Semiclassical Asymptotics of Orthogonal Polynomials, Riemann-Hilbert Problem, and Universality in the Matrix Model

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Abstract. We derive semiclassical asymptotics for the orthogonal polynomials on the line with the weight $\exp(-NV(z))$, where $V(z) = \frac{tz^2}{2} + \frac{gz^4}{4}$, $g > 0$, $t < 0$, is a double-well quartic polynomial. Simultaneously we derive semiclassical asymptotics for the recursive coefficients of the orthogonal polynomials. The proof of the asymptotics is based on the analysis of the appropriate matrix Riemann-Hilbert problem. As an application of the semiclassical asymptotics, we prove the universality of the local distribution of eigenvalues in the matrix model with the double-well quartic interaction in the presence of two cuts.

Contents

1. Main Result

Let

$$V(z) = \frac{tz^2}{2} + \frac{gz^4}{4}, \qquad g > 0, \qquad t < 0, \tag{1.1}$$

be a double-well quartic polynomial, and let

$$P_n(z) = z^n + \dots, \qquad n = 0, 1, 2, \dots, \tag{1.2}$$

be orthogonal polynomials on a line with the weight $e^{-NV(z)}$,

$$\int_{-\infty}^{\infty} P_n(z) P_m(z)\, e^{-NV(z)} dz = h_n \delta_{mn}. \tag{1.3}$$

The polynomials $P_n(z)$ satisfy the basic recursive equation

$$zP_n(z) = P_{n+1}(z) + R_n P_{n-1}(z), \tag{1.4}$$

where

$$R_n = \frac{h_n}{h_{n-1}}. \tag{1.5}$$

In addition, integration by parts gives

$$\begin{aligned} P_n'(z) &= NR_n[t + g(R_{n-1} + R_n + R_{n+1})]P_{n-1}(z) \\ &\quad + (NR_{n-2}R_{n-1}R_n)P_{n-3}(z), \qquad (') \equiv \frac{d}{dz}\,. \end{aligned} \tag{1.6}$$

Since $P_n'(z) = nz^{n-1} + \dots$, this implies the Freud equation

$$n = NR_n[t + g(R_{n-1} + R_n + R_{n+1})]. \tag{1.7}$$

(cf. [Fre]). From (1.5) and (1.7) it follows that

$$0 < R_n < \frac{-t + \sqrt{t^2 + 4\lambda g}}{2g}\,, \qquad \lambda = \frac{n}{N}\,. \tag{1.8}$$

Let

$$\psi_n(z) = \frac{1}{\sqrt{h_n}}\, P_n(z) e^{-NV(z)/2}\,. \tag{1.9}$$

Then

$$\int_{-\infty}^{\infty} \psi_n(z)\psi_m(z)\, dz = \delta_{nm}. \tag{1.10}$$

In this work we prove the semiclassical asymptotics for the functions $\psi_n(z)$ and for the coefficients R_n in the limit when $N, n \to \infty$ in such a way that there exists $\varepsilon > 0$ such that the ratio $\lambda = n/N$ satisfies the inequalities

$$\varepsilon < \lambda < \lambda_{\rm cr} - \varepsilon, \qquad \lambda = \frac{n}{N}, \tag{1.11}$$

where

$$\lambda_{\rm cr} = \frac{t^2}{4g}. \tag{1.12}$$

Denote

$$\lambda' = \frac{n+\frac{1}{2}}{N}.$$

In what follows the potential function

$$U_0(z) = z^2 \left[\frac{(gz^2+t)^2}{4} - \lambda' g \right],$$

is important. Introduce the turning points z_1 and z_2 as zeros of $U_0(z)$,

$$z_{1,2} = \left(\frac{-t \mp 2\sqrt{\lambda' g}}{g} \right)^{1/2}. \tag{1.13}$$

The condition (1.11) implies that z_1 and z_2 are real for large N, and $z_2 > z_1 > C\sqrt{\varepsilon}$. We prove the following main theorem.

Theorem 1.1. *Assume that $N, n \to \infty$ in such a way that (1.11) holds. Then there exists $C = C(\varepsilon) > 0$ such that*

$$\left| R_n - \frac{-t - (-1)^n \sqrt{t^2 - 4\lambda g}}{2g} \right| \le C N^{-1}, \qquad \lambda = \frac{n}{N}. \tag{1.14}$$

In addition, for every $\delta > 0$, in the interval $z_1 + \delta < z < z_2 - \delta$,

$$\psi_n(z) = \frac{2C_n\sqrt{z}}{\sqrt{\sin\phi}} \left\{ \cos\left[\frac{\left(n+\frac{1}{2}\right)}{2} \left(\frac{\sin 2\phi}{2} - \phi \right) + \frac{\pi - (-1)^n \chi}{4} \right] + O(N^{-1}) \right\}, \tag{1.15}$$

where

$$\phi = \arccos x, \qquad \chi = \arccos y,$$

and

$$x = \frac{gz^2+t}{2\sqrt{\lambda' g}}, \qquad y = \frac{2\sqrt{\lambda' g} - tx}{2\sqrt{\lambda' g}\, x - t} = \frac{-tgz^2 - t^2 + 4\lambda' g}{2\sqrt{\lambda' g}\, g z^2}, \qquad \lambda' = \frac{n+\frac{1}{2}}{N}. \tag{1.16}$$

If $z > z_2 + \delta$ or $0 \le z < z_1 - \delta$, then

$$\psi_n(z) = (-1)^\sigma \frac{C_n \sqrt{z}}{\sqrt{\sinh \phi}} \exp\left\{ -\frac{\left(n+\frac{1}{2}\right)}{2} \left[\frac{\sinh(2\phi)}{2} - \phi \right] + \frac{(-1)^n \chi}{4} + O\left(\frac{1}{N(1+|z|)} \right) \right\}, \tag{1.17}$$

where

$$\sigma = \frac{1 - \operatorname{sign}(z - z_1)}{2} \left[\frac{n}{2} \right], \qquad \left[\frac{n}{2} \right] = l, \quad \textit{if} \quad n = 2l \quad \textit{or} \quad 2l+1,$$

$$\phi = \cosh^{-1} |x|, \qquad \chi = \cosh^{-1} |y|$$

and x, y are given by (1.16).

If $z_k - \delta \le z \le z_k + \delta$, $\quad k = 1, 2$, then

$$\psi_n(z) = \frac{D_n\, z}{\sqrt{|\varphi_N'(z)|}} \left[\operatorname{Ai}\left(N^{2/3} \varphi_N(z) \right) + O(N^{-1}) \right], \tag{1.18}$$

where $\operatorname{Ai}(z)$ is the Airy function, $\varphi_N(z)$ is an analytic function on $[z_k - \delta, z_k + \delta]$ such that for $(z - z_k^{(N)})(-1)^k \ge 0$,

$$\varphi_N(z) = \left[\frac{3}{2} \left| \int_{z_k^{(N)}}^{z} \sqrt{U_N(v)}\, dv \right| \right]^{2/3}, \qquad k = 1, 2, \tag{1.19}$$

where $z_k^{(N)} = z_k + O(N^{-1})$ is the closest to z_k zero of the polynomial

$$U_N(z) = U_0(z) + N^{-1} \left(\frac{t}{2} + gR_n \right) = z^2 \left[\frac{(gz^2 + t)^2}{4} - \lambda' g \right] + N^{-1} \left(\frac{t}{2} + gR_n \right). \tag{1.20}$$

The constant factor C_n in (1.15) and (1.17) is

$$C_n = \frac{1}{2\sqrt{\pi}} \left(\frac{g}{\lambda} \right)^{1/4} (1 + O(N^{-1})), \tag{1.21}$$

and D_n in (1.18) is

$$D_n = N^{1/6} \sqrt{g}\, (-1)^{\sigma_0} (1 + O(N^{-1})), \qquad \sigma_0 = (2 - k) \left[\frac{n}{2} \right]. \tag{1.22}$$

Finally, h_n in (1.3) is

$$h_n = 2\pi \sqrt{R_n} \exp\left[\frac{Nt^2}{4g} - \frac{N\lambda}{2} \left(1 + \ln \frac{g}{\lambda} \right) + O(N^{-1}) \right]. \tag{1.23}$$

The asymptotic formulae (1.15), (1.17) and (1.18) is an extension of the classical Plancherel–Rotach asymptotics of the Hermite polynomials (see [PR] and [Sze]), to the orthogonal polynomials with respect to the weight $e^{-NV(z)}$ where $V(z)$ is the quartic polynomial (1.1). These formulae are extended into the complex plane in z as well (see section 10 below). We derive the formula (1.15) from the semiclassical formula

$$\psi_n(z) = \frac{z\sqrt{g/\pi}}{|U_0(z)|^{1/4}} \left[\cos\left(N\int_{z_2}^{z} |U_N(v)|^{1/2}\,dv + \frac{\pi}{4}\right) + O(N^{-1})\right]. \tag{1.24}$$

where $U_N(z)$ is defined in (1.20). Asymptotics (1.14) of the coefficients R_n is a Freud's type asymptotics. For the homogeneous function $V(z)=|z|^\alpha$ and some its generalizations, the asymptotics of R_n is obtained in the papers of Freud [Fre], Nevai [Nev1], Magnus [Mag1,Mag2], Lew and Quarles [LQ], Máté, Nevai, and Zaslavsky [MNZ]. Semiclassical asymptotics of the functions $\psi_n(z)$ is proven for $V(z) = z^4$ by Nevai [Nev1] and for $V(z)=z^6$ by Sheen [She]. See also somewhat weaker asymptotic results for general homogeneous $V(z)$ in the works of Lubinsky and Saff [LS], Lubinsky, Mhaskar, and Saff [LMS], Levin and Lubinsky [LL], Rahmanov [Rah], and others. Application of these asymptotics to random matrices is discussed in the work of Pastur [Pas]. The distribution of zeros and related problems for orthogonal polynomials corresponding to general homogeneous $V(z)$ are studied in the recent work [DKM] by Deift, Kriecherbauer, and McLaughlin. Many results and references on the asymptotics of orthogonal polynomials are given in the comprehensive review article [Nev2] of Nevai. The problem of finding asymptotics of R_n for a quartic nonconvex polynomial is discussed in [Nev2,3], and it is known as "Nevai's problem".

The equation (1.14) shows that if $0<\lambda<\lambda_{\rm cr}$ then

$$\begin{aligned} \lim_{N\to\infty;\ (2m)/N\to\lambda} R_{2m} = L(\lambda) = \frac{-t-\sqrt{t^2-4\lambda g}}{2g}, \\ \lim_{N\to\infty;\ (2m+1)/N\to\lambda} R_{2m+1} = R(\lambda) = \frac{-t+\sqrt{t^2-4\lambda g}}{2g}. \end{aligned} \tag{1.25}$$

Both $L(\lambda)$ and $R(\lambda)$ satisfy the quadratic equation

$$gu^2+tu+\lambda=0,$$

so that, when n grows, R_n jumps back and forth from one sheet of the parabola to another (see Fig.1). At $\lambda=\lambda_{\rm cr}$ the two sheets merge, i.e., $L(\lambda_{\rm cr})=R(\lambda_{\rm cr})$. For $\lambda>\lambda_{\rm cr}$,

$$\lim_{N\to\infty;\ n/N\to\lambda} R_n = Q(\lambda), \tag{1.26}$$

where $u = Q(\lambda)$ satisfies the quadratic equation

$$3gu^2 + tu - \lambda = 0$$

(which follows from the Freud equation (1.7) if we put $u = R_{n-1} = R_n = R_{n+1}$). We consider semiclassical asymptotics for $\lambda > \lambda_{\rm cr}$ and in the vicinity of $\lambda_{\rm cr}$ (double scaling limit) in a separate work. The difference in the asymptotics between the cases $\lambda < \lambda_{\rm cr}$ and $\lambda > \lambda_{\rm cr}$ is that for $\lambda < \lambda_{\rm cr}$ the function $\psi_n(z)$ is concentrated on two intervals, or two cuts, $[-z_2, -z_1]$ and $[z_1, z_2]$, and it is exponentially small outside of these intervals, while for $\lambda > \lambda_{\rm cr}$, z_1 becomes pure imaginary, and $\psi_n(z)$ is concentrated on one cut $[-z_2, z_2]$. The transition from two-cut to one-cut regime is discussed in physical works by Cicuta, Molinari, and Montaldi [CMM], Crnković and Moore [CM], Douglas, Seiberg, Shenker [DSS], Periwal and Shevitz [PeS], and others.

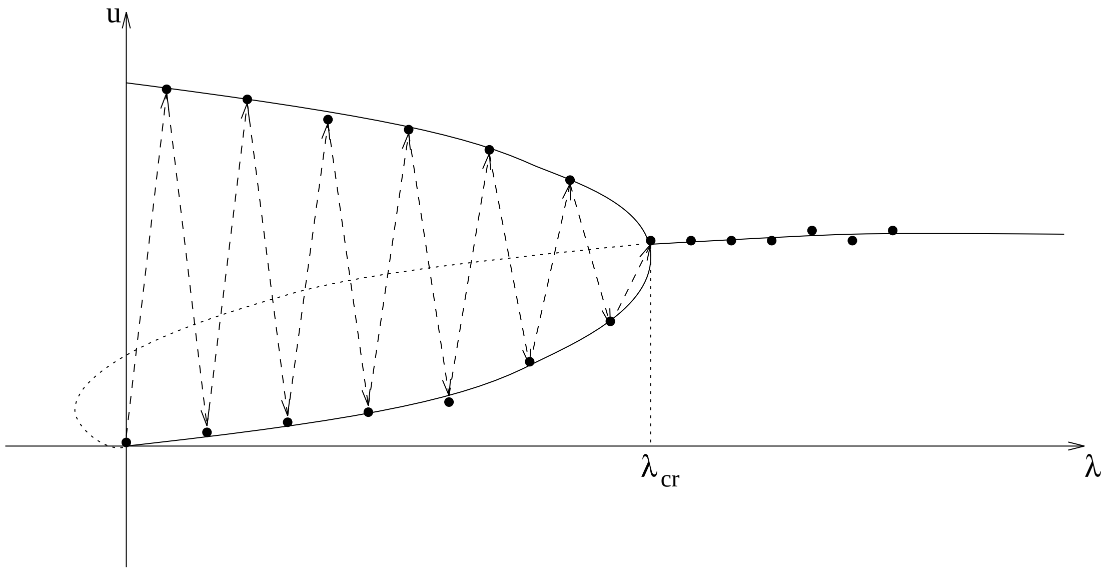

Fig.1. The qualitative behaviour of the recurrence coefficients

A general ansatz on the structure of the semiclassical asymptotics of the functions $\psi_n(z)$ for a "generic" polynomial $V(z)$ is proposed in the work [BZ] of Brézin and Zee.

They consider n close to N, $n = N + O(1)$, and they suggest that for these n's,

$$\psi_n(z) = \frac{1}{\sqrt{f(z)}} \cos\big(N\zeta(z) - (N-n)\varphi(z) + \chi(z)\big), \tag{1.27}$$

with some fuctions $f(z)$, $\zeta(z)$, $\varphi(z)$, and $\chi(z)$. This fits in well the asymptotics (1.15), except for the factor $(-1)^n$ at χ in (1.15), which is related to the two-cut structure of $\psi_n(z)$.

Equation (1.7) also appears in the planar Feynman diagram expansions of Hermitian matrix models, which were introduced and studied in the classical papers [BIPZ], [BIZ], [IZ] by Brézin, Bessis, Itzykson, Parisi, and Zuber and in the well-known recent works by Brézin, Kazakov [BK], Duglas, Shenker [DS], and Gross, Migdal [GM] devoted to the matrix models for 2D quantum gravity (see also [Dem] and [Wit]). In fact, it is the latter context that broadened the interest to the Freud equation (1.7) and brought in the area new powerful analytic methods from the theory of integrable systems. It turns out [FIK1,2] that equation (1.7) admits 2×2 matrix Lax pair representation (see equation (3.15) below), which allows one to identify Freud equation (1.7) as a discrete Painlevé I equation and imbeds it in the framework of the Isomonodromy Deformation Method suggested in 1980 by Flaschka and Newell [FN] and by Jimbo, Miwa, and Ueno [JMU] (about analytical aspects of the method see, e.g., [IN] and [FI]). The relevant Riemann-Hilbert formalism for (1.7) was developed in [FIK1,2] as well. It was used in [FIK1-3] together with the Isomonodromy Method for the asymptotic analysis of the solution of (1.7), which is related to the double-scaling limit in the 2D quantum gravity studied in [BK], [DS], [GM].

The solution of (1.7) which is analysed in [FIK] is different from the one associated to the orthogonal polynomials (1.3). It corresponds to the system of orthogonal polynomials on the certain rays in complex domain. Nevertheless, the basic elements of the Riemann-Hilbert isomonodromy scheme suggested in [FIK] can be easiely extended (not the concrete analysis of course) to the other systems of semiclassical orthogonal polynomials (see e.g. [FIK4]). The proof of Theorem 1.1 is based on the approach of [FIK] combined with the Nonlinear Steepest Descent Method proposed recently by Deift and Zhou [DZ] for analyzing the asymptotics of oscillatory matrix Riemann-Hilbert problems. We appeal to the Deift-Zhou method in the section 10.5 where we construct explicitly and then justify rigorously the asymptotic solution of the master Riemann-Hilbert problem associated to the orthogonal polynomials (1.3) (see problem **(i-iii)** and (5.16-18) below). The use of the method of [DZ] rather than the original approach of [FIK] at this point of the proof simplifies it dramatically (see Appendix 1 for more details).

The paper is organized as follows:

In the next section we use the results listed in Theorem 1.1 for proving the universality of the local distribution of eigenvalues in the matrix model with quartic potential.

In sections 3-5 we reproduce in a slightly different way and with more details the results of [FIK] concerning the Lax pair representation of equation (1.7) and the matrix Riemann-Hilbert reformulation of the orthogonal polynomial system (1.3). In particular, we show that there is an exact and simple relation between orthogonal polynomials $P_n(z)$ and the 2×2 matrix-valued function $\Psi_n(z)$ which solves the following matrix Riemann-Hilbert problem on a line (the problem (5.16-18) below) :

(i) $\Psi_n(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$, and it has a jump at the real line.

(ii) $$\Psi_n(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k}{z^k}\right) e^{-\left(\frac{NV(z)}{2} - n\ln z + \lambda_n\right)\sigma_3}, \qquad z \to \infty,$$

$$\lambda_n = \frac{1}{2}\ln h_n, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & R_n^{-1/2} \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ R_n^{1/2} & 0 \end{pmatrix}.$$

(iii) $$\Psi_{n+}(z) = \Psi_{n-}(z) S, \qquad \operatorname{Im} z = 0, \qquad S = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix}.$$

As it is explained at the end of section 5, in the setting of the Riemann-Hilbert problem **(i-iii)** the real quantities R_n and λ_n *are not the given data.* They are evaluated via the solution $\Psi_n(z)$, which is determined by conditions **(i-iii)** uniquely without any prior specification of R_n and λ_n.

Simultaneously, the function $\Psi_n(z)$ satisfies the Lax pair (see equation (3.15) below) whose second equation is the linear differential equation:

$$\frac{d\Psi_n(z)}{dz} = NA_n(z)\Psi_n(z), \tag{1.6$'$}$$

$$A_n(z) = \begin{pmatrix} -(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n) & R_n^{1/2}[t + gz^2 + g(R_n + R_{n+1})] \\ -R_n^{1/2}[t + gz^2 + g(R_{n-1} + R_n)] & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n \end{pmatrix}.$$

The jump matrix S in **(iii)** constitutes the only nontrivial Stokes matrix (for more details see sections 4, 5, and 10.1-3) corresponding to the system (1.6$'$) with R_n generated by the orthogonal polynomials (1.3). This reduces the problem of the asymptotic analysis of the quantities $P_n(z)$ and R_n to the asymptotic solution of the matrix Riemann-Hilbert problem **(i-iii)**, i.e. to the asymptotic solution of the corresponding inverse monodromy problem for differential equation (1.6$'$).

We do not assume that the reader is well-familiar to the general monodromy theory of the systems of linear ordinary differential equations (whose comprehensive modern exposition can be found in the book [Sib] of Sibuya). Therefore, we try to make the paper as much self-contained as possible. In fact, the sections 4,5 below give an elementary introduction into the central for the monodromy theory concept of the *Stokes Phenomena*.

Short but important section 6 provides a formal asymptotic ansatz indicated in (1.14) for the recurrence coefficients R_n. Using this ansatz and reducing the matrix differential equation $(1.6')$ to the scalar Schrödinger equation (see the equation (7.1) below) we derive in the sections 7–9 all the asymptotic formulas announced in Theorem 1.1. The analysis in these sections is based on the semiclassical technique, and it is formal since we have not yet proven the asymptotic formula (1.14) for R_n.

The proof of (1.14) together with the rigorous evaluation of the asymptotics of the orthogonal polynomials $P_n(z)$ on the whole complex plane z, which include again the asymptotic equations (1.15-23), is given in section 10. The main objective of this section is to build up and then to justify the asymptotic solution of the matrix Riemann-Hilbert problem **(i-iii)**. To this end we first need the general monodromy theory for the basic matrix differential equation $(1.6')$ under the only assumption that R_{n-1}, R_n, and R_{n+1} are real numbers satisfying the Freud equation (1.7). We refer the reader to the monograph [Sib] for the general background on the monodromy theory for the systems of ordinary differential equations with rational coefficients. The particular case of the system $(1.6')$, (1.7) is spelled out in [FIK2]. For the reader's convinience we repeat, again with more details, the corresponding derivations in the subsections 10.1-3. Following [FIK], we formulate the general matrix Riemann-Hilbert problem (see the equations (10.15) below) on the six rays in complex domain z, which is equivalent to the inverse monodromy problem for the system $(1.6')$ and which includes the orthogonal polynomial Riemann-Hilbert problem **(i-iii)** as a particular triangular case (subsection 10.3). After that, in the subsection 10.4 we solve asymptotically the *direct monodromy problem* for the equation $(1.6')$ *assuming* for R_n the ansatz (1.14). Our analysis in this subsection is based on the version of complex WKB method which was recently suggested in [Kap] for asymptotic solution of the direct monodromy problems for the 2×2 systems with rational coefficients. The results obtained in subsection 10.4 extend the asymptotic formulae for $P_n(z)$ found in the sections 7–9 into the whole complex plane z. They are also used in the subsection 10.5 for introducing an *explicit* matrix-valued function (see the equation (10.135) below) which is shown (Proposition 10.3) to solve asymptotically the basic Riemann-Hilbert problem **(i-iii)**. This provides us with the asymptotic solution of the *inverse monodromy problem* for equation $(1.6')$ related to the orthogonal polynomials (1.3), which proves estimate (1.14) and completes the proof of Theorem 1.1. The basic ideas and technique used in the subsection 10.5 are those of the Deift-Zhou nonlinear steepest descent method [DZ]. Finally, in the Appendix 1 we present an alternative approach to the solution of the orthogonal polynomial inverse monodromy problem. It is based directly on the principle result of the subsection 10.4, i.e., on the fact that the monodromy data of the model system (1.6′), (1.14) are close to the genuine monodromy data (matrix S from **(iii)**) corresponding to the orthogonal polynomials (1.3). This is the original ideology used in [FIK] for the orthogonal polynomial problem considered there.

As has already been mentioned above, the Freud equation (1.7) has a meaning of the discrete Painlevé I equation. We refer the reader to the papers [FIZ], [NPCQ], [GRP], [Mag3,4], [Meh2] for more on the subject. As it was first noticed by Kitaev, the equation (1.7) can be also interpreted as the Backlund-Schlezinger transform of the classical Painlevé IV equation so that the coefficients R_n coincide, in fact, with the special PIV function (see [FIK1,3] for more details). This PIV function, in turn, can be expressed in terms of certain $n \times n$ determinants involving the parabolic cylinder functions (see [Mag4]). In the present paper however we do not use these algebraic by their nature connections to the modern Painlevé theory. We use its analytical methods.

2. Universality of the Local Distribution of Eigenvalues in the Matrix Model

Theorem 1.1 can be applied to proving the universality of the local distribution of eigenvalues in the matrix model with quartic potential. The matrix model is defined as follows. Let $M = (M_{jk})_{j,k=1,\dots N}$ be a Hermitian random matrix, with the probability distribution

$$\mu_N(dM) = Z_N^{-1} e^{-N \operatorname{Tr} V(M)} dM, \tag{2.1}$$

where

$$V(M) = a_0 + a_1 M + \cdots + a_{2p} M^{2p}, \qquad a_{2p} > 0,$$

is a polynomial,

$$dM = \prod_{j<k} (d\operatorname{Re} M_{jk}\, d\operatorname{Im} M_{jk}) \prod_j dM_{jj},$$

is the Lebesgue measure on the space of Hermitian matrices, and

$$Z_N = \int e^{-N \operatorname{Tr} V(M)} dM$$

is the grand partition function. Let $\lambda_1 \leq \cdots \leq \lambda_N$ be eigenvalues of M. Consider the distribution function of the eigenvalues,

$$F_N(z) = N^{-1}\,\mathrm{E}\,\#\,\{j\colon \lambda_j \leq z\}.$$

and the density function

$$p_N(z) = F_N'(z).$$

In the matrix model we are interested in the following problems:

(1) To calculate the limit density $p(z) = \lim_{N\to\infty} p_N(z)$.

(2) To calculate the limit local distribution (scaling limit) of eigenvalues at regular points, where $p(z)$ is positive, and at end-points, where $p(z)$ vanishes.

(3) To calculate the free energy

$$f(a_0,\ldots,a_{2p}) = -\lim_{N\to\infty} \frac{\log Z_N(a_0,\ldots,a_{2p})}{N^2}$$

and to find the points of nonanaliticity of f (critical points) in the space of the parameters $a_0,\ldots,a_{2p}$. To calculate the critical asymptotics of the recursive coefficients R_n and of the local distribution of eigenvalues (double scaling limit).

Dyson [Dys] (see also [Meh1] and [TW1]) proves a formula which expresses the correlations between the eigenvalues of M in terms of orthogonal polynomials. Namely, the m-point correlation function is written as

$$K_{Nm}(z_1,\ldots,z_m) = \frac{1}{m!}\,\det\left(Q_N(z_j,z_k)\right)_{j,k=1,\ldots,m} \tag{2.2}$$

where

$$Q_N(z,w) = \sum_{j=1}^{N} \psi_j(z)\psi_j(w), \tag{2.3}$$

and $\psi_j(z)$ is defined in (1.9). When $m=1$ the correlation function reduces to the function $Np_N(z)$, hence

$$p_N(z) = N^{-1}\sum_{j=1}^{N} \psi_j^2(z).$$

By the Christoffel-Darboux formula (see, e.g., [Sze]), the kernel $Q_N(z,w)$ can be written as

$$Q_N(z,w) = \frac{\sqrt{R_{N+1}}\,[\psi_{N+1}(z)\psi_N(w) - \psi_N(z)\psi_{N+1}(w)]}{z-w}, \tag{2.4}$$

and

$$p_N(z) = \frac{\sqrt{R_{N+1}}\left[\psi'_{N+1}(z)\psi_N(z) - \psi'_N(z)\psi_{N+1}(z)\right]}{N}\,. \tag{2.5}$$

The formula (1.24) is valid in a complex neighborhood of the interval $[z_1+\delta, z_2-\delta]$ and this allows us to differentiate it. We will assume that

$$t < t_{\rm cr} = -2\sqrt{g}$$

(two-cut case), hence we can use $n = N$ in the asymptotic formulae (1.15)–(1.18). For the sake of brevity we rewrite (1.24), (1.15) as

$$\psi_n = \frac{Cz}{\sqrt{\zeta_z}}\cos(N\zeta + \eta), \tag{2.6}$$

where

$$\begin{aligned}
&C = \sqrt{g/\pi}\,; \qquad \zeta = \zeta(z;\lambda') = \int_{z_2}^{z} |U_0(v;\lambda')|^{1/2}dv + \frac{\pi}{4N}\,;\\
&\zeta_z = \frac{\partial\zeta(z;\lambda')}{\partial z} = |U_0(z;\lambda')|^{1/2}\,; \qquad U_0(z;\lambda') = z^2\left[\frac{(gz^2+t)^2}{4} - \lambda' g\right]\,;\\
&\eta = -\frac{(-1)^n}{4}\,\chi(z;\lambda') = -\frac{(-1)^n}{4}\arccos y, \qquad y = \frac{2\sqrt{\lambda' g} - tx}{2\sqrt{\lambda' g}\,x - t}, \qquad x = \frac{gz^2+t}{2\sqrt{\lambda' g}}\,,
\end{aligned} \tag{2.7}$$

and we drop terms of the order of N^{-1}. In addition, (1.24) gives that modulo terms of the order of N^{-1},

$$\psi_{n\pm1} = \frac{Cz}{\sqrt{\zeta_z}}\cos(N\zeta \pm \xi - \eta), \tag{2.8}$$

where

$$\xi = \frac{\partial\zeta(z;\lambda')}{\partial\lambda'} = -\frac{1}{2}\arccos x, \qquad x = \frac{gz^2+t}{2\sqrt{\lambda' g}}\,. \tag{2.9}$$

The finctions ψ_n satisfy the recursive equation

$$z\psi_n = \sqrt{R_{n+1}}\,\psi_{n+1} + \sqrt{R_n}\,\psi_{n-1}$$

(see (1.4)), hence from (2.6) and (2.8) we obtain that

$$\begin{aligned}
z\cos(N\zeta-\eta)\cos(2\eta) &- z\sin(N\zeta-\eta)\sin(2\eta)\\
&= \sqrt{R_{n+1}}\,\cos(N\zeta-\eta)\cos\xi - \sqrt{R_{n+1}}\,\sin(N\zeta-\eta)\sin\xi\\
&+ \sqrt{R_n}\,\cos(N\zeta-\eta)\cos\xi + \sqrt{R_n}\,\sin(N\zeta-\eta)\sin\xi.
\end{aligned}$$

Equating the coefficients at $\cos(N\zeta-\eta)$ and $\sin(N\zeta-\eta)$, we obtain that

$$\begin{aligned}
z\cos 2\eta &= \left(\sqrt{R_{n+1}} + \sqrt{R_n}\right)\cos\xi,\\
z\sin 2\eta &= \left(\sqrt{R_{n+1}} - \sqrt{R_n}\right)\sin\xi.
\end{aligned} \tag{2.10}$$

These formulae can be checked directly from (1.14), (2.7) and (2.9). Differentiating (2.6) and (2.8) in z, we get that

$$\begin{aligned} \psi_n' &= -Cz\sin(N\zeta+\eta)\,N\sqrt{\zeta_z}+O(1), \\ \psi_{n+1}' &= -Cz\sin(N\zeta+\xi-\eta)\,N\sqrt{\zeta_z}+O(1), \end{aligned}$$

hence by (2.5), modulo terms of the order of N^{-1},

$$\begin{aligned} p_N &= \sqrt{R_{N+1}}\,C^2z^2\left[-\sin(N\zeta+\xi-\eta)\cos(N\zeta+\eta)+\cos(N\zeta+\xi-\eta)\sin(N\zeta+\eta)\right] \\ &= \sqrt{R_{N+1}}\,C^2z^2\sin(2\eta-\xi)=\sqrt{R_{N+1}}\,C^2z^2(\sin 2\eta\cos\xi-\cos 2\eta\sin\xi), \end{aligned} \tag{2.11}$$

and by (2.10),

$$\begin{aligned} p_N &= \sqrt{R_{N+1}}\,C^2z\left[(\sqrt{R_{N+1}}-\sqrt{R_N})\sin\xi\cos\xi-(\sqrt{R_{N+1}}+\sqrt{R_N})\sin\xi\cos\xi\right] \\ &= -\sqrt{R_{N+1}R_N}\,C^2z\sin 2\xi. \end{aligned}$$

Since modulo terms of the order of N^{-1},

$$R_{N+1}R_N=\frac{1}{g}; \qquad C^2=\frac{g}{\pi}; \qquad \sin 2\xi=\sin(-\arccos x)=-\sqrt{1-x^2},$$

we obtain that

$$p_N=\frac{\sqrt{g}}{\pi}\,z\sqrt{1-x^2}+O(N^{-1}).$$

Substituting the value of x gives that

$$p_N(z)=p(z)+O(N^{-1}),$$

where

$$p(z)=\frac{1}{\pi}\,|U_0(z;1)|^{1/2}=\frac{|z|}{\pi}\left[g-\left(\frac{gz^2+t}{2}\right)^2\right]^{1/2}=\frac{g|z|}{2\pi}\sqrt{(z^2-z_1^2)(z_2^2-z^2)} \tag{2.12}$$

and

$$z_{1,2}=\left(\frac{-t\mp 2\sqrt{g}}{g}\right)^{1/2}. \tag{2.13}$$

This gives an explicit formula for the limiting density $p=p(z)$ of eigenvalues (integrated density of states). In a completely different approach, based on the Coulomb gas representation of the matrix model, this formula is derived in the work [BPS] of Boutet de Monvel, Pastur, and Shcherbina, as an application of the proven in [BPS] variational principle for the integrated density of states.

The scaling limit of the correlation function $K_{Nm}(z_1,\dots,z_m)$ at a regular point z, where $p(z)>0$, is defined as

$$K_m(u_1,\dots,u_m)=\lim_{N\to\infty}\big[Np(z)\big]^{-m}K_{Nm}\left(z+\frac{u_1}{Np(z)},\dots,z+\frac{u_m}{Np(z)}\right).$$

Observe that $K_m(u_1,\dots,u_m)$ is the limiting m-point correlation function of the rescaled eigenvalues

$$\mu_j=Np(z)(\lambda_j-z).$$

The rescaling reduces the mean value of the spacing $\mu_{j+1}-\mu_j$ to 1. From Dyson's formula (2.2),

$$K_m(u_1,\dots,u_m)=\frac{1}{m!}\det\big(Q(u_j,u_k)\big)_{j,k=1,\dots,m},\tag{2.14}$$

where

$$Q(u,v)=\lim_{N\to\infty}\big[Np(z)\big]^{-1}Q_N\left(z+\frac{u}{Np(z)},z+\frac{v}{Np(z)}\right).\tag{2.15}$$

By (2.4),

$$\big[Np(z)\big]^{-1}Q_N\left(z+\frac{u}{Np(z)},z+\frac{v}{Np(z)}\right)=\frac{\sqrt{R_{N+1}}}{u-v}T_N\left(z+\frac{u}{Np(z)},z+\frac{v}{Np(z)}\right),\tag{2.16}$$

where

$$T_N(z,w)=\psi_{N+1}(z)\psi_N(w)-\psi_N(z)\psi_{N+1}(w).$$

By (2.6) and (2.8), modulo terms of the order of N^{-1},

$$\begin{aligned}\psi_N\left(z+\frac{u}{Np(z)}\right)&=\frac{Cz}{\sqrt{\zeta_z}}\cos(N\zeta+\alpha+\eta),\qquad \alpha=\frac{\zeta_z u}{Np(z)},\\ \psi_{N+1}\left(z+\frac{u}{Np(z)}\right)&=\frac{Cz}{\sqrt{\zeta_z}}\cos(N\zeta+\alpha+\xi-\eta),\end{aligned}\tag{2.17}$$

hence

$$\begin{aligned}T_N\left(z+\frac{u}{Np(z)},z+\frac{v}{Np(z)}\right)&=\frac{C^2z^2}{\zeta_z}\,[\cos(N\zeta+\alpha+\xi-\eta)\cos(N\zeta+\beta+\eta)\\ &\quad-\cos(N\zeta+\alpha+\eta)\cos(N\zeta+\beta+\xi-\eta)]\\ &=\frac{C^2z^2}{2\zeta_z}\,[\cos(\alpha+\xi-\beta-2\eta)-\cos(\alpha-\xi-\beta+2\eta)]\\ &=\frac{C^2z^2}{\zeta_z}\,\sin(2\eta-\xi)\sin(\alpha-\beta),\end{aligned}\tag{2.18}$$

where

$$\alpha = \frac{\zeta_z u}{p(z)} = \frac{|U_0(z)|^{1/2} u}{|U_0(z)|^{1/2}\pi^{-1}} = \pi u, \qquad \beta = \pi v\,. \tag{2.19}$$

By (2.11) and (2.12),

$$\sqrt{R_{N+1}}\, C^2 z^2 \sin(2\eta - \xi) = p(z) = \frac{1}{\pi}\sqrt{|U_0(z;1)|} = \zeta_z(z;1),$$

hence (2.18) implies that

$$\begin{aligned}\sqrt{R_{N+1}}\, T_N\left(z + \frac{u}{Np(z)}, z + \frac{v}{Np(z)}\right) &= \sqrt{R_{N+1}}\,\frac{C^2 z^2}{\zeta_z}\,\sin(2\eta-\xi)\sin(\alpha-\beta)\\ &= \frac{\sin(\alpha-\beta)}{\pi} = \frac{\sin\pi(u-v)}{\pi}\,,\end{aligned}$$

and, by (2.15), (2.16),

$$Q(u,v) = \frac{\sin\pi(u-v)}{\pi(u-v)}\,.$$

This proves the Dyson sine-kernel for the local distribution of eigenvalues at a regular point z. In a completely different approach, the sine-kernel at regular points is proved in [PS].

Remark. It follows from the Dyson sine-kernel, due to the Gaudin formula (see, e.g., [Meh1]), that the spacing distribution of eigenvalues is determined by the Fredholm determinant $\det(1 - Q(x,y))_{x,y\in J}$. The asymptotics of this determinant as $|J| \to \infty$ has been studied intensively since the classical works by des Cloizeaux, Dyson, Gaudin, Mehta, and Widom (see [Meh1] for the history of the subject). The Riemann-Hilbert approach to this asymptotics has been developed in the paper [DIZ].

At the endpoints of the spectrum we use the semiclassical asymptotics (1.18), and it leads to the Airy kernel (cf. the papers of Bowick and Brézin [BB], Forrester [For], Moore [Moo], and Tracy and Widom [TW2], where the Airy kernel is discussed for the Gaussian matrix model and some other related models, and, in addition, some nonrigorous arguments are given for general matrix models). Consider for the sake of definiteness $z = z_2$.

By (1.18),

$$\psi_n = \frac{DN^{1/6}z}{\sqrt{\varphi_N'}}\left[\mathrm{Ai}\left(N^{2/3}\varphi_N\right) + O(N^{-1})\right], \qquad D = \sqrt{g}, \tag{2.20}$$

where φ_N is defined in (1.19). From (1.19),

$$\sqrt{\varphi_N}\,\frac{\partial\varphi_N}{\partial\lambda'} = \frac{\partial}{\partial\lambda'}\int_{z_2^{(N)}}^{z}\sqrt{U_N(v)}\,dv.$$

This allows us to derive from (2.20) that

$$\begin{aligned}\psi_n &= \frac{DN^{1/6}z}{\sqrt{\varphi_0'}}\left[\mathrm{Ai}\,(N^{2/3}\varphi_0 + N^{-1/3}\omega) + O(N^{-1})\right],\\ \psi_{n\pm 1} &= \frac{DN^{1/6}z}{\sqrt{\varphi_0'}}\left[\mathrm{Ai}\,(N^{2/3}\varphi_0 \pm N^{-1/3}\rho - N^{-1/3}\omega) + O(N^{-1})\right],\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}&\varphi_0 = \varphi_0(z;\lambda') = \left(\frac{3}{2}\int_{z_2}^{z}\sqrt{U_0(v;\lambda')}\,dv\right)^{2/3},\\ &\rho = \rho(z;\lambda') = \frac{\xi(z;\lambda')}{\sqrt{\varphi_0(z;\lambda')}}; \qquad \omega = \omega(z;\lambda') = \frac{\eta(z;\lambda')}{\sqrt{\varphi_0(z;\lambda')}},\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\xi(z;\lambda') &= -\frac{\cosh^{-1}x}{2}, \qquad x = \frac{gz^2+t}{2\sqrt{\lambda' g}};\\ \eta(z;\lambda') &= -\frac{(-1)^n}{4}\cosh^{-1}y, \qquad y = \frac{2\sqrt{\lambda' g} - tx}{2\sqrt{\lambda' g}x - t}.\end{aligned} \tag{2.23}$$

The formulae (2.22), (2.23) define the functions $\varphi_0(z;\lambda')$, $\rho(z;\lambda')$ and $\omega(z;\lambda')$ for $z \ge z_2$. It is easy to check that these functions are analytic in z at $z = z_2$, and they can be continued analytically to the interval $z > z_1$. In addition,

$$\begin{aligned}&U_0(z_2;\lambda') = 0, \qquad \frac{\partial U_0}{\partial z}(z_2;\lambda') = \varkappa = 2(\lambda')^{1/2}g^{3/2}z_2^3;\\ &\varphi_0(z_2;\lambda') = 0, \qquad \frac{\partial \varphi_0}{\partial z}(z_2;\lambda') = \varkappa^{1/3} = 2^{1/3}(\lambda')^{1/6}g^{1/2}z_2;\\ &\rho(z_2) = -2^{-2/3}(\lambda')^{-1/3};\\ &\omega(z_2) = -\frac{(-1)^n}{4}\,2^{1/3}(\lambda')^{-1/3}z_1 z_2^{-1}.\end{aligned} \tag{2.24}$$

We will consider

$$z = z_2 + N^{-2/3}\alpha, \qquad w = z_2 + N^{-2/3}\beta, \tag{2.26}$$

where α and β are fixed.

Substitution of (2.21) into the recursive equation

$$z\psi_n = \sqrt{R_{n+1}}\psi_{n+1} + \sqrt{R_n}\psi_{n-1}$$

gives the equations

$$\begin{aligned}z_2 &= \sqrt{R_{n+1}} + \sqrt{R_n},\\ z_2\omega &= \sqrt{R_{n+1}}(\rho - \omega) + \sqrt{R_n}(-\rho - \omega),\end{aligned} \tag{2.27}$$

from where

$$(\sqrt{R_{n+1}}+\sqrt{R_n})\,2\omega=(\sqrt{R_{n+1}}-\sqrt{R_n})\,\rho. \tag{2.28}$$

Similarly,

$$z_1=(-1)^n(\sqrt{R_{n+1}}-\sqrt{R_n}),$$

hence

$$2\omega=\frac{(-1)^n z_1\rho}{z_2}, \tag{2.29}$$

which agrees with (2.24).

Substituting the formulae (2.21) into (2.4) and throwing away terms of the lower order, we obtain that

$$\begin{aligned}Q_N(z,w)&=\frac{\sqrt{R_{N+1}}\,D^2N^{1/3}z_2^2}{(z-w)\varphi_0'}\\&\times\Big[\operatorname{Ai}\big(N^{2/3}\varphi_0(z)+N^{-1/3}\rho-N^{-1/3}\omega\big)\operatorname{Ai}\big(N^{2/3}\varphi_0(w)+N^{-1/3}\omega\big)\\&-\operatorname{Ai}\big(N^{2/3}\varphi_0(z)+N^{-1/3}\omega\big)\operatorname{Ai}\big(N^{2/3}\varphi_0(w)+N^{-1/3}\rho-N^{-1/3}\omega\big)\Big],\end{aligned} \tag{2.30}$$

where φ_0', ρ and ω are taken at z_2. Taking the linear part of Ai we obtain that

$$Q_N(z,w)=\frac{\sqrt{R_{N+1}}\,D^2N^{1/3}z_2^2}{(z-w)\varphi_0'}\,[\operatorname{Ai}(u)\operatorname{Ai}'(v)-\operatorname{Ai}'(u)\operatorname{Ai}(v)]\,(2\omega-\rho)N^{-1/3},$$

where

$$u=\varphi_0'\,\alpha,\qquad v=\varphi_0'\,\beta.$$

By (2.28) and (2.24), modulo terms of the order of $N^{-1/3}$,

$$\sqrt{R_{N+1}}\,(2\omega-\rho)=\sqrt{R_{N+1}}\,\frac{(-2\sqrt{R_n})}{\sqrt{R_{N+1}}+\sqrt{R_n}}\,(-2^{-2/3})=2^{1/3}g^{-1/2}z_2^{-1},$$

hence

$$Q_N(z,w)=N^{2/3}2^{1/3}g^{1/2}z_2\,\frac{\operatorname{Ai}(u)\operatorname{Ai}'(v)-\operatorname{Ai}'(u)\operatorname{Ai}(v)}{u-v}+O(N^{1/3}).$$

Thus,

$$\lim_{N\to\infty}\frac{1}{cN^{2/3}}Q_N\left(z_2+\frac{u}{cN^{2/3}},z_2+\frac{v}{cN^{2/3}}\right)=\frac{\operatorname{Ai}(u)\operatorname{Ai}'(v)-\operatorname{Ai}'(u)\operatorname{Ai}(v)}{u-v},$$

where

$$c=\varphi_0'(z_2;1)=2^{1/3}g^{1/2}z_2.$$

This proves the Airy kernel at the endpoint z_2. The endpoint z_1 is treated similarly.

3. The Lax Pair for the Freud Equation

Let

$$\psi_n(z) = \frac{1}{\sqrt{h_n}} P_n(z) e^{-NV(z)/2}. \tag{3.1}$$

Then

$$\int_{-\infty}^{\infty} \psi_n(z)\psi_m(z)\, dz = \delta_{nm}. \tag{3.2}$$

Recursive equation for $\psi_n(z)$ follows from (1.4):

$$z\psi_n(z) = R_{n+1}^{1/2}\psi_{n+1}(z) + R_n^{1/2}\psi_{n-1}(z). \tag{3.3}$$

In addition,

$$\begin{aligned}
\psi_n'(z) = &- \left(N\,\frac{g}{2}\,R_{n+1}^{1/2}R_{n+2}^{1/2}R_{n+3}^{1/2}\right)\psi_{n+3}(z) \\
&- \left[N\,\frac{t}{2}\,R_{n+1}^{1/2} + N\,\frac{g}{2}\,R_{n+1}^{1/2}(R_n + R_{n+1} + R_{n+2})\right]\psi_{n+1}(z) \\
&+ \left[N\,\frac{t}{2}\,R_n^{1/2} + N\,\frac{g}{2}\,R_n^{1/2}(R_{n-1} + R_n + R_{n+1})\right]\psi_{n-1}(z) \\
&+ \left(N\,\frac{g}{2}\,R_{n-2}^{1/2}R_{n-1}^{1/2}R_n^{1/2}\right)\psi_{n-3}(z)
\end{aligned} \tag{3.4}$$

Let

$$\vec{\Psi}_n(z) = \begin{pmatrix} \psi_n(z) \\ \psi_{n-1} \end{pmatrix}. \tag{3.5}$$

Then combining (3.3) with (3.4), one can obtain (cf. (3.1-7) in [FIK2]) that

$$\begin{cases} \vec{\Psi}_{n+1}(z) = U_n(z)\vec{\Psi}_n(z), \\ \vec{\Psi}_n'(z) = NA_n(z)\vec{\Psi}_n(z), \end{cases} \tag{3.6}$$

where

$$U_n(z) = \begin{pmatrix} R_{n+1}^{-1/2} z & -R_{n+1}^{-1/2} R_n^{1/2} \\ 1 & 0 \end{pmatrix}, \tag{3.7}$$

and

$$A_n(z) = \begin{pmatrix} -(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n) & R_n^{1/2}[t + gz^2 + g(R_n + R_{n+1})] \\ -R_n^{1/2}[t + gz^2 + g(R_{n-1} + R_n)] & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n \end{pmatrix}. \tag{3.8}$$

Observe that

$$\operatorname{tr} A_n(z) = 0$$

and

$$\det A_n(z) = -\left(\frac{tz}{2} + \frac{gz^3}{2}\right)^2 + gR_n\left(t + gR_{n-1} + gR_n + gR_{n+1}\right)z^2 + R_n\theta_{n-1}\theta_n, \quad (3.9)$$

where

$$\theta_n = t + gR_n + gR_{n+1}. \quad (3.10)$$

Due to (1.7), we can rewrite $\det A_n(z)$ as

$$\det A_n(z) = -\left(\frac{tz}{2} + \frac{gz^3}{2}\right)^2 + \frac{gnz^2}{N} + R_n\theta_{n-1}\theta_n. \quad (3.9')$$

Compatibility condition of the equations (3.6) is

$$U_n'(z) = NA_{n+1}(z)U_n(z) - NU_n(z)A_n(z). \quad (3.11)$$

Restricting this equation to the matrix element $U'_{n,11}(z)$ we obtain that

$$J_{n+1} - J_n = 1, \quad (3.12)$$

where

$$J_n = NR_n[t + g(R_{n-1} + R_n + R_{n+1})].$$

Hence $J_n = n+\text{const}$. Since $J_0 = 0$, in fact, $J_n = n$. This means, that the compatibility condition (3.10), together with the initial value $J_0 = 0$ imply (1.7), and thus the equations (3.6) give the Lax pair for the nonlinear difference Freud equation (1.7). In addition, the equation (3.12) gives the recursive equation

$$R_{n+1}\theta_n\theta_{n+1} = R_n\theta_{n-1}\theta_n + \frac{\theta_n}{N}. \quad (3.13)$$

The equations (3.6) have two linear independent solutions and $\vec{\Psi}_n(z)$ is one of them. We will consider another solution,

$$\vec{\Phi}_n(z) = \begin{pmatrix} \varphi_n(z) \\ \varphi_{n-1}(z) \end{pmatrix},$$

and the 2×2 matrix

$$\Psi_n(z) = \begin{pmatrix} \psi_n(z) & \varphi_n(z) \\ \psi_{n-1}(z) & \varphi_{n-1}(z) \end{pmatrix} \quad (3.14)$$

which satisfies the same equations,

$$\begin{cases} \Psi_{n+1}(z) = U_n(z)\Psi_n(z), \\ \Psi_n'(z) = NA_n(z)\Psi_n(z). \end{cases} \quad (3.15)$$

To define $\vec{\Phi}_n(z)$ we consider an arbitrary, linearly independent with $\vec{\Psi}_1(z)$, solution of the differential equation $\vec{\Phi}_1'(z) = NA_1(z)\vec{\Phi}_1(z)$, and then define $\vec{\Phi}_n(z)$, $n \ge 2$, with the help of the recursive equation $\vec{\Phi}_{n+1}(z) = U_n(z)\vec{\Phi}_n(z)$. The equation (3.11) then leads to the differential equation $\vec{\Phi}_n'(z) = NA_n\vec{\Phi}_n(z)$ for $n \ge 2$. The equation $\vec{\Phi}_{n+1}(z) = U_n(z)\vec{\Phi}_n(z)$ means that $\varphi_n(z)$ satisfies the recursive equation (3.3), i.e.,

$$z\varphi_n(z) = R_{n+1}^{1/2}\varphi_{n+1}(z) + R_n^{1/2}\varphi_{n-1}(z). \tag{3.16}$$

Since $\operatorname{tr} A_n(z) = 0$, the second equation in (3.15) implies that

$$\det \Psi_n(z) = C \neq 0 \tag{3.17}$$

is independent of z, i.e.,

$$\psi_n(z)\varphi_{n-1}(z) - \psi_{n-1}(z)\varphi_n(z) = C. \tag{3.18}$$

This enables us to find $\varphi_n(z)$. Namely,

$$\begin{aligned} \varphi_n'(z) &= Na_{11}(z)\varphi_n(z) + Na_{12}(z)\varphi_{n-1}(z), \\ \psi_n'(z) &= Na_{11}(z)\psi_n(z) + Na_{12}(z)\psi_{n-1}(z), \end{aligned} \tag{3.19}$$

where $A_n(z) = \big(a_{ij}(z)\big)_{i,j=1,2}$, hence

$$\psi_n(z)\varphi_n'(z) - \varphi_n(z)\psi_n'(z) = Na_{12}(z)[\psi_n(z)\varphi_{n-1}(z) - \psi_{n-1}(z)\varphi_n(z)] = CNa_{12}(z),$$

and

$$\left(\frac{\varphi_n(z)}{\psi_n(z)}\right)' = \frac{CNa_{12}(z)}{\psi_n^2(z)}\,.$$

This gives

$$\varphi_n(z) = CN\psi_n(z)\int_{z_0}^{z} \frac{a_{12}(u)}{\psi_n^2(u)}\,du, \qquad n \ge 1. \tag{3.20}$$

In a similar way we get

$$\varphi_{n-1}(z) = CN\psi_{n-1}(z)\int_{z_0}^{z} \frac{a_{21}(u)}{\psi_{n-1}^2(u)}\,du, \qquad n \ge 1. \tag{3.21}$$

It is useful to note that (3.18) allows to express $\varphi_{n-1}(z)$ through $\varphi_n(z)$:

$$\varphi_{n-1}(z) = \frac{\psi_{n-1}(z)}{\psi_n(z)}\,\varphi_n(z) + \frac{C}{\psi_n(z)}. \tag{3.22}$$

The system of two differential equations of the first order, $\vec{\Psi}_n' = NA_n\vec{\Psi}_n$, can be reduced to one equation of the second order (cf. [Sho]). Namely, from the first equation of the system we can express ψ_{n-1} in terms of ψ_n,

$$\psi_{n-1} = N^{-1}\frac{1}{a_{12}}\,\psi_n' - \frac{a_{11}}{a_{12}}\,\psi_n. \tag{3.23}$$

and then we can substitute this expression into the second equation of the system, which gives

$$\psi_n'' - \frac{a_{12}'}{a_{12}}\,\psi_n' + N^2(a_{11}a_{22} - a_{12}a_{21})\,\psi_n - Na_{12}\left(\frac{a_{11}}{a_{12}}\right)'\psi_n = 0. \tag{3.24}$$

With the help of the substitution

$$\psi_n = a_{12}^{1/2}\zeta_n \tag{3.25}$$

we reduce (3.24) to the Schrödinger equation

$$-\zeta_n'' + N^2U\zeta_n = 0, \tag{3.26}$$

where

$$U = -(a_{11}a_{22} - a_{12}a_{21}) + N^{-1}\left(a_{11}' - a_{11}\frac{a_{12}'}{a_{12}}\right) - N^{-2}\left[\frac{a_{12}''}{2a_{12}} - \frac{3(a_{12}')^2}{4a_{12}^2}\right]. \tag{3.27}$$

By (3.8),

$$\begin{aligned} -a_{11} = a_{22} &= \frac{tz}{2} + \frac{gz^3}{2} + gzR_n, \\ a_{12} &= R_n^{1/2}(\theta_n + gz^2), \qquad a_{21} = -R_n^{1/2}(\theta_{n-1} + gz^2), \end{aligned} \tag{3.28}$$

which gives

$$\begin{aligned} U(z) = &\left[\frac{g^2z^6}{4} + \frac{tgz^4}{2} + \left(\frac{t^2}{4} - \frac{n}{N}g\right)z^2 - R_n\theta_{n-1}\theta_n\right] \\ &- N^{-1}\left[\frac{t}{2} + \frac{3gz^2}{2} + gR_n - \frac{gz^2(t + gz^2 + 2gR_n)}{gz^2 + \theta_n}\right] + N^{-2}\left[\frac{g(2gz^2 - \theta_n)}{(gz^2 + \theta_n)^2}\right]. \end{aligned} \tag{3.29}$$

It is convenient to write $U(z)$ as

$$U(z) = U_0(z) + U_1(z) + U_2(z), \tag{3.30}$$

where

$$\begin{aligned} U_0(z) &= z^2\left[\left(\frac{gz^2 + t}{2}\right)^2 - \lambda' g\right], \qquad \lambda' = \frac{n + \frac{1}{2}}{N}, \\ U_1(z) &= N^{-1}\left(\frac{t}{2} + gR_n\right), \\ U_2(z) &= -R_n\theta_{n-1}\theta_n - N^{-1}\left[\frac{\theta_n(t + gz^2 + 2gR_n)}{gz^2 + \theta_n}\right] + N^{-2}\left[\frac{g(2gz^2 - \theta_n)}{(gz^2 + \theta_n)^2}\right]. \end{aligned} \tag{3.31}$$

To simplify some formulae below we will use the substitution

$$\psi_n(z) = \left(z^2 + \frac{\theta_n}{g}\right)^{1/2} \zeta_n(z), \tag{3.32}$$

rather than (3.25). These two substitutions differ by a constant factor and lead to the same equation (3.26) on $\zeta_n(z)$.

4. The Stokes Phenomenon

Consider the sectors

$$\Omega_j = \left\{ z\colon \frac{\pi}{8} + \frac{\pi(j-1)}{2} - \varepsilon < \arg z < \frac{3\pi}{8} + \frac{\pi(j-1)}{2} + \varepsilon \right\}, \qquad j = 1,2,3,4, \tag{4.1}$$

$\varepsilon > 0$, on a complex plane where the function

$$\psi_n(z) = \frac{1}{\sqrt{h_n}} P_n(z)\, e^{-N\left(\frac{t}{4}z^2 + \frac{g}{8}z^4\right)} \tag{4.2}$$

goes to infinity as $z \to \infty$. Let us take in (3.20) $z_0 \to \infty$ along the bisector of Ω_j,

$$b_j = \left\{ z\colon \arg z = \frac{\pi}{4} + \frac{\pi(j-1)}{2} \right\}, \tag{4.3}$$

and consider the corresponding solution (3.20),

$$\begin{aligned} \varphi_{nj}(z) &= CN\psi_n(z) \int_{\omega_j\infty}^{z} \frac{a_{12}(u)}{\psi_n^2(u)}\, du \\ &= CNh_n^{1/2} P_n(z)\, e^{-N\left(\frac{t}{4}z^2 + \frac{g}{8}z^4\right)} \\ &\times \int_{\omega_j\infty}^{z} \frac{R_n^{1/2}[t + gu^2 + g(R_n + R_{n+1})]}{P_n^2(u)}\, e^{N\left(\frac{t}{2}u^2 + \frac{g}{4}u^4\right)} du \end{aligned} \tag{4.4}$$

where

$$\omega_j = e^{i\left(\frac{\pi}{4} + \frac{\pi(j-1)}{2}\right)}, \qquad j = 1,2,3,4.$$

The solution $\varphi_{nj}(z) \to 0$ as $z \to \infty$ in Ω_j. Evaluating the integral in (4.4) with the help of the Laplace method, we obtain the asymptotic expansion of $\varphi_{nj}(z)$ in Ω_j:

$$\varphi_{nj}(z) \sim C_n z^{-n-1} e^{\frac{NV(z)}{2}} \left(1 + \sum_{k=1}^{\infty} \frac{\gamma_{2k}}{z^{2k}}\right), \qquad \gamma_{2k} = \gamma_{2k}(n), \tag{4.5}$$

where

$$C_n = -Ch_n^{1/2} R_n^{1/2}. \tag{4.6}$$

In the similar way we get from the equation (3.21) the asymptotic expansion of $\varphi_{n-1,j}(z)$,

$$\varphi_{n-1,j}(z) \sim C_{n-1} z^{-n} e^{\frac{NV(z)}{2}} \left(1 + \sum_{k=1}^{\infty} \frac{\gamma_{2k}}{z^{2k}}\right),$$

where

$$C_{n-1} = -Ch_{n-1}^{1/2} R_n^{1/2}. \tag{4.7}$$

¿From (4.6), (4.7),

$$\frac{C_n}{C_{n-1}} = \frac{h_n^{1/2}}{h_{n-1}^{1/2}} \tag{4.8}$$

[which is compatible with (3.16)]. The initial constant C_0 is a free parameter. We put $C_0 = h_0^{1/2}$. Then (4.8) gives $C_n = h_n^{1/2}$, so that

$$\varphi_{nj}(z) \sim h_n^{1/2} z^{-n-1} e^{\frac{NV(z)}{2}} \left(1 + \sum_{k=1}^{\infty} \frac{\gamma_{2k}}{z^{2k}}\right). \tag{4.9}$$

As a matter of fact, this asymptotic expansion holds in a bigger domain,

$$\Sigma_j = \left\{ -\frac{\pi}{8} + \frac{\pi(j-1)}{2} + \varepsilon < \arg z < \frac{5\pi}{8} + \frac{\pi(j-1)}{2} - \varepsilon \right\}, \qquad \varepsilon > 0. \tag{4.10}$$

The matrix-valued function

$$\Psi_{nj}(z) = \begin{pmatrix} \psi_n(z) & \varphi_{nj}(z) \\ \psi_{n-1}(z) & \varphi_{n-1,j}(z) \end{pmatrix} = \left(\vec{\Psi}_n(z), \vec{\Phi}_{nj}(z)\right) \tag{4.11}$$

is an entire function of z, and according to (4.2) and (4.9) it has the asymptotic expansion

$$\Psi_{nj}(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k}{z^k}\right) e^{-\left(\frac{NV(z)}{2} - n\ln z + \lambda_n\right)\sigma_3}, \qquad z \to \infty, \qquad z \in \Sigma_j, \tag{4.12}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.13}$$

is the Pauli matrix,

$$\lambda_n = \frac{1}{2} \ln h_n, \tag{4.14}$$

and Γ_k are some 2×2 matrices which depend on n and j. From (4.2) and (4.9) we get

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & R_n^{-1/2} \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ R_n^{1/2} & 0 \end{pmatrix}. \tag{4.15}$$

5. The Riemann – Hilbert Problem

Since both $\vec{\Phi}_{nj}(z)$ for different j and $\vec{\Psi}_n(z)$ satisfy the same differential equation (3.6), they are linearly dependent,

$$\vec{\Phi}_{n,j+1}(z) = q\vec{\Phi}_{nj}(z) + s\vec{\Psi}_n(z),$$

where j is defined $\mod 4$. The domains Σ_j and Σ_{j+1} intersect and in the intersection the functions $\vec{\Phi}_{nj}(z)$ and $\vec{\Phi}_{n,j+1}(z)$ grow to infinity and have the same asymptotic expansion (4.12). On the other hand $\vec{\Psi}_n(z)$ goes to zero in this intersection, hence $q = 1$, so that

$$\vec{\Phi}_{n,j+1}(z) = \vec{\Phi}_{nj}(z) + s\vec{\Psi}_n(z). \tag{5.1}$$

Since both $\vec{\Phi}_{nj}(z)$ and $\vec{\Psi}_n(z)$ satisfy the same recursive equation (3.6), the coefficient s does not depend on n, but in general it depends on j, $s = s_j$. The equation (5.1) implies that

$$\varphi_{n,j+1}(z) = \varphi_{nj}(z) + s_j\psi_n(z). \tag{5.2}$$

We can rewrite (5.1) in matrix form as

$$\Psi_{n,j+1}(z) = \Psi_{nj}(z)\, S_j, \tag{5.3}$$

where

$$S_j = \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix}. \tag{5.4}$$

To determine s_j consider (5.2) at $n = 0$. The formula (3.21) reads for $n = 1$,

$$\begin{aligned}\varphi_{0j}(z) &= CN\psi_0(z)\int_{\omega_j\infty}^{z} \frac{a_{21}(u)}{\psi_0^2(u)}\, du \\ &= C'Ne^{-N\left(\frac{t}{4}z^2+\frac{g}{8}z^4\right)}\int_z^{\omega_j\infty} (t+gu^2+gR_1)e^{N\left(\frac{t}{2}u^2+\frac{g}{4}u^4\right)}du.\end{aligned}$$

¿From the asymptotics (4.9) we get $C' = h_0^{1/2}$, hence

$$\varphi_{0j}(z) = h_0^{1/2}e^{-N\left(\frac{t}{4}z^2+\frac{g}{8}z^4\right)}\int_z^{\omega_j\infty} N\,(t+gu^2+gR_1)e^{N\left(\frac{t}{2}u^2+\frac{g}{4}u^4\right)}du. \tag{5.5}$$

Putting $z = 0$ in (5.2) we get

$$s_j = \frac{\varphi_{0,j+1}(0) - \varphi_{0j}(0)}{\psi_0(0)}\,.$$

Since

$$\psi_0(0) = h_0^{-1/2}$$

[see (4.2)] and

$$\varphi_{0j}(0) = h_0^{1/2} \int_0^{\omega_j \infty} N\left(t + gu^2 + gR_1\right) e^{N\left(\frac{t}{2}u^2 + \frac{g}{4}u^4\right)}\, du$$

we obtain that

$$s_j = h_0 \int_{\omega_j \infty}^{\omega_{j+1}\infty} N\left(t + gu^2 + gR_1\right) e^{N\left(\frac{t}{2}u^2 + \frac{g}{4}u^4\right)}\, du, \qquad j = 1, 2, 3, 4. \tag{5.6}$$

The change of variable $u \to -u$ gives

$$s_3 = -s_1, \qquad s_4 = -s_2. \tag{5.7}$$

Another way to compute s_j is to use the Cauchy type integral.

The function

$$y_{nj}(z) = e^{-\frac{NV(z)}{2}} \varphi_{nj}(z), \qquad j = 1, 2, 3, 4, \tag{5.8}$$

is an entire function of z and in Σ_j it has the asymptotics

$$y_{nj}(z) = h_n^{1/2} z^{-n-1} \left(1 + \sum_{k=1}^{\infty} \frac{\gamma_{2k}}{z^{2k}}\right) \tag{5.9}$$

In addition,

$$y_{n,j+1}(z) = y_{nj}(z) + s_j e^{\frac{-NV(z)}{2}} \psi_n(z). \tag{5.10}$$

Observe that the domain Σ_j contains the j-th quadrant,

$$\Delta_j = \left\{ z\colon \frac{(j-1)\pi}{2} \le \arg z \le \frac{j\pi}{2} \right\},$$

hence (5.9) holds in Δ_j. This allows us to solve (5.10) with the help of the Cauchy type integral. Namely,

$$\begin{aligned} y_n(z) &= \frac{s_4}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{NV(u)}{2}} \psi_n(u)}{u - z}\, du + \frac{s_1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-\frac{NV(u)}{2}} \psi_n(u)}{u - z}\, du \\ &= \frac{s_4}{2\pi i} \int_{-\infty}^{\infty} \frac{h_n^{-1/2} P_n(u) e^{-NV(u)}}{u - z}\, du + \frac{s_1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{h_n^{-1/2} P_n(u) e^{-NV(u)}}{u - z}\, du \end{aligned} \tag{5.11}$$

where $y_n(z)$ is a piece-wise analytic function which coincides with $y_{nj}(z)$ in the quadrant Δ_j. Expanding

$$\frac{1}{u-z} = -\frac{1}{z}\sum_{k=0}^{\infty}\frac{u^k}{z^k},$$

we obtain from (5.11) the asymptotic expansion of $y_n(z)$ as $z \to \infty$,

$$y_n(z) \sim \frac{1}{z}\sum_{k=0}^{\infty}\frac{s_4 a_{nk} + s_1 b_{nk}}{z^k}, \tag{5.12}$$

where

$$\begin{aligned} a_{nk} &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} h_n^{-1/2} P_n(u) u^k e^{-NV(u)}\, du \\ b_{nk} &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h_n^{-1/2} P_n(u) u^k e^{-NV(u)}\, du \end{aligned} \tag{5.13}$$

Since by (5.9), $y_n(z) = O(z^{-n-1})$,

$$s_4 a_{nk} + s_1 b_{nk} = 0, \qquad k = 0, 1, \ldots, n-1,$$

but $a_{nk} = 0$ for $k = 0, 1, \ldots, n-1$ in virtue of the orthogonality property (1.3), hence $s_1 b_{nk} = 0$ for these k's. Let us take $n = 2$ and $k = 0$. In this case

$$i\int_{-i\infty}^{i\infty} P_2(u) e^{-NV(u)}\, du = -\int_{-\infty}^{\infty} P_2(iu) e^{-NV(iu)}\, du$$

is obviously positive hence $b_{20} \neq 0$. This implies

$$s_1 = s_3 = 0. \tag{5.14}$$

By (5.2) this means that

$$\varphi_{n2}(z) = \varphi_{n1}(z), \qquad \varphi_{n4}(z) = \varphi_{n3}(z).$$

Let us find s_4. For the sake of simplicity we redenote it by s. ¿From (5.9) we know that

$$y_0(z) = h_0^{1/2}(z^{-1} + \ldots),$$

hence by (5.12), $s a_{00} = h_0^{1/2}$. In addition, by (5.13),

$$a_{00} = -\frac{1}{2\pi i}\, h_0^{1/2}.$$

This gives

$$s = -2\pi i. \tag{5.15}$$

Now we can formulate the Riemann–Hilbert problem. Define

$$\Psi_{n+}(z) = \begin{pmatrix} \psi_n(z) & \varphi_{n1}(z) \\ \psi_{n-1}(z) & \varphi_{n-1,1}(z) \end{pmatrix}$$

and

$$\Psi_{n-}(z) = \begin{pmatrix} \psi_n(z) & \varphi_{n3}(z) \\ \psi_{n-1}(z) & \varphi_{n-1,3}(z) \end{pmatrix}.$$

Let

$$\Psi_n(z) = \begin{cases} \Psi_{n+}(z), & \text{if} \quad \operatorname{Im} z \geq 0, \\ \Psi_{n-}(z), & \text{if} \quad \operatorname{Im} z \leq 0. \end{cases}$$

Then $\Psi_n(z)$ has the asymptotic expansion

$$\Psi_n(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k}{z^k} \right) e^{-\left(\frac{NV(z)}{2} - n \ln z + \lambda_n\right)\sigma_3}, \qquad z \to \infty, \tag{5.16}$$

where $\lambda_n = \frac{1}{2} \ln h_n$ and

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & R_n^{-1/2} \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ R_n^{1/2} & 0 \end{pmatrix}. \tag{5.17}$$

On the real line

$$\Psi_{n+}(z) = \Psi_{n-}(z) S, \qquad \operatorname{Im} z = 0, \tag{5.18}$$

where

$$S = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix}. \tag{5.19}$$

In addition,

$$\det \Psi_n(z) = R_n^{-1/2}. \tag{5.20}$$

It must be emphasized that in the setting of the Riemann-Hilbert problem (5.16-18) the real quantities R_n and λ_n *are not the given data.* They are evaluated via the solution $\Psi_n(z)$. Indeed, suppose that $\tilde{\Psi}_n(z)$ is another function satisfying (5.16-18) with perhaps another $\tilde{R}_n$ and another $\tilde{\lambda}_n$. Consider the matrix ratio:

$$X^*(z) = [e^{\lambda_n \sigma_3} \Gamma_0^{-1} \Psi_n(z)][e^{\tilde{\lambda}_n \sigma_3} \tilde{\Gamma}_0^{-1} \tilde{\Psi}_n(z)]^{-1}.$$

Since the jump matrix S for the both $\Psi_n(z)$ and $\tilde{\Psi}_n(z)$ is the same, function $X^*(z)$ has no jump across the real line. Therefore, it is an entire function equals I at $z = \infty$. Hence

$$X^*(z) \equiv I,$$

or

$$[e^{\lambda_n \sigma_3}\Gamma_0^{-1}\Psi_n(z)][e^{\tilde{\lambda}_n \sigma_3}\tilde{\Gamma}_0^{-1}\tilde{\Psi}_n(z)]^{-1} \equiv I.$$

Substituting into this identity asymptotic expansions (5.16) and equating the terms of order z^{-1} we come up with the matrix equation,

$$e^{\lambda_n \sigma_3}\Gamma_0^{-1}\Gamma_1 e^{-\lambda_n \sigma_3} = e^{\tilde{\lambda}_n \sigma_3}\tilde{\Gamma}_0^{-1}\tilde{\Gamma}_1 e^{-\tilde{\lambda}_n \sigma_3},$$

whose (12) and (21) entries imply

$$\lambda_n = \tilde{\lambda}_n \qquad \text{and} \qquad R_n = \tilde{R}_n.$$

6. Formal Asymptotic Expansion for R_n

We expect that

$$\lim_{n,N\to\infty:\ n/N\to\lambda} R_n = \begin{cases} R(\lambda) & \text{if} \quad n = 2k+1, \\ L(\lambda) & \text{if} \quad n = 2k. \end{cases} \tag{6.1}$$

¿From (1.7) we get the equations

$$\begin{aligned} \lambda &= R[t + g(2L + R)], \\ \lambda &= L[t + g(2R + L)]. \end{aligned} \tag{6.2}$$

Equating the expressions on the right we obtain that

$$(R - L)\,[t + g(R + L)] = 0.$$

We assume that $R \neq L$ hence

$$t + g(R + L) = 0. \tag{6.3}$$

Combining this with (6.2) we obtain

$$\lambda = gRL, \tag{6.4}$$

so that R, L are solutions of the quadratic equation

$$u^2 + \frac{t}{g}u + \frac{\lambda}{g} = 0, \tag{6.5}$$

which are

$$R, L = \frac{-t \pm \sqrt{t^2 - 4\lambda g}}{2g} \tag{6.6}$$

We can find an asymptotic expansion of R_n in powers of N^{-2}. Put

$$R_n = \begin{cases} R(n/N) & \text{if} \quad n = 2k+1, \\ L(n/N) & \text{if} \quad n = 2k. \end{cases}$$

Then (1.7) is equivalent to

$$\begin{aligned} \lambda &= R(t + gR + 2gL + N^{-2} g\Delta L), \\ \lambda &= L(t + gL + 2gR + N^{-2} g\Delta R), \end{aligned} \tag{6.7}$$

where

$$\Delta f(\lambda) = \frac{f(\lambda - \frac{1}{N}) - 2f(\lambda) + f(\lambda + \frac{1}{N})}{(1/N)^2}.$$

Let us substitute the expansions

$$L(\lambda) = L_0(\lambda) + N^{-2} L_1(\lambda) + \ldots; \qquad R(\lambda) = R_0(\lambda) + N^{-2} R_1(\lambda) + \ldots$$

into (6.7) and equate coefficients at powers of N^{-2}. Then the equations on $L_0(\lambda), R_0(\lambda)$ coincide with (6.2), hence $L_0(\lambda), R_0(\lambda)$ are given by (6.6). Equating coefficients at N^{-2} we obtain the system of equations

$$\begin{cases} (R_0 + L_0)R_1 + 2R_0 L_1 = -R_0 \Delta L_0, \\ 2L_0 R_1 + (R_0 + L_0)L_1 = -L_0 \Delta R_0. \end{cases} \tag{6.8}$$

Solving this system we find R_1, L_1 and so on. In what follows the quantity

$$\theta_n = t + gR_n + gR_{n+1}$$

plays an important role. It follows from the asymptotic formula for R_n that

$$\theta_n = \frac{(-1)^{n+1} g}{N \left(t^2 - \frac{4gn}{N}\right)^{1/2}} + O(N^{-2}). \tag{6.9}$$

7. The Bohr–Sommerfeld Quantization Condition

To find a semiclassical formula for $\psi_n(z)$ we use the Schrödinger equation

$$-\zeta_n''(z)+N^2U(z)\zeta_n(z)=0, \tag{7.1}$$

where

$$\psi_n(z)=\left(z^2+\frac{\theta_n}{g}\right)^{1/2}\zeta_n(z), \tag{7.2}$$

(see (3.26) and (3.32)). By (3.29),

$$\begin{aligned} U(z)=&\left[\frac{g^2z^6}{4}+\frac{tgz^4}{2}+\left(\frac{t^2}{4}-\frac{n}{N}g\right)z^2-R_n\theta_{n-1}\theta_n\right]\\ &-N^{-1}\left[\frac{t}{2}+\frac{3gz^2}{2}+gR_n-\frac{gz^2(t+gz^2+2gR_n)}{gz^2+\theta_n}\right]+N^{-2}\left[\frac{g(2gz^2-\theta_n)}{(gz^2+\theta_n)^2}\right]. \end{aligned} \tag{7.3}$$

Turning points for (7.1) are to be found as real zeros of $U(z)$. To simplify calculations we make the assumption that there exists $C>0$ such that

$$|\theta_n|\le CN^{-1}. \tag{7.4}$$

This assumption is motivated by the equation (6.9) and will be justified later. Neglecting terms of the order of N^{-1} we derive from (7.3) the following equation on positive turning points $z_2>z_1>0$:

$$(gz^2+t)^2-4\lambda g=0,\qquad \lambda=\frac{n}{N},$$

which gives

$$z_{1,2}=\left(\frac{-t\mp 2\sqrt{\lambda g}}{g}\right)^{1/2}+O(N^{-1}).$$

The condition

$$0<\frac{n}{N}<\lambda_{\rm cr}=\frac{t^2}{4g}$$

guarantees that the zeros $z_{1,2}$ are real. A semiclassical solution to the Schrödinger equation (7.1) is

$$\zeta_n(z)=\begin{cases}\dfrac{C}{\sqrt[4]{U(z)}}\exp\left[-N\displaystyle\int_{z_2}^{z}\sqrt{U(v)}\,dv\right], & \text{if}\quad z>z_2+\varepsilon,\\ \dfrac{2C}{\sqrt[4]{-U(z)}}\cos\left[N\displaystyle\int_{z_2}^{z}\sqrt{-U(v)}\,dv+\frac{\pi}{4}\right], & \text{if}\quad z_1+\varepsilon<z<z_2-\varepsilon,\end{cases} \tag{7.5}$$

The orthogonal polynomial $P_n(z)$ is either an even function or an odd one, and it has n real zeros. If $n = 2k$ or $2k+1$, then k of these zeros are positive, k negative and one zero is equal to 0 if n is odd. The Bohr–Sommerfeld quantization rule is then

$$N \int_{z_1}^{z_2} \sqrt{-U(z)}\, dz = \pi \left(k + \frac{1}{2}\right), \qquad k = \left[\frac{n}{2}\right]. \tag{7.6}$$

This rule comes from the fact that the function $\psi_n(z)$ has k zeros on the interval $[z_1, z_2]$ and it decays in a neighborhood of the points z_1, z_2 outside of this interval. To evaluate the integral on the left in (7.6) we rewrite $U(z)$ as

$$U(z) = U_0(z) + U_1 + U_2(z),$$

where

$$\begin{aligned} U_0(z) &= \frac{g^2 z^6}{4} + \frac{tgz^4}{2} + \left(\frac{t^2}{4} - \lambda' g\right) z^2, \qquad \lambda' = \frac{n+\frac{1}{2}}{N}, \\ U_1 &= N^{-1}\left(\frac{t}{2} + gR_n\right) \end{aligned} \tag{7.7}$$

and

$$U_2(z) = -R_n\theta_{n-1}\theta_n - N^{-1}\left[\frac{\theta_n(t + gz^2 + 2gR_n)}{gz^2 + \theta_n}\right] + N^{-2}\left[\frac{g(2gz^2 - \theta_n)}{(gz^2+\theta_n)^2}\right]. \tag{7.8}$$

Our assumption (7.4) shows that $U_2(z)$ is of the order of N^{-2}, and we will neglect this term in the calculation of the integral on the left in (7.6). The term U_1 is of the order of N^{-1} and we will consider it as a small perturbation of $U_0(z)$. Let γ be a contour in a complex plane enclosing the interval $[z_1, z_2]$. Then

$$\int_{z_1}^{z_2} \sqrt{-U(z)}\, dz = \frac{1}{2}\oint_\gamma \sqrt{-U(z)}\, dz = \frac{1}{2}\oint_\gamma \sqrt{-U_0(z)}\, dz - \frac{1}{4}\oint_\gamma \frac{U_1}{\sqrt{-U(z)}}\, dz + O(N^{-2}).$$

Let z_{10} and z_{20} be zeros of $U_0(z)$. Then

$$\begin{aligned} \frac{1}{2}\oint_\gamma \sqrt{-U_0(z)}\, dz &= \int_{z_{10}}^{z_{20}} \sqrt{-U_0(z)}\, dz = \int_{z_{10}}^{z_{20}} (z/2)\sqrt{4\lambda' g - (gz^2+t)^2}\, dz \\ &= \frac{\lambda'}{2}\left(x\sqrt{1-x^2} + \arcsin x\right)\Big|_{-1}^{1}, \qquad x = \frac{t + gz^2}{2(\lambda' g)^{1/2}}, \end{aligned} \tag{7.9}$$

which gives

$$\frac{1}{2}\oint_\gamma \sqrt{-U_0(z)}\, dz = \frac{\pi\lambda'}{2}. \tag{7.10}$$

Similarly,

$$\frac{1}{2}\oint_\gamma \frac{dz}{\sqrt{-U_0(z)}} = \int_{z_{10}}^{z_{20}} \frac{dz}{\sqrt{-U_0(z)}} = \int_{z_{10}}^{z_{20}} \frac{2dz}{z\sqrt{4\lambda' g-(gz^2+t)^2}}\,dz$$
$$= \frac{1}{\sqrt{t^2-4\lambda' g}} \arcsin\left(\frac{2\sqrt{\lambda' g}-tx}{2\sqrt{\lambda' g}\,x-t}\right)\Bigg|_{-1}^{1} = \frac{\pi}{\sqrt{t^2-4\lambda' g}}, \tag{7.11}$$

¿From (7.10) and (7.11) we obtain that modulo terms of the order of N^{-1},

$$N\int_{z_1}^{z_2}\sqrt{-U(z)}\,dz = \frac{\pi\left(n+\frac{1}{2}\right)}{2} - \frac{\left(\frac{t}{2}+gR_n\right)\pi}{2\sqrt{t^2-4\lambda' g}} \tag{7.12}$$

and hence by (7.5)

$$\pi\left(k+\frac{1}{2}\right) = \frac{\pi\left(n+\frac{1}{2}\right)}{2} - \frac{\left(\frac{t}{2}+gR_n\right)\pi}{2\sqrt{t^2-4\lambda' g}} \tag{7.13}$$

This gives

$$R_n = \frac{-t-(-1)^n\sqrt{t^2-4\lambda' g}}{2g} + O(N^{-1}), \tag{7.14}$$

or replacing λ' for λ,

$$R_n = \frac{-t-(-1)^n\sqrt{t^2-4\lambda g}}{2g} + O(N^{-1}), \qquad \lambda = \frac{n}{N}. \tag{7.15}$$

The error term in this formula is uniform in λ in the interval $0 \le \lambda \le \lambda_{\rm cr} - \varepsilon$.

Let us calculate the semiclassical formula for $\psi_n(z)$. Assume first that $z > z_2+\varepsilon$, $\varepsilon > 0$. Then modulo terms of the order of N^{-1},

$$\zeta_n(z) = \frac{C}{\sqrt[4]{U_0(z)}} \exp\left\{-N\int_{z_{20}}^{z}\left[\sqrt{U_0(v)} + \frac{U_1}{2\sqrt{U_0(v)}}\right]dv\right\}. \tag{7.16}$$

Observe that

$$\int_{z_{20}}^{z}\sqrt{U_0(v)}\,dv = \int_{z_{20}}^{z}\frac{v}{2}\sqrt{(gv^2+t)^2-4\lambda' g}\,dv = \int_1^x \lambda'\sqrt{u^2-1}\,du$$
$$= \frac{\lambda'}{2}\left[x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right], \qquad x = \frac{gz^2+t}{2\sqrt{\lambda' g}}. \tag{7.17}$$

and

$$\int_{z_{20}}^{z}\frac{dv}{\sqrt{U_0(v)}} = \int_{z_{20}}^{z}\frac{2vdv}{v^2\sqrt{(gv^2+t)^2-4\lambda' g}} = \int_1^x \frac{du}{(2\sqrt{\lambda' g}\,u-t)\sqrt{u^2-1}}$$
$$= \frac{\ln(y+\sqrt{y^2-1})}{\sqrt{t^2-4\lambda' g}}, \qquad y = \frac{2\sqrt{\lambda' g}-tx}{2\sqrt{\lambda' g}\,x-t}.$$

In addition, by (7.14),

$$\frac{U_1}{2\sqrt{t^2-4\lambda' g}} = \frac{N^{-1}\left(\frac{t}{2}+gR_n\right)}{2\sqrt{t^2-4\lambda' g}} = -\frac{N^{-1}(-1)^n}{4}\,.$$

This gives

$$\begin{aligned}\int_{z_2}^{z} \sqrt{U(v)}\,dv = \frac{\lambda'}{2}\left[x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right] \\ - \frac{N^{-1}(-1)^n}{4}\ln\left(y+\sqrt{y^2-1}\right) + O(N^{-1}|z|^{-2}), \qquad z > z_2+\varepsilon.\end{aligned} \tag{7.18}$$

In addition,

$$\sqrt[4]{U_0(z)} = \sqrt{z/2}\,\sqrt[4]{(gz^2+t)^2-4\lambda' g} = \sqrt[4]{\lambda' g}\,\sqrt{z}\,\sqrt[4]{x^2-1},$$

hence

$$\begin{aligned}\zeta_n(z) = \frac{C_n}{\sqrt{z}\,\sqrt[4]{(x^2-1)}}\exp\Bigg\{-\frac{\left(n+\frac{1}{2}\right)}{2}\left[x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right] \\ + \frac{(-1)^n}{4}\ln\left(y+\sqrt{y^2-1}\right) + O(N^{-1}|z|^{-2})\Bigg\}.\end{aligned} \tag{7.19}$$

where

$$x = \frac{gz^2+t}{2\sqrt{\lambda' g}}\,, \qquad y = \frac{2\sqrt{\lambda' g}-tx}{2\sqrt{\lambda' g}\,x-t} = \frac{-tgz^2-t^2+4\lambda' g}{2\sqrt{\lambda' g}\,gz^2}\,, \qquad \lambda' = \frac{n+\frac{1}{2}}{N}\,. \tag{7.20}$$

In virtue of (7.4), the substitution (7.2) gives

$$\psi_n(z) = z\zeta_n(z)\,[1+O(N^{-1}|z|^{-2})], \tag{7.21}$$

hence

$$\begin{aligned}\psi_n(z) = \frac{C_n\sqrt{z}}{\sqrt[4]{x^2-1}}\exp\Bigg\{-\frac{\left(n+\frac{1}{2}\right)}{2}\left[x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right] \\ + \frac{(-1)^n}{4}\ln\left(y+\sqrt{y^2-1}\right) + O(N^{-1}|z|^{-2})\Bigg\}.\end{aligned} \tag{7.22}$$

In the interval $z_1+\varepsilon < z < z_2-\varepsilon,\ \varepsilon > 0$, the semiclassical solution is

$$\psi_n(z) = \frac{2C_n\sqrt{z}}{\sqrt{\sin\phi}}\cos\left\{\frac{\left(n+\frac{1}{2}\right)}{2}\left[\frac{\sin(2\phi)}{2}-\phi\right] + \frac{\pi-(-1)^n\chi}{4} + O(N^{-1})\right\}, \tag{7.23}$$

where

$$\phi = \arccos x, \qquad \chi = \arccos y,$$

and x, y are defined in (7.20). The error term $O(N^{-1})$ in (7.22) and (7.23) is uniform in n assuming that for some $\varepsilon > 0$,

$$|z - z_1|, \ |z - z_2| \geq \varepsilon.$$

The formula (7.22) can be written also in the form

$$\psi_n(z) = \frac{C_n\sqrt{z}}{\sqrt{\sinh\phi}} \exp\left\{-\frac{\left(n+\frac{1}{2}\right)}{2}\left[\frac{\sinh(2\phi)}{2} - \phi\right] + \frac{(-1)^n\chi}{4} + O(N^{-1})\right\}, \tag{7.24}$$

where

$$\phi = \cosh^{-1} x, \qquad \chi = \cosh^{-1} y.$$

In this form the formulae (7.23) and (7.24) are similar to the classical Plancherel–Rotach formulae for the Hermite polynomials (see [PR] and [Sz]).

In the interval $0 \leq z \leq z_1 - \varepsilon$ the semiclassical solution is

$$\begin{aligned}\psi_n(z) = \frac{C_n\sqrt{z}}{\sqrt[4]{(x^2-1)}} \exp\Bigg\{&-\frac{\left(n+\frac{1}{2}\right)}{2}\left[|x|\sqrt{x^2-1} - \ln\left(|x| + \sqrt{x^2-1}\right)\right] \\ &+ \frac{(-1)^n}{4}\ln\left(|y| + \sqrt{y^2-1}\right) + O(N^{-1})\Bigg\},\end{aligned} \tag{7.25}$$

where x and y are defined in (7.20). This formula coincides with (7.22) when $z > z_2$, so it can be used both when $0 \leq z \leq z_1 - \varepsilon$ and $z > z_2 + \varepsilon$. It can be rewritten in the form (7.24) as well, with

$$\phi = \cosh^{-1}|x|, \qquad \chi = \cosh^{-1}|y|. \tag{7.26}$$

A formula combining (7.23) with (7.24) can be obtained with the help of the Airy function (see next section).

8. Semiclassical Approximation Near Turning Point

To construct a semiclassical approximate solution to the Schrödinger equation (7.1) near the turning point z_2, we are looking for the solution in the form (cf. [Ble])

$$\zeta_n(z) = \frac{C}{\sqrt{\phi'(z)}} \operatorname{Ai}\left(N^{2/3}\phi(z)\right). \tag{8.1}$$

where $\operatorname{Ai}(z)$ is the Airy function which satisfies the model equation $\operatorname{Ai}''(z) - z \operatorname{Ai}(z) = 0$. Then (7.1) reduces to the equation

$$(\phi')^2\phi = U + \frac{1}{N^2}\left(\frac{\phi'''}{2\phi'} - \frac{3\phi''}{4(\phi')^2}\right).$$

Now we solve this equation iteratively. In the zeroth order we get

$$\phi(z) = \left[\frac{3}{2}\int_{z_2}^{z} \sqrt{U(v)}dv\right]^{2/3}.$$

The last formula defines a function $\phi(z)$ analytic at $z = z_2$ with

$$\phi'(z_2) = [U'(z_2)]^{1/3} > 0. \tag{8.2}$$

The Airy function can be written as

$$\operatorname{Ai}(-z) = \frac{1}{\sqrt{\pi\xi'(z)}} \cos\left[\xi(z) - \frac{\pi}{4}\right], \qquad z \geq 0, \tag{8.3}$$

and

$$\operatorname{Ai}(z) = \frac{1}{2\sqrt{\pi\eta'(z)}} \exp[-\eta(z)], \qquad z \geq 0, \tag{8.4}$$

where $\xi(z)$ and $\eta(z)$ are analytic functions with the asymptotics

$$\begin{aligned} \xi(z) &\sim \frac{2}{3}z^{3/2}\left(1 + \frac{5}{32}z^{-3} + \dots\right) = \frac{2}{3}z^{3/2}\left(1 + \sum_{j=1}^{\infty}\alpha_j z^{-3j}\right), \\ \eta(z) &\sim \frac{2}{3}z^{3/2}\left(1 - \frac{5}{32}z^{-3} + \dots\right) = \frac{2}{3}z^{3/2}\left(1 + \sum_{j=1}^{\infty}(-1)^j\alpha_j z^{-3j}\right), \qquad z \to \infty. \end{aligned} \tag{8.5}$$

Combining (8.1) with (8.4) we obtain that

$$\begin{aligned} \zeta_n(z) &= \frac{C}{2\sqrt{\pi\eta'(N^{2/3}\phi(z))\phi'(z)}} \exp\left[-\eta(N^{2/3}\phi(z))\right] \\ &= \frac{CN^{-1/6}}{2\sqrt{\pi\Phi'(z)}} \exp[-N\Phi(z)], \end{aligned} \tag{8.6}$$

where

$$\Phi(z)=N^{-1}\eta(N^{2/3}\phi(z)). \tag{8.7}$$

Assume that $z\ge z_2+\varepsilon>0$. Then from (8.5),

$$\Phi(z)=\frac{2}{3}\,\phi(z)^{3/2}+O(N^{-2})=\int_{z_2}^{z}\sqrt{U(v)}\,dv+O(N^{-2}). \tag{8.8}$$

Hence

$$\zeta_n(z)=\frac{CN^{-1/6}}{2\sqrt{\pi\sqrt{U(z)}}}\exp\left[-N\int_{z_2}^{z}\sqrt{U(v)}\,dv+O(N^{-1})\right]. \tag{8.9}$$

(cf. (7.3)). Combining this formula with (7.19) we obtain an asymptotics of $\zeta_n(z)$ as $z\to\infty$:

$$\begin{aligned}\zeta_n(z)=\frac{CN^{-1/6}(1+\delta_n)}{2\sqrt{\pi\sqrt{U(z)}}}\exp\Big\{&-\frac{(n+\frac{1}{2})}{2}\left[x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right]\\&+\frac{(-1)^n}{4}\ln\left(y+\sqrt{y^2-1}\right)+O(N^{-1}|z|^{-2})\Big\},\end{aligned} \tag{8.10}$$

with some $\delta_n=O(N^{-2})$, which does not depend on z, and

$$x=\frac{gz^2+t}{2\sqrt{\lambda' g}}\,,\qquad y=\frac{2\sqrt{\lambda' g}-tx}{2\sqrt{\lambda' g}\,x-t}\,,\qquad \lambda'=\frac{n+\frac{1}{2}}{N}$$

(cf. (7.18)), hence by (7.21),

$$\begin{aligned}\psi_n(z)=\frac{CN^{-1/6}(1+\delta_n)z}{2\sqrt{\pi\sqrt{U(z)}}}\exp\Big\{&-\frac{(n+\frac{1}{2})}{2}\left[x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right]\\&+\frac{(-1)^n}{4}\ln\left(y+\sqrt{y^2-1}\right)+O(N^{-1}|z|^{-2})\Big\}.\end{aligned} \tag{8.11}$$

As $z\to\infty$,

$$\begin{aligned}U(z)&=\frac{g^2z^6}{4}\,(1+O(|z|^{-2})),\\x\sqrt{x^2-1}&=x^2-\frac{1}{2}+\cdots=\frac{(gz^2+t)^2}{4\lambda' g}-\frac{1}{2}+O(|z|^{-2}),\\\ln\left(x+\sqrt{x^2-1}\right)&=\ln(2x)+O(|z|^{-2})=2\ln z+\frac{1}{2}\,\ln\frac{g}{\lambda'}+O(|z|^{-2}),\\\ln\left(y+\sqrt{y^2-1}\right)&=\ln\left(\frac{-t+\sqrt{t^2-4\lambda' g}}{2\sqrt{\lambda' g}}\right)+O(|z|^{-2}),\end{aligned} \tag{8.12}$$

hence

$$-\frac{\left(n+\frac{1}{2}\right)}{2}\,x\sqrt{x^2-1}=-\frac{N(gz^2+t)^2}{8g}+\frac{N\lambda'}{4}+O(N|z|^{-2})$$
$$=-\frac{NV(z)}{2}-\frac{Nt^2}{8g}+\frac{N\lambda'}{4}+O(N|z|^{-2}), \tag{8.13}$$
$$\frac{\left(n+\frac{1}{2}\right)}{2}\ln\left(x+\sqrt{x^2-1}\right)=\left(n+\frac{1}{2}\right)\ln z+\frac{N\lambda'}{4}\ln\frac{g}{\lambda'}+O(N|z|^{-2}),$$

and (8.11) gives the asymptotics

$$\psi_n(z)=\frac{CN^{-1/6}(1+\delta_n)}{\sqrt{2\pi g}}\exp\left[-\frac{NV(z)}{2}+n\ln z+\gamma_n+O(N|z|^{-2})\right], \qquad z\to+\infty, \tag{8.14}$$

where

$$\gamma_n=-\frac{Nt^2}{8g}+\frac{N\lambda'}{4}\left(1+\ln\frac{g}{\lambda'}\right)+\frac{(-1)^n}{4}\,\ln\left(\frac{-t+\sqrt{t^2-4\lambda' g}}{2\sqrt{\lambda' g}}\right). \tag{8.15}$$

By (3.1),

$$\psi_n(z)=h_n^{-1/2}\exp\left[-\frac{NV(z)}{2}+n\ln z+O(|z|^{-2})\right].$$

Comparing this with (8.13) we get

$$C=h_n^{-1/2}\sqrt{2\pi g}N^{1/6}(1+\delta_n)^{-1}\exp(-\gamma_n). \tag{8.16}$$

By (8.1) and (8.2),

$$\psi_n(z_2)=\frac{C(z_2^2+\theta_{n-1})^{1/2}}{|U'(z_2)|^{1/6}}\,\mathrm{Ai}\,(0)\,, \tag{8.17}$$

hence

$$\psi_n(z_2)=h_n^{-1/2}\sqrt{2\pi g}N^{1/6}(1+\delta_n)^{-1}\exp(-\gamma_n)\,\frac{(z_2^2+\theta_{n-1})^{1/2}}{|U'(z_2)|^{1/6}}\,\mathrm{Ai}\,(0)\,. \tag{8.18}$$

Our next step is to get a similar connection formula for $\varphi_n(z)$.

9. Connection Formula Between Turning Point and Infinity

To compute the connection formula for $\varphi_n(z)$ we construct a semiclassical approximation of this function near z_2. Let

$$\alpha = e^{2\pi i/3}. \tag{9.1}$$

Consider the following semiclassical approximate solution to (7.1):

$$\zeta_n(z) = \frac{C'}{\sqrt{\phi'(z)}} \operatorname{Ai}\left(N^{2/3}\alpha^{-1}\phi(z)\right), \tag{9.2}$$

where

$$\phi(z) = \left[\frac{3}{2}\int_{z_2}^{z} \sqrt{U(v)}dv\right]^{2/3}. \tag{9.3}$$

Let L_1 be a curve starting at z_2, such that

$$L_1 = \{z\colon \alpha^{-1}\phi(z) \geq 0\}. \tag{9.4}$$

Since $\phi(z)$ is analytic at z_2 and $\phi'(z_2) > 0$ [see (8.2)], the tangent line to L_1 at z_2 forms an angle $2\pi/3$ with the positive half-axis. By (8.4) on L_1,

$$\begin{aligned}\zeta_n(z) &= \frac{C'}{2\sqrt{\pi\eta'(N^{2/3}\alpha^{-1}\phi(z))\phi'(z)}} \exp[-\eta(N^{2/3}\alpha^{-1}\phi(z))] \\ &= \frac{C'N^{-1/6}}{2\sqrt{\pi\alpha\Phi'(z)}} \exp[-N\Phi(z)],\end{aligned} \tag{9.5}$$

where

$$\Phi(z) = N^{-1}\eta(N^{2/3}\alpha^{-1}\phi(z)). \tag{9.6}$$

Applying the asymptotics (8.5) of $\eta(z)$, we get that on S_1,

$$\Phi(z) = -\frac{2}{3}\,\phi(z)^{3/2} + O(N^{-2}) = -\int_{z_2}^{z} \sqrt{U(v)}\,dv + O(N^{-2}). \tag{9.7}$$

Hence

$$\begin{aligned}\zeta_n(z) &= \frac{C'N^{-1/6}}{2\sqrt{\pi(-\alpha)\sqrt{U(z)}}} \exp\left[N\int_{z_2}^{z} \sqrt{U(v)}\,dv + O(N^{-2})\right] \\ &= \frac{C'N^{-1/6}e^{\pi i/6}}{2\sqrt{\pi\sqrt{U(z)}}} \exp\left[N\int_{z_2}^{z} \sqrt{U(v)}\,dv + O(N^{-2})\right]\end{aligned} \tag{9.8}$$

.

[cf. (7.3)]. Combining this formula with (7.19) we obtain an asymptotics of $\zeta_n(z)$ on S_1:

$$\begin{aligned}\zeta_n(z) = \frac{C' N^{-1/6} e^{\pi i/6}(1+\delta_n')}{2\sqrt{\pi\sqrt{U(z)}}} \exp\Bigg\{ \frac{(n+\frac{1}{2})}{2}\left[x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right] \\ - \frac{(-1)^n}{4}\ln\left(y+\sqrt{y^2-1}\right) + O(N^{-1}|z|^{-2})\Bigg\}.\end{aligned} \tag{9.9}$$

At infinity L_1 is approaching the ray $\arg z = \pi/4$. From (9.9) and (8.12) it follows that

$$\begin{aligned}\varphi_n(z) = \frac{C' N^{-1/6} e^{\pi i/6}(1+\delta_n')}{\sqrt{2\pi g}} \exp\left[\frac{NV(z)}{2} - (n+1)\ln z - \gamma_n + O(N|z|^{-2})\right], \\ z\to\infty, \qquad z\in L_1,\end{aligned} \tag{9.10}$$

By (4.9),

$$\varphi_n(z) = h_n^{1/2}\exp\left[\frac{NV(z)}{2} - (n+1)\ln z + O(|z|^{-2})\right].$$

Comparing this with (9.10) we get

$$C' = h_n^{1/2}\sqrt{2\pi g} N^{1/6} e^{-\pi i/6}(1+\delta_n')^{-1}\exp(\gamma_n). \tag{9.11}$$

By (9.2),

$$\varphi_n(z_2) = \frac{C'(z_2^2+\theta_{n-1})^{1/2}}{|U'(z_2)|^{1/6}}\,\mathrm{Ai}\,(0)\,, \tag{9.12}$$

hence

$$\varphi_n(z_2) = h_n^{1/2}\sqrt{2\pi g} N^{1/6} e^{-\pi i/6}(1+\delta_n')^{-1}\exp(\gamma_n)\frac{(z_2^2+\theta_{n-1})^{1/2}}{|U'(z_2)|^{1/6}}\,\mathrm{Ai}\,(0)\,. \tag{9.13}$$

Let us compare this expression with (8.18). By the equation (5.18) of the Riemann–Hilbert problem,

$$\varphi_n(z) - \overline{\varphi_n(z)} = -2\pi i\,\psi_n(z), \qquad \mathrm{Im}\, z = 0, \tag{9.14}$$

hence taking $z = z_2$ we derive from (8.18) and (9.13) the following equation on h_n:

$$h_n = 2\pi\,\exp\left[-2\gamma_n + O(N^{-1})\right]. \tag{9.15}$$

Substituting the value (8.15) of γ_n we get

$$h_n = 2\pi\,\exp\left[\frac{Nt^2}{4g} - \frac{N\lambda'}{2}\left(1+\ln\frac{g}{\lambda'}\right) - \frac{(-1)^n}{2}\,\ln\left(\frac{-t+\sqrt{t^2-4\lambda' g}}{2\sqrt{\lambda' g}}\right) + O(N^{-1})\right]. \tag{9.16}$$

Since

$$\frac{d}{d\lambda}\left[\lambda\left(1+\ln\frac{g}{\lambda}\right)\right]=\ln\frac{g}{\lambda},$$

we get that

$$\frac{N\lambda'}{2}\left(1+\ln\frac{g}{\lambda'}\right)=\frac{N\lambda}{2}\left(1+\ln\frac{g}{\lambda}\right)+\frac{1}{4}\,\ln\frac{g}{\lambda}+O(N^{-1}).$$

If n is odd, then

$$-\frac{1}{2}\,\ln\frac{g}{\lambda}+\ln\left(\frac{-t+\sqrt{t^2-4\lambda g}}{2\sqrt{\lambda g}}\right)=\ln\left(\frac{-t+\sqrt{t^2-4\lambda g}}{2g}\right)=R_n.$$

If n is even, then

$$\begin{aligned}-\frac{1}{2}\,\ln\frac{g}{\lambda}-\ln\left(\frac{-t+\sqrt{t^2-4\lambda g}}{2\sqrt{\lambda g}}\right)&=-\ln\left(\frac{-t+\sqrt{t^2-4\lambda g}}{2\lambda}\right)\\&=\ln\left(\frac{-t-\sqrt{t^2-4\lambda g}}{2g}\right)=R_n.\end{aligned}$$

In both cases this implies that

$$h_n=2\pi\sqrt{R_n}\exp\left[\frac{Nt^2}{4g}-\frac{N\lambda}{2}\left(1+\ln\frac{g}{\lambda}\right)+O(N^{-1})\right]. \tag{9.17}$$

¿From (8.16) and (9.15) we obtain that

$$C=N^{1/6}\sqrt{g}(1+O(N^{-1})), \tag{9.18}$$

hence by (8.1),

$$\psi_n(z)=\frac{D_n z}{\sqrt{\varphi_N(z)}}\;\mathrm{Ai}\left(N^{2/3}\varphi_N(z)+O(N^{-1})\right),$$

where

$$D_n=N^{1/6}\sqrt{g}\,(1+O(N^{-1}))$$

and $\varphi_N(z)$ is defined in (1.19). By (8.11) and (9.18),

$$\begin{aligned}\psi_n(z)=\frac{C_n\sqrt{z}}{\sqrt{x^2-1}}\;\exp\Bigg\{&-\frac{n+\frac{1}{2}}{2}\,\left[x\sqrt{x^2-1}-\ln(x+\sqrt{x^2-1})\right]\\&+\frac{(-1)^n}{4}\,\ln(y+\sqrt{y^2-1})+O(N^{-1}(1+|z|)^{-2})\Bigg\},\end{aligned}$$

where

$$C_n=\frac{1}{2\sqrt{\pi}}\,\left(\frac{g}{\lambda}\right)^{1/4}(1+O(N^{-1})).$$

This gives (1.17) and (1.21).

10. Proof of the Main Theorem: Asymptotic Riemann–Hilbert Problem

We start this part of the paper with the sketch (following [FIK2,4]) of the general monodromy theory for 2×2 matrix equation

$$\Psi'(z) = NA(z)\Psi(z) \tag{10.1}$$

with matrix $A(z)$ of the form (cf. (3.8))

$$A(z) = \begin{pmatrix} -(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n) & R_n^{1/2}(gz^2 + \theta_n) \\ -R_n^{1/2}(gz^2 + \theta_{n-1}) & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n \end{pmatrix}, \qquad \theta_n = t + gR_n + gR_{n+1}. \tag{10.2}$$

Quantities R_{n-1}, R_n, and R_{n+1} are *not* supposed to be necessarily related to any system of orthogonal polynomials. Now they are arbitrary real numbers satisfying only one condition, the Freud equation

$$\frac{n}{N} = R_n(t + gR_{n-1} + gR_n + gR_{n+1})\,, \tag{10.3}$$

where the integers n, N are fixed.

10.1. Direct Monodromy Problem

For the basic definitions and concepts related to the general monodromy theory of systems of ordinary differential equations with rational coefficients we refer the reader to the monograph [Sib] (see also [JMU]). Observe that $A(z)$ is a cubic polynomial in z and this implies the existence of some special solutions to (10.1). Namely, given equation (10.1) and an *arbitrary* real number λ_n, there exist eight canonical matrix solutions $\Psi_j(z)$, $j = 1, 2, ..., 8$, to (10.1), which are uniquely determined by the following asymptotic expansion at $z = \infty$:

$$\Psi_j(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k}{z^k}\right) e^{-\left(\frac{NV(z)}{2} - n\ln z + \lambda_n\right)\sigma_3}, \tag{10.4}$$

$$z \to \infty, \qquad \left|\arg z - \left(-\frac{\pi}{8} + \frac{\pi(j-1)}{4}\right)\right| < \frac{\pi}{4} - \varepsilon, \qquad \varepsilon > 0, \tag{10.5}$$

where as before

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad V(z) = \frac{tz^2}{2} + \frac{gz^4}{4},$$

and

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & R_n^{-1/2} \end{pmatrix}.$$

The pronounced statement follows from the general theory. Nevertheless, let us comment on $\Psi_j(z)$. To that end consider the vector equation

$$\vec{\Psi}'(z) = NA(z)\vec{\Psi}(z).$$

The claim is that, for a given λ_n and for a given $j = 1, 2, 3, 4$, this equation has a unique solution $\vec{\Psi}_j(z)$ which goes to zero as $z \to \infty$ along the ray $\arg z = \pi(j-1)/2$ and has the following asymptotics at $z = \infty$:

$$\vec{\Psi}_j(z) \sim \left(\sum_{k=0}^{\infty} \frac{\vec{\Gamma}_k}{z^k}\right) e^{-\left(\frac{NV(z)}{2} - n\ln z + \lambda_n\right)},$$

$$z \to \infty, \qquad \left|\arg z - \frac{\pi(j-1)}{2}\right| < \frac{3\pi}{8} - \varepsilon, \qquad \varepsilon > 0,$$

with

$$\vec{\Gamma}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In addition, there is another unique solution $\vec{\Phi}_j(z)$ which goes to zero as $z \to \infty$ along the ray $\arg z = (-\pi/4) + (\pi(j-1)/2)$ and which has the following asymptotics at $z = \infty$:

$$\vec{\Phi}_j(z) \sim \left(\sum_{k=0}^{\infty} \frac{\vec{\Theta}_k}{z^k}\right) e^{\frac{NV(z)}{2} - n\ln z + \lambda_n},$$

$$z \to \infty, \qquad \left|\arg z + \frac{\pi}{4} - \frac{\pi(j-1)}{2}\right| < \frac{3\pi}{8} - \varepsilon, \qquad \varepsilon > 0,$$

with

$$\vec{\Theta}_0 = \begin{pmatrix} 0 \\ R_n^{-1/2} \end{pmatrix}.$$

To get a matrix canonical solution we combine the vector solutions as follows:

$$\Psi_{2j-1}(z) = \left(\vec{\Psi}_j(z), \vec{\Phi}_j(z)\right), \qquad \Psi_{2j}(z) = \left(\vec{\Psi}_j(z), \vec{\Phi}_{j+1}(z)\right), \qquad j = 1, 2, 3, 4,$$

and this produces the matrix solutions $\Psi_j(z)$ satisfying (10.4). We will discuss semiclassical asymptotics for $\Psi_j(z)$ as $N \to \infty$ in the section 10.4. It is worth mentioning that the existence of the canonical solutions $\Psi_j(z)$ follows from a semiclassical analysis of the equation (10.1) as $z \to \infty$. This analysis does not require the Freud equation (10.3). However, the advantage of (10.3) is that in this case the number

$$n = NR_n(t + gR_{n-1} + gR_n + gR_{n+1})$$

in the asymptotics (10.4) is integral, and $e^{n \ln z} = z^n$ is an entire function. Otherwise we have to indicate the branch of $\ln z$ in (10.4).

Later on we will be especially interested in the case of equation (10.1) with R_n given by the equations (10.32) below. The exsitence of the canonical solutions $\Psi_j(z)$ corresponding to that case will appear as a by-product of Theorem 10.2 in the section 10.4. Note that the coefficients Γ_k in (10.4) are some elementary matrix functions of the parameters $R_{n-1},\ R_n$, and R_{n+1} which do not depend on j. In addition, the coefficients $\Gamma_k,\ k \geq 1$, are uniquely determined as soon as Γ_0 is fixed. Due to the symmetry relation

$$A(-z) = -\sigma_3 A(z)\sigma_3\,, \tag{10.6}$$

all Γ_{2l} are diagonal while all Γ_{2l+1} are off–diagonal. In particular,

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ R_n^{1/2} & 0 \end{pmatrix}.$$

The equation (10.6) implies that

$$\Psi_{j+4}(-z) = (-1)^n \sigma_3 \Psi_j(z)\sigma_3.$$

Since $\overline{A(z)} = A(\overline{z})$, we also have that

$$\overline{\Psi_2(z)} = \Psi_1(\overline{z}), \qquad \overline{\Psi_3(z)} = \Psi_8(\overline{z}), \qquad \overline{\Psi_4(z)} = \Psi_7(\overline{z}), \qquad \overline{\Psi_5(z)} = \Psi_6(\overline{z}).$$

Having defined the canonical solutions, we can introduce the *Stokes Matrices* as

$$S_j = \Psi_j{}^{-1}(z)\Psi_{j+1}(z), \qquad j = 1, ..., 8\,; \qquad \Psi_9 \equiv \Psi_1. \tag{10.7}$$

Since all $\Psi_j(z)$ satisfy the same matrix differential equation (11.1), the matrices S_j do not depend on z. The matrices S_j obey the following general constraints:

$$S_{j+4} = \sigma_3 S_j \sigma_3, \quad \overline{S_1} = S_1{}^{-1}, \quad \overline{S_2} = S_8{}^{-1}, \quad \overline{S_3} = S_7{}^{-1}, \quad S_1 S_2 \dots S_8 = I\,, \tag{10.8}$$

where the bar means the complex conjugation of matrix elements, and

$$S_{2l+1} = \begin{pmatrix} 1 & s_{2l+1} \\ 0 & 1 \end{pmatrix}, \quad S_{2l} = \begin{pmatrix} 1 & 0 \\ s_{2l} & 1 \end{pmatrix}. \tag{10.9}$$

The first four constraints in (10.8) are expressed in terms of s_j as follows:

$$s_{j+4} = -s_j, \qquad \operatorname{Re} s_1 = \operatorname{Im} s_3 = 0, \qquad s_4 = \overline{s_2}.$$

The constraint $S_1S_2\dots S_8 = I$ is equivalent to the equation

$$s_1s_2 + s_1s_4 + s_3s_4 - s_2s_3 + s_1s_2s_3s_4 = 0.$$

If we change λ_n for $\lambda_n + c$ in (10.4), then the solution $\Psi_j(z)$ is changed to $\Psi_j(z)\exp(-c\sigma_3)$, and hence the element s_j are changed as follows:

$$s_{2l+1} \to s_{2l+1}e^{-2c}, \qquad s_{2l} \to s_{2l}e^{2c}.$$

Assuming that $s_1 \neq 0$ we can fix the normalization constant λ_n if we put

$$s_1 = -2\pi i \tag{10.10}$$

(cf. (5.19)). The algebraic equations (10.8)–(10.10) indicate that the set $\{S_j\}$ of *Monodromy Data* of the differential equation (10.1) can be parametrized by three real parameters, s_3, $\operatorname{Re} s_2$, $\operatorname{Im} s_2$, satisfying the equation

$$s_3\left(|s_2|^2 - \frac{1}{\pi}\operatorname{Im} s_2\right) + 2\operatorname{Re} s_2 = 0. \tag{10.11}$$

Proposition 10.1. *The monodromy map,*

$$\left\{R_{n-1}, R_n, R_{n+1} : \frac{n}{N} = R_n(t + gR_{n-1} + gR_n + gR_{n+1})\right\} \Longrightarrow$$

$$\Longrightarrow \left\{s_2, s_3 : \; s_3\left(|s_2|^2 - \frac{1}{\pi}\mathrm{Im}\ s_2\right) + 2\mathrm{Re}\ s_2 = 0\right\}$$

is one-to-one.

Proof. Consider two systems,

$$\Psi'(z) = NA(z)\Psi(z) \quad \text{and} \quad \tilde{\Psi}'(z) = N\tilde{A}(z)\tilde{\Psi}(z),$$

from the class (10.1-3) whose monodromy data coincide. Let $\Psi_j(z)$ and $\tilde{\Psi}_j(z)$ be the corresponding canonical matrix solutions. Put

$$F(z) = \tilde{\Psi}_1(z)\Psi_1^{-1}(z).$$

Since the basic monodromy equation (10.7) has the same l.h.s., regardless which of the two sets of the canonical solutions is taken, we have that

$$F(z) = \tilde{\Psi}_2(z)\Psi_2^{-1}(z) = \tilde{\Psi}_3(z)\Psi_3^{-1}(z) = \dots. = \tilde{\Psi}_8(z)\Psi_8^{-1}(z).$$

This implies that the asymptotic equation,

$$F(z) = F_\infty + O\left(z^{-1}\right), \quad F_\infty = e^{(\lambda_n - \tilde{\lambda}_n)\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & \frac{R_n^{1/2}}{\tilde{R}_n^{1/2}} \end{pmatrix},$$

$$z \to \infty, \quad -\frac{3\pi}{8} < \arg z < \frac{\pi}{8},$$

which follows from (10.4) and (10.5) ($j = 1$), is valid in the whole neighborhood of $z = \infty$. A prior, $F(z)$ is an entire function. Therefore, we conclude that $F(z)$ is in fact a constant diagonal matrix, i.e.

$$F(z) \equiv F_\infty.$$

This in turn yields the equation,

$$A(z) = F_\infty^{-1} \tilde{A}(z) F_\infty, \qquad \forall\, z,$$

or, component-wise,

$$\frac{tz}{2} + \frac{gz^3}{2} + gzR_n = \frac{tz}{2} + \frac{gz^3}{2} + gz\tilde{R}_n, \qquad \forall\, z, \tag{10.12}$$

$$gz^2 + \theta_n = (gz^2 + \tilde{\theta}_n)e^{2(\tilde{\lambda}_n - \lambda_n)}, \qquad \forall\, z, \tag{10.13}$$

$$gz^2 + \theta_{n-1} = (gz^2 + \tilde{\theta}_{n-1})e^{2(\lambda_n - \tilde{\lambda}_n)} \frac{\tilde{R}_n}{R_n}, \qquad \forall\, z. \tag{10.14}$$

¿From equation (10.12) it follows that

$$R_n = \tilde{R}_n.$$

After that, equations (10.13) and (10.14) imply

$$\tilde{\lambda}_n = \lambda_n, \quad R_{n\pm1} = \tilde{R}_{n\pm1},$$

which completes the proof of the Proposition.

For what follows it is useful to introduce a piecewise analytic matrix function $\Psi(z)$ on a complex plane, which coincides with the function $\Psi_j(z)$ in the sector

$$\left\{ z \in \mathbb{C}\colon\ \frac{\pi(j-2)}{4} \le \arg z \le \frac{\pi(j-1)}{4} \right\}, \qquad j = 1, \ldots, 8.$$

The function $\Psi(z)$ has the asymptotics (10.4),

$$\Psi(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k}{z^k} \right) e^{-\left(\frac{NV(z)}{2} - n\ln z + \lambda_n\right)\sigma_3}, \qquad z \to \infty,$$

and it is two–valued on the rays

$$r_j = \{z \in \mathbb{C} : \arg z = \frac{\pi(j-1)}{4}\},$$

where its two values are related by the equation (cf.10.7),

$$\Psi_+(z) = \Psi_-(z) S_j, \qquad z \in r_j, \qquad j = 1, \dots, 8,$$

assuming that the orientation on r_j is from 0 to ∞. This describes the Riemann–Hilbert problem for which $\Psi(z)$ is a solution. The Riemann-Hilbert problem is depicted in Fig.2.

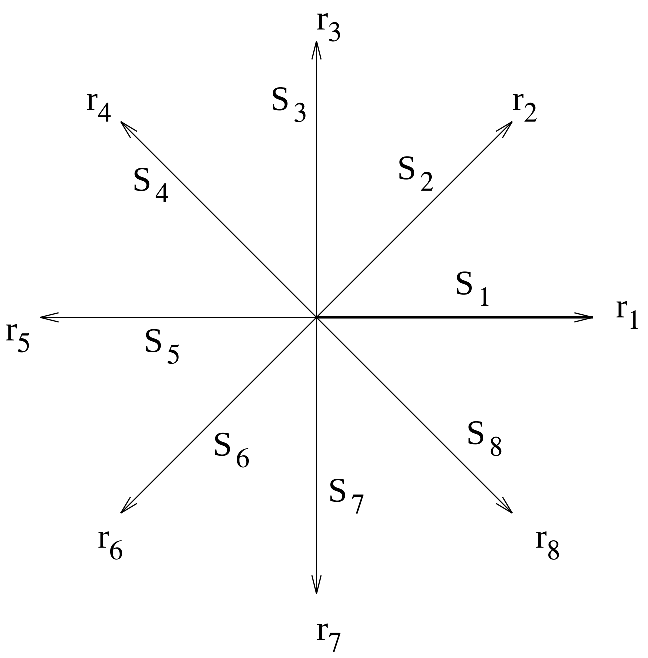

Fig. 2. The Riemann-Hilbert problem for the function Ψ(z)

Performing the gauge transformation,

$$\Psi(z) \to \Phi(z) = e^{\lambda_n \sigma_3} \Gamma_0^{-1} \Psi(z),$$

we formulate the following *normalized* Riemann-Hilbert problem, which is associated to the system (10.1) (cf. (3.9-10) in [FIK2]):

$$\Phi(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{\Theta_k}{z^k} \right) e^{-\left(\frac{NV(z)}{2} - n \ln z\right)\sigma_3}, \qquad z \to \infty, \tag{10.15a}$$

$$\Phi_+(z) = \Phi_-(z)S_j, \qquad z \in r_j, \qquad j = 1, \dots, 8, \tag{10.15b}$$
$$r_j = \{z \in \mathbb{C} : \arg z = \frac{\pi(j-1)}{4}\}.$$

Now we are ready to discuss the inverse monodromy problem: how to reconstruct a differential equation by the monodromy data.

10.2. Inverse Monodromy Problem

Assume that $\Phi(z)$ is a solution of the Riemann–Hilbert problem (10.15a,b) . Put

$$Y(z) = \Phi(z)e^{W(z)\sigma_3}, \tag{10.16}$$

where we denote for the sake of brevity,

$$W(z) = \frac{NV(z)}{2} - n \ln z.$$

Then

$$\Phi(z) = Y(z)e^{-W(z)\sigma_3}, \tag{10.17}$$

and by (10.15a),

$$Y(z) \sim I + \sum_{k=1}^{\infty} \frac{\Theta_k}{z^k}, \qquad z \to \infty. \tag{10.18}$$

Next lemma, which is of course just a particular case of the corresponding general construction (see e.g., [JMU]), shows that the Riemann-Hilbert problem implies a polynomial matrix differential equation on $\Phi(z)$. We shall use the following usual notations: If

$$B(z) \sim \sum_{k=-\infty}^{m} b_k z^k, \qquad z \to \infty,$$

we denote by

$$\Big\{B(z)\Big\}_+ = \sum_{k=0}^{m} b_k z^k,$$

the polynomial part of $B(z)$ at infinity.

Lemma 10.1. *Assume that $\Phi(z)$ is a solution of the RH problem (10.15a,b). Then $\Phi(z)$ satisfies the polynomial 2×2 matrix differential equation (10.1) with*

$$A(z) = -(1/2)\Big\{Y(z)V'(z)\sigma_3 Y^{-1}(z)\Big\}_+, \tag{10.19}$$

where $Y(z)$ is defined in (10.16).

Proof. Observe that $\det \Phi(z)$ is an entire function, since

$$\det \Phi_+(z) = \det \Phi_-(z) \det S_j = \det \Phi_-(z).$$

In addition,

$$\lim_{z\to\infty} \det \Phi(z) = \lim_{z\to\infty} \det Y(z) = \det I = 1.$$

Hence

$$\det \Phi(z) \equiv 1 \neq 0.$$

We want to check that $\Phi(z)$ satisfies a matrix differential equation. Define

$$Q(z) = \Phi'(z)\Phi^{-1}(z).$$

Then by (10.15b),

$$Q_+(z) = \Phi'_+(z)\Phi_+^{-1}(z) = \Phi'_-(z)S_jS_j^{-1}\Phi_-^{-1}(z) = Q_-(z),$$

so that $Q(z)$ is an entire matrix-valued function. By (10.17),

$$\begin{aligned} Q(z) &= \left[Y'(z)e^{-W(z)\sigma_3} - Y(z)W'(z)\sigma_3 e^{-W(z)\sigma_3}\right] e^{W(z)\sigma_3}Y^{-1}(z) \\ &= \left[Y'(z) - Y(z)W'(z)\sigma_3\right] Y^{-1}(z), \end{aligned}$$

hence $Q(z)$ grows polynomially at infinity, and hence $Q(z)$ is a polynomial,

$$Q(z) = \left\{\left[Y'(z) - Y(z)W'(z)\sigma_3\right] Y^{-1}(z)\right\}_+ = -(N/2)\left\{Y(z)V'(z)\sigma_3 Y^{-1}(z)\right\}_+ .$$

Thus we get a polynomial differential equation on $\Phi(z)$,

$$\Phi'(z) = Q(z)\Phi(z),$$

with

$$Q(z) = -(N/2)\left\{Y(z)V'(z)\sigma_3 Y^{-1}(z)\right\}_+ .$$

Lemma 10.1 is proved.

Solution $\Phi(z)$ of the Riemann-Hilbert problem (10.15a,b) is uniquely defined (if it exists) by the set of the Stokes matrices $\{S_j,\ j = 1, \dots, 8\}$, which we assume satisfy the restrictions (10.8-10). A straightforward calculation based on the asymptotic series (10.18) and on the symmetry $z \to -z$, leads to the following representation for the matrix $A(z)$ in (10.19) (cf.(10.1)):

$$A(z) = \begin{pmatrix} -(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n) & \beta_+(gz^2 + \theta_n) \\ -\beta_-(gz^2 + \theta_{n-1}) & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n \end{pmatrix}, \qquad \theta_n = t + gR_n + gR_{n+1}, \quad (10.20)$$

where the real parameters $R_n, R_{n\pm 1}$, and $\beta_\pm$, are given by the equations:

$$\beta_+ = (\Theta_1)_{12}, \qquad \beta_- = (\Theta_1)_{21}, \qquad R_n = \beta_-\beta_+,$$
$$R_{n+1} = \beta_+^{-1}(\Theta_3)_{12} - (\Theta_2)_{22}, \qquad R_{n-1} = \beta_-^{-1}(\Theta_3)_{21} - (\Theta_2)_{11},$$

and $(\Theta_k)_{jl}$ denote the entries of the matrix coefficients Θ_k in the asymptotic series (10.18). Moreover, substituting (10.17), (10.18) into (10.1) and equating the terms of order z^{-1} in (10.1), one can easily see that the quantities $R_n, R_{n\pm1}$, satisfy the Freud equation (10.3).

Observe now that the gauge transformation,

$$\Phi(z) \to \Psi(z) = \begin{pmatrix} \beta_+^{-1/2} & 0 \\ 0 & \beta_-^{-1/2} \end{pmatrix} \Phi(z),$$

brings the matrix (10.20) to the form indicated in (10.2). Hence we can state the following theorem (cf. Theorem 3.1 in [FIK4]), which reduces the inverse monodromy problem for the system (10.1) to the analysis of the Riemann-Hilbert problem (10.15a,b).

Theorem 10.1. *Assume that Riemann-Hilbert problem (10.15a,b) is solvable. Assume also that its solution $\Phi(z)$ satisfies the condition $(\Theta_1)_{12}(\Theta_1)_{21} \neq 0$ (generic case). Then,* {**i**} *there exists a unique differential equation (10.1) whose set of monodromy data coincides with the given set $\{S_j, j = 1, ..., 8\}$;* {**ii**} *the corresponding parameters R_{n-1}, R_n, R_{n+1}, and λ_n can be evaluated in terms of the matrix coefficients of the series (10.18) according to the equations*

$$R_n = (\Theta_1)_{12}(\Theta_1)_{21}, \qquad R_{n+1} = (\Theta_1)_{12}^{-1}(\Theta_3)_{12} - (\Theta_2)_{22}\,,$$
$$R_{n-1} = (\Theta_1)_{21}^{-1}(\Theta_3)_{21} - (\Theta_2)_{11}\,, \qquad \lambda_n = \frac{1}{2}\ln(\Theta_1)_{12}\,;$$

{**iii**} *the corresponding canonical solutions $\Psi_j(z)$, $j = 1, \ldots, 8$, are given by the formula:*

$$\Psi_j(z) = \begin{pmatrix} (\Theta_1)_{12}^{-1/2} & 0 \\ 0 & (\Theta_1)_{21}^{-1/2} \end{pmatrix} \Phi(z), \qquad \frac{\pi(j-2)}{4} \le \arg z \le \frac{\pi(j-1)}{4}\,.$$

10.3. Triangular case. Orthogonal polynomials.

Let us suppose that in (10.15b) all Stokes matrices with even indices are trivial, i.e.

$$S_{2l} = I,$$

and consider the matrix function,

$$Y^*(z) = \Phi(z) e^{\frac{NV(z)}{2}\sigma_3}.$$

This function satisfies the following RH problem on the cross $L = \mathbb{R} \bigcup i\mathbb{R}$:

$$Y^*_+(z) = Y^*_-(z) \begin{pmatrix} 1 & se^{-NV(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in L\,, \tag{10.21}$$

$$Y^*(z) \sim \begin{pmatrix} z^n + O(z^{n-1}) & O(z^{-n-1}) \\ O(z^{n-1}) & z^{-n} + O(z^{-n-1}) \end{pmatrix}, \qquad z \to \infty\,. \tag{10.22}$$

The Riemann-Hilbert problem (10.21-22) is depicted in Fig.3. We assume that the cross L is oriented in a natural way, i.e., from $-\infty$ to $+\infty$, and from $-i\infty$ to $+i\infty$ (see Fig.3) so that in (10.21)

$$s = s_1 = -2\pi i \quad \text{if} \quad z \in \mathbb{R}, \qquad s = s_3 \quad \text{if} \quad z \in i\mathbb{R}\,.$$

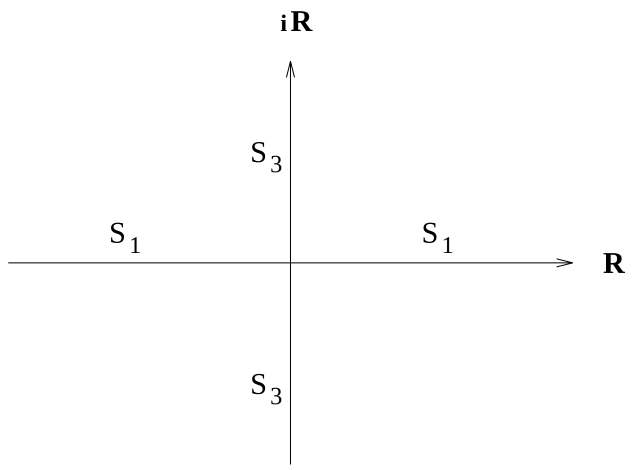

Fig. 3. The Riemann-Hilbert problem for the function $Y^*(z)e^{-NV(z)\frac{\sigma_3}{2}}$

The RH problem (10.21)–(10.22) is triangular and hence (cf. [FIK4], section 3.4) it can be solved in a closed form. The 11 and 21 components of (10.21) yield $(Y_+^*(z))_{11} = (Y_-^*(z))_{11}$ and $(Y_+^*(z))_{21} = (Y_-^*(z))_{21}$. Using these equations and (10.22), we find that

$$(Y^*(z))_{11} = P_n(z), \qquad (Y^*(z))_{21} = Q_{n-1}(z) \tag{10.23}$$

where $P_n(z)$ and $Q_{n-1}(z)$ are some polynomials of the degree n and $n-1$, respectively, such that

$$P_n(z) = z^n + \dots \tag{10.24}$$

By (10.21),

$$(Y_+^*(z))_{12} - (Y_-^*(z))_{12} = se^{-NV(z)}(Y_-^*(z))_{11},$$
$$(Y_+^*(z))_{22} - (Y_-^*(z))_{22} = se^{-NV(z)}(Y_-^*(z))_{21},$$

which together with (10.23) provide us with the following representation for the solution $Y^*(z)$ of the problem (10.21,22):

$$Y^*(z) = \begin{pmatrix} P_n(z) & \frac{1}{2\pi i}\int_L \frac{e^{-NV(\mu)}P_n(\mu)d\mu}{\mu - z} \\ Q_{n-1}(z) & \frac{1}{2\pi i}\int_L \frac{e^{-NV(\mu)}Q_{n-1}(\mu)d\mu}{\mu - z} \end{pmatrix} \tag{10.25}$$

where

$$\int_L = s_1 \int_{-\infty}^{+\infty} + s_3 \int_{-i\infty}^{+i\infty}$$

It remains to notice that the asymptotic condition (10.22) is satisfied iff

$$\int_L \mu^l e^{-NV(\mu)} P_n(\mu) d\mu = 0, \qquad l = 0, 1, \dots, n-1, \tag{10.26}$$

and

$$-\frac{1}{2\pi i}\int_L \mu^l e^{-NV(\mu)} Q_{n-1}(\mu) d\mu = \delta_{l,n-1}, \qquad l = 0, 1, \dots, n-1. \tag{10.27}$$

These equations imply that $P_n(z)$ are orthogonal polynomials on the cross L with respect to the measure $e^{-NV(z)}dz$:

$$\int_L P_n(z)P_m(z)e^{-NV(z)}dz = -2\pi i h_n \delta_{n,m}, \qquad P_n(z) = z^n + \dots, \tag{10.28}$$

and

$$Q_{n-1}(z) = \frac{1}{h_{n-1}} P_{n-1}(z) \tag{10.29}$$

The equations (10.26)–(10.27), together with the normalization condition (10.24), determine the polynomials $P_n(z)$ and $Q_{n-1}(z)$ uniquely assuming the nondegeneracy condition

$$\det \left\| \left\{ \int \mu^k \mu^j e^{-NV(\mu)} d\mu \right\}_{j,k=1,\dots,n-1} \right\| \neq 0\,. \tag{10.30}$$

This condition holds for generic s_3 and for the case (1.3) of our principal interest, i.e., when

$$s_3 = 0\,. \tag{10.31}$$

We note also that the function $Y^*(z)$ relates to the orthogonal polynomial Ψ-function, $\Psi_n(z)$ (which was introduced in the section 4), by the equation:

$$\Psi_n(z) = \begin{pmatrix} h_n^{-1/2} & 0 \\ 0 & h_{n-1}^{1/2} \end{pmatrix} Y_n^*(z) e^{-\frac{NV(z)}{2}\sigma_3}.$$

The corresponding Riemann-Hilbert problem is exactly our main problem (5.16-18).

Remark. The technique used in this section was first suggested by Fokas, Mugan, and Ablowitz [FMA] for analyzing the explicit solutions of the Painlevé equations. In [FIK4] it was applied to the case of an arbitrary even polynomial $V(z)$. In fact, using the same idea one can reduce the analysis of an *arbitrary* system of the orthogonal polynomials $\{P_n(z)\}$ on some contour L with some weight $\omega(z)$ to the analysis of the relevant 2×2 matrix Riemann-Hilbert problem. The RH problem is formulated for a 2×2 matrix function $Y^*(z)$ which is analytic outside of the contour L, normalized by the asymptotic condition

$$Y^*(z) z^{-n\sigma_3} \to I\,, \qquad z \to \infty\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and whose boundary values $Y^*_{\pm}(z)$ satisfy equation:

$$Y^*_+(z) = Y^*_-(z) \begin{pmatrix} 1 & -2\pi i \omega(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in L.$$

This Riemann-Hilbert problem can be also used to explain the appearance (see, e.g., [ASM]) of the KP-type hierarchies in the matrix models. Indeed, let us assume that the weight function $\omega(\lambda)$ is of the form

$$\omega(z) = \exp\left(\sum_{k=1}^{m} t_k z^k\right)$$

and put

$$\Psi(z) = Y^*(z) \exp\left(\frac{1}{2}\sum_{k=1}^{m} t^k z^k \sigma_3\right).$$

Then, the same arguments as the ones we used for proving Lemma 10.1 yield (cf. (1.12-16) in [FIK4]) the system of linear differential and difference equations for the function $\Psi(z) \equiv \Psi(z; n, t_1, t_2, t_3, \dots)$,

$$\begin{aligned} \Psi(z; n+1) &= U_n(z)\Psi(z; n), \\ \partial_z \Psi(z) &= A_n(z)\Psi(z), \\ \partial_{t_k}\Psi(z) &= V_n^{(k)}(z)\Psi(z), \qquad k = 1, 2, 3, \dots, \end{aligned}$$

where $U_n(z)$, $A_n(z)$, and $V_n^{(k)}(z)$ are polynomial on z (for their exact expressions in terms of the corresponding R_n see [FIK4]). The first two equations constitute the Lax pair for the relevant Freud equation (cf. (3.11)):

$$U_n'(z) = A_{n+1}(z)U_n(z) - U_n(z)A_n(z)\,.$$

The compatability conditions of the third equations with the different k generate the KP-type hierarchy of the integrable PDEs; the compatability condition of the second and the third equations produces the Virasoro-type constraints; the compatability condition of the first and the third equations is related to the Toda-type hierarchy and vertex operators.

10.4. Asymptotic Solution of the Direct Monodromy Problem. Complex WKB Analysis

Our aim now is to study the direct monodromy problem for the system (10.1). We will assume that the numbers R_{n-1}, R_n and R_{n+1} are given by the equations

$$R_n = \frac{-t - 2\alpha_n}{2g}, \qquad R_{n\pm1} = \frac{-t + 2\alpha_n}{2g} \mp \frac{1}{2N\alpha_n}, \tag{10.32a}$$

where

$$\alpha_n = \frac{(-1)^n}{2}\sqrt{t^2 - \frac{4gn}{N}}.$$

It is easy to check that the formulae (10.32a) are consistent with the Freud equation (10.3), and that they imply the equations

$$\theta_n = -\frac{g}{2N\alpha_n}, \qquad \theta_{n-1} = \frac{g}{2N\alpha_n}, \tag{10.32b}$$

for the quantities $\theta_{n,n-1}$ in (10.2).

We are interested in the semiclassical asymptotics for solutions of the equation (10.1) in the case when $n, N \to \infty$ in such a way that n/N approaches a limit such that

$$0 < \lim_{N\to\infty} \frac{n}{N} < \lambda_{\rm cr} = \frac{t^2}{4g}\,. \tag{10.33}$$

We denote

$$\lambda = n/N$$

and we derive a semiclassical asymptotics for $\Psi(z)$, which is uniform in the interval $\varepsilon \le \lambda \le \lambda_{\rm cr} - \varepsilon$ for every fixed $\varepsilon > 0$. The condition (10.32) is motivated by the formal asymptotic expansion for *genuine* R_n discussed in the section 6.

We shall follow the standard scheme (see [IN]) of the asymptotic analysis in the framework of the Isomonodromy Method.

The **first step** is a construction of relevant WKB-solutions of the system (10.1). According to the general WKB-method (see, e.g., [Was] and [Fed]), the WKB solutions have the asymptotics

$$\Psi^{\rm WKB}(z) \sim T(z) e^{-N\int_{z^0}^{z} \mu(u)du\sigma_3}, \qquad N \to \infty, \tag{10.34}$$

where,

$$\mu(z) = \sqrt{-\det A(z)}$$

is an eigenvalue of the matrix $A(z)$, and $T(z)$ is the matrix of eigenvectors, so that $T(z)$ diagonalizes the matrix $A(z) = \{a_{jk}(z)\}_{j,k=1,2}$. The lower limit of integration z^0 is an arbitrary number which can be chosen differently for different solutions. The Freud equation (10.3) simplifies the formula for $\det A(z)$ to

$$\det A(z) = -\left(\frac{tz}{2} + \frac{gz^3}{2}\right)^2 + \frac{gnz^2}{N} + R_n\theta_{n-1}\theta_n, \qquad \theta_n = t + gR_n + gR_{n+1},$$

(cf. (3.9′)). The condition (10.32) implies that

$$\theta_{n-1}, \theta_n = O(N^{-1}),$$

hence (see the proof of the theorem 10.2 below) we can neglect the term $R_n\theta_{n-1}\theta_n$ in $\det A(z)$ and take

$$\mu(z) = \sqrt{\left(\frac{tz}{2} + \frac{gz^3}{2}\right)^2 - \frac{gnz^2}{N}} = \frac{z}{2}\sqrt{(t+gz^2)^2 - 4\lambda g}\,. \tag{10.35}$$

To formulate rigorous statements concerning the WKB-solutions, we need to indroduce some standard ingredients of the complex WKB-method:

(a) *Stokes' lines* are defined by the equations

$$\operatorname{Re} \int_{z_k}^{z} \mu(u)du = 0, \tag{10.36}$$

where $z_k,\ k=0,1,2,3,4$, are zeros of $\mu(z)$, i.e., the *turning points* of the system (10.1). From (10.35) we get that

$$z_0=0, \quad z_{1,2}=\left(\frac{-t\mp 2\sqrt{\lambda g}}{g}\right)^{1/2}, \quad z_3=-z_1, \quad z_4=-z_2. \tag{10.37}$$

Note that the condition $0<\lambda\le\lambda_{\rm cr}-\varepsilon$ implies that $z_{1,2}$ are real and

$$C\sqrt{\varepsilon}<z_1<z_2.$$

The curves (10.36) are asymptotic to the rays

$$r_j^0=\left\{z\in\mathbb{C}:\arg z=\frac{\pi(2j-3)}{8}\right\}, \quad j=1,2,\dots,8.$$

We shall denote by γ_j a Stokes line which is asymptotic to the ray $r_j^0,\ j=1,\dots,8$, and such that:

(i) $\gamma_{3,8}$ come out from the turning point z_1;

(ii) $\gamma_{1,2}$ come out from the turning point z_2;

(iii) $\gamma_{4,7}$ come out from the turning point $-z_1$;

and

(iv) $\gamma_{5,6}$ come out from the turning point$-z_2$,

as indicated in Fig.4. Four additional Stokes lines, coming out from the turning point $z_0=0$ and asymptotic to the rays $r_j^0,\ j=3,4,7,8$, we shall denote by $\gamma_j^0,\ j=3,4,7,8$, respectively (see again Fig.4).

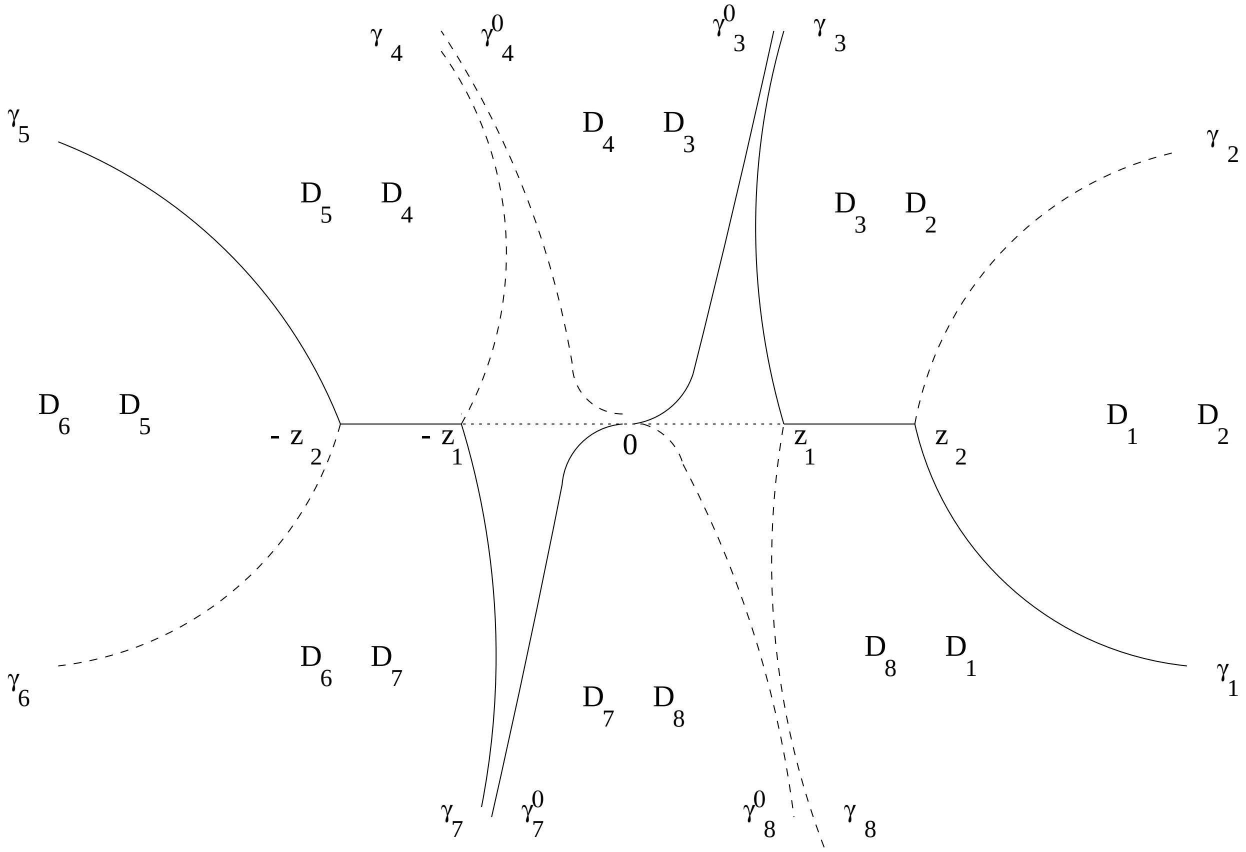

Fig. 4. The canonical WKB domains D_j

(b) *The Canonical Domains* D_j are defined as open domains whose boundaries are:

$$\begin{aligned}
\partial D_1 &= \gamma_8 \cup [z_1, z_2] \cup \gamma_2 \, , \\
\partial D_2 &= \gamma_1 \cup [z_1, z_2] \cup \gamma_3 \, , \\
\partial D_3 &= \gamma_2 \cup [0, z_2] \cup \gamma_4^0 \, , \\
\partial D_4 &= \gamma_3^0 \cup [-z_2, 0] \cup \gamma_5 \, , \\
\partial D_5 &= \gamma_4 \cup [-z_2, -z_1] \cup \gamma_6 \, , \\
\partial D_6 &= \gamma_5 \cup [-z_2, -z_1] \cup \gamma_7 \, , \\
\partial D_7 &= \gamma_6 \cup [-z_2, 0] \cup \gamma_8^0 \, , \\
\partial D_8 &= \gamma_7^0 \cup [0, z_2] \cup \gamma_1 \, ,
\end{aligned}$$

They are depicted in Fig.4 above. Observe that the domains D_j and D_{j+1} are overlapping, and D_j contains the Stokes line γ_j. A rigorous version of (10.34), which simultaneously proves the existence of the canonical solutions (10.4) for the system (10.1), (10.32), can be now formulated as the following theorem:

Theorem 10.2. *Let D_j, $j = 1, 2, \ldots 8$, be the introduced canonical domains. Assume that the quantities R_{n-1}, R_n and R_{n+1} are defined as in (10.32a). Then, in each region D_j there exists a WKB-solution $\Psi_j^{\mathrm{WKB}}(z)$ of the equation (10.1), which satisfies, under the condition (10.33), the following asymptotic equation:*

$$\Psi_j^{\mathrm{WKB}}(z) = T_0(z)\left\{I + O\left(\frac{1}{N(1+|z|^2)}\right)\right\} e^{-N\int_{z_{k(j)}}^{z}\mu(u)du\sigma_3}, \tag{10.38}$$

where $z_{k(j)}$ is the initial point of the Stokes line γ_j, $\mu(z)$ is given by the equation (10.35) and

$$T_0(z) = \sqrt{\frac{\mu - a_{11}}{2\mu}}\begin{pmatrix} 1 & \frac{a_{12}^0}{\mu - a_{11}} \\ \frac{a_{12}^0}{\mu - a_{11}} & 1 \end{pmatrix}, \tag{10.39}$$

where

$$a_{11} = \alpha_n z - \frac{gz^3}{2}, \qquad a_{12}^0 = R_n^{1/2} g z^2. \tag{10.39a}$$

The asymptotics (10.38) is uniform in $z \in K$ for any closed $K \subset D_j$ such that $dist\{K, \partial D_j\} > 0$.

Remark 10.1. The integral in the exponent in the r.h.s. of (10.38) can be easily evaluated (cf. (7.17)). In fact, we have:

$$\begin{aligned}\int_{z_{k(1,2)}}^{z}\mu(u)du &= \int_{z_{k(5,6)}}^{-z}\mu(u)du = \int_{z_2}^{z}\mu(u)du \\ &= \frac{\lambda}{2}\left[x\sqrt{x^2-1} - \ln(x + \sqrt{x^2-1})\right] \equiv \xi(z), \qquad z \in D_1 \cup D_2,\end{aligned} \tag{10.40}$$

and

$$\int_{z_{k(3)}}^{z}\mu(u)du = \int_{z_{k(7)}}^{-z}\mu(u)du = \int_{z_1}^{z}\mu(u)du = \xi(z) + i\lambda\frac{\pi}{2}, \qquad z \in D_3, \tag{10.41}$$

$$\int_{z_{k(8)}}^{z}\mu(u)du = \int_{z_{k(4)}}^{-z}\mu(u)du = \int_{z_1}^{z}\mu(u)du = \xi(z) - i\lambda\frac{\pi}{2}, \qquad z \in D_8, \tag{10.42}$$

where

$$x = \frac{t + gz^2}{2\sqrt{\lambda g}}.$$

Note also, that by (10.35) and (10.39a),

$$\mu(z) = z(\lambda g)^{1/2}(x^2-1)^{1/2} = z\left[\left(\alpha_n - \frac{gz^2}{2}\right)^2 - R_n g^2 z^2\right]^{1/2} = [a_{11}^2 - (a_{12}^0)^2]^{1/2}, \tag{10.43}$$

and the variable x we use in this section differs from the variable x in (1.16) by the replacement, $\lambda' \to \lambda = \frac{n}{N}$. We fix the branches of the involved multivalued functions by the conditions

$$-\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \quad -\pi < \arg\sqrt{x^2-1} < \pi, \quad -\pi < \arg(x+\sqrt{x^2-1}) < \pi, \tag{10.44}$$

and we consider $\sqrt{x^2-1}$ as a single-valued analytic function on $\mathbb{C}\backslash([-z_2,-z_1]\cup[z_1,-z_2])$.

Proof of Theorem 10.2. We follow the WKB-scheme as given in [Kap], taking also into accouint some specific features of the system (10.1), (10.32). To avoid some inessential technical diversities we assume that n is even. Let us try to diagonalize the system (10.1) with the help of the substitution

$$\Psi(z) = T(z)\Phi(z), \qquad T(z) = \big(t_{ij}(z)\big)_{i,j=1,2}.$$

Then Φ satisfies the equation

$$\Phi' = NB\Phi, \tag{10.45}$$

where

$$B = T^{-1}AT - \frac{T^{-1}T'}{N}.$$

The elements of B are

$$\begin{aligned}
b_{11} &= \Delta^{-1}\big[a_{11}(t_{11}t_{22}+t_{12}t_{21}) + a_{12}t_{21}t_{22} - a_{21}t_{11}t_{12} - N^{-1}(t'_{11}t_{22} - t_{12}t'_{21})\big],\\
b_{12} &= \Delta^{-1}\big[2a_{11}t_{12}t_{22} + a_{12}t_{22}^2 - a_{21}t_{12}^2 - N^{-1}(t_{22}t'_{12} - t_{12}t'_{22})\big],\\
b_{21} &= \Delta^{-1}\big[-2a_{11}t_{11}t_{21} - a_{12}t_{21}^2 + a_{21}t_{11}^2 - N^{-1}(t_{11}t'_{21} - t'_{11}t_{21})\big],\\
b_{22} &= \Delta^{-1}\big[-a_{11}(t_{11}t_{22}+t_{12}t_{21}) - a_{12}t_{21}t_{22} + a_{21}t_{11}t_{12} - N^{-1}(t_{11}t'_{22} - t'_{12}t_{21})\big],
\end{aligned} \tag{10.46}$$

where

$$\Delta = t_{11}t_{22} - t_{12}t_{21}.$$

and $A(z) = \big(a_{ij}(z)\big)_{i,j=1,2}$. The condition $b_{12}=0$ leads to the Riccati equation (cf. [Kap]),

$$\frac{h'}{N} = a_{12} + 2a_{11}h - a_{21}h^2, \qquad h = \frac{t_{12}}{t_{22}}. \tag{10.47}$$

We are looking for a solution of this equation as

$$h(z) = h_0(z) + \frac{h_1(z)}{N} + O(N^{-2}), \tag{10.48}$$

where $h_0(z)$ and $h_1(z)$ are analytic functions near $z=0$.

¿From the equations (10.2) and (10.32) we obtain that

$$\begin{aligned} a_{11} &= -a_{22} = \alpha_n z - \frac{gz^3}{2}, \\ a_{12} &= R_n^{1/2} g z^2 - \frac{R_n^{1/2} g}{2\alpha_n N}, \\ a_{21} &= -R_n^{1/2} g z^2 - \frac{R_n^{1/2} g}{2\alpha_n N}. \end{aligned} \tag{10.49}$$

Let us substitute (10.48) and (10.49) into the Riccati equation (10.47) and equate the terms of the zeroth and first order in N^{-1}. In the zeroth order this produces the quadratic equation

$$h_0^2 + \frac{2\left(\alpha - \frac{gz^2}{2}\right) h_0}{R^{1/2} g z} + 1 = 0, \tag{10.50}$$

which has two solutions, one of the order of z^{-1} as $z \to 0$ and another of the order of z. Taking into account the assumed evennest of n, equation (10.43) and convention (10.44), the regular at $z = 0$ solution of (10.50) can be written as,

$$h_0 = -\frac{R_n^{1/2} g z}{\alpha_n - \frac{gz^2}{2} - \left[\left(\alpha_n - \frac{gz^2}{2}\right)^2 - R_n g^2 z^2\right]^{1/2}} = \frac{a_{12}^0(z)}{\mu(z) - a_{11}(z)}. \tag{10.51}$$

In the first order in N^{-1} the Riccati equation (10.47) produces a linear equation on h_1, with the solution

$$h_1 = \frac{h_0' + (1 - h_0^2) R_n^{1/2} g\,(2\alpha_n)^{-1}}{2\left(\alpha_n z - \frac{gz^3}{2}\right) + 2R_n^{1/2} g z^2 h_0}. \tag{10.52}$$

The neccessary and sufficient condition for h_1 to be analytic at 0 is that the numerator vanishes at $z = 0$,

$$\left[h_0' + \frac{(1 - h_0^2) R_n^{1/2} g}{2\alpha_n} \right]_{z=0} = 0. \tag{10.53}$$

¿From (10.51),

$$h_0(0) = 0, \qquad h_0'(0) = -\frac{R_n^{1/2} g}{2\alpha_n},$$

hence (10.53) holds.

The condition $b_{21} = 0$ leads to another Riccati equation,

$$\frac{h'}{N} = a_{21} - 2a_{11} h - a_{12} h^2, \qquad h = \frac{t_{21}}{t_{11}}.$$

This equation can be solved in the same way as the equation (10.47), and we get the solution

$$h(z)=h_0(z)-\frac{h_1(z)}{N}+O(N^{-2}),$$

where h_0 and h_1 are the same as in (10.48). A straightforward analysis of the equations (10.51) and (10.52) lead to the following proposition.

Proposition 10.2. *The functions $h_0(z)$ and $h_1(z)$ are holomorphic everywhere on $\mathbb{C}\setminus\big([-z_2,-z_1]\cup[z_1,z_2]\big)$, and they satisfy the following global estimates:*

$$|h_0(z)|\le\frac{C|z|}{1+|z|^2}\,,\qquad |h_0'(z)|\le\frac{C}{1+|z|^2}\,,\tag{10.54}$$

$$|h_1(z)|\le\frac{C|z|}{1+|z|^4}\,,\qquad |h_1'(z)|\le\frac{C}{1+|z|^4}\,,\tag{10.55}$$

where the positive constant $C=C(t,g,\lambda)$ is uniform in $z\in\mathbb{C}\setminus\big([-z_2,-z_1]\cup[z_1,z_2]\big)$.

Let us now take

$$T(z)=\begin{pmatrix}1 & h_0(z)+\frac{h_1(z)}{N}\\ h_0(z)-\frac{h_1(z)}{N} & 1\end{pmatrix},\tag{10.56}$$

and analyse the analytic properties of the corresponding matrix $B(z)$ in (10.45). Consider first the determinant of $T(z)$,

$$\Delta=t_{11}t_{22}-t_{12}t_{21}=1-h_0^2+\frac{h_1^2}{N^2}.$$

¿From (10.51) and (10.43),

$$1-h_0^2=1-\frac{(a_{12}^0)^2}{(\mu-a_{11})^2}=\frac{\mu^2-2\mu a_{11}+a_{11}^2-(a_{12}^0)^2}{(\mu-a_{11})^2}=\frac{2\mu^2-2\mu a_{11}}{(\mu-a_{11})^2}=\frac{2\mu}{\mu-a_{11}}.\tag{10.57}$$

This implies that there exist positive constants, $c_0=c_0(t,g,\lambda)$ and $C_0=C_0(t,g,\lambda)$, such that

$$c_0\frac{1+|z|^2}{|z^2-z_1^2|^{1/2}|z^2-z_2^2|^{1/2}}\le\frac{1}{|1-h_0^2(z)|}\le C_0\frac{1+|z|^2}{|z^2-z_1^2|^{1/2}|z^2-z_2^2|^{1/2}},\tag{10.58}$$

for all $z\in\mathbb{C}\setminus\big([-z_2,-z_1]\cup[z_1,z_2]\big)$. Observe that $\mu(z)-a_{11}(z)\neq 0,\quad\forall z\neq 0$. The two-sided inequality (10.58), in turn, implies that, given $\rho>0$, there exist positive constants, $c=c(t,g,\lambda,\rho)$ and $C=C(t,g,\lambda,\rho)$, such that,

$$c\le\frac{1}{|1-h_0^2(z)|}\le C,\qquad\forall z\in\mathbb{C}\setminus\big([-z_2,-z_1]\cup[z_1,z_2]\big),\quad |z\pm z_{1,2}|\ge\rho.\tag{10.59}$$

The estimates (10.59), (10.55), and the equation

$$\Delta^{-1} = \frac{1}{1-h_0^2(z)}\left(1+\frac{h_1^2}{N^2(1-h_0^2)}\right)^{-1}, \tag{10.60}$$

imply that,

$$\Delta^{-1}(z) = \frac{1}{1-h_0^2(z)} + O\left(\frac{|z|^2}{N^2(1+|z|^8)}\right), \qquad N \to \infty, \tag{10.61}$$

uniformly for $z \in \mathbb{C} \setminus \big([-z_2, -z_1] \cup [z_1, z_2]\big), \quad |z \pm z_{1,2}| \geq \rho.$

We are ready now to formulate a principal analytical statement concerning the matrix $B(z)$, which is a direct consequence of the Proposition 10.2 and the equations (10.46), (10.61):

Lemma 10.2. *Let the matrix $T(z)$ be chosen as in (10.56) and let*

$$\Omega_\rho = \left\{z \in \mathbb{C} \setminus \big([-z_2, -z_1] \cup [z_1, z_2]\big) : |z \pm z_{1,2}| \geq \rho > 0\right\}.$$

Then there exist positive constants $N_0 = N_0(t, g, \lambda, \rho)$ and $C = C(t, g, \lambda, \rho)$ such that

(a) *the matrix function $B(z)$ in (10.45) is holomorphic in Ω_ρ for all $N \geq N_0$,*

(b) *for all $N \geq N_0$ and $z \in \Omega_\rho$, the following inequalities take place:*

$$\left| b_{jj} - (-1)^j \mu - \frac{h_0' h_0}{(1-h_0^2)N} \right| < \frac{C|z|}{N^2(1+|z|^4)}, \quad j = 1, 2,$$

$$|b_{jk}| < \frac{C}{N^2(1+|z|^4)}, \quad j \neq k; \quad j, k = 1, 2,$$

where μ is given by the equations (10.35) or (10.43).

Lemma 10.2 allows us to represent $B(z)$ as

$$B(z) = -\mu\sigma_3 + \frac{h_0' h_0}{(1-h_0^2)N} + R(z), \tag{10.62}$$

where the matrix-valued function $R(z)$ is holomorphic in Ω_ρ for all $N > N_0$, and it satisfies the estimate

$$|R(z)| \leq \frac{C(1+|z|)}{N^2(1+|z|^4)}, \qquad z \in \Omega_\rho, \quad N > N_0. \tag{10.63}$$

The representation (10.62) suggests the following formula for the WKB-solution of the system (10.45):

$$\Phi(z) = \frac{1}{\sqrt{1-h_0^2(z)}} \chi(z) e^{-N\int_{z_0}^{z} \mu(u)du\sigma_3}, \tag{10.64}$$

where the matrix-valued function $\chi(z)$ is looked for as a solution of the integral equation,

$$\chi(z) = I - N \int_{\gamma(z)} e^{N \int_z^\eta \mu(u) du \sigma_3} R(\eta) \chi(\eta) e^{-N \int_z^\eta \mu(u) du \sigma_3} d\eta. \qquad (10.65)$$

Here, $\gamma(z) = \{\gamma_{lk}(z)\}_{l,k=1,2}$ is a matrix of the *canonical paths*, which are defined (cf. [Fed]) by the conditions:

$\{i\}$. Each γ_{lk} is a simple contour starting at z and ending at ∞.

$\{ii\}$. $\mathrm{Re}\int_z^\eta \mu(u)du$ strictly decreases (increases) as η goes from z to ∞ along $\gamma_{12}(z)$ ($\gamma_{21}(z)$, respectively).

The matrix equation (10.65) should be understood as the system of four scalar equations:

$$\chi_{lk}(z) = \delta_{lk} - N \int_{\gamma_{lk}(z)} e^{2(k-l)N \int_z^\eta \mu(u) du} \left(R(\eta)\chi(\eta)\right)_{lk} d\eta, \quad l, k = 1, 2. \qquad (10.65')$$

Let D_j be one of the canonical domains and $K \subset D_j$, $\mathrm{dist}\,\{K, \partial D_j\} > 0$ be a closed set, as in the formulation of the theorem. Being a canonical domain means exactly (cf. [Fed]) that for each $z \in D_j$ there exists a matrix $\gamma(z)$ of the indicated above canonical paths, such that:

(a) $\gamma_{lk}(z) \subset D_j$, $\forall\, l, k$,

(b) for any two points $z, z' \in D_j$, the following equation takes place:

$$\gamma_{l,k}(z) - \gamma_{l,k}(z') + [z', z] = \partial\Omega_{l,k}(z, z'),$$

for some bounded $\Omega_{l,k}(z, z') \subset D_j$. In other words, any two $\gamma_{l,k}(z), \gamma_{l,k}(z')$ have the same infinite part.

Notice also that without loss of generality we may assume that the closed set K satisfies the following conditions:

(1) K contains in its interior an infinite part of the Stokes line γ_j.

(2) $\gamma_{lk}(z) \subset K$, $\forall z \in K$ and $\forall\, l, k = 1, 2$.

(3) $K \subset \Omega_\rho$ for a sufficiently small ρ.

Due to the conditions (2), (3), the condition $\{ii\}$ on the canonical paths, and the properties (a), (b) of a canonical domain, the integral operator in (10.65) is well-defined on the Banach space of holomorphic and bounded in K matrix-valued functions. In addition, due to the estimate (10.63) and, again, the property $\{ii\}$ of the canonical paths, there exists a positive constant $N_0 = N_0(t, g, \lambda, K)$ such that the $C(K)$-norm of the integral operator (10.65) can be estimated by a positive number $\tau < 1$ for all $N > N_0$. This, in turn, means that there exists a solution $\chi_j(z)$ of the equation (10.65), which is holomorphic in D_j and uniformly bounded in every closed set $K \subset D_j$:

$$|\chi_j(z)| \leq M = M(t, g, \lambda, K), \qquad z \in K, \quad N \geq N_0.$$

The last estimate can be easily improved by using the equation (10.65) as an integral *representation* for the holomorphic matrix-valued function $\chi_j(z)$. In fact, taking again into account (10.63) we conclude that

$$\chi_j(z) = I + O\left(\frac{1}{N(1+|z|^2)}\right), \tag{10.66}$$

uniformly on K.

Let us now define the function $\Psi_{j,K}^{\mathrm{WKB}}(z)$ by the equation

$$\Psi_{j,K}^{\mathrm{WKB}}(z) = T_0(z)\chi_j(z)e^{-N\int_{z_{k(j)}}^{z}\mu(u)du\sigma_3}, \qquad z \in K, \quad N \geq N_0, \tag{10.67}$$

where $T_0(z)$ is the matrix indicated in (10.39). Due to (10.51) and (10.57) we have that

$$T_0(z) = (1 - h_0^2(z))^{-1/2}T(z).$$

Hence, the function $\Psi_{j,K}^{\mathrm{WKB}}(z)$ (see eqs. (10.64), (10.45)) satisfies the equation (10.1) for $z \in K$ and $N \geq N_0$. In addition, it has a universal (independent on K and N_0) behaviour as $z \to \infty$, $z \in K$. Indeed, for the function $\xi(z)$ in (10.40-42) we have that, as $z \to \infty$,

$$\xi(z) = \frac{g}{8}z^4 + \frac{t}{4}z^2 - \lambda \ln z + C_n + O\left(\frac{1}{z^2}\right) = \frac{V(z)}{2} - \lambda \ln z + C_n + O\left(\frac{1}{z^2}\right), \tag{10.68}$$

where

$$C_n = \frac{t^2}{8g} - \frac{\lambda}{4}\ln\frac{g}{\lambda} - \frac{\lambda}{4}.$$

Simultaneously, from (10.43) we have that

$$\mu(z) = z^3\frac{g}{2} + z\frac{t}{2} + O\left(\frac{1}{z}\right). \tag{10.69}$$

Taking into account the equations (10.40-42) and (10.39), we derive from the estimates (10.68), (10.69), and (10.66) that, for any $N > N_0$, the following asymptotic equation takes place:

$$\Psi^{\mathrm{WKB}}_{j,K}(z) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} 0 & R_n^{1/2} \\ R_n^{1/2} & 0 \end{pmatrix} + O\left(\frac{1}{z^2}\right) \right\} e^{-\left(N\frac{V(z)}{2} - n\ln z + NC_n^j\right)\sigma_3}, \tag{10.70}$$

as $z \to \infty$, $\qquad z \in K$. In (10.70) we use the notations

$$C_n^{1,2} = C_n, \qquad C_n^{5,6} = C_n + i\lambda\pi, \tag{10.71}$$

$$C_n^{3,4} = C_n + i\lambda\frac{\pi}{2}, \tag{10.72}$$

$$C_n^{7,8} = C_n - i\lambda\frac{\pi}{2}. \tag{10.73}$$

To complete the proof of the theorem we now define $\Psi^{\mathrm{WKB}}_j(z)$ as a *unique* solution of equation (10.1) satisfying the same asymptotic condition at $z \to \infty$ as in (10.70):

$$\Psi^{\mathrm{WKB}}_j(z) e^{\left(N\frac{V(z)}{2} - n\ln z + NC_n^j\right)\sigma_3} \to I,$$

as

$$z \to \infty, \qquad z \in D_j.$$

Since K contains an infinite part of the Stokes line γ_j (see the condition (1) above), and due to the uniqueness theorem, the functions $\Psi^{\mathrm{WKB}}_j(z)$ and $\Psi^{\mathrm{WKB}}_{j,K}(z)$ should coincide for $z \in K$ and $N > N_0$:

$$\Psi^{\mathrm{WKB}}_j(z) = \Psi^{\mathrm{WKB}}_{j,K}(z) = T_0(z)\chi_j(z) e^{-N\int_{z_{k(j)}}^{z} \mu(u)du\sigma_3}, \qquad z \in K, \quad N \geq N_0. \tag{10.74}$$

the estimate (10.38) for the function $\Psi^{\mathrm{WKB}}_j(z)$ follows from (10.74) and (10.66). This completes the proof of the Theorem 10.2 for n even. The case of n odd is treated similarly.

Remark 10.2. Comparing to the general isomonodromy WKB-scheme developed in [Kap], the presented above analysis of a *particular system (10.1), (10.32)* has an important advantage. In spite of the fact that $z = 0$ is a double turning point of the system (10.1), the matrix remainder $R(z)$ is analytic at $z = 0$. This means that the WKB-asymptotics (10.38) is actually valid up to the point $z = 0$. More precisely, for $j = 3, 4, 7$, and 8, the condition

$$K \subset D_j, \quad \mathrm{dist}\,\{K, \partial D_j\} > 0,$$

on the closed set K can be replaced by a weaker condition,

$$K \subset D_j \cup \left(\{z : |z| \le z_1\} \cap \bar{D}_j \right), \quad \mathrm{dist}\,\{K, \partial D_j - \gamma^0_{k(j)} \cup [0, z_1)\} > 0,$$

for $j = 3, 8$, where $k(3) = 4,\ k(8) = 7$, and

$$K \subset D_j \cup \left(\{z : |z| \le z_1\} \cap \bar{D}_j \right), \quad \mathrm{dist}\,\{K, \partial D_j - (-z_1, 0\,] \cup \gamma^0_{k(j)}\} > 0,$$

for $j = 4, 7$, where $k(4) = 3,\ k(7) = 8$. It is worth emphasizing that the analyticity of $R(z)$ at $z = 0$ is the result of using the exact terms of the order of N^{-1} in the ansatz (10.32) for the quantities $R_{n\pm 1}$.

Remark 10.3. The condition Dist $\{K, \partial D_j\}$ on the closed domain K can be weakened to

$$\mathrm{dist}\,\left\{K, \partial D_j \setminus \left(\gamma^0_3 \cup \gamma^0_4 \cup \gamma^0_7 \cup \gamma^0_8 \cup (-z_1, z_1)\,\right)\right\} > cN^{-4+\varepsilon}, \quad c, \varepsilon > 0.$$

This weaker condition secures the existence of a WKB solution of the equation (10.1), with a somewhat weaker than in (10.38) estimate of the error term.

The analyticity of the matrix $R(z)$, defined in (10.62), at $z = 0$ enables us to use the same equation (10.45) to construct a *local* solution of the system (10.1) near the double turning point $z = 0$. Indeed, due to Proposition 10.2 there exists a positive number $\rho_0 = \rho_0(t, g, \lambda) < z_1$ such that $R(z)$ is holomorphic in the disk

$$U_0 = \{z \in \mathbb{C} : |z| < \rho_0\}, \tag{10.75}$$

and it satisfies there the uniform estimate

$$|R(z)| \le \frac{C}{N^2}, \qquad z \in U_0, \quad C = C(t, g, \lambda). \tag{10.76}$$

Assuming $z \in U_0$, let us define the function

$$\Psi_0^{\mathrm{TP}}(z) = T_0(z)\chi_0(z) e^{-N\int_0^z \mu(u)du\sigma_3},$$

where, instead of (10.65), $\chi_0(z)$ is a solution of the Volterra integral equation

$$\chi_0(z) = I + N \int_0^z e^{N\int_z^\eta \mu(u)du\sigma_3} R(\eta)\chi_0(\eta) e^{-N\int_z^\eta \mu(u)du\sigma_3} d\eta.$$

By similar considerations we prove that the matrix function $\chi_0(z)$ exists, it is holomorphic in U_0, and it satisfies the estimate

$$|\chi_0(z) - I| \le e^{\sigma(z)} - 1, \quad \sigma(z) = CN \int_0^z |R(\eta)| d\eta, \tag{10.77}$$

in the star-shaped region

$$W_0 = \{z \in U_0 : |N \operatorname{Re} \int_0^z \mu(u)du| \leq \text{const}\}, \tag{10.78}$$

which is depicted in Fig.5.

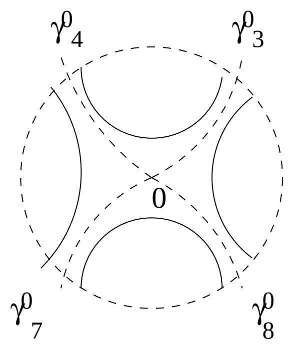

Fig. 5. The region W_0

The positive constant C in (10.77) depends on W_0 only. From (10.77) and (10.76) it follows that

$$\chi_0(z) = I + O\left(\frac{1}{N}\right),$$

as $N \to \infty$, uniformly in $z \in W_0$. Hence we have the following result.

Theorem 10.3. *Let U_0 and W_0 be the disk and the star-shaped region defined by the equations (10.75) and (10.78), respectively. Assume that the quantities R_{n-1}, R_n and R_{n+1} are defined as in (10.32). Then, there exists a turning point solution $\Psi_0^{\mathrm{TP}}(z)$ of the equation (10.1), which is holomorphic in U_0 and satisfies, under the conditions (10.33), the following asymptotic equation:*

$$\Psi_0^{\mathrm{TP}}(z) = T_0(z)\left\{I + O\left(\frac{1}{N}\right)\right\} e^{-N\int_0^z \mu(u)du\sigma_3}, \tag{10.79}$$

where, for even n, $T_0(z)$ and $\mu(z)$ are the same as in Theorem 10.2. For odd n, one should replace $T_0(z)$ by the matrix,

$$T_0^{\mathrm{odd}}(z) = \sqrt{\frac{\mu + a_{11}}{2\mu}} \begin{pmatrix} -\frac{a_{12}^0}{\mu + a_{11}} & 1 \\ 1 & -\frac{a_{12}^0}{\mu + a_{11}} \end{pmatrix}.$$

The asymptotics (10.79) is uniform in $z \in W_0$.

Note, that for odd n we have that

$$T_0(z) = \mp i T_0^{\mathrm{odd}}(z), \qquad z \in U_0 \cap \{\pm \mathrm{Im}\, z > 0\}.$$

Our **second step** is a semiclassical analysis of the equation (10.1) near the turning points $\pm z_{1,2}$, where the WKB formulae (10.38) do not work (see Remark 10.3). The points $\pm z_{1,2}$ are simple turning points of the equation (i.e., the simple roots of $\mu^2(z)$). Due to the symmetry $z \to -z$, it is enough to consider the points z_1, z_2 only.

Let $z_k,\ k = 1$ or 2, is one of the simple turning points under consideration. Following [Kap] (see also [Ble] and the semiclassical analysis performed in the section 8 above), consider the analytic change of the variable z, in some neighborhood of the point z_k, given by the equation

$$z \to w_k = \left(\frac{3}{2} \int_{z_k^N}^{z} \nu(z) dz \right)^{\frac{2}{3}}, \qquad k = 1, 2, \tag{10.80}$$

where

$$\begin{aligned}
\nu^2 &= -\det A + \frac{1}{N}\left(a_{11}' - a_{11}\frac{a_{12}'}{a_{12}} \right) - \frac{1}{2N^2}\left[\frac{a_{12}''}{a_{12}} - \frac{3}{2}\left(\frac{a_{12}'}{a_{12}}\right)^2 \right] \\
&= \left[\frac{g^2 z^6}{4} + \frac{t g z^4}{2} + \left(\frac{t^2}{4} - \frac{n}{N} g \right) z^2 - R_n \theta_{n-1}\theta_n \right] \\
&\quad - N^{-1}\left[\frac{t}{2} + \frac{3 g z^2}{2} + g R_n - \frac{g z^2 (t + g z^2 + 2 g R_n)}{g z^2 + \theta_n} \right] + N^{-2}\left[\frac{g(2 g z^2 - \theta_n)}{(g z^2 + \theta_n)^2} \right].
\end{aligned}$$

and the point z_k^N is determined by the conditions

$$\nu^2(z_k^N) = 0, \qquad z_k^N - z_k = O\left(N^{-1}\right).$$

Observe that ν^2 is nothing else than the potential $U(z)$ in the Schrödinger equation (3.26) [cf. (3.27)], and w_k is just the zeroth order approximation of the function $\phi(z)$ in (8.1). In fact, as in the section 7 above, we can neglect terms of the order of N^{-2} in ν^2 and put

$$\nu^2(z) = \left[\frac{g^2 z^6}{4} + \frac{t g z^4}{2} + \left(\frac{t^2}{4} - \frac{n}{N} g \right) z^2 \right] + \frac{1}{N}\left(\frac{t}{2} + g R_n - \frac{g z^2}{2} \right). \tag{10.81}$$

The rigorous justification of this cuttof follows immediately from Theorem 10.4 below.

Since,

$$w_k'(z) = \frac{\nu(z)}{w_k^{1/2}(z)} \to \mathrm{const} \neq 0 \qquad \text{as} \quad z \to z_k^N,$$

each branch of the multivalued function (10.80) indeed defines (for sufficiently large N) a nonsingular holomorphic change of variable in some *finite* disk

$$U_k = \left\{ z \in \mathbb{C} : |z - z_k^N| < \rho_k \right\}, \tag{10.82}$$

where ρ_k satisfy the inequalities

$$0 < \rho_1 < \min\{z_1, z_2 - z_1\}, \quad 0 < \rho_2 < z_2 - z_1.$$

We shall fix the branch of $w_k(z)$ by the equation

$$w_k(z) = e^{\frac{i\pi}{3}(2-k)} z_k (4g^3\lambda)^{1/6}(z - z_k) + O\left(N^{-1} + |z - z_k|^2\right), \qquad z \to z_k,\ N \to \infty. \tag{10.83}$$

Supplementing the change of variable $z \to w_k$ by the gauge transformation (see [Kap])

$$\Psi(z) = V_k(z)\Phi(w_k(z)),$$

where

$$V_k(z) = \begin{pmatrix} \sqrt{\frac{a_{12}}{w_k'}} & 0 \\ -\frac{1}{\sqrt{w_k' a_{12}}}\left[a_{11} - \frac{1}{2N}\left(\frac{a_{12}'}{a_{12}} - \frac{w_k''}{w_k'}\right)\right] & \sqrt{\frac{w_k'}{a_{12}}} \end{pmatrix}, \tag{10.84}$$

we bring the equation (10.1) to the form

$$\frac{d}{dw}\Phi(w) = \left\{ N \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix} + O\left(\frac{1}{N}\right) \right\} \Phi(w), \quad w \equiv w_k,$$

so that we can approximate it in the neighborhood U_k by the model equation

$$\frac{d}{dw}\Phi_0(w) = N \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix} \Phi_0(w), \quad w \equiv w_k. \tag{10.85}$$

The latter can be solved in terms of the Airy or Bessel functions. We shall choose the fundamental solution of (10.85) in the form

$$\Phi_0(w) = N^{\frac{1}{6}\sigma_3} \begin{pmatrix} y_1(N^{2/3}w) & y_2(N^{2/3}w) \\ y_1'(N^{2/3}w) & y_2'(N^{2/3}w) \end{pmatrix} \tag{10.86}$$

where $(') = \frac{d}{d\zeta}$ and

$$y_{1,2}(\zeta) = \zeta^{1/2} H_{1/3}^{(1,2)}\left(\frac{2i}{3}\zeta^{3/2}\right), \qquad i = e^{-\frac{3i\pi}{2}},$$

or in terms of the Airy function,

$$y_k(\zeta) = 2\sqrt{3}\left[(-1)^{k-1}\,\mathrm{Ai}\,(\zeta) + (k-1)\,\mathrm{Ai}\,(\zeta e^{2\pi i/3})\, e^{-2\pi i/3}\right], \qquad k = 1, 2.$$

The corresponding rigorous statement can be formulated as follows (see [Kap]).

Theorem 10.4. *Let $w_k(z), V_k(z)$, and $\Phi_0(w)$ be given by (10.80,81), (10.84), and (10.86), respectively. Assume that the quantities R_{n-1}, R_n and R_{n+1} are defined as in (10.32). Then, in each disk U_k, $k = 1, 2$, there exists a turning point solution $\Psi_k^{\mathrm{TP}}(z)$ of the equation (10.1), which satisfies, under the conditions (10.33), the estimate*

$$\Psi_k^{\mathrm{TP}}(z) = V_k(z)\left(I + O(N^{-1})\right)\Phi_0(w_k(z)), \tag{10.87}$$

uniformly in the star-shaped region

$$W_k = \left\{z \in U_k : |\mathrm{Re}\, N w_k^{3/2}(z)| \le \mathrm{const}\right\}, \tag{10.88}$$

The region W_k for $k = 1$ is depicted in Fig.6.

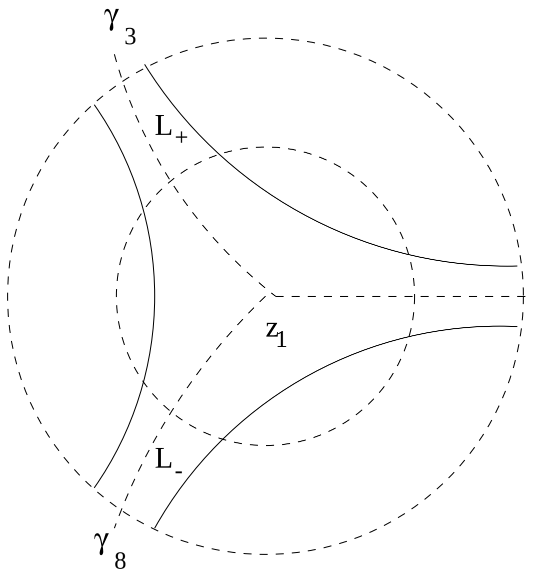

Fig. 6. The region W_1

Remark 10.4. One can also construct the *canonical* turning point solutions $\Psi_{k,l}^{\mathrm{TP}}(z)$, $l = 1, 2, 3$, which are specified by the asymptotic equations

$$\Psi_{k,l}^{\mathrm{TP}}(z) = V_k(z)\left(I + O(N^{-1})\right)\Phi_{0,l}(w_k(z))\sigma_{k,l}, \qquad N \to \infty, \tag{10.89}$$

where

$$\sigma_{1,1} = \sigma_1, \quad \sigma_{k,l} = I, \ (k,l) \neq (1,1),$$

and

$$z \in W_{k,l} = \left\{ z \in U_k : \frac{2\pi}{3}\left(l - \frac{5}{2}\right) + \varepsilon < \arg w_k(z) < \frac{2\pi}{3}\left(l - \frac{1}{2}\right) - \varepsilon \right\}, \quad l = 1,2,3.$$

The turning point domains $W_{k,l}$ are depicted in Fig.7.

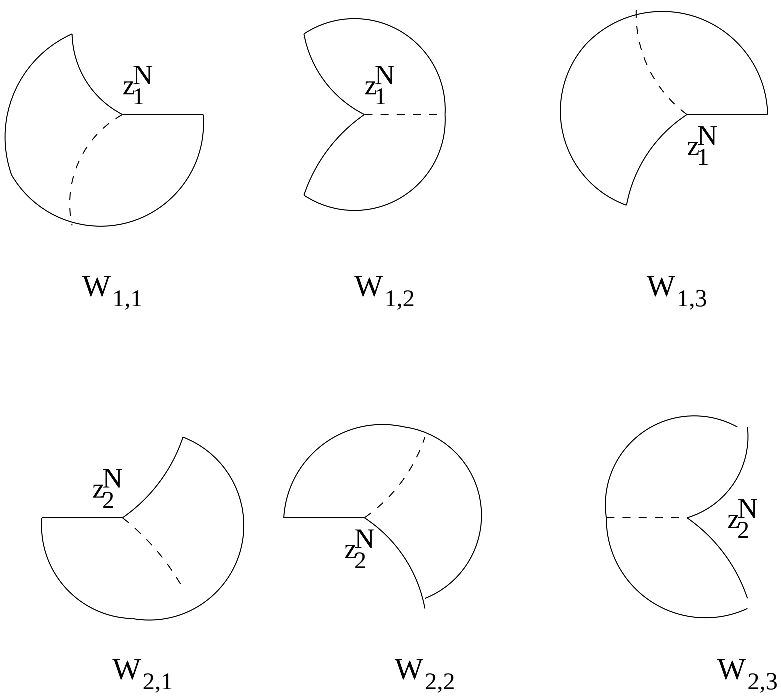

Fig. 7. The canonical Turning Point Domains $W_{k,l}$

The canonical solutions $\Phi_{0,l}(w)$, $l = 1,2,3$, of the model system (10.85) are given by the same formula (10.86) as for $\Phi_0(w)$ but with a different choice of the fundamental Airy functions $y_{1,2}(\zeta)$. We choose $y_{1,2}(\zeta) \equiv y_{1,2}^{(l)}(\zeta)$ in such a way that the asymptotics

$$\Phi_{0,l}(w) = w^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(I + O(N^{-1})\right) e^{-\frac{2}{3}Nw^{3/2}\sigma_3} \tag{10.90}$$

takes place when

$$Nw^{3/2} \to \infty, \qquad \frac{2\pi}{3}\left(l - \frac{5}{2}\right) + \varepsilon < \arg w < \frac{2\pi}{3}\left(l - \frac{1}{2}\right) - \varepsilon, \quad l = 1,2,3.$$

In fact,

$$
\begin{aligned}
\Phi_{0,1}(w) &= \sqrt{\frac{\pi}{3}}\,\Phi_0(w)\begin{pmatrix} e^{-i\frac{\pi}{3}} & e^{i\frac{\pi}{6}} \\ -e^{i\frac{\pi}{3}} & 0 \end{pmatrix}, \\
\Phi_{0,2}(w) &= \sqrt{\frac{\pi}{3}}\,\Phi_0(w)\begin{pmatrix} e^{-i\frac{\pi}{3}} & 0 \\ -e^{i\frac{\pi}{3}} & -e^{-i\frac{\pi}{6}} \end{pmatrix}, \\
\Phi_{0,3}(w) &= \sqrt{\frac{\pi}{3}}\,\Phi_0(w)\begin{pmatrix} e^{-i\frac{\pi}{3}} & 0 \\ 0 & -e^{-i\frac{\pi}{6}} \end{pmatrix}.
\end{aligned}
\tag{10.91}
$$

We are ready now to proceed with the **final step** in the asymptotic solution of the direct monodromy problem under consideration: evaluation of the Stokes matrices $\{S_j\}$ corresponding to the equation (10.1). According to the general scheme presented at the beginning of this section, we only need to calculate the Stokes parameters s_2, s_3 and to choose the normalization parameter λ_n in such a way that the equation (10.10) for the Stokes multiplier s_1 holds. To that end, it is convenient to evaluate the Stokes matrices between the canonical solutions Ψ_3, Ψ_4, and Ψ_8. Indeed, if we managed to evaluate the (constant in z !) matrix

$$
\begin{aligned}
Q = \Psi_3^{-1}(z)\Psi_8(z) = S_2^{-1}S_1^{-1}S_8^{-1} &= \begin{pmatrix} 1 - s_1 s_4 & -s_1 \\ s_4 - s_2 + s_1 s_2 s_4 & 1 + s_1 s_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 + 2\pi i \bar{s}_2 & 2\pi i \\ \bar{s}_2 - s_2 - 2\pi i |s_2| & 1 - 2\pi i s_2 \end{pmatrix}
\end{aligned}
\tag{10.92}
$$

then we would know the parameter s_2. Similarly, if we evaluated the matrix

$$
S_3 = \Psi_3^{-1}(z)\Psi_4(z) = \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix},
\tag{10.93}
$$

we would get the parameter s_3. Let us evaluate first the matrix Q.

The matrix Q admits the following factorization:

$$
Q = \Lambda_3^{-1}\Pi_3\Pi_8^{-1}\Lambda_8
\tag{10.94}
$$

where the constant in z matrices Λ_3, Π_3, Π_8, and Λ_8 are defined as

$$
\Lambda_{3,8} = \left[\Psi_{3,8}^{\mathrm{WKB}}(z)\right]^{-1}\Psi_{3,8}(z)
$$

and

$$
\Pi_{3,8} = \left[\Psi_{3,8}^{\mathrm{WKB}}(z)\right]^{-1}\Psi_1^{\mathrm{TP}}(z).
$$

Consider first the matrices $\Lambda_{3,8}$, which virtually have already been evaluated during the proof of Theorem 10.2. Indeed, it follows from (10.74) and (10.70) that

$$\Psi_j^{\mathrm{WKB}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} 0 & R_n^{1/2} \\ R_n^{1/2} & 0 \end{pmatrix} + O\left(\frac{1}{z^2}\right) \right\} e^{-\left(N\frac{V(z)}{2} - n\ln z + NC_n^j\right)\sigma_3}, \quad (10.95)$$

as $z \to \infty$, $N \geq N_0$, and

$$\frac{\pi}{8} + \varepsilon < \arg z < \frac{5\pi}{8} - \varepsilon,$$

for Ψ_3^{WKB}, and

$$-\frac{5\pi}{8} + \varepsilon < \arg z < -\frac{\pi}{8} - \varepsilon,$$

for Ψ_8^{WKB}. Formulae for C_n^j are given in (10.72), (10.73), and (10.68).

Comparing the asymptotics (10.95) with (10.4) and taking into account the equation

$$\Psi_j(z) = \Psi_j(ze^{2\pi i}),$$

i.e., the triviality of the monodromy group of the differential equation (10.1), we obtain the following *exact* formulae for the matrices $\Lambda_{3,8}$:

$$\Lambda_j = \begin{pmatrix} 1 & 0 \\ 0 & R_n^{-1/2} \end{pmatrix} e^{(NC_n^j - \lambda_n)\sigma_3}, \quad j = 3, 8, \quad N \geq N_0. \quad (10.96)$$

To evaluate the connection matrix Π_3 let us notice that for all z such that

$$\frac{1}{2}\rho_1 < |z - z_1^N| < \rho_1, \qquad 0 \leq \arg(z - z_1^N) \leq \pi, \quad (10.97)$$

the function $w_1(z)$ satisfies the following asymptotic equation:

$$\frac{2}{3} N w_1^{3/2}(z) = N\xi(z) + \frac{i\pi n}{2} - d(z) + ip + q + O\left(N^{-1}\right), \quad (10.98)$$

where

$$\begin{aligned} p &= \frac{\pi}{4}\left(1 + (-1)^n\right), \\ q &= \frac{1}{4}(-1)^n \ln \frac{1 - a + \sqrt{a^2 - 1}}{a - 1 + \sqrt{a^2 - 1}} \equiv \frac{1}{8}\ln\frac{g}{\lambda} + \frac{1}{4}\ln R_n, \\ d(z) &= \frac{1}{4}\ln \frac{(x + \sqrt{x^2 - 1} + a - (-1)^n\sqrt{a^2 - 1})^2}{2(x + a)}, \qquad a = -\frac{t}{2\sqrt{\lambda g}}, \end{aligned}$$

and $\xi(z)$ is defined in (10.40). From (10.98) it follows that the intersection of the half-annulus (10.97) and the turning point domain W_1 (see (10.88)) is not empty and it contains (for sufficiently large N) an arc of the Stokes line γ_3, where $\mathrm{Re}\,\xi(z) = 0$. We shall denote this arc by L_+ (see Fig.6).

Let us choose a closed set K in such a way that $L_+ \subset K \subset D_3$. Then by Theorem 10.2, in D_3 there exists a WKB-solution $\Psi_3^{\mathrm{WKB}}(z)$ which satisfies the asymptotics (10.38) on K. Let us compare $\Psi_3^{\mathrm{WKB}}(z)$ with $\Psi_1^{\mathrm{TP}}(z)$ on the Stokes line γ_3, or more precisely on the arc L_+. For all $z \in L_+$ the Bessel functions in (10.86) have a large argument, and they can be replaced by their known asymptotics. Taking into account that on L_+, $\arg \zeta^{3/2}$ is close to $\frac{3\pi}{2}$ we obtain the formula

$$\Phi_0(w_1(z)) = \sqrt{\frac{3}{\pi}}\, w_1^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left(I + O(N^{-1})\right) e^{-\frac{2}{3}N w_1^{3/2}} \begin{pmatrix} e^{i\frac{\pi}{3}} & 0 \\ 0 & -e^{i\frac{\pi}{6}} \end{pmatrix} \tag{10.99}$$

as $N \to \infty$ and $z \in L_+$. Noticing that

$$w_1' = w_1^{-1/2}\mu + O\left(N^{-1}\right), \tag{10.100}$$

we can rewrite on L_+ the gauge factor (10.84) as

$$V_1(z) = \sqrt{\frac{a_{12}^0}{\mu}} \left[\begin{pmatrix} 1 & 0 \\ -\frac{a_{11}}{a_{12}^0} & \frac{\mu}{a_{12}^0} \end{pmatrix} + O\left(N^{-1}\right) \right] w_1^{1/4\sigma_3}, \tag{10.101}$$

where

$$a_{12}^0 = R_n^{1/2} g z^2.$$

The formulae (10.99) and (10.101) yield the asymptotic equation

$$\begin{aligned} \Psi_1^{\mathrm{TP}}(z) = \sqrt{\frac{3a_{12}^0}{\pi\mu}} & \begin{pmatrix} 1 & 1 \\ -\frac{\mu+a_{11}}{a_{12}^0} & \frac{\mu-a_{11}}{a_{12}^0} \end{pmatrix} \left\{I + O\left(N^{-1}\right)\right\} \\ & \times e^{-\frac{2}{3}N w_1^{3/2}\sigma_3} \begin{pmatrix} e^{i\frac{\pi}{3}} & 0 \\ 0 & -e^{i\frac{\pi}{6}} \end{pmatrix}, \end{aligned} \tag{10.102}$$

as $N \to \infty$ and $z \in L_+$. The function $d(z)$ in (10.98) can be rewritten as

$$d(z) = \frac{1}{2}\ln\frac{\mu - a_{11}}{a_{12}^0} - \frac{1}{8}\ln\frac{\lambda}{16g} + \frac{1}{2}\ln\frac{R_n^{1/2}}{2},$$

which leads to the algebraic identity,

$$\begin{aligned} \sqrt{\frac{a_{12}^0}{\mu}} \begin{pmatrix} 1 & 1 \\ -\frac{\mu+a_{11}}{a_{12}^0} & \frac{\mu-a_{11}}{a_{12}^0} \end{pmatrix} e^{d(z)\sigma_3} &= \sqrt{\frac{\mu - a_{11}}{2\mu}} \begin{pmatrix} 1 & \frac{a_{12}^0}{\mu-a_{11}} \\ \frac{a_{12}^0}{\mu-a_{11}} & 1 \end{pmatrix} \\ \times \left(\frac{\lambda}{16g}\right)^{-\frac{\sigma_3}{8}} R_n^{\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} &= T_0(z)\sqrt{2}\left(\frac{\lambda}{g}\right)^{-\frac{\sigma_3}{8}} R_n^{\frac{\sigma_3}{4}}, \end{aligned} \tag{10.103}$$

where in the last equality we took into account (10.39) as well. The equations (10.103) and (10.98) allow us to rewrite the formula (10.102) in the form

$$\Psi_1^{\mathrm{TP}}(z) = \sqrt{\frac{6}{\pi}} T_0(z) \left\{ I + O\left(N^{-1}\right) \right\} e^{\left(-N\xi - i\frac{\pi n}{2} - ip\right)\sigma_3} \begin{pmatrix} e^{i\frac{\pi}{3}} & 0 \\ 0 & -e^{i\frac{\pi}{6}} \end{pmatrix},$$

or, in view of (10.42),

$$\Psi_1^{\mathrm{TP}}(z) = \sqrt{\frac{6}{\pi}} T_0(z) \left\{ I + O\left(N^{-1}\right) \right\} e^{\left(-N\int_{z_{k(3)}}^{z} \mu(u)du - ip\right)\sigma_3} \begin{pmatrix} e^{i\frac{\pi}{3}} & 0 \\ 0 & -e^{i\frac{\pi}{6}} \end{pmatrix}, \quad (10.104)$$

as $N \to \infty$ and $z \in L_+$.

Comparing the equation (10.104) with the WKB-formula (10.38), $j = 3$, we end up with the following estimate for the connection matrix Π_3 :

$$\Pi_3 = \sqrt{\frac{6}{\pi}} e^{-ip\sigma_3} \begin{pmatrix} e^{i\frac{\pi}{3}} & 0 \\ 0 & -e^{i\frac{\pi}{6}} \end{pmatrix} + O(N^{-1}). \quad (10.105)$$

The evaluation of the matrix Π_8 is carried out along the same lines. However, there are some technical yet important differences:

(i) Because of our choice (see (10.44) and (10.83)) of the branches of the functions $\xi(z)$ and $w_k(z)$, the equation (10.98) should be replaced by the equation

$$-\frac{2}{3} N w_1^{3/2}(z) = N\xi(z) - \frac{i\pi n}{2} - d(z) - ip + q + O\left(N^{-1}\right), \quad (10.106)$$

which holds now in the semiannulus

$$\frac{1}{2}\rho_1 < |z - z_1^N| < \rho_1, \qquad -\pi \le \arg(z - z_1^N) \le 0. \quad (10.107)$$

(ii) On the arc L_-, which is an intersection of the Stokes line γ_8 with the semiannulus (10.107) (see again Fig.6), $\arg \zeta^{3/2}$ is close to $-\frac{\pi}{2}$. This means that instead of (10.99) we have now that

$$\Phi_0(w_1(z)) = \sqrt{\frac{3}{\pi}} w_1^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(I + O(N^{-1})\right) e^{\frac{2}{3} N w_1^{3/2}} \begin{pmatrix} e^{-i\frac{\pi}{6}} & -e^{i\frac{\pi}{6}} \\ 0 & -e^{-i\frac{\pi}{3}} \end{pmatrix}, \quad (10.108)$$

as $N \to \infty$, $\quad z \in L_-$.

(iii) Since we always assume that $\mu(z)$ is defined according to (10.44), the right hand side of (10.100) gains factor $e^{i\pi}$, which in turn replaces the formula (10.101) by the one

$$V_1(z) = -i\sqrt{\frac{a_{12}^0}{\mu}} \left[\begin{pmatrix} 1 & 0 \\ -\frac{a_{11}}{a_{12}^0} & \frac{-\mu}{a_{12}^0} \end{pmatrix} + O\left(N^{-1}\right) \right] w_1^{1/4\sigma_3}. \quad (10.109)$$

The completion of the derivation is exactly the same as in the case of Π_3. The formulae (10.108), (10.109), and (10.106) yield the asymptotic equation

$$\Psi_1^{\mathrm{TP}}(z) = -i\sqrt{\frac{6}{\pi}}\,T_0(z)\left\{I + O\left(N^{-1}\right)\right\}\, e^{\left(-N\int_{z_{k(8)}}^{z}\mu(u)du+ip\right)\sigma_3}\begin{pmatrix} e^{-i\frac{\pi}{6}} & -e^{i\frac{\pi}{6}} \\ 0 & -e^{-i\frac{\pi}{3}} \end{pmatrix}, \tag{10.110}$$

as $N \to \infty$ and $z \in L_-$. In obtaining (10.110) we also took into account (10.42). Comparing the equation (10.110) with the WKB-formula (10.38), $j = 8$, we get the following estimate for the connection matrix Π_8:

$$\Pi_8 = \sqrt{\frac{6}{\pi}}\,e^{ip\sigma_3}\begin{pmatrix} e^{-2i\frac{\pi}{3}} & -e^{-i\frac{\pi}{3}} \\ 0 & e^{i\frac{\pi}{6}} \end{pmatrix} + O(N^{-1}). \tag{10.111}$$

To complete the evaluation of the matrix Q we only need to substitute the formulae (10.96), (10.105), and (10.111) into the equation (10.94) and take into account the definitions (see (10.71-73)) of constants C_n^j. This gives us the following representation for the entries q_{jk} of matrix Q :

$$q_{11} = \overline{q_{22}} = -e^{-i\pi n - 2ip} + O(N^{-1}) = 1 + O(N^{-1}), \tag{10.112}$$

$$q_{12} = ie^{-2NC_n + \frac{1}{4}\ln\frac{g}{\lambda} + 2\lambda_n - 2q}\left(1 + O(N^{-1})\right), \tag{10.113}$$

$$q_{21} = e^{2NC_n - \frac{1}{4}\ln\frac{g}{\lambda} - 2\lambda_n + 2q}O(N^{-1}), \tag{10.114}$$

where the last equality in (10.112) is due to the exact formula for p given in (10.98)

¿From (10.112) we conclude (see (10.92)) that

$$s_2 = O(N^{-1}). \tag{10.115}$$

The normalization condition $s_1 = -2\pi i = -q_{12}$ and (10.113) imply the asymptotic equation

$$\lambda_n = NC_n - \frac{1}{8}\ln\frac{g}{\lambda} + q + \frac{1}{2}\ln 2\pi + O(N^{-1}) \tag{10.116}$$

for the normalization parameter λ_n, which in turn transforms (10.114) into the estimate

$$q_{21} = O(N^{-1}), \tag{10.117}$$

We will improve this estimate significantly later on.

Remark 10.5. If instead of (10.32) we had assumed the weaker asymptotic conditions

$$R_n = O(1), \qquad \theta_n, \theta_{n-1} = O\left(N^{-1}\right),$$

all the above analysis would have gone through. The only difference would be in the expressions for the quantities p and q: instead of the equations given in (10.98) we would have that

$$p = \frac{\pi}{4}\left(1 - \sqrt{\frac{g}{\lambda}}\frac{R_n + t/2g}{\sqrt{a^2-1}}\right)$$

and

$$q = -\frac{1}{4}\sqrt{\frac{g}{\lambda}}\frac{R_n + t/2g}{\sqrt{a^2-1}}\ln\frac{1-a+\sqrt{a^2-1}}{a-1+\sqrt{a^2-1}}.$$

Imposing *afterwords* the constraint $s_2 = 0 \Longrightarrow q_{11} = 1$, we would come again to the equation (10.112), but this time as an equation *for* R_n, which obviously leads to ansatz (10.32). Hence *the equation* $s_2 = 0$ *is identified with the Bohr-Sommerfeld quantization condition from the section 7 above (cf. (7.12-15)).*

The only monodromy parameter which is left is s_3. For its evaluation we, just like in the case of the matrix Q, shall use the factorization

$$S_3 = \Lambda_3^{-1}\Pi_{03}\Pi_{04}^{-1}\Lambda_4 \tag{10.118}$$

where

$$\Lambda_{3,4} = \left[\Psi_{3,4}^{\mathrm{WKB}}(z)\right]^{-1}\Psi_{3,4}(z)$$

and

$$\Pi_{03,04} = \left[\Psi_{3,4}^{\mathrm{WKB}}(z)\right]^{-1}\Psi_0^{\mathrm{TP}}(z).$$

The matrices $\Lambda_{3,4}$ are given by the same equation (10.96) as we have for the ones $\Lambda_{3,8}$. In addition, due to (10.116) we obtain that

$$\Lambda_3 = \Lambda_4 = \begin{pmatrix} 1 & 0 \\ 0 & R^{-1/2}\end{pmatrix} e^{\left(\frac{1}{8}\ln\frac{g}{\lambda} + i\frac{n\pi}{2} - \frac{1}{2}\ln 2\pi - q\right)\sigma_3}\left(I + O(N^{-1})\right). \tag{10.119}$$

The evaluation of the connection matrices $\Pi_{03,04}$ is very simple since the asymptotic representation (10.38) for the WKB-solutions $\Psi_{3,4}^{\mathrm{WKB}}(z)$ given in Theorem 10.2 is virtually the same as the asymptotic representation (10.79) for the turning point solution $\Psi_0^{\mathrm{TP}}(z)$ given in Theorem 10.3. This means that the matching procedure this time is absolutely trivial (again, a benefit of the initial ansatz (10.32) !). Indeed, comparing the equations (10.38) with (10.79) in the intersections of the turning point domain W_0 and the Stokes lines $\gamma_{3,4}^0$ (we recall that the WKB-formula (10.38) is valid up to the point z=0) we obtain that

$$\Pi_{03} = \Pi_{04} = e^{-N\int_0^{z_1}\mu(u)du\sigma_3}\left(I + O\left(N^{-1}\right)\right) = e^{N\delta_0\sigma_3}\left(I + O\left(N^{-1}\right)\right), \tag{10.120}$$

where

$$\delta_0 = \frac{\lambda}{2}\left[a\sqrt{a^2-1} - \ln(a+\sqrt{a^2-1})\right] > 0,$$

and n is supposed to be even. For odd n, one has to put i in front of the r.h.s. of the last equation in (10.120).

The formulae (10.118-120) lead to the asymptotic equation

$$S_3 = e^{N\delta_0\sigma_3}\left(I + O\left(N^{-1}\right)\right)e^{-N\delta_0\sigma_3}, \tag{10.121}$$

which in turn implies the following estimate for the Stokes parameter s_3:

$$s_3 = s^0 e^{2N\delta_0}, \qquad s^0 = O\left(N^{-1}\right). \tag{10.122}$$

Along with the factorization (10.94) one can suggest another factorization of the matrix Q, which uses the possibility of connecting the canonical solutions Ψ_3 and Ψ_8 through the double turning point $z = 0$:

$$Q = \Lambda_3^{-1}\Pi_{03}\Pi_{08}^{-1}\Lambda_8,$$

where

$$\Pi_{08} = \left[\Psi_8^{\mathrm{WKB}}(z)\right]^{-1}\Psi_0^{\mathrm{TP}}(z).$$

The calculation of the matrix Π_{08} is similar to the one of the matrices $\Pi_{03,04}$, and it leads to an estimate similar to (10.121):

$$Q = e^{N\delta_0\sigma_3}\left(I + O\left(N^{-1}\right)\right)e^{-N\delta_0\sigma_3}.$$

This equation provides us with the following improvement of the asymptotics (10.117):

$$q_{21} = q^0 e^{-2N\delta_0}, \qquad q^0 = O\left(N^{-1}\right). \tag{10.123}$$

Remark 10.6. This analysis also gives the following representation of the canonical solutions:

$$\Psi_j(z) = \Psi_j^{\mathrm{WKB}}(z)\Lambda_j = \Psi^{\mathrm{TP}}_{2,l_j^{(2)}}(z)\Pi_{2,j}^{-1}\Lambda_j, \qquad j = 1,2, \quad l_j^{(2)} = j\,; \tag{10.124a}$$

$$\Psi_j(z) = \Psi_j^{\mathrm{WKB}}(z)\Lambda_j = \Psi^{\mathrm{TP}}_{1,l_j^{(1)}}(z)\Pi_{1,j}^{-1}\Lambda_j, \qquad j = 8,3, \quad l_8^{(1)} = 1,\ l_3^{(1)} = 3\,, \tag{10.124b}$$

where the WKB-solutions $\Psi_j^{\mathrm{WKB}}(z)$ and the canonical turning point solutions $\Psi_{k,l}^{\mathrm{TP}}(z)$ are described in Theorem 10.2 and Remark 10.4, respectively. The matrices Λ_j are given in (10.96) and the matrices $\Pi_{k,j}$ satisfy the asymptotic equations (cf. the evaluation of the matrices Π_3 and Π_8 above)

$$\begin{aligned} &\Pi_{1,3} = 2^{\frac{1}{2}}e^{-ip\sigma_3} + O\left(N^{-1}\right), \qquad \Pi_{1,8} = -i2^{\frac{1}{2}}e^{ip\sigma_3} + O\left(N^{-1}\right), \\ &\Pi_{2,1} = \Pi_{2,2} = 2^{\frac{1}{2}}I + O\left(N^{-1}\right), \end{aligned} \tag{10.125}$$

and

$$e^{-N\xi(z)\sigma_3}\Pi_{k,j}e^{N\xi(z)\sigma_3} = O(1)\,; \qquad N\to\infty, \quad z\in W_{k,l_j^{(k)}}\cap\left\{z: |z-z_k|\geq \frac{1}{2}\rho_k\right\}.$$

Similar representations for the rest of the canonical solutions can be obtained by the use of the symmetry, $z\to -z$. Let us emphasize that all the matrices Λ_j and $\Pi_{k,j}$ do not depend on z.

10.5. Asymptotic Solution of the Inverse Monodromy Problem. Completion of the Proof of the Main Theorem

The results obtained above in the analysis of the direct monodromy problem for the model differential equation (10.1), (10.32) can be used for the construction of an asymptotic solution of the main Riemann-Hilbert problem (5.16-18) via two different ways. In the first approach (cf. [FIK4]) one construct the asymptotic solution of (5.16 -18) with the help of the canonical solutions of the system (10.1), (10.32) and use the fact that, due to the equations (10.124), the asymptotic behaviour of these canonical solutions is under a complete control. Taking also into account the estimates (10.115), (10.122), and (10.123) for the model monodromy problem, it is easy to see that the "dressed" monodromy data, i.e., the matrices

$$e^{-N\xi(z)\sigma_3}S_k e^{N\xi(z)\sigma_3},$$

corresponding to the model and to the main systems are close to each other. Using then some general results concerning the matrix Riemann-Hilbert problem, one proves that the asymptotic solution approaches the genuine solution of (5.16-18) as $N\to\infty$. The details are given in the Appendix 1.

The second approach is based on the Deift-Zhou method [DZ]. In this approach we define explicitly, using the semiclassical formulae of the section 10.4, a piecewise analytic function $\Psi^0(z)$ which is expected to approximate the solution $\Psi_n(z)$ of the basic Riemann-Hilbert problem (5.16-18) uniformly on the whole z-plane. Similar to the first scheme, the function $\Psi^0(z)$ is closely related to the canonical solutions of the equation (10.1).

In this section we follow the second scheme. The first step in this scheme is to define explicitly a matrix-valued picewise analytic function $\Psi^0(z)$. Define the number

$$R_n^0 = \frac{-t-(-1)^n(t^2-4\lambda g)^{1/2}}{2g}, \qquad \lambda = \frac{n}{N}, \tag{10.125a}$$

and the matrix

$$A^0 = A^0(z) = \begin{pmatrix} a_{11}^0 & a_{12}^0 \\ a_{21}^0 & a_{22}^0 \end{pmatrix},$$

where

$$
\begin{aligned}
&a^0_{11} = -\left(\frac{tz}{2} + \frac{gz^3}{2} + gzR^0_n\right) = \alpha_n z - \frac{gz^3}{2}, \qquad \alpha_n = \frac{(-1)^n(t^2-4\lambda g)^{1/2}}{2}, \\
&a^0_{12} = (R^0_n)^{1/2} gz^2, \\
&a^0_{21} = -a^0_{12}, \qquad a^0_{22} = -a^0_{11}.
\end{aligned} \tag{10.125b}
$$

Define then the disks

$$U^0_k = \{z : |z - z_k| \le \rho_k\}, \quad k = 1, 2,$$

where

$$0 < \rho_1 < \min\{z_1, z_2 - z_1\}, \qquad 0 < \rho_2 < z_2 - z_1.$$

and

$$z_{1,2} = \left(\frac{-t \mp 2\sqrt{\lambda g}}{g}\right)^{1/2}, \quad \lambda = \frac{n}{N},$$

are the zeros of the μ-function,

$$\mu(z) = \sqrt{-\det A^0(z)} = \frac{z}{2}\sqrt{(t+gz^2)^2 - 4\lambda g} = \frac{zg}{2}\sqrt{(z^2-z_1^2)(z^2-z_2^2)}. \tag{10.126}$$

Let

$$\xi(z) \equiv \int_{z_2}^{z} \mu(u)du = \frac{\lambda}{2}\left[x\sqrt{x^2-1} - \ln(x+\sqrt{x^2-1})\right], \tag{10.127}$$

where

$$x = \frac{t+gz^2}{2\sqrt{\lambda g}}.$$

(cf. (10.40-42)). Observe that

$$\mu(z) = z\sqrt{\lambda g}\sqrt{x^2-1},$$

(cf. (10.43-44)). We fix the branches of the multivalued functions in (10.126-7) by the inequalities

$$-\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \quad -\pi < \arg\sqrt{x^2-1} < \pi, \quad -\pi < \arg(x+\sqrt{x^2-1}) < \pi.$$

This defines $\mu(z)$ as an analytic function on $\mathbb{C}$, with two cuts $[-z_2, -z_1]$ and $[z_1, z_2]$. The function $\xi(z)$ is analytic as well, and it has one cut $(-\infty, z_2]$.

Let us now choose the radii ρ_1 and ρ_2 in such a way that $\rho_1 + \rho_2 > z_2 - z_1$. Then

$$\emptyset \ne \partial U^0_1 \cap \partial U^0_2 = \{z_0, \bar{z}_0\}, \quad \operatorname{Im} z_0 > 0.$$

Introduce the arcs (see Fig.8)

$$C_2^u = \{z : |z - z_2| = \rho_2,\ 0 \le \arg(z - z_2) \le \arg(z_0 - z_2)\}\,, \quad C_2^d = \tau^*(C_2^u),$$
$$C_1^u = \{z : |z - z_1| = \rho_1,\ \arg(z_0 - z_1) \le \arg(z - z_1) \le \pi\}\,, \quad C_1^d = \tau^*(C_1^u),$$

where

$$\tau^* : z \to \tau^*(z) = \bar{z}.$$

All the arcs are supposed to be positively oriented with respect to the corresponding disk U_k^0. Together with the segment $l_0 = [\bar{z}_0, z_0]$ they form the boundaries for simply connected subsets Ω_1 and Ω_2 of the disks U_1^0 and U_2^0, respectively:

$$\partial\Omega_1 = l_0 \cup C_1^u \cup C_1^d,$$
$$\partial\Omega_2 = l_0 \cup C_2^d \cup C_2^u.$$

The segment l_0 is supposed to be positevely oriented with respect to Ω_1. We shall also take the following convention: given any set $\Omega \subset \mathbb{C}$ we denote by Ω^u and Ω^d the closed upper and lower parts of set Ω with respect to the real axis. We illustrate the above geometrical constructions in Fig.8.

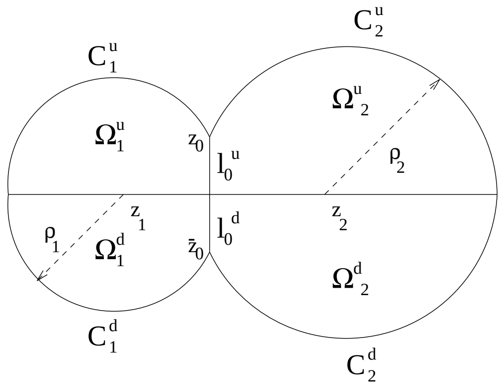

Fig. 8. The sets $\Omega_{1,2}^{u,d}$

Consider the region

$$\mathbb{C}^* = \mathbb{C} \setminus \left(\Omega_1 \cup \Omega_2 \cup \tau(\Omega_1) \cup \tau(\Omega_2)\right), \quad \tau : z \to \tau(z) = -z.$$

The WKB-analysis performed in the section 10.4 (see Theorem 10.2) suggests the following explicit matrix-valued function as a candidate for the asymptotic representation of the solution of the orthogonal polynomial Riemann-Hilbert problem (5.16-18) in the domain $\mathbb{C}^*$:

$$\Psi^{\rm WKB}_{\rm formal}(z) = T_0(z)e^{-N\int_{z_2}^{z}\mu(u)du\sigma_3} = T_0(z)e^{-N\xi(z)\sigma_3}, \tag{10.128}$$

where

$$T_0(z) = \sqrt{\frac{\mu - a^0_{11}}{2\mu}}\begin{pmatrix} 1 & \frac{a^0_{12}}{\mu - a_{11}} \\ \frac{a^0_{12}}{\mu - a_{11}} & 1 \end{pmatrix}.$$

$\mu(z)$ and $\xi(z)$ are defined in (10.126) and (10.127), respectively, and a^0_{11}, a^0_{12} are the elements (1.125b) of the matrix A^0. It is very important for us that both the function $T_0(z)$ and $e^{-N\xi(z)\sigma_3}$ have the same multiplicative jump, $(-1)^n$, over the interval $[-z_1, z_1]$, so that their product, $\Psi^{\rm WKB}_{\rm formal}(z)$, is a *single-valued* analytic function on $\mathbb{C}^*$, which, in addition, satisfies the symmetry equation

$$\Psi^{\rm WKB}_{\rm formal}(z) = (-1)^n\sigma_3\Psi^{\rm WKB}_{\rm formal}(-z)\sigma_3.$$

Later we will define the function $\Psi^0(z)$ in $\mathbb{C}^*$ in terms of $\Psi^{\rm WKB}_{\rm formal}(z)$ (see the formula (10.135) below).

To construct $\Psi^0(z)$ in the sets Ω_k and $\tau(\Omega_k)$ we use the turning points analysis of the section 10.4 (see Theorem 10.4 and Remark 10.4). To that end, we recall the definition (10.80-83) of the change-of-variable functions $w_{1,2}(z)$:

$$w_k = \left(\frac{3}{2}\int_{z_k^N}^{z}\nu(z)dz\right)^{\frac{2}{3}}, \quad k = 1, 2, \tag{10.129}$$

$$\nu^2(z) = \mu^2(z) + \frac{1}{N}\left(\frac{t}{2} + gR_n - \frac{gz^2}{2}\right),$$

$$\nu^2(z_k^N) = 0, \quad z_k^N - z_k = O\left(N^{-1}\right), \quad z_{1,2} = \left(\frac{-t \mp 2\sqrt{\lambda g}}{g}\right)^{1/2},$$

$$w_k(z) = e^{\frac{i\pi}{3}(2-k)}z_k(4g^3\lambda)^{1/6}(z - z_k) + O\left(N^{-1} + |z - z_k|^2\right), \qquad z \to z_k, \quad N \to \infty.$$

Observe that the function $\varphi_N(z)$ of Theorem 1.1 is given by the equation

$$\varphi_N(z) = \begin{cases} w_2(z), & \text{for} \quad k = 2, \\ e^{2\pi i/3}w_1(z), & \text{for} \quad k = 1. \end{cases}$$

For sufficiently large N, the sets U^0_k and $\Omega_k \subset U^0_k$ are both included into the domain of analyticity of the function $w_k(z)$, $k = 1, 2$. Therefore, we can define holomorphic in $z \in \Omega_k$ matrix-valued functions $\Psi^u_k(z)$ and $\Psi^d_k(z)$ by the equations:

$$\Psi^u_1(z) = 2^{-\frac{1}{2}}V^0_1(z)\Phi_{0,3}(w_1(z))\Lambda_0 e^{(ip+i\frac{n\pi}{2})\sigma_3}, \qquad z \in \Omega_1, \tag{10.130a}$$

$$\Psi^d_1(z) = 2^{-\frac{1}{2}}V^0_1(z)\Phi_{0,1}(w_1(z))i\sigma_1\Lambda_0 e^{-(ip+i\frac{n\pi}{2})\sigma_3}, \qquad z \in \Omega_1 \tag{10.130b}$$

$$\Psi^u_2(z) = 2^{-\frac{1}{2}}V^0_2(z)\Phi_{0,2}(w_2(z))\Lambda_0, \qquad z \in \Omega_2, \tag{10.130c}$$

$$\Psi^d_2(z) = 2^{-\frac{1}{2}}V^0_2(z)\Phi_{0,1}(w_2(z))\Lambda_0, \qquad z \in \Omega_2, \tag{10.130d}$$

(cf. (10.89), (10.124), (10.125)), where $w_{1,2}(z)$ are given in (10.129), then,

$$p = \frac{\pi}{4}\,(1 + (-1)^n),$$

and the rest of the objects in (10.130a-d) is defined as follows:

1. Model canonical solutions $\Phi_{0,l}(z)$ (cf. (10.86), (10.91)) **:**

$$\Phi_{0,1}(w) = \sqrt{\frac{\pi}{3}}\Phi_0(w)\begin{pmatrix} e^{-i\frac{\pi}{3}} & e^{i\frac{\pi}{6}} \\ -e^{i\frac{\pi}{3}} & 0 \end{pmatrix}, \qquad \Phi_{0,2}(w) = \sqrt{\frac{\pi}{3}}\Phi_0(w)\begin{pmatrix} e^{-i\frac{\pi}{3}} & 0 \\ -e^{i\frac{\pi}{3}} & -e^{-i\frac{\pi}{6}} \end{pmatrix},$$
$$\Phi_{0,3}(w) = \sqrt{\frac{\pi}{3}}\Phi_0(w)\begin{pmatrix} e^{-i\frac{\pi}{3}} & 0 \\ 0 & -e^{-i\frac{\pi}{6}} \end{pmatrix}, \quad \Phi_0(w) = N^{\frac{1}{6}\sigma_3}\begin{pmatrix} y_1(N^{2/3}w) & y_2(N^{2/3}w) \\ y_1'(N^{2/3}w) & y_2'(N^{2/3}w) \end{pmatrix},$$
$$y_{1,2}(\zeta) = \zeta^{1/2}H^{(1,2)}_{1/3}(\frac{2}{3}i\zeta^{3/2}), \qquad i = e^{-\frac{3i\pi}{2}}, \qquad (') = \frac{d}{d\zeta}\,.$$

2. Gauge factors $V_k^0(z)$ (cf. (10.84)) **:**

$$V_k^0(z) = \sqrt{\frac{a_{12}^0}{w_k'}}\begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0}{a_{12}^0} & \frac{w_k'}{a_{12}^0} \end{pmatrix}, \qquad (') = \frac{d}{dz},$$

where a_{11}^0 and a_{12}^0 are given in (10.125b).

3. Constant matrix Λ_0 (cf. (10.119)) **:**

$$\Lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & (R_n^0)^{-1/2} \end{pmatrix} e^{(\frac{1}{8}\ln\frac{g}{\lambda} - q - \frac{1}{2}\ln 2\pi)\sigma_3},$$
$$q = \frac{1}{4}(-1)^n \ln\frac{1 - a + \sqrt{a^2-1}}{a - 1 + \sqrt{a^2-1}} \equiv \frac{1}{8}\ln\frac{g}{\lambda} + \frac{1}{4}\ln R_n^0, \qquad a = -\frac{t}{2\sqrt{\lambda g}}.$$

We note that

$$\Lambda_0 = (R_n^0)^{-1/4}(2\pi)^{-\frac{\sigma_3}{2}},$$

therefore the *exact* equation

$$\Psi_k^u(z) = \Psi_k^d(z)\begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix}, \tag{10.131}$$

takes place.

An elementary analysis leads to the following equations (cf. (10.98), (10.106)) for the functions $w_{1,2}(z)$:

$$\frac{2}{3}Nw_1^{3/2}(z) = N\xi(z) + \frac{i\pi n}{2} - d(z) + ip + q + O\left(N^{-1}\right), \qquad (10.132a)$$

$$z \in (\partial\Omega_1)^u, \quad \frac{\pi}{3} \le \arg w_1 \le \frac{4\pi}{3}\,;$$

$$-\frac{2}{3}Nw_1^{3/2}(z) = N\xi(z) - \frac{i\pi n}{2} - d(z) - ip + q + O\left(N^{-1}\right), \qquad (10.132b)$$

$$z \in (\partial\Omega_1)^d, \quad -\frac{2\pi}{3} \le \arg w_1 \le \frac{\pi}{3}\,;$$

$$\frac{2}{3}Nw_2^{3/2}(z) = N\xi(z) - d(z) + q + O\left(N^{-1}\right), \qquad (10.132c)$$

$$z \in \partial\Omega_2, \quad -\pi \le \arg w_2 \le \pi\,,$$

where

$$d(z) = \frac{1}{4}\ln\frac{(x+\sqrt{x^2-1}+a-(-1)^n\sqrt{a^2-1})^2}{2(x+a)}.$$

Simultaneously, we have (cf. (10.100)) that

$$\begin{aligned} w_1' &= w_1^{-1/2}\mu + O\left(N^{-1}\right), \quad z \in (\partial\Omega_1)^u, \\ w_1' &= e^{i\pi}w_1^{-1/2}\mu + O\left(N^{-1}\right), \quad z \in (\partial\Omega_1)^d, \\ w_2' &= w_2^{-1/2}\mu + O\left(N^{-1}\right), \quad z \in \partial\Omega_2, \end{aligned}$$

which implies the following equations (cf. (10.101), (10.109)) for the gauge factors $V_{1,2}^0(z)$:

$$V_1^0(z) = \sqrt{\frac{a_{12}^0}{\mu}}\left[\begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0}{a_{12}^0} & \frac{\mu}{a_{12}^0}\end{pmatrix} + O\left(N^{-1}\right)\right]w_1^{1/4\sigma_3}, \quad z \in (\partial\Omega_1)^u, \qquad (10.133a)$$

$$V_1^0(z) = -i\sqrt{\frac{a_{12}^0}{\mu}}\left[\begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0}{a_{12}^0} & \frac{-\mu}{a_{12}^0}\end{pmatrix} + O\left(N^{-1}\right)\right]w_1^{1/4\sigma_3}, \quad z \in (\partial\Omega_1)^d, \qquad (10.133b)$$

$$V_2^0(z) = \sqrt{\frac{a_{12}^0}{\mu}}\left[\begin{pmatrix} 1 & 0 \\ -\frac{a_{11}^0}{a_{12}^0} & \frac{\mu}{a_{12}^0}\end{pmatrix} + O\left(N^{-1}\right)\right]w_2^{1/4\sigma_3}, \quad z \in \partial\Omega_2. \qquad (10.133c)$$

Finally, taking into account the asymptotics of the Bessel functions and the algebraic identity (see (10.103)),

$$\sqrt{\frac{a_{12}^0}{\mu}}\begin{pmatrix} 1 & 1 \\ -\frac{\mu+a_{11}^0}{a_{12}^0} & \frac{\mu-a_{11}^0}{a_{12}^0}\end{pmatrix}e^{d(z)\sigma_3} = T_0(z)\sqrt{2}\left(\frac{\lambda}{g}\right)^{-\frac{\sigma_3}{8}}(R_n^0)^{\frac{\sigma_3}{4}} = T_0(z)2^{\frac{1}{2}}e^{q\sigma_3},$$

we derive from (10.132-133) the following crucial asymptotic properties of the functions $\Psi_k^{u,d}(z)$ (cf. (10.104), (10.110)):

(i) on the boundary of the sets $\Omega_k,\ k=1,2$, except for the segment l_0, the asymptotic behaviour of $\Psi_k^{u,d}(z)$, as $N\to\infty$, matches that of the WKB-ansatz (10.128). Namely, we have:

$$\Psi_k^{u,d}(z)=T_0(z)\left\{I+O\left(N^{-1}\right)\right\}e^{-N\xi(z)\sigma_3}\Lambda_0\,,\quad N\to\infty, \tag{10.134a}$$

uniformly in $z\in\overline{(\partial\Omega_k)^{u,d}\setminus l_0^{u,d}}$ where the bar means the closure;

(ii) on the segment l_0^u (l_0^d) the functions $\{\Psi_k^u(z)\}_{k=1,2}$ ($\{\Psi_k^d(z)\}_{k=1,2}$, respectively) have the following asymptotics independent of k :

$$\Psi_k^u(z)=T_0(z)\left\{I+O\left(N^{-1}\right)\right\}e^{-N\xi(z)\sigma_3}\begin{pmatrix}1&0\\ i&1\end{pmatrix}\Lambda_0\,,\quad N\to\infty, \tag{10.134b}$$

uniformly in $z\in l_0^u$, and

$$\Psi_k^d(z)=T_0(z)\left\{I+O\left(N^{-1}\right)\right\}e^{-N\xi(z)\sigma_3}\begin{pmatrix}1&0\\ -i&1\end{pmatrix}\Lambda_0\,,\quad N\to\infty, \tag{10.134c}$$

uniformly in $z\in l_0^d$.

Put now

$$\Psi^0(z)=\begin{cases}T_0(z)e^{-N\xi(z)\sigma_3}\Lambda_0 & \text{for}\quad z\in\mathbb{C}\setminus\left(\Omega_1\cup\Omega_2\cup\Omega_{-1}\cup\Omega_{-2}\right)\equiv\mathbb{C}^*,\\ \Psi_k^{u,d}(z) & \text{for}\quad z\in\Omega_k^{u,d},\quad k=1,2,\\ \Psi_{-k}^{u,d}(z) & \text{for}\quad z\in\Omega_{-k}^{u,d},\quad k=1,2,\end{cases} \tag{10.135}$$

where we use the notations

$$\Omega_{-k}^{u,d}=\tau(\Omega_k^{u,d}),\qquad \Psi_{-k}^{u,d}(z)=(-1)^n\sigma_3\Psi_k^{d,u}(-z)\sigma_3,\quad k=1,2.$$

We would like to recall that the product $T_0(z)e^{-N\xi(z)\sigma_3}$ is a single-valued analytic function on $\mathbb{C}^*$ (cf. the above discussion related to the equation (10.128)). The function $\Psi^0(z)$ is presented in Fig.9.

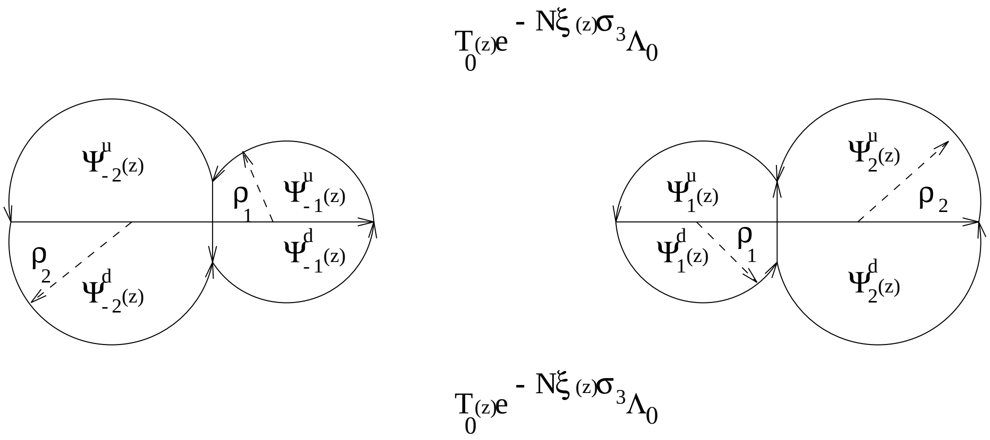

Fig. 9. The function $\Psi^0(z)$ and contour (10.136)

The function $\Psi^0(z)$ is a piecewise analytic function, which satisfies the symmetry relation

$$\Psi^0(z) = (-1)^n \sigma_3 \Psi^0(-z)\sigma_3.$$

It has a jump across the contour

$$\Sigma_1 \cup \Sigma_{-1} \cup [z_1 - \rho_1, z_2 + \rho_2] \cup [-z_2 - \rho_2, -z_1 + \rho_1] \tag{10.136}$$

(see Fig.9), where

$$\Sigma_1 = \partial\Omega_1 \cup \partial\Omega_2 = l_0 + C_1^u + C_1^d + C_2^d + C_2^u,$$
$$\Sigma_{-1} = \tau(\Sigma_1), \qquad \tau(z) = -z,$$

and we assume that the intervals $[z_1 - \rho_1, z_2 + \rho_2]$ and $[-z_2 - \rho_2, -z_1 + \rho_1]$ are oriented from the left to the right. Let

$$M(z) = [\Psi^0_-(z)]^{-1}\Psi^0_+(z)$$

be the corresponding jump matrix. Due to the equation (10.131),

$$M(z) = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix} = S_1 \quad \text{for} \quad z \in [z_1 - \rho_1, z_2 + \rho_2] \cup [-z_2 - \rho_2, -z_1 + \rho_1]. \tag{10.137}$$

On the rest of the contour (10.136) the matrix $M(z)$ does depend on z. Indeed, the functions $\Psi_k^{u,d}(z)$ and $T_0(z)e^{-N\xi(z)\sigma_3}\Lambda_0$ are not exact solutions of the system (10.1) (they are only approximate solutions, cf. (10.124a,b)). Nevertheless, the equations (10.134a,b,c) and the inequality $\operatorname{Re}\xi(z) \le 0,\ z \in l_0$, give the following uniform asymptotics:

$$e^{-N\xi(z)\sigma_3} M(z) e^{N\xi(z)\sigma_3} = I + O\left(N^{-1}\right), \qquad z \in \Sigma_1 \cup \Sigma_{-1}. \tag{10.138}$$

Observe (cf. (10.68-70)) that as $z \to \infty$, the function $\Psi^0(z)$ has the asymptotics (10.4), i.e.,

$$\Psi^0(z) \sim \left(\sum_{k=0}^{\infty} \frac{\Gamma_k^0}{z^k}\right) e^{-\left(\frac{NV(z)}{2} - n\ln z + \lambda_n^0\right)\sigma_3},$$

with

$$\Gamma_0^0 = \begin{pmatrix} 1 & 0 \\ 0 & (R_n^0)^{-1/2} \end{pmatrix}, \qquad \Gamma_1^0 = \begin{pmatrix} 0 & 1 \\ (R_n^0)^{1/2} & 0 \end{pmatrix}, \tag{10.139a}$$

and

$$\lambda_n^0 = NC_n - \frac{1}{8}\ln\frac{g}{\lambda} + q + \frac{1}{2}\ln 2\pi, \qquad C_n = \frac{t^2}{8g} - \frac{\lambda}{4}\ln\frac{g}{\lambda} - \frac{\lambda}{4}. \tag{10.139b}$$

Let us summarize our construction. The matrix function $\Psi^0(z)$ is given by the *explicit* formulae (10.135), and these formulae imply that $\Psi_{11}^0(z)$ satisfies the asymptotic equations (1.15-18) of Theorem 1.1. Hence to prove Theorem 1.1 it remains to prove the following proposition.

Proposition 10.3. *The solution $\Psi_n(z)$ of the Riemann-Hilbert problem (5.16-18) satisfies the asymptotic equation*

$$\Psi_n(z) = \Pi\left\{I + O\left(\frac{1}{(1+|z|)N}\right)\right\}\Psi^0(z),$$

as

$$N \to \infty, \qquad 0 < \varepsilon < \frac{n}{N} \equiv \lambda < \lambda_{cr} - \varepsilon, \qquad \lambda_{\text{cr}} = \frac{t^2}{4g},$$

uniformly in $z \in \mathbb{C}$, where Π is a diagonal, constant in z, matrix such that

$$\Pi = I + O(N^{-1}).$$

Proof. Let us enlarge the contour (10.136) by the whole real line, i.e., let us consider the contour (see Fig.10 below)

$$\Sigma_1 \cup \Sigma_{-1} \cup \mathbb{R} \tag{10.140}$$

(the real line $\mathbb{R}$ is supposed to be oriented from the left to the right) and put

$$Z^0(z) = \Psi^0(z) e^{N\xi(z)\sigma_3}.$$

The function $Z^0(z)$ satisfies the following Riemann-Hilbert problem on the contour (10.140):

$$Z^0(\infty) = e^{(NC_n - \lambda_n^0)\sigma_3}\Gamma_0^0, \tag{10.141a}$$

$$Z_+^0(z) = Z_-^0(z) G^0(z), \qquad z \in \Sigma_1 \cup \Sigma_{-1} \cup \mathbb{R}, \tag{10.141b}$$

where the jump matrix $G^0(z)$ is given by the equations

$$G^0(z) = e^{-N\xi_-(z)\sigma_3} M(z) e^{N\xi_+(z)\sigma_3}, \tag{10.142a}$$

$$z \in \Sigma_1 \cup \Sigma_{-1} \cup [z_1 - \rho_1, z_2 + \rho_2] \cup [-z_2 - \rho_2, -z_1 + \rho_1],$$

$$G^0(z) = I, \qquad z \in (-\infty, -z_2 - \rho_2] \cup [z_2 + \rho_2, \infty), \tag{10.142b}$$

$$G^0(z) = e^{-N\xi_-(z)\sigma_3} I e^{N\xi_+(z)\sigma_3} = (-1)^n I, \qquad z \in [-z_1 + \rho_1, z_1 - \rho_1]. \tag{10.142c}$$

In the last two equations we have taken into account that

$$e^{N\xi_+(z)} = e^{N\xi_-(z)}, \qquad z \in (-\infty, -z_2 - \rho_2] \cup [z_2 + \rho_2, \infty),$$

$$e^{N\xi_+(z)} = (-1)^n e^{N\xi_-(z)}, \qquad z \in [-z_1, z_1].$$

The Riemann-Hilbert problem (10.141a,b) is depicted in Fig.10.

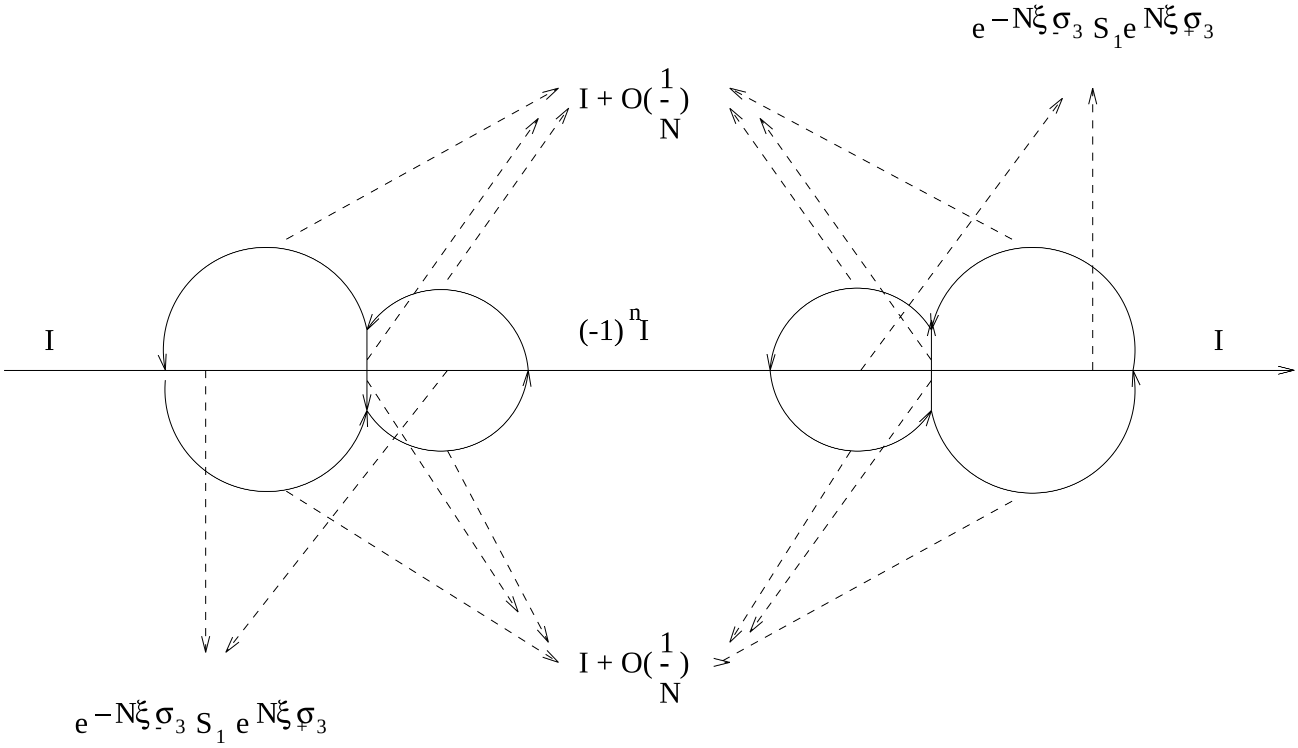

Fig. 10. The Riemann-Hilbert problem for the function $Z^0(z)$

The equations (10.137) and (10.142a) give that

$$G^0(z) = e^{-N\xi_-(z)\sigma_3} S_1 e^{N\xi_+(z)\sigma_3}, \qquad z \in [z_1 - \rho_1, z_2 + \rho_2] \cup [-z_2 - \rho_2, -z_1 + \rho_1] \quad (10.143)$$

(see Fig.10). Simultaneously, the equations (10.138) and (10.142a-c) yield the following uniform estimates:

$$|I - G^0(z)| \le CN^{-1}, \qquad z \in \Sigma_1 \cup \Sigma_{-1}, \qquad (10.144a)$$

$$|I - G^0(z) e^{-N\xi_+(z)\sigma_3} S_1^{-1} e^{N\xi_-(z)\sigma_3}| \le Ce^{-2N \operatorname{Re} \xi(z)}, \qquad (10.144b)$$

$$z \in (-\infty, -z_2 - \rho_2] \cup [-z_1 + \rho_1, z_1 - \rho_1] \cup [z_2 + \rho_2, \infty).$$

Observe now that $\operatorname{Re} \xi(z) \ge \delta > 0$ for all $z \in (-\infty, -z_2 - \rho_2] \cup [-z_1 + \rho_1, z_1 - \rho_1] \cup [z_2 + \rho_2, \infty)$. Therefore, the formulae (10.144) imply the inequalities

$$|I - G^0(z)| \le CN^{-1}, \qquad (10.145)$$

and

$$|I - G^0(z) e^{-N\xi_+(z)\sigma_3} S_1^{-1} e^{N\xi_-(z)\sigma_3}| \le Ce^{-2N\delta}, \qquad (10.146)$$

which hold uniformly on

$$\Sigma' = \Sigma_1 \cup \Sigma_{-1},$$

and

$$\Sigma'' = (-\infty, -z_2 - \rho_2] \cup [-z_1 + \rho_1, z_1 - \rho_1] \cup [z_2 + \rho_2, \infty),$$

respectively. In addition, from (10.144) we have a uniform estimate for the corresponding L_2-norms as well:

$$||I - G^0||_{L_2(\Sigma')} \le CN^{-1}, \tag{10.147a}$$

$$||I - G^0 e^{-N\xi_+\sigma_3} S_1^{-1} e^{N\xi_-\sigma_3}||_{L_2(\Sigma'')} \le Ce^{-2N\delta}. \tag{10.147b}$$

As indicated in Fig.10, the only pieces of the contour (10.140) where the matrix $I - G^0(z)$ is not uniformly controllable, are the intervals $[z_1 - \rho_1, z_2 + \rho_2]$ and $[-z_2 - \rho_2, -z_1 + \rho_1]$. On the parts $[z_1, z_2]$ and $[-z_2, -z_1]$ of these intervals the matrix $I - G^0(z)$ oscillates. The principal point is that on the "bad" set $[z_1 - \rho_1, z_2 + \rho_2] \cup [-z_2 - \rho_2, -z_1 + \rho_1]$ the matrix $G^0(z)$ *coincides* with the jump matrix of the orthogonal polynomial Riemann-Hilbert problem (5.16-18).

Let us consider now the orthogonal polynomial Riemann-Hilbert problem (5.16-18). Keeping the notations λ_n, Γ_0, etc., for the corresponding objects associated with the orthogonal polynomials, let us put

$$Z(z) = e^{(\lambda_n - \lambda_n^0)\sigma_3} \Gamma_0^0 \Gamma_0^{-1} \Psi_n(z) e^{N\xi(z)\sigma_3}.$$

The matrix function $Z(z)$ solves the Riemann-Hilbert problem (10.141a,b) for the contour (10.140) with the *same* normalization yet different jump conditions. Denoting by $G(z)$ the jump matrix corresponding to $Z(z)$, we have that

$$G(z) = e^{-N\xi_-(z)\sigma_3} S_1 e^{N\xi_+(z)\sigma_3}, \qquad z \in \mathbb{R},$$

and

$$G(z) = I, \qquad z \in \Sigma_1 \cup \Sigma_{-1}.$$

Hence we can rewrite the inequalities (10.145), (10.146) as a single inequality,

$$|I - G^0(z)[G(z)]^{-1}| \le CN^{-1}, \tag{10.148}$$

and this inequality is valid uniformly on the contour

$$\Sigma^0 = \Sigma' \cup \Sigma''.$$

(see Fig.11). Similarly, the inequalities (10.147) imply that

$$||I - G^0 G^{-1}||_{L_2(\Sigma^0)} \leq C N^{-1}. \tag{10.149}$$

Simultaneously, we have that

$$G(z) = G^0(z), \qquad z \in [z_1 - \rho_1, z_2 + \rho_2] \cup [-z_2 - \rho_2, -z_1 + \rho_1]. \tag{10.150}$$

Let now $X(z)$ be the matrix ratio,

$$X(z) = Z(z)[Z^0(z)]^{-1}.$$

The equation (10.150) shows that $X(z)$ has jumps only on the contour Σ^0 where it solves the following Riemann-Hilbert problem:

$$\begin{aligned} &X(\infty) = I, \\ &X_+(z) = X_-(z) F^0(z), \qquad z \in \Sigma^0, \end{aligned} \tag{10.151}$$

where

$$F^0(z) = Z_-^0(z) G(z) [G^0(z)]^{-1} [Z_-^0(z)]^{-1}. \tag{10.152}$$

This Riemann-Hilbert problem, whose jump matrix $F^0(z)$ is under a complete control, is depicted in Fig.11.

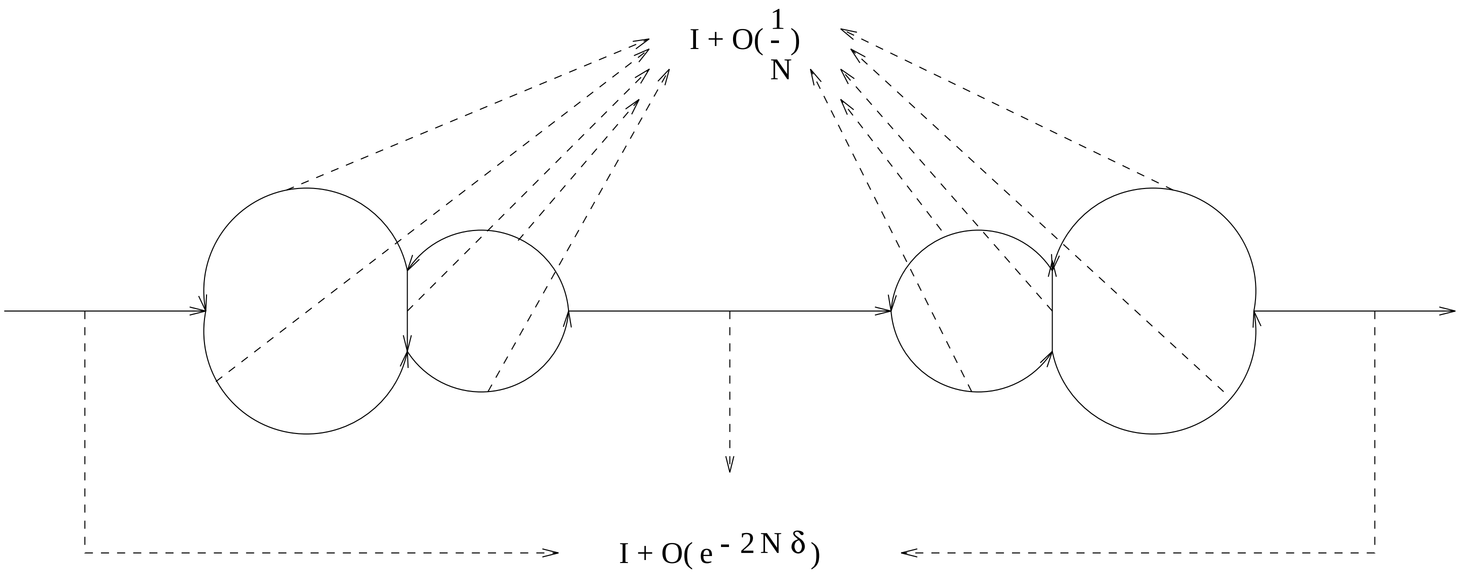

Fig. 11. The Riemann-Hilbert problem for the function $X^0(z)$ and the contour Σ^0

Similar to the asymptotic problems considered in [DZ2] and [DIZ], the crucial fact here is that

$$|I - [F^0(z)]^{-1}| = O(N^{-1}), \tag{10.153}$$

uniformly in $z \in \Sigma^0$ (see Fig.11). Indeed, from the definition (10.135) of the function $\Psi^0(z)$ and the estimates (10.134b,c) it follows that

$$|Z^0_-(z)| \equiv |\Psi^0_-(z) e^{N\xi_-(z)\sigma_3}| \leq C, \qquad \forall z \in \Sigma^0. \tag{10.154}$$

This equation, together with (10.148), imply obviously (10.153). Similarly, from (10.149) we obtain the L_2-estimate

$$||I - (F^0)^{-1}||_{L_2(\Sigma^0)} = O(N^{-1}). \tag{10.155}$$

Remark 10.7. The contour $\Sigma^0 = \Sigma^0(\rho_{1,2})$ is not rigid. Namely, as follows from the *explicit* formula (10.152) for the jump matrix $F^0(z)$, this matrix-valued function is analytic and preserves its asymptotic behaviour (as $N \to \infty$) in a neighborhood of the contour Σ^0. This means that in the Riemann-Hilbert problem (10.151) one can always slightly deform the contour Σ^0. One of the possible deformations is presented in Fig.12.

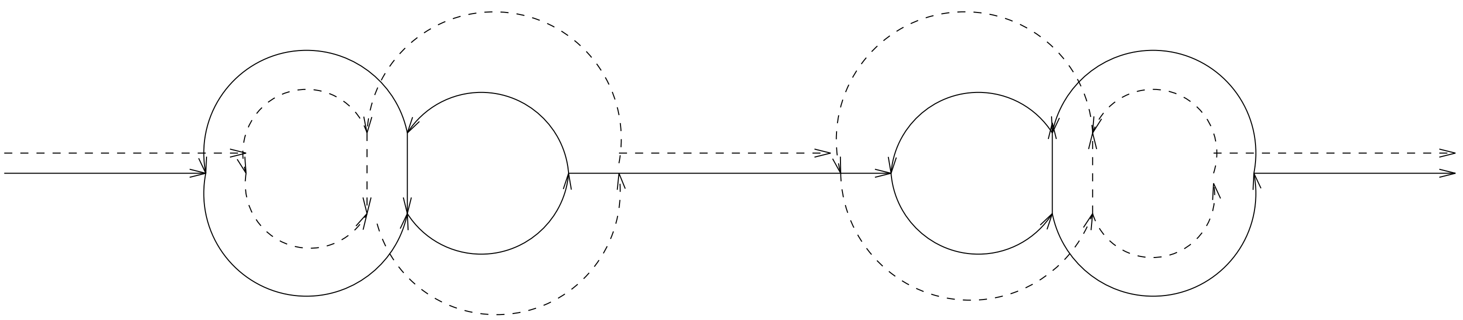

Fig. 12. An admissible deformation of the contour Σ^0

By a standard technique in the theory of the Riemann-Hilbert problem (see e.g. [CG]; see also [BDT] and [DZ]), the solution $X^0(z)$ of the Riemann-Hilbert problem (10.151) is given by the formula

$$X^0(z) = I + \frac{1}{2\pi i} \int_{\Sigma^0} \rho^0(\nu) \left[I - (F^0(\nu))^{-1}\right] \frac{d\nu}{\nu - z}, \tag{10.156}$$

where $\rho^0(z) \equiv X^0_+(z)$ solves the equation

$$\rho^0 = I + C_+[\rho^0\left(I - (F^0)^{-1}\right)],$$

in $L_2(\Sigma^0)$, and C_+ is the Cauchy operator. We note that $X^0(z)$ is built up from the functions, which satisfy the asymptotic condition (10.4). Therefore, $I - \rho^0(z) \sim Cz^{-1}$ as $z \to \infty$, and hence *a priori* it belongs to the space $L_2(\Sigma^0)$.

The L_2-boundness of the operator C_+ (see e.g. [LiS]; see also [BDT] and [Zh]), together with the estimates (10.153, 155) imply the equation

$$||I - \rho^0||_{L_2(\Sigma^0)} = O(N^{-1}).$$

This equation together with (10.153) and (10.156) show that

$$X^0(z) = I + O\left(\frac{1}{(1+|z|)N}\right),$$

or

$$e^{(\lambda_n - \lambda_n^0)\sigma_3}\Gamma_0^0\Gamma_0^{-1}\Psi_n(z) = \left\{I + O\left(\frac{1}{(1+|z|)N}\right)\right\}\Psi^0(z), \tag{10.157}$$

uniformly in $z \in K$ for any closed $K \subset \mathbb{C} \setminus \Sigma^0$. Replacing $\Psi_n(z)$ and $\Psi^0(z)$ in (10.157) by their asymptotic series from (10.4) ($z \to \infty$, $\quad$ dist $\{z, \Sigma^0\} \geq \varepsilon > 0$) we obtain that

$$e^{(\lambda_n - \lambda_n^0)\sigma_3}\Gamma_0^{-1}\Gamma_1 e^{-(\lambda_n - \lambda_n^0)\sigma_3} - (\Gamma_0^0)^{-1}\Gamma_1^0 = O\left(N^{-1}\right),$$

where Γ_1 and Γ_1^0 are the matrix coefficients of the z^{-1}-terms in the series for $\Psi_n(z)$ and $\Psi^0(z)$, respectively (see (4.15) and (10.139a)). Considering (12) and (21) entries in this matrix equation and taking into account (10.139b) and (10.125a), we end up with the desired estimates for the normalization constant λ_n and the recurrence coefficients R_n corresponding to the system of the orthogonal polynomials in question:

$$\lambda_n = \lambda_n^0 + O\left(N^{-1}\right) = NC_n - \frac{1}{8}\ln\frac{g}{\lambda} + q + \frac{1}{2}\ln 2\pi + O\left(N^{-1}\right), \tag{10.158a}$$

$$R_n = R_n^0 + O\left(N^{-1}\right) = \frac{-t - (-1)^n\sqrt{t^2 - 4\lambda g}}{2g} + O\left(N^{-1}\right). \tag{10.158b}$$

In turn, these asymptotic formulae allow us to rewrite the asymptotic equation (10.117) for the orthogonal polynomial Ψ-function as

$$\Psi_n(z) = \Pi\left\{I + O\left(\frac{1}{(1+|z|)N}\right)\right\}\Psi^0(z), \qquad z \in K \subset \mathbb{C}\setminus\Sigma^0, \tag{10.158c}$$

where Π is a constant diagonal matrix satisfying the estimate

$$\Pi = I + O(N^{-1}).$$

To complete the proof of Proposition 10.3 we only need to notice that the domain of validity of the asymptotic equation (10.158c) can be made the whole z-plane because of the discussed in Remark 10.7 flexibility in the choice of the contour Σ^0. The Proposition 10.3 and hence the main Theorem 1.1. are proven.

Remark 10.8. The kernel $I - [F^0(\nu)]^{-1}$ in the integral representation (10.156) is small (due to the estimate (10.153)), and it is given explicitly in terms of elementary and Bessel's functions. Therefore, the equation (10.156) enables (cf. [DZ3] and [DIZ]) to obtain the full asymptotic expansion for all the three main objects, λ_n, R_n, and $P_n(z)$.

Remark 10.9. The asymptotic formulae (1.15–17) follow from (10.158c) and the definition (10.135) of Ψ^0.

Appendix 1: An Alternative Asymptotic Analysis of the Inverse Monodromy Problem.

Let us denote by γ_+ the anti-Stokes line

$$\operatorname{Im}\int_{z_1}^{z}\mu(z)dz = 0, \tag{A1.1}$$

which passes through the point z_1 and which has the rays $\arg z = \pm\frac{\pi}{4}$ as its asymptotes. The image of γ_+ with respect to the map $z \to -z$ we denote by γ_-. We shall also denote by $\gamma_\pm^u$ and $\gamma_\pm^d$ the upper and lower parts of the $\gamma_\pm$, respectively.

Assume now that $\gamma_\pm$ are oriented from the bottom to the top and include them as a part of the *steepest descent* contour (anti-Stokes graph)

$$\Sigma = \gamma_+ \cup \gamma_- \cup \mathbb{R} \cup i\mathbb{R}$$

where $\mathbb{R}$ is oriented from the left to the right and $i\mathbb{R}$ – from the bottom to the top. Let also $\Psi(z)$ be (cf. Section 10.1) a piecewise analytic function defined on $\mathbb{C}\setminus\Sigma$, which coincides with $\Psi_j(z)$ on the Stokes line γ_j. Define

$$Z(z) = \Psi(z)e^{N\xi(z)\sigma_3} \tag{A1.2}$$

Following the basic idea of the Deift-Zhou method [DZ], the inverse monodromy problem (10.15) can be reformulated, in terms of $Z(z)$, as the following Riemann-Hilbert problem on the contour Σ:

$$Z(\infty) = e^{(NC_n - \lambda_n)\sigma_3}\Gamma_0, \tag{A1.3}$$

$$Z_+(z) = Z_-(z)G(z), \qquad z \in \Sigma, \tag{A1.4}$$

where (see (10.4))

$$\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & R_n^{-1/2} \end{pmatrix}$$

and the jump matrix $G(z)$ is defined by the equations

$$G(z) = \begin{cases} e^{-N\xi_-(z)}S_1 e^{N\xi_+(z)}, & z \in (-\infty, -z_1] \cup [z_1, \infty), \\ e^{-N\xi_-(z)}S_2 e^{N\xi_+(z)}, & z \in \gamma_+^u \cup \gamma_-^d, \\ e^{-N\xi_-(z)}S_3 e^{N\xi_+(z)}, & z \in i\mathbb{R}, \\ e^{-N\xi_-(z)}S_4 e^{N\xi_+(z)}, & z \in \gamma_-^u \cup \gamma_+^d, \\ e^{-N\xi_-(z)}Q^{-1} e^{N\xi_+(z)}, & z \in [0, z_1]. \\ e^{-N\xi_-(z)}\sigma_3 Q \sigma_3 e^{N\xi_+(z)}, & z \in [-z_1, 0]. \end{cases}$$

The Riemann-Hilbert problem (A1.3-4) is depicted in Fig.13.

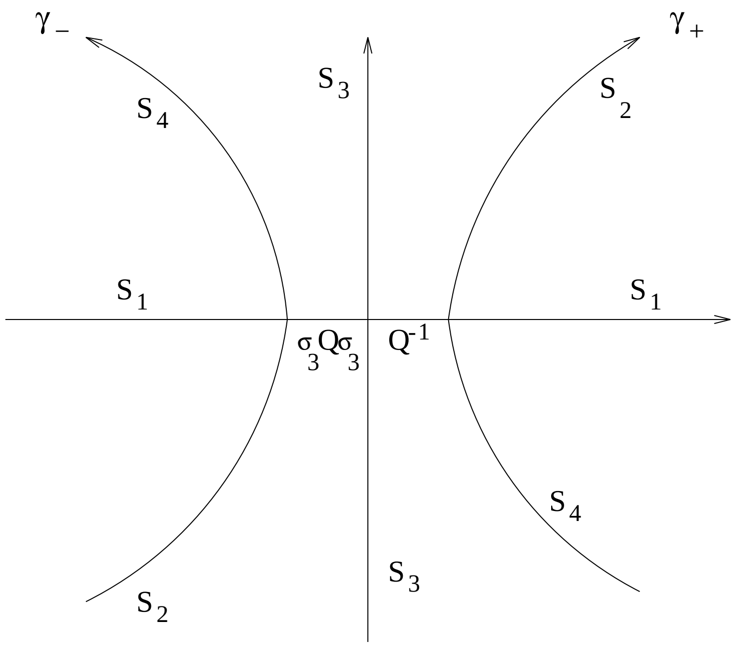

Fig. 13. The Riemann-Hilbert problem for the function $Z(z)e^{-N\xi(z)\sigma_3}$

It is worth noticing that:

$$e^{N\xi_+(z)} = e^{N\xi_-(z)}, \quad z \in \gamma_+ \cup \gamma_- \cup i\mathbb{R} \cup (-\infty, -z_2] \cup [z_2, \infty),$$
$$e^{N\xi_+(z)} = (-1)^n e^{N\xi_-(z)}, \quad z \in [-z_1, z_1],$$

and that the triangular structure of the jump matrices corresponding to the infinite branches of Σ is consistent with the exponential decay of the exponent $e^{N\xi(z)}$. On the contrary, the matrix $G(z)$ oscillates on the intervals $[\pm z_1, \pm z_2]$ and the entry $G_{21}(z)$ exponentially grows (if $q_{21} \neq 0$) on the interval $[-z_1, z_1]$.

Before passing to the asymptotic analysis of the problem (A1.3-4) let us make three more comments:

(1). Because of the normalization condition (10.10), any monodromy RH problem of the type (A1.3-4) has the same jump matrices on $(-\infty, -z_1] \cup [z_1, \infty)$.

(2). On the line $i\mathbb{R}$, Re $\xi(z)$ approches its minimum at $z = 0$ and this minimum coincides with the positive parameter δ_0 in (10.120):

$$\operatorname{Re}\xi(z) \geq \delta_0, \qquad z \in i\mathbb{R}. \tag{A1.5}$$

(3). On the segment $[-z_1, z_1]$, Re $\xi(z)$ approches its maximum at $z = 0$:

$$\operatorname{Re}\xi(z) \leq \delta_0, \qquad z \in [-z_1, z_1]. \tag{A1.6}$$

Let us denote by $Z^*(z), G^*(z)$, etc., the corresponding objects assosiated with the model equation (10.1), (10.32), keeping the notation $Z(z), G(z)$, etc., for the orthogonal polynomial RH-problem (5.16-18). Let us also introduce an auxiliary function,

$$\hat{Z}(z) = e^{(\lambda_n - \lambda_n^*)\sigma_3}\Gamma_0^*\Gamma_0^{-1}Z(z).$$

Consider now the matrix ratio

$$X(z) = \hat{Z}(z)Z^{*-1}(z).$$

The matrix-valued function $X(z)$ solves the following RH problem (see Fig.14 below):

$$X(\infty) = I, \tag{A1.7}$$
$$X_+(z) = X_-(z)F(z), \qquad z \in \Sigma^* = \Sigma - (-\infty, -z_1] \cup [z_1, \infty), \tag{A1.8}$$

where

$$F(z) = Z_-^*(z)G(z)G^{*-1}(z)Z_-^{*-1}(z),$$

and it has no jump on $(-\infty, -z_1] \cup [z_1, \infty)$, which follows from the comment (1) above.

Our aim now is to show that (cf. (10.153)):

$$|I - F^{-1}(z)| = O(N^{-2/3}), \tag{A1.9}$$

uniformly in $z \in \Sigma^*$. To that end we first notice that the equations (10.124) yield the following estimates for the canonical solutions of the equations (10.1), (10.32) :

$$|\Psi_{2,3}^*(z)e^{N\xi(z)\sigma_3}| < CN^{1/6}, \quad z \in \gamma_+^u \cap \{z : |z - z_1| \leq \rho_1\}, \tag{A1.10a}$$
$$|\Psi_{2,3}^*(z)e^{N\xi(z)\sigma_3}| < C, \quad z \in \gamma_+^u \cap \{z : |z - z_1| \geq \rho_1\}, \tag{A1.10b}$$
$$|\Psi_{1,8}^*(z)e^{N\xi(z)\sigma_3}| < CN^{1/6}, \quad z \in \gamma_+^d \cap \{z : |z - z_1| \leq \rho_1\}, \tag{A1.11a}$$
$$|\Psi_{1,8}^*(z)e^{N\xi(z)\sigma_3}| < C, \quad z \in \gamma_+^d \cap \{z : |z - z_1| \geq \rho_1\}, \tag{A1.11b}$$
$$|\Psi_{3,8}^*(z)e^{N\xi(z)\sigma_3}| < CN^{1/6}, \quad z \in [z_1 - \rho_1, z_1], \tag{A1.12}$$
$$|\Psi_3^*(z)e^{N\xi(z)\sigma_3}| < C, \quad z \in i\mathbb{R} \cup [0, z_1], \tag{A1.13}$$
$$|\Psi_8^*(z)e^{N\xi(z)\sigma_3}| < C, \quad z \in -i\mathbb{R} \cup [0, z_1], \tag{A1.14}$$

where

$$\rho_1 < \min\{z_1, z_2 - z_1\},$$

and similar estimates involving $\Psi_{4,5,6,7}^*$, the line γ_-, and the point $-z_1$ (use of the symmetry $z \to -z$).

When z runs along Σ^*, the quantity $|Z_-^*(z)|$ always coincides with the l.h.s. of one of the inequalities (A1.10-14). This immediately yields (see the inequalities (A1.10-12)) the estimate

$$|Z_-^*(z)| \leq CN^{1/6} \qquad (A1.15),$$

uniformly on $\gamma_+ \cup \gamma_- \cup [-z_1, -z_1 + \rho_1] \cup [z_1 - \rho_1, z_1]$. Taking into account (10.115), we conclude from (A1.15) that uniformly on $\gamma_\pm$ we have that

$$|I - F(z)| \leq CN^{-2/3}e^{-2N|\mathrm{Re}\,\xi(z)|}, \tag{A1.16}$$

On the imaginary line, due to (A1.13-14), estimate (A1.15) can be replaced by the estimate,

$$|Z_-^*(z)| \leq C. \tag{A1.17}$$

By virtue of (10.122), this implies that inequality

$$|I - F(z)| \leq CN^{-1}e^{-2N(\mathrm{Re}\,\xi(z) - \delta_0)} \tag{A1.18}$$

holds on $i\mathbb{R}$. Note, that because of (A1.5), there is enough of the exponential decay in the factor $e^{-2N\xi(z)}$ to balance the exponential growth in s_3.

An exponential growth of the entry G_{21} on the interval $[-z_1, z_1]$ is compensated by the exponential decay of q^0 (see (10.123)). More precisely, on the part, $[-z_1, z_1] \setminus [-z_1 + \rho_1, z_1 - \rho_1]$, of the interval $[-z_1, z_1]$ we have (see (A1.15), (10.123), and (A1.6)) that

$$|I - F(z)| \leq CN^{-2/3} e^{-c_0 N}, \quad c_0 > 0, \tag{A1.19}$$

while on the part, $[-z_1 + \rho_1, z_1 - \rho_1]$ we can again use (A1.13-14) and replace the estimate (A1.15) by (A1.17), which gives for this part that

$$|I - F(z)| \leq CN^{-1}. \tag{A1.20}$$

The inequalities (A1.16, 18-20) prove the asymptotic formula (A1.9). Notice, that the same inequalities yield the estimate for the corresponding L_2-norm of $I - F^{-1}(z)$:

$$||I - F^{-1}||_{L_2(\Sigma^*)} = O(N^{-2/3}). \tag{A1.21}$$

The more detailed description of the asymptotic behaviour of $F(z)$ with respect to the different pieces of the contour Σ^* is given in Fig.14.

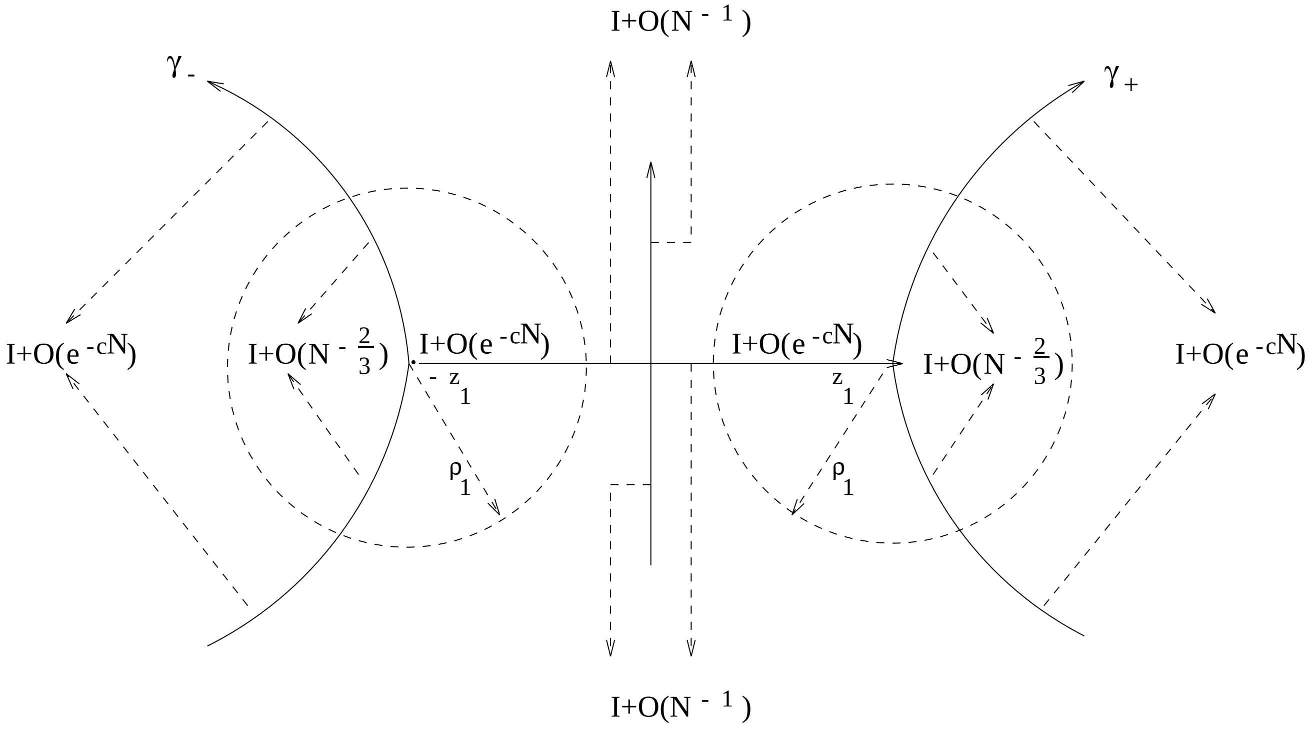

Fig. 14. The Riemann-Hilbert problem for the function X(z)

Exactly the same as in Sec.10.5 arguments based on the Cauchy integral representation for the function $X(z)$,

$$X(z) = I + \frac{1}{2\pi i}\int_{\Sigma^*} \rho(\nu)\left(I - F^{-1}(\nu)\right)\frac{d\nu}{\nu - z}, \tag{A1.22}$$

$$\rho(z) = X_+(z), \qquad \rho = I + C_+[\rho\left(I - F^{-1}\right)],$$

show that the estimates (A1.9), (A1.21) imply that

$$X(z) = I + O\left(\frac{1}{(1+|z|)N^{2/3}}\right),$$

uniformly in $z \in K$ for any closed $K \subset \mathbb{C} \setminus \Sigma^*$. This estimate, in turn, leads to the asymptotic equations

$$\lambda_n = \lambda_n^* + O\left(\frac{1}{N^{2/3}}\right) = NC_n - \frac{1}{8}\ln\frac{g}{\lambda} + q + \frac{1}{2}\ln 2\pi + O\left(N^{-2/3}\right), \tag{A1.23a}$$

$$R_n = R_n^* + O\left(N^{-2/3}\right) = \frac{-t - (-1)^n\sqrt{t^2 - 4\lambda g}}{2g} + O\left(N^{-2/3}\right), \tag{A1.23b}$$

$$\Psi(z) \equiv \Psi_n(z) = \Pi^*\left\{I + O\left(\frac{1}{(1+|z|)N^{2/3}}\right)\right\}\Psi^*(z), \qquad z \in K \subset \mathbb{C} \setminus \Sigma^*. \tag{A1.23c}$$

The constant diagonal matrix Π^* in (A1.23c) satisfies the estimate

$$\Pi^* = I + O(N^{-2/3}).$$

Denote

$$\hat\rho(z) = \rho(z)(I - F^{-1}(z)),$$

the density function in the integral representation (A1.22). One can show that:

(a) at the points of self-intersection of the contour Σ^*, i.e., at the points $s = 0,\ \pm z_1$, we have the equation

$$\sum_{j=1}^{n_0(s)} \lim_{z\to s,\, z\in\Sigma_j^*(s)} \rho(z) = 0; \qquad s = 0,\ \pm z_1,$$

where $\Sigma_j^*(s),\ j = 1, \dots, n_0(s)$, are the smooth pieces of the contour Σ^* that meet at the point s.

(b) $\hat\rho(z)$ has a small $H^1(\Sigma^*)$-norm.

Having the properties (a), (b) and using some general facts proven in [BDT], one can easily extend the estimate (A1.23c) onto the whole z-plane.

We would like to notice that the method, presented in this appendix, gives a somewhat weaker estimate of the error term in the equations (A1.23a,b,c) than we proved in the section 10.5. To improve the error term in (A1.23a,b,c) to the one of the order of N^{-1} (within the Riemann-Hilbert problem (A1.7-8)) we need an impovement of the asymptotic analysis of the direct monodromy problem in the section 10.4. Namely, we have to use the ansatz (10.32) for R_n and perform the WKB-calculations up to higher order terms in N^{-1} (see e.g. [Kap]). There are no principle difficulties in this approach, however, technically it is quite involved. In the main text we circumvented this difficulty by using more advanced (and simultaniously more explicit !) Deift-Zhou technique for constructing the model solution $\Psi^0(z)$ for the orthogonal polynomial Riemann-Hilbert problem (5.16-18).

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