n-\ell}$. Clearly, whenever $A \in \mathcal{A}^k$, then

$$\vartheta^{l-1-\ell} \leq |T^\ell A| \leq \vartheta'^{k-1-\ell}.$$

Suppose we have a squiggle of some length s. If we approximate by a locally constant function which is constant on intervals of lengths $< s/3$ say, then the conditional variation can only be produced by those intervals which contain corners (kinks) of the squiggle (of which there are four in each squiggle). Hence if we only consider squiggles of size $\geq \alpha_{m-1}$ we get

$$\mathrm{Var}_{T^\ell A \cap ([0,1]\setminus H_m)}(\log T', \mathcal{R}_\ell) \leq \mathrm{diam}\,(\mathcal{R}_\ell) 4 N_m 2 |T^\ell A|(1 - |H_m|), \tag{5}$$

(since all squiggles of lengths $\leq \alpha_m$ dwell on H_m) where N_m is the total number of squiggles of lengths $\geq \alpha_{m-1}$. We estimate ($\beta > 1$ large enough):

$$N_m = \sum_{k=2}^{m-1} M_k \sim \sum_{k=2}^{m-1} \frac{1}{k^2 \alpha_k} \leq \frac{1}{m\alpha_{m-1}}.$$

On the other hand for squiggles $\leq \alpha_m$ we get as before

$$\mathrm{Var}_{T^\ell A \cap H_m}(\log T', \mathcal{R}_\ell) \leq \frac{4}{3}|T^\ell A| \cdot |H_m| \sim \frac{4}{3m}|T^\ell A| \tag{6}$$

(the variation of a squiggle is 4/3 times its length). As $|T^\ell A| \leq \vartheta'^{k-1-\ell}$ if $A \in \mathcal{A}^k$, we now obtain from equations (5) and (6)

$$\begin{aligned}\mathrm{Var}_{T^\ell A}(\log T', \mathcal{R}_\ell) &\leq \frac{4}{3m}|T^\ell A| + 8|T^\ell A| N_m \vartheta'^{n-\ell} \\ &\leq \vartheta'^{k-1-\ell}\frac{1}{m}\left(\frac{4}{3} + \frac{4\vartheta'^{n-\ell}}{\alpha_{m-1}}\right),\end{aligned}$$

as $N_m \le 1/m\alpha_{m-1}$. Hence, since by assumption $\vartheta'^{n-\ell}\alpha_{m-1}^{-1} \le 1$, (as $\ell < k = [n/2]$) we get

$$\sum_{\ell=0}^{k-2} \mathrm{Var}_{T^\ell A}(\log T', \mathcal{R}_\ell) \le \frac{6}{m}\sum_{\ell=0}^{k-2} \vartheta'^{k-1-\ell} \le \frac{6}{m}\frac{\vartheta'}{1-\vartheta'}. \tag{7}$$

(iv) Let us now complete the estimate of the variation in our example. As above, let $A_\varphi \in \mathcal{A}^k$ be the ranges of the inverse branches $\varphi \in S_k$ of T^k, and consider on $T^\ell A_\varphi$ the partition $\mathcal{Q}_\ell$ induced by $T^\ell\mathcal{A}^n$. (As above, $|T_n'|$ and $|T_n'\hat\varphi_\ell|$ are both constant on the atoms of $\mathcal{Q}_\ell$.) In every atom $Q \in \mathcal{Q}_\ell$ there is (by the mean value theorem) a point y_Q so that $|T'(y_Q)| = |T_n'|_Q|$. The points $\{y_Q : Q \in \mathcal{Q}_\ell\}$ form a partition of $T^\ell A_\varphi$ which we shall call $\mathcal{R}_\ell$. One obtains for $\ell = 0, \ldots, k-1$:

$$\mathrm{var}_{T^\ell A_\varphi} \log\frac{|T'|}{|T_n'\hat\varphi_\ell|} = \mathrm{Var}_{T^\ell A_\varphi}(\log|T'|, \mathcal{R}_\ell),$$

and

$$\mathrm{diam}\,\mathcal{R}_\ell \le 2\,\mathrm{diam}\,\mathcal{Q}_\ell \le 2\,\mathrm{diam}\,T^\ell\mathcal{A}^n \le 2\,\mathrm{diam}\,\mathcal{A}^{n-\ell} \le 2\vartheta'^{n-\ell}.$$

Therefore, if we use the estimates (4) and (7) which we deduced above, we obtain:

$$\begin{aligned}
\mathrm{var}_{A_\varphi} \log\left|\frac{(T^k)'}{(T_n^k)'}\right| &= \mathrm{var}_{A_\varphi} \sum_{\ell=0}^{k-1}\log\left|\frac{T'T^\ell}{T_n'T_n^\ell}\right| \\
&\ge \mathrm{var}_{A_\varphi} \log\left|\frac{T'T^{k-1}}{T_n'T_n^{k-1}}\right| - \mathrm{var}_{A_\varphi}\sum_{\ell=0}^{k-2}\log\left|\frac{T'T^\ell}{T_n'T_n^\ell}\right| \\
&= \mathrm{Var}_{T^{k-1}A_\varphi}(\log T', \mathcal{R}_{k-1}) - \sum_{\ell=0}^{k-2}\mathrm{Var}_{T^\ell A_\varphi}(\log T', \mathcal{R}_\ell) \\
&\ge \frac{1}{m} - \frac{6}{m}\frac{\vartheta'}{1-\vartheta'}.
\end{aligned}$$

If we choose P large enough so that $\frac{6\vartheta'}{1-\vartheta'} \le \frac{1}{2}$, then

$$\mathrm{var}_{A_\varphi} \log\left|\frac{(T^k)'}{(T_n^k)'}\right| \ge \frac{1}{2m}.$$

This finally allows us to estimate the term IV from below by

$$\mathrm{IV} \ge c_1\frac{1}{2m}\sum_{\varphi\in S_k}\frac{h_n\varphi}{|(T_n^k)'\tilde\varphi|} \ge c_1\frac{1}{2m}\inf h_n \ge \frac{c_1}{4m}\inf h,$$

for large enough n and some positive constant c_1. Consequently, since we chose $k = [n/2]$, we obtain

$$\mathrm{var}\,(h - h_n) \ge IV - (I + II + III) \ge \frac{c_2}{m} - c(\vartheta^k + \kappa^n + n\vartheta^{n-k}) \ge \frac{c_3}{m(n)} \ge \frac{c_4}{\log n}$$

for some positive constants c_2, c_3, c_4.

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