# INTERNAL LIFSHITS TAILS FOR RANDOM PERTURBATIONS OF PERIODIC SCHRÖDINGER OPERATORS

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Abstract. Let $H$ be a $\Gamma$-periodic Schrödinger operator acting on $L^2(\mathbb{R}^d)$ and consider the random Schrödinger operator $H_\omega = H + V_\omega$ where $V_\omega(x) = \sum_{\gamma\in\Gamma} \omega_\gamma V(x-\gamma)$ (here $V$ is a positive potential and $(\omega_\gamma)_{\gamma\in\Gamma}$ a collection of positive i.i.d random variables). We prove that, at the edge of a gap of $H$ that is not filled in for $H_\omega$, the integrated density of states of $H_\omega$ has a Lifshits tail behaviour if and only if the integrated density of states of $H$ is non-degenerate.

Résumé. Soient $H$ un opérateur de Schrödinger $\Gamma$-périodique agissant sur $L^2(\mathbb{R}^d)$, $V$ un potentiel positif et $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ une famille de variables aléatoires i.i.d positives. Considérons l'opérateur de Schrödinger aléatoire $H_\omega = H + V_\omega$ où $V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma)$. On montre que, au bord d'une lacune spectrale de $H$ qui n'est pas comblé pour $H_\omega$ la densité d'états intégrée de $H_\omega$ a un comportement asymptotique de Lifshits si et seulement si la densité d'états intégrée de $H$ est non dégénérée au bord de cette lacune.

## 0. Introduction

This paper is devoted to the study of Lifshits tails for random perturbations of periodic Schrödinger operators. We prove that, at the edge of a gap, a true Lifshits tail for the random operator occurs if and only if the density of states of the background periodic operator has a non-degenerate behaviour.

To be more precise, let us state our main result in a restricted setting. Consider $W$ a bounded $\mathbb{Z}^d$-periodic potential on $\mathbb{R}^d$ and define $H = -\Delta + W$ to be a periodic Schrödinger operator. Let $\sigma = \sigma(H)$ be the spectrum of $H$ and assume that $\sigma$ has a gap below energy 0 of length at least $\delta$ that is,

- for some $a > 0$ and $\delta > 0$, $\sigma \cap [0,a) = [0,a)$ and $\sigma \cap [-\delta, 0) = \emptyset$.

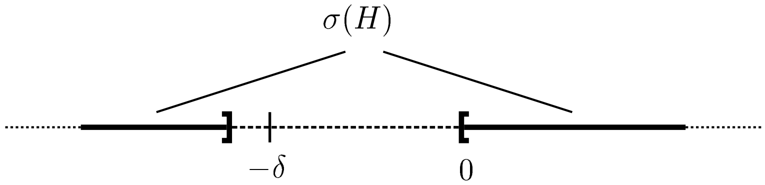


Figure 1.

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We then consider a random Schrödinger operator of the form

$$H_\omega = H + V_\omega = H + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(\cdot - \gamma),$$

where it is assumed that

- $V$ is a non-negative, bounded and compactly supported potential.
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a collection of independent identically uniformly distributed random variables on $[0,1]$.

Let $\Sigma = \sigma(H_\omega)$ be the almost sure spectrum of $H_\omega$. We know that 0 belongs to $\Sigma$ (see [13]). Let us now assume that $\Sigma$ has a gap below 0 that is

- for some $\delta' > 0$, $\Sigma \cap [-\delta', 0) = \emptyset$.

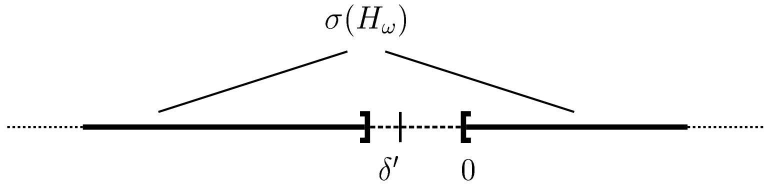


Figure 2.

Let $n(E)$ be the integrated density of states of $H$, and $N(E)$ the integrated density of states of $H_\omega$. Then we prove that

- 0 is continuity point for $N(E)$.
- under no further assumption on $n$, one has

$$\liminf_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} \geq -\frac{d}{2}.$$

- one has the equivalence

$$\lim_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} = -\frac{d}{2} \Longleftrightarrow \lim_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} = \frac{d}{2}.$$

Remark 0.1. When $n(E)$ satisfies

$$\lim_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} = \frac{d}{2},$$

we will say that $n$ is *non-degenerate* at 0. The reasons for this terminology will be obvious from the results of Proposition 1.1.

In our general theorem, the assumptions made on $W$, $V$ and $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are much less restrictive than in the example given above. In particular, $W$ and $V$ may admit local singularities; $V$ need not be of compact support; the common distribution of the $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is not of real importance as long as it does not vanish to quickly near 0. If the common essential support of the $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ happens not to be connected, then the assumption on the open gap below 0 will have to be strengthened somewhat.

The above results hold at the upper edge of a gap if we replace the positivity assumptions by negativity assumptions. Let us underline the fact that the positivity of the

$(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is not really important; as they are bounded, one can always shift them by some constant to make them positive. Then of course, one has to change $W$ that is $H$, the background periodic operator.

Before giving precise results, let us briefly sketch the ideas used to prove them. The upper and lower bound are proved separately. To prove the lower bound, we first show that $N(E)-N(0)$ can be lower bounded by the integrated density of states (at energy $E$) of some well chosen discrete operator. Then, for any cube $\Lambda$ large enough, we construct approximated eigenfunctions associated energies in $[0,E]$ and the discrete operator restricted to the cube $\Lambda$. We manage to construct enough quasi-modes to get a sufficiently sharp estimate of the number of eigenvalues in $[0,E]$. Normalizing this counting function by the volume of $\Lambda$ and averaging over the disorder, we get the announced lower bound.

The main idea used to prove the upper bound is to show that, if $n(E)$ has a non-degenerate behaviour at energy 0 then $N(E)$ may be compared with the integrated density of states of some well chosen discrete Anderson model. To see this, we first show that the zeros of the Floquet eigenvalues (that vanish somewhere) are non-degenerate under the non-degeneracy assumption for the periodic density of states.

The second step consists in showing that one can localize the density of states in energy; more precisely, using periodic approximations, we show that, for $E$ close to 0, $N(E)$ may be estimated from above by the density of states of some reduced bounded random operator called $H^0_\omega$ that is acting on $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$ (for some $n\in\mathbb{N}$). This reduced operator is coming in quite naturally as it is nothing but the random operator $H_\omega$ conjugated with $\Pi^0$ the spectral projection for $H$ on the band starting at 0. $H^0_\omega$ may be decomposed into $H^0_\omega=\Pi^0H_\omega\Pi^0=\Pi^0H\Pi^0+\Pi^0V_\omega\Pi^0=H^0+V^0_\omega$.

To perform the third step we notice that if a vector minimizes $H^0_\omega$, it has to minimize $H^0$ hence to be concentrated near the zeros of the Floquet eigenvalues of $H$. But these zeros are non-degenerate hence isolated; so near these zeros, $H^0$ resembles the usual discrete Laplace operator. On the other hand, if the vector concentrates near a point $\theta^0$ then, in the kernel $V^0_\omega(\theta,\theta')$, we may freeze $\theta=\theta'=\theta^0$. A close look at $V^0_\omega(\theta^0,\theta^0)$ shows that this kernel resembles the kernel of the usual Anderson potential. A fact that is crucially used is that, as the zeros of the $(E_j)_{j\in N}$ are non-degenerate, it is possible to bound the operator of kernel $V^0_\omega(\theta,\theta')-V^0_\omega(\theta^0,\theta^0)$ by $(H^0)^{1/2}$.

Once we reduced ourselves to the Anderson model, we apply well known results for the Lifshits tails for this model ([27], [21] or [28]).

One sees that, in a way, this scheme adopts a point of view dual to the one usually adopted ([12], [14], [22], etc); indeed, instead of using the fact that, in order to have energy close to 0, a vector must be spread out in space, we use its localization in momentum.

When one tries to push both of these schemes through, the main difficulties come from the fact that the objects one has to work with (especially the Floquet eigenvalues and eigenvectors) are not smooth. Nevertheless, it is possible to replace them by smooth objects near the points where Floquet eigenvalues vanish. As all the important quantities concentrate there, this will be sufficient.

Though we did not develop this here, a look at the methods used in this paper will easily convince the reader that many of these results extend to the case of Schrödinger operators with periodic magnetic fields for which one can develop a Floquet theory; the only condition needed to ensure this is that the magnetic flux through a cell of the lattice is rational multiple of $\pi$ ([29]). In this case, the magnetic translations play the role the usual translations play here.

As a last remark, we would like to cite the main application of the Lifshits tail behaviour that is exponential localization for random Schrödinger operators. In a Fröhlich-Spencer type proof of localization, the Lifshits behaviour can be used to initiate the induction procedure (for the bottom of the spectrum this was done in [18]). Our new results can, for example, be used in conjunction with [2] or [15] to prove localization at internal band edges.

To end this introduction, let us give a few notations used in the body of the paper followed by the table of contents:

- for $E \subset \mathbb{R}^d$, $\chi_E$ denotes the characteristic function of $E$.
- $(e_j)_{1\leq j\leq d}$ is a basis of $\mathbb{R}^d$. Unless otherwise stated, coordinates of vectors in $\mathbb{R}^d$ will be meant in this base.
- $\Gamma = \bigoplus_{1\leq j\leq d} \mathbb{Z}e_j$ is an integer lattice in $\mathbb{R}^d$.
- On $\mathbb{R}^d$, we define the norm $|x|_\Gamma = \sum_{j=1}^d |x_j|$ if $x = \sum_{j=1}^d x_j e_j$.
- Using the basis $(e_j)_{1\leq j\leq d}$, we identify $(\Gamma, |\cdot|_\Gamma)$ with the lattice $(\mathbb{Z}^d, |\cdot|_1)$ (where $|\cdot|_1$ is the usual $\ell^1$-norm on $\mathbb{R}^d$).
- $C_0 = \left\{ \sum_{\leq j\leq d} x_j e_j;\ \forall 1\leq j\leq d,\ -1/2 < x_j \leq 1/2 \right\}$ is the fundamental cell of $\Gamma$.
- $\mathbb{T} = \mathbb{R}^d/\Gamma$ is the quotient torus generated by $\Gamma$.
- $\Gamma^* = \{\gamma^* \in (\mathbb{R}^d)^*;\ \forall \gamma \in \Gamma,\ \gamma\cdot\gamma^* \in 2\pi\mathbb{Z}\}$ is $\Gamma$'s dual lattice.
- $(e_j^*)_{1\leq j\leq d}$ is the dual basis of $(e_j)_{1\leq j\leq d}$. So by means of this basis, $\Gamma^*$ is identified with $2\pi\mathbb{Z}^d$. For $\theta \in (\mathbb{R}^d)^*$, we write $\theta = (\theta_j)_{1\leq j\leq d}$ when $\theta = \sum_{j=1}^d \theta_j e_j^*$
- $C_0^*$ is the fundamental cell of $\Gamma^*$ and $\mathbb{T}^*$ the quotient torus generated by $\Gamma^*$.
- For $v \in \mathbb{R}^d$, $\tau_v :\ L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the "translation by $v$" operator i.e for $\varphi \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $(\tau_v\varphi)(x) = \varphi(x - v)$.
- $\mathcal{D}'_\theta$ is the space $\theta$-quasiperiodic distribution in $\mathbb{R}^d$ i.e the space of distributions $u \in \mathcal{D}'(\mathbb{R}^d)$ such that, for any $\gamma \in \Gamma$, we have $\tau_\gamma u = e^{-i\gamma\cdot\theta}u$.
- $H^k_{\text{loc}}(\mathbb{R}^d)$ is the space of distributions that locally belong to $H^k(\mathbb{R}^d)$ and we define $\mathcal{H}^k_\theta = H^k_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{D}'_\theta$.
- For $k = 0$, we define $\mathcal{H}_\theta = \mathcal{H}^0_\theta \subset L^2(C_0)$; equipped with the $L^2$-norm over $C$, it is a Hilbert space. The scalar product will be denoted by $\langle\cdot,\cdot\rangle_\theta$.
- $\text{tr}(\cdot)$ will denote the trace of operators and $\|\cdot\|_{\text{tr}}$ the trace-class norm. When the context makes it obvious, we will not specify the Hilbert space on which the trace is taken. $\text{tr}_{\mathcal{H}_\theta}$ will denote the trace in $\mathcal{H}_\theta$.
- For $(\theta,\theta') \in (\mathbb{T}^*)^2$, we define $T_{\theta\to\theta'} : \mathcal{H}_\theta \to \mathcal{H}_{\theta'}$ defined by $(T_{\theta\to\theta'}u)(x) = e^{ix\cdot(\theta'-\theta)}u(x)$. It is an isometry and $(T_{\theta\to\theta'})^* = T_{\theta'\to\theta}$.

- $\mathcal{H}_\theta$ may be view as fibre bundle over $\mathbb{T}^*$, denoted by $\mathcal{H}$. $\mathcal{H}$ is Hilbert space with the norm $\displaystyle\frac{1}{Vol(\mathbb{T}^*)}\int_{\mathbb{T}^*}\|u(.,\theta)\|^2_{L^2(C_0)}d\theta$.
- $\mathbb{E}(f)$ (resp. $\mathbb{P}(\Omega)$) will denote the expectation of a function $f$ (resp. the probability of an event $\Omega$). The probabilistic structure we refer to will always be obvious from the context.

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## 1. Some preliminary considerations on periodic Schrödinger operators

We will first quickly review some standard facts from the Floquet theory for periodic Schrödinger operators. Basic references where this material may be found are [29], [24] and [25].

Let $W$ be a $\Gamma$-periodic potential in $L^p_{\text{loc}}(\mathbb{R}^d)$ (where $p=2$ if $d\leq 3$, $p>2$ if $d=4$ and $p>d/2$ if $d\geq 5$) and $H=-\Delta+W$ be a periodic Schrödinger operator acting on $L^2(\mathbb{R}^d)$. Then, by [24], $H$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. Its unique self-adjoint extension will also be denoted by $H$.

1.1. **The Floquet decomposition.** As $W$ is $\Gamma$-periodic, we know that, for any $\gamma \in \Gamma$,

$$\tau_\gamma \circ H \circ \tau_\gamma^* = \tau_\gamma \circ H \circ \tau_{-\gamma} = H.$$

For $\theta \in \mathbb{R}^d$ and $u \in \mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, we define

$$(Uu)(x,\theta) = \sum_{\gamma\in\Gamma} e^{i\gamma\cdot\theta} u(x-\gamma).$$

$U$ can be extended as a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$. Its inverse is given by the following formula:

$$\text{for } v \in \mathcal{H}, \quad (U^*v)(x) = \frac{1}{Vol(\mathbb{T}^*)} \int_{\mathbb{T}^*} v(x,\theta) d\theta.$$

One sees that, for $1 \leq j \leq d$, $[\partial_j, U] = 0$ (here $\partial_j$ denotes the $j$-th partial derivative). Moreover, as $W$ is periodic, $[W, U] = 0$. Hence, $H$ admits the Floquet decomposition

$$UHU^* = \int^{\oplus} H_\theta d\theta, \tag{1}$$

where $H_\theta$ is the differential operator $H$ acting on $\mathcal{H}_\theta$ with domain $\mathcal{H}_\theta^2$. $H_\theta$ is self-adjoint. As $H$ is elliptic, we know that $H_\theta$ has a compact resolvent; hence its spectrum is discrete. Let us denote its eigenvalues (also called Floquet eigenvalues of $H$) by

$$E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_n(\theta) \leq \ldots$$

The functions $(\theta \mapsto E_n(\theta))_{n\in\mathbb{N}}$ are Lipschitz continuous and one has

$$E_n(\theta) \to +\infty \text{ as } n \to +\infty \text{ (uniformly in } \theta).$$

Let $\sigma$ denote the spectrum of $H$. It is purely absolutely continuous and given by

$$\sigma = \bigcup_{n\in\mathbb{N}} E_n(\mathbb{T}^*).$$

1.2. **Wannier basis.** Let $\mathcal{E} \subset L^2(\mathbb{R}^d)$ be a closed subspace $\Gamma$-translation invariant i.e if $\Pi^{\mathcal{E}}$ is the orthogonal projector on $\mathcal{E}$, one has for $\gamma \in \Gamma$,

$$\Pi^{\mathcal{E}} = \tau_\gamma \Pi^{\mathcal{E}} \tau_\gamma^*.$$

The Floquet decomposition of $\Pi^{\mathcal{E}}$ tells us then that

$$\Pi_{\mathcal{E}} = \int_{\mathbb{T}^*}^{\oplus} \Pi_\theta^{\mathcal{E}} d\theta,$$

where $\Pi_\theta^{\mathcal{E}}$ is $\Pi^{\mathcal{E}}$ acting on $\mathcal{H}_\theta$. $\Pi_\theta^{\mathcal{E}}$ still is an orthogonal projector. The family of projectors $(\Pi_\theta^{\mathcal{E}})_{\theta\in\mathbb{T}^*}$ is continuous in $\theta$. Hence it is of constant rank. For $\theta \in \mathbb{T}^*$, we can find an orthonormal system of vectors $(u_n(\cdot,\theta))_{n\in N}$ (where $N \subset \mathbb{N}$ is a set of indices independent of $\theta$) that spans the range of $\Pi_\theta^{\mathcal{E}}$.

So transporting $(u_n(\cdot,\theta))_{n\in N}$ over to $L^2(\mathbb{R}^d)$ using $U^*$, we see that there exists an orthonormal system of vectors $(u_{n,0})_{n\in N}$ such that, if for $\gamma \in \Gamma$, we set $u_{n,\gamma} = \tau_\gamma(u_{n,0})$, then, $(u_{n,\gamma})_{(n\in N;\ \gamma\in\Gamma)}$ is an orthonormal basis for $\mathcal{E}$. We will call it a *Wannier basis* of $\mathcal{E}$. The vectors $(u_{n,0})_{n\in N}$ are the *Wannier generators* for $\mathcal{E}$.

We will say that $\mathcal{E} \subset H^2(\mathbb{R}^d)$, a $\Gamma$-translation invariant subspace, is of *finite energy* if $\Pi^{\mathcal{E}} H \Pi^{\mathcal{E}}$ is a bounded operator. If so, $\mathcal{E}$ has only finitely many Wannier generators.

1.3. **The density of states.** One defines $n(E)$, the *integrated density of states* of $H$ in the following way (cf [25] or [29]), for $E \in \mathbb{R}$,

$$n(E) = \frac{1}{(2\pi)^d} \sum_{n\in\mathbb{N}} \int_{\{\theta\in\mathbb{T}^*;\ E_n(\theta)\leq E\}} d\theta. \tag{2}$$

It is a continuous, positive, increasing function. It is constant in the gaps of the spectrum of $H$ and its growth points are exactly the points of $\sigma$.

We now give a result that relates the behaviour of the Floquet eigenvalues at the edge of the spectrum to the behaviour of the integrated density of states. We assume that $\sigma$ has a gap below energy 0 of length at least $\delta$ that is
(H1):

- for some $a > 0$ and $\delta > 0$, $\sigma \cap [0, a] = [0, a]$ and $\sigma \cap [-\delta, 0) = \emptyset$.

Then there exists a finite set of indices $J_0 \subset \mathbb{N}$ such that

- for $j \in J_0$, $E_j(\theta)$ vanishes at some point $\theta \in \mathbb{T}^*$.
- for $j \notin J_0$, $E_j(\theta)$ takes its values in $(-\infty, -\delta)$ or $(b, +\infty)$ for some $b > 0$.

The Floquet eigenvalues $(E_j)_{j\in J_0}$ will be said to *generate the band at 0.* One has the

**Proposition 1.1.** *One has*

$$\limsup_{E\to 0^+} \frac{\log(n(E) - n(0))}{\log E} \leq \frac{d}{2}. \tag{3}$$

*The following statements are equivalent:*

1. $\displaystyle\limsup_{E\to 0^+} \frac{\log(n(E) - n(0))}{\log E} = \frac{d}{2}$.
2. $\displaystyle\lim_{E\to 0^+} \frac{\log(n(E) - n(0))}{\log E}$ *exists and is equal to* $d/2$.
3. *Let $E_j$ be any of the Floquet eigenvalues generating the band at 0 and $\theta^0 \in \mathbb{T}^*$ be such that $E_j(\theta^0) = 0$, then $\theta^0$ is non-degenerate minimum of $E_j$ that is, there exist $C > 1$ and $\delta > 0$ such that*

$$\forall |\theta - \theta^0| \leq \delta,\ \frac{1}{C}|\theta - \theta^0|^2 \leq E_j(\theta) \leq C|\theta - \theta^0|^2.$$

*Hence, the (real) zeros of the Floquet eigenvalues that generate the band at 0 are isolated, non-degenerate and there are only finitely many of them.*

Remark 1.1. Of course, the same result holds if $H$ admits a gap above energy 0. One then replaces the non-degenerate minimum result by a non-degenerate maximum result.

The correspondence between the non-degeneracy of the Floquet eigenvalues and the non-degeneracy of the density of states comes from the fact that both behaviours are extremal in their own class. The extremality of the non-degenerate behaviour of the density of states is given by (3) and the equivalent points 1., 2. and 3. of Proposition 1.1.

## 2. Lifshits tails behaviour for random Schrödinger operators

In the sequel, $H$ will be a periodic Schrödinger operator as above that satisfies (H.1). We will now consider a random Schrödinger of the form

$$H_\omega = H + V_\omega = H + \sum_{\gamma\in\Gamma} \omega_\gamma V(\cdot - \gamma), \tag{4}$$

where we assume that
(H.2)

- $0 \not\equiv V \in \ell^1(L^p(\mathbb{R}^d))$ (where $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p > d/2$ if $d \geq 5$) (see [24] or [11]).
- $V$ is non-negative.
- $(\omega_\gamma)_{\gamma\in\Gamma}$ is a collection of independent identically distributed bounded random variables.

By [11], we know that $H_\omega$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$, and we denote by $H_\omega$ its self-adjoint extension. By [23] or [11], we know that its integrated density of states is well defined; it is a non random, non increasing function and right continuous. Let us denote it by $N(E)$. Its distributional derivative is a positive measure of support $\Sigma$. $N(E)$ is constant in the gaps of $\Sigma$. $N(E^-)$ will denote the left-limit of $N$ at energy $E$.

We will first under some assumptions on $V$ and $(\omega_\gamma)_{\gamma\in\Gamma}$ state a general result that links Lifshits tails behaviour for the integrated density of states of $H_\omega$ to the non-degenerate behaviour of the integrated density of states of $H$. We will then use this result to recover some well known results for Lifshits tails in dimension 1 and in any dimension at the bottom of the spectrum. We will also derive some new results.

### 2.1. The main theorem.

To prove this result we will need a more precise assumption on the behaviour of $V$ at infinity that is
(H.2bis)

- there exist $g_+ \geq g_- \geq 0$, $g_+ \in L^p(C_0)$ and $0 \not\equiv g_- \in L^p(C_0)$ (here $p$ is taken as in (H.2)) such that,
  1. either, for some $\nu \in (d, d+2)$, one has, for any $\gamma \in \Gamma$ and almost every $x \in C_0$,
  $$g_-(x) \leq V(x+\gamma)\cdot(1+\mid\gamma\mid)^\nu \leq g_+(x),$$
  2. or, one has, for any $\gamma \in \Gamma$ and almost every $x \in C_0$,
  $$V \not\equiv 0 \text{ and } 0 \leq V(x+\gamma)\cdot(1+\mid\gamma\mid)^{d+2} \leq g_+(x).$$

We will also need to know more about the random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ that is we assume
(H.3)

- the i.i.d random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ are bounded non negatively valued. Their essential support contains 0 and is not reduced to this single point.
- $\displaystyle\limsup_{\varepsilon\to0^+} \frac{\log|\log\mathbb{P}(\{\omega_0\leq\varepsilon\})|}{|\log\varepsilon|} = 0.$

Remark 2.1. The positivity assumption on the $(\omega_\gamma)_{\gamma\in\Gamma}$ is not necessary as using their boundedness we may always shift them to be positive. The price to pay for that is a change in the periodic background potential.

The second point of the previous assumption ensures that the common probability measure of the i.i.d random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ does not vanish too fast (actually exponentially fast) at 0.

Under assumption (H.1)-(H.3), we know that $\Sigma$ the almost sure spectrum of $H_\omega$ contains some interval of the form $[0,a)$ $(a>0)$ ([13]). Now we want to study Lifshits tails near 0. Therefore we need to know that 0 is the edge of a gap for $\Sigma$. Hence we assume that
(H.4)

- for some $\delta'>0$, $\Sigma\cap[-\delta',0)=\emptyset$.

To get our strongest result, we will need a slightly stronger assumption that can be summed up as follows: 0 should stay the edge of a gap when the random perturbation is turned on. More precisely, for $t\in[0,1]$, we define the random Schrödinger operator $H_{t,\omega}=H+tV_\omega$ and $\Sigma_t$ to be its almost sure spectrum, then, we assume that
(H.4s)

- for some $\delta'>0$, for any $t\in[0,1]$, $\Sigma(t)\cap[-\delta',0)=\emptyset$.

Remark 2.2. First let us notice that, if we assume that the support of the common probability measure of the $(\omega_\gamma)_{\gamma\in\Gamma}$ is connected (i.e is an interval) then (H.4s) is equivalent (H.4). Indeed, by [13], if the support of the common probability measure of the $(\omega_\gamma)_{\gamma\in\Gamma}$ is connected then $\Sigma(t)\subset\Sigma(t')$ for $0\le t\le t'\le 1$, $\Sigma(t)$ is continuous in $t$ and $\cup_{t\in[0,1]}\Sigma(t)=\Sigma$. In general (H.4) and (H.4s) are not equivalent, the connectedness of the support being necessary as can be seen from discrete examples (e.g [28]).

More generally, assumption (H.4s) can always be replaced by a stronger assumption on the size of $V_\omega$ of the form

- for some $\delta'>0$, almost surely $\Pi_-V_\omega\Pi_-\le(\delta-\delta')\Pi_-$.

Here $\Pi_-$ is the spectral projection associated to $H$ and the interval $(-\infty,0)$. This will ensure that the gap below energy 0 will stay open for almost every $\omega$. Notice that as $\Pi_-$ is of finite energy (i.e $\Pi_-H\Pi_-$ is a bounded operator) and as $V_\omega$ is uniformly relatively $H$-bounded, $\Pi_-V_\omega\Pi_-$ is a bounded family of operators.

We have the

**Theorem 2.1.** *Let $H$ and $H_\omega$ be constructed as above, and assume that (H.1), (H.2) and (H.3) hold. Then, we show that*

- *if (H.4) hold, then*
  - *if $V$ satisfies (H.2bis.2) then we have*

$$\liminf_{E\to0^+}\frac{\log|\log(N(E)-N(0^-))|}{\log E}\ge-\frac{d}{2}$$

*and*

$$\limsup_{E\to0^+}\frac{\log(n(E)-n(0))}{\log E}<\frac{d}{2}\Longrightarrow\liminf_{E\to0^+}\frac{\log|\log(N(E)-N(0^-))|}{\log E}>-\frac{d}{2}$$

– *if $V$ satisfies (H.2bis.1) then we have*

$$\liminf_{E\to 0^+} \frac{\log|\log(N(E)-N(0^-))|}{\log E} \geq -\frac{d}{\nu - d}$$

*and*

$$\limsup_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} < \frac{d}{2} \Longrightarrow \liminf_{E\to 0^+} \frac{\log|\log(N(E)-N(0^-))|}{\log E} > -\frac{d}{\nu - d}$$

- *if (H.4s) holds then 0 is a continuity point for $N$ and*
  – *if $V$ satisfies (H.2bis.2) then we have*

$$\lim_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} = -\frac{d}{2} \Longleftrightarrow \lim_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} = \frac{d}{2}$$

  – *if $V$ satisfies (H.2bis.1) then we have*

$$\lim_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} = -\frac{d}{\nu - d} \Longleftrightarrow \lim_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} = \frac{d}{2}$$

Remark 2.3. The same result holds if 0 is the upper edge of a band of $H$. One just has to replace the positivity assumption on $V$ by a negativity assumption and make the obvious changes in assumptions (H.2bis), (H.4) and (H.4s).
We also get a result if we remove the assumption about the not too fast decay of the common probability measure of the $(\omega_\gamma)_{\gamma\in\Gamma}$ near 0 (see Remark 7.1).

2.2. **Some applications.** We now apply Theorem 2.1 to get the Lifshits tails asymptotics for some concrete examples of random Schrödinger operators.

2.2.1. *In dimension 1.* Assume $d = 1$ and let $H$ be a periodic Schrödinger operator as above. It is a well known fact that, in the one-dimensional situation, all the bands of the spectrum of $H$ are simple and that their extrema are non-degenerate ([7]). Hence we will be able to apply Theorem 2.1 in this case. For convenience let us rephrase the assumptions in this particular case. Let $H_\omega = H + V_\omega$ be the random Schrödinger operator constructed as in (4) where

- $V$ satisfies (H.2bis).
- $(\omega_\gamma)_{\gamma\in\Gamma}$ are i.i.d random variables with essential support exactly the interval $[-p, p]$.
- $\limsup_{\varepsilon\to 0^+} \left|\frac{\log|\log \mathbb{P}(\{\omega_0 \leq -p+\varepsilon\})|}{\log\varepsilon}\right| + \left|\frac{\log|\log \mathbb{P}(\{\omega_0 \geq p-\varepsilon\})|}{\log\varepsilon}\right| = 0.$

We show the

**Corollary 2.1** ([22]). *Assume $d = 1$ and $H$ and $H_\omega$ constructed as above. Let $E_0$ be an edge of an open gap of $\Sigma$, the almost sure spectrum of $H_\omega$. Then, we have that*

- *if $V$ satisfies (H.2bis.2) then*

$$\lim_{\substack{E\in\sigma(H_\omega)\\ E\to E_0}} \frac{\log|\log(|\,N(E)-N(E_0)\,|)|}{\log E} = -\frac{1}{2} \tag{5}$$

- *if $V$ satisfies (H.2bis.1) then*

$$\lim_{\substack{E\in\sigma(H_\omega)\\ E\to E_0}} \frac{\log|\log(|\,N(E)-N(E_0)\,|)|}{\log E} = -\frac{1}{\nu-1} \tag{6}$$

Remark 2.4. This result improves slightly on [22] as we allow a lower decrease rate at infinity and local singularities for the single-site potential $V$. One can also use Theorem 2.1 to get results when the support of the $(\omega_\gamma)_{\gamma\in\Gamma}$ is not connected.

**Proof.** Define the periodic Schrödinger operators $H^- = H - p\sum_{\gamma\in\Gamma} V(\cdot-\gamma)$ resp. $H^+ = H + p\sum_{\gamma\in\Gamma} V(\cdot-\gamma)$ and $\sigma^-$ resp. $\sigma^+$ their spectra. We have $H_\omega = H^+ + V_\omega^+ = H^- + V_\omega^-$ where

$$V_\omega^-(x) = \sum_{\gamma\in\Gamma}\omega_\gamma^- V(x-\gamma),\ \omega^- = \omega_\gamma + p,\ V_\omega^+(x) = \sum_{\gamma\in\Gamma}\omega_\gamma^+ V(x-\gamma) \text{ and } \omega^+ = \omega_\gamma - p.$$

Let $E_0$ be an edge of an open gap of $\Sigma$. Then we claim that:

- if for some $a>0$, $[E_0,E_0+a]\subset\Sigma$ and $[E_0-a,E_0)\cap\Sigma=\emptyset$ then, for some $a'>0$, we have $[E_0,E_0+a]\subset\sigma^-$ and $[E_0-a,E_0)\cap\sigma^-=\emptyset$.
- if for some $a>0$, $[E_0-a,E_0]\subset\Sigma$ and $(E_0,E_0+a]\cap\Sigma=\emptyset$ then, for some $a'>0$, we have $[E_0-a,E_0]\subset\sigma^+$ and $(E_0,E_0+a]\cap\sigma^+=\emptyset$.

Both claims being proved the same way, let us only prove the first. Assume that $[E_0,E_0+a]\subset\Sigma$ and $[E_0-a,E_0)\cap\Sigma=\emptyset$. By [13], we know that

- for $H_{\text{per}}$ an admissible periodic realization of $H_\omega$, we have $[E_0-a,E_0)\cap\sigma(H_{\text{per}})=\emptyset$ (where $\sigma(H_{\text{per}})$ is the spectrum of $H_{\text{per}}$).
- for any $\varepsilon>0$, there exists $H_\varepsilon$ an admissible periodic realization of $H_\omega$ such that $[E_0,E_0+\varepsilon]\subset\sigma(H_\varepsilon)\neq\emptyset$.

Pick $\varepsilon>0$ and write $H_\varepsilon = H+V_\varepsilon$. Then, if we set $W_\varepsilon = -(V_\varepsilon + p\sum_{\gamma\in\Gamma}V(\cdot-\gamma))$ then for any $t\in[0,1]$, $H_\varepsilon+tW_\varepsilon$is an admissible periodic realization of $H_\omega$ (as the support of the random variables is an interval). So that for any $t\in[0,1]$, $[E_0-a,E_0)\cap\sigma(H_\varepsilon+tW_\varepsilon)=\emptyset$. Hence, as the edges of the bands of the spectrum of $H_\varepsilon+tW_\varepsilon$ are continuous and non-increasing functions of $t$ ($W_\varepsilon$ is non-positive), we get that for any $t\in[0,1]$, $[E_0,E_0+\varepsilon]\subset\sigma(H_\varepsilon+tW_\varepsilon)\neq\emptyset$. For $t=1$, we get that $[E_0,E_0+\varepsilon]\subset\sigma(H^-)\neq\emptyset$. This now holds for any $\varepsilon>0$. As the spectrum of $H^-$ is closed, it implies that $E_0\in\sigma^-$. The claim then follows from the fact that the bands of $\sigma^-$ are never reduced to single points.

The claim shows that assumptions (H.1)-(H.3) and (H.4s) are fulfilled. As we are working in dimension 1, the integrated density of states of states is always non-degenerate at the edges of open gaps; so applying Theorem 2.1 proves Corollary 2.1.

□

2.2.2. *At the bottom of the spectrum in any dimension.* Let the dimension $d$ now be arbitrary. Then we have

**Corollary 2.2** ([14]). *Let $H$ and $H_\omega$ be constructed as in (4) and assume that 0 is the infimum of the spectrum of $H$. Furthermore assume that (H.2bis) and (H.3) are satisfied. Then, we have*

- *if $V$ satisfies (H.2bis.2),*

$$\lim_{E\to 0^+} \frac{\log|\log(N(E))|}{\log E} = -\frac{d}{2} \tag{7}$$

- *if $V$ satisfies (H.2bis.1),*

$$\lim_{E\to 0^+} \frac{\log|\log(N(E))|}{\log E} = -\frac{d}{\nu - d} \tag{8}$$

Remark 2.5. This result actually improves on the result proved in [14] as we do not need any symmetry assumption on the background potential $W$.

**Proof.** It is a well known fact that, in any dimension, the bottom of the spectrum of a periodic Schrödinger operator is given by a simple Floquet eigenvalue, and that its minimum is non-degenerate ([1], [3] or [20]). As we are dealing with the bottom of the spectrum, (H.4s) is obviously satisfied. We then apply Theorem 2.1 to get Corollary 2.2. □

2.2.3. *Some new results.* We will not present detailed results but only indicate other cases of periodic Schrödinger operators where it is known that their integrated density of states has a non-degenerate behaviour at some edge of a band.

The first result concerns operators in the large coupling constant limit (or in the semi-classical limit) and operators in the tight-binding approximation. In the large coupling constant limit, consider a periodic Schrödinger operator of the form

$$H_\lambda = -\Delta + \lambda W$$

where

- $\lambda$ is a non-negative coupling constant.
- $W$ is non-negative $\mathbb{Z}^d$-periodic potential that admits 0 as its minimum.
- 0 is attained at the points of $\mathbb{Z}^d$ only and it is non-degenerate minimum of $W$.

Then it was proved in [16] that, for $\lambda$ large, the first Floquet eigenvalue of $H_\lambda$ is simple and that it is Morse function. Moreover we know that there is a gap of size $\lambda/C$ that separates the first band from the rest of the spectrum. So adding to $H$ a random potential constructed as in (4) that satisfies (H.2bis) and such that the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$) are $o(\lambda)$, we define $H_{\omega,\lambda}$. We then check that, for $\lambda$ large enough, we will be able to apply Theorem 2.1 to $H_{\omega,\lambda}$ at both edges of the first band. So we get the Lifshits tail asymptotic at both edges of the first band of $H_{\omega,\lambda}$.
In the tight-binding limit, the model is the following; let

$$H_\lambda = -\Delta + \sum_{\gamma\in\mathbb{Z}^d} W(x - \lambda\gamma)$$

where

- $\lambda$ is a non-negative scaling constant.
- $W$ is non-positive smooth sufficiently fast decreasing potential (see [5]).

- the infimum of the spectrum of the operator $H = -\Delta + W$ is a simple eigenvalue denoted by $E_0$.

Then, in [5], it was proved that, for $\lambda$ large, the first Floquet eigenvalue of $H_\lambda$ is simple and has non-degenerate extrema. Moreover there is a gap of size $1/C$ (for some $C > 0$ independent of $\lambda$) that separates the first band from the rest of the spectrum. So with the adequate assumptions on $V_\omega$, in this case too, we prove that the integrated density of states of $H_{\omega,\lambda}$ exhibits a Lifshits tail asymptotic at both edges of the first band of $H_{\omega,\lambda}$ for $\lambda$ large enough.

To end this section, let us say that, in [6], it was proved that in dimension $d = 2$, for a given lattice $\Gamma$ and a number $N$ fixed, generically, all the Floquet eigenvalues of a $\Gamma$-periodic operator $H = -\Delta + W$ (for $W$ small enough) are Morse functions. Using this in conjunction with the results of [30] stating that in dimension 2, there are at most a finite number of open gaps, we get that, generically for small potentials, the density of states of $H$ has a non-degenerate behaviour at the edges of a gap. Hence, by Theorem 2.1, for sufficiently small random perturbation of $H$ (satisfying the assumptions given above), the integrated density of states exhibits Lifshits tail asymptotic behaviour at all the edges of the spectrum.

Remark 2.6. To our knowledge, the behaviour of the integrated density of states of a generic periodic Schrödinger operator at the edges of its spectrum is a still open question (in dimension larger than 1) (see [19]).

## 3. The proof of Proposition 1.1

We recall that we supposed that $0 \in \sigma$ and that, for some $a > 0$ and $b > 0$, $\sigma \cap [0, a) = [0, a)$ and $\sigma \cap (-b, 0) = \emptyset$. We recall that the Floquet eigenvalues of $H$ are indexed by integers and ordered increasingly. We can then partition the set of indices $\mathbb{N}$ into $\mathbb{N} = J_- \cup J_0 \cup J_+$ where

- $J_0$ is the set of indices of the Floquet eigenvalues of $H$ that vanish at some point in $\mathbb{T}^*$. It is a finite set of consecutive integers.
- $J_-$ is the set of indices less than $\inf J_0$.
- $J_+$ is the set of indices larger than $\sup J_0$.

To simplify the notations, we now change the indexation of the Floquet eigenvalues. The eigenvalues of $H_\theta$ denoted by $(E_j(\theta))_{j \in J_0}$ that vanish somewhere in $\mathbb{T}^*$ are indexed in increasing order and enumerated as follows: $E_1(\theta) \le \cdots \le E_n(\theta)$. The other Floquet eigenvalues will obviously not play a role in this section; so we do not bother to reindex them.

Let $\theta^0$ be a point in $\mathbb{T}^*$ where some $E_j$ vanishes. Define $1 \le q \le N$ to be such that

- for $1 \le k \le q$, $E_k(\theta^0) = 0$,
- for $q + 1 \le j \le n$, $E_j(\theta^0) > 0$.

For the sake of simplicity and without reducing the generality of our discussion, in the sequel, we will assume $\theta^0 = 0$. So, for $\theta$ close to 0, the only Floquet eigenvalues of $H$ close to 0 are the $(E_j)_{1 \le j \le q}$.

Following [29], locally for $\theta$ close to 0, we can reduce the study of $H_\theta$ at energies close to 0 to the study of a $q \times q$ matrix that is analytic in $\theta$, hermitian when $\theta$ is real.

Indeed let $(u_j(\cdot,0))_{1\le j\le q}$ be a set of orthonormal eigenvectors for the eigenvalue 0 for $H_0$. Define the projector

$$\Pi_\theta = \int_{\mathcal{C}_0} (z-H_\theta)^{-1}dz. \tag{9}$$

(where $\mathcal{C}_0$ is a closed path around 0 that does not enclose any other eigenvalue of $H_0$). Orthonormalizing the system $(\Pi_\theta(T_{0\to\theta}u_j(\cdot,0)))_{1\le j\le q}$, we get a system $(v_j(\cdot,\theta))_{1\le j\le q}$ that is an orthonormal basis of the spectral space of $H_\theta$ associated to energies close to 0 when $\theta$ is close to 0. Let $M(\theta)$ denote the matrix of $H_\theta$ in the basis $(v_j(\cdot,\theta))_{1\le j\le q}$. As $H_\theta$ and the $(v_j(\cdot,\theta))_{1\le j\le q}$ are analytic in $\theta$ so is $M(\theta)$. Moreover, for $\theta$ close to 0, $E$ (close to 0) is an eigenvalue of $H_\theta$ if and only if $E$ is an eigenvalue for $M(\theta)$.
We can perform a Taylor expansion near 0 for $M(\theta)$

$$M(\theta) = M + \sum_{1\le j\le d}\theta_j M^j + \sum_{1\le j<k\le d}\theta_j\theta_k M^{j,k} + O(|\theta|^3).$$

All the coefficients are $q\times q$ hermitian matrices. Moreover $M(0)=0$ and as 0 is a minimum for the spectrum of $M(\theta)$, for $1\le j\le d$, we know that $M^j=0$. So that

$$M(\theta) = \sum_{1\le j<k\le d}\theta_j\theta_k M^{j,k} + O(|\theta|^3).$$

Let us write $\theta = r\sigma$ in polar coordinates; then

$$\sum_{1\le j<k\le d}\theta_j\theta_k M^{j,k} = r^2\sum_{1\le j<k\le d}\sigma_j\sigma_k M^{j,k} = r^2 M^{(2)}(\sigma).$$

As 0 is a minimum for the spectrum of $M(\theta)$, we get that, for any $\sigma\in\mathbb{S}^{d-1}$, $M^{(2)}(\sigma)\ge 0$. If for any $\sigma\in\mathbb{S}^{d-1}$ we have $M^{(2)}(\sigma)>0$, then as the sphere $\mathbb{S}^{d-1}$ is compact, we know that, for some $C>1$ and $\theta$ real close enough to 0,

$$\frac{1}{C}\sum_{1\le j\le d}\theta_j^2 \le M(\theta) \le C\sum_{1\le j\le d}\theta_j^2.$$

Hence, for some $r>0$ and $C>1$, we have for $1\le j\le q$ and $|\theta|\le r$

$$\frac{1}{C}\sum_{1\le j\le d}\theta_j^2 \le E_j(\theta) \le C\sum_{1\le j\le d}\theta_j^2. \tag{10}$$

Let us now assume that for some $\sigma^0\in\mathbb{S}^{d-1}$, $M^{(2)}(\sigma^0)$ has a non trivial kernel. By performing a linear change of variables in $\theta$, we may assume that $\sigma^0=(1,0,\dots,0)$. Let $f\in\mathbb{C}^q$ be a unit vector such that $M^{(2)}(\sigma^0)f=0$. Then we consider the analytic function $m(\theta)=\langle M(\theta)f,f\rangle_{\mathbb{C}^q}$. By assumption, 0 is a local minimum for $m$ and we know that, along $\sigma^0$, $m$ vanishes at least to third order. Let us write a Taylor expansion for $m$ at 0, we set $\theta=(\theta_1,\theta')$,

$$0\le m(\theta) = q(\theta') + m_0\theta_1^3 + \theta_1^2\cdot l(\theta') + \theta_1\cdot q'(\theta') + p(\theta') + O(|\theta|^4)$$

where $q$ and $q'$ are quadratic forms in $\theta'$, $l$ is a linear form in $\theta'$ and $p$ is homogeneous of order 3.

We notice that $m_0 = 0$ as 0 is a minimum for $m$. Then, as for some $C > 0$ we have $|\theta_1^2 \cdot l(\theta')| \leq C(\theta_1^4 + |\theta'|^2)$, we immediately get that, for some $C > 0$ and $\theta$ close to 0

$$0 \leq m(\theta) \leq C(\theta_1^4 + |\theta'|^2).$$

$E_1(\theta)$ being the smallest eigenvalue of $M(\theta)$, by the minimax principle, it is upper bounded by $m(\theta)$; so there exists $C > 0$ and $r > 0$, for $|\theta| < r$, we get

$$0 \leq E_1(\theta) \leq C(\theta_1^4 + |\theta'|^2). \tag{11}$$

By definition

$$\begin{aligned} n(E) - n(0) &\geq \mathrm{Vol}\{\theta \in \mathbb{T}^*;\ |\theta| < r \text{ and } E_1(\theta) \leq E\} \\ &\geq \mathrm{Vol}\{\theta \in \mathbb{T}^*;\ |\theta| < r \text{ and } C(\theta_1^4 + |\theta'|^2) \leq E\} \\ &\geq C E^{(d-1)/2+1/4} \text{ for } E \text{ small enough.} \end{aligned}$$

Hence

$$\limsup_{E \to 0^+} \frac{\log(n(E) - n(0))}{\log E} \leq \frac{d-1}{2} + \frac{1}{4} = \frac{d}{2} - \frac{1}{4} < \frac{d}{2}. \tag{12}$$

Now, if $n(E)$ has a non-degenerate behaviour at energy 0 then, for all $\theta$ such that some $E_j(\theta) = 0$, we have (10). Otherwise we would have (12). On the other hand, if all the eigenvalues $(E_j)_{j \in N}$ admit only non-degenerate minima, it is clear that the integrated density of states has a non-degenerate behaviour at 0.

Moreover, if $E_1(\theta)$ admits at least one minimum that is degenerate, then (12) holds. So that we always have

$$\limsup_{E \to 0^+} \frac{\log(n(E) - n(0))}{\log E} \leq \frac{d}{2}.$$

This ends the proof of Proposition 1.1.

□

Remark 3.1. If we assume that $\limsup_{E \to 0^+} \frac{\log(n(E) - n(0))}{\log E} < \frac{d}{2}$ then we know that there exists a $\theta^0 \in \mathbb{T}^*$ that is a degenerate minimum for $E_1(\theta)$. Again, assume that $\theta^0 = 0$ and that $M^{(2)}((1, 0, \ldots, 0))$ has a non trivial kernel. We write $\theta = (\theta_1, \theta')$. We will now construct a vector $f(\theta)$ such that, for $\theta$ close to 0, we have

- $\theta \mapsto f(\theta)$ is linear.
- $|\theta_1| \leq \|f(\theta)\|_{\mathbb{C}^q}$.
- for some $C > 0$,

$$\|M(\theta) f(\theta)\|_{\mathbb{C}^q} \leq C(|\theta_1| \cdot |\theta'|^2 + |\theta'|^3 + |\theta|^4).$$

The notations being the same as above, we may write:

$$M^{(2)}(\sigma) = \sigma_1^2 N^{(0)} + \sigma_1 N^{(1)}(\sigma') + N^{(2)}(\sigma'),$$

where $\sigma' = (\sigma_2, \ldots, \sigma_d)$ and $N^{(1)}(\sigma')$ (resp. $N^{(2)}(\sigma')$) are symmetric matrices linear in $\sigma'$ (resp. quadratic in $\sigma'$).

For any $\sigma$, $M^{(2)}(\sigma)$ is positive; hence for any $u \in \mathbb{C}^q$, we have

$$|\langle N^{(1)}(\sigma') u, u\rangle| \leq 4 \langle N^{(2)}(\sigma') u, u\rangle \langle N^{(0)} u, u\rangle.$$

This now gives that, as a quadratic form, $N^{(1)}(\sigma')$ is vanishing on $\ker N^{(0)}$, the kernel of $N^{(0)}$; in other words, the image of the $\ker N^{(0)}$ under $N^{(1)}(\sigma')$ is orthogonal to the kernel of $N^{(0)}$. Let $v^0$ be a non zero vector in $\ker N^{(0)}$. Then we may set

$$v(\sigma) = \sigma_1 \cdot v^0 + v^1(\sigma') \text{ where } v^1(\sigma') = -\left[N^{(0)}\right]^{-1} N^{(1)}(\sigma')v^0.$$

(here $[N^{(0)}]^{-1}$ is the inverse of $N^{(0)}$ on $\ker(N^{(0)})^{\perp}$).
$v(\sigma)$ is linear in $\sigma$ and, as $\langle [N^{(0)}]^{-1}N^{(1)}(\sigma')v^0, v^0\rangle = 0$, for some $C > 0$, it satisfies

$$|\sigma_1| \leq |v(\sigma)| \leq C|\sigma|$$

Moreover, we compute

$$\begin{aligned} M^{(2)}(\sigma)(\sigma_1 v^0 + v^1(\sigma')) &= (\sigma_1^2 N^{(0)} + \sigma_1 N^{(1)}(\sigma') + N^{(2)}(\sigma'))(\sigma_1 \cdot v^0 + v^1(\sigma')) \\ &= \sigma_1^2(N^{(0)}v^1(\sigma') + N^{(1)}(\sigma')v^0) + \sigma_1 N^{(2)}(\sigma')v^0 + \sigma_1 N^{(1)}(\sigma')v^1(\sigma') \\ &= O(|\sigma_1| \cdot |\sigma'|^2 + |\sigma'|^3). \end{aligned}$$

Now as $M(\theta) = M^{(2)}(\theta) + O(|\theta|^3)$, if we set $f(\theta) = \theta_1 \cdot v^0 + v^1(\theta')$ then $f(\theta)$ satisfies the announced properties.

To end this section, we will construct a set of vectors in $L^2(\mathbb{T}^*, \mathcal{H}_\theta)$ that spans the spectral space associated to $H$ and the Floquet eigenvalues $(E_j)_{j\in J_0}$ and that are analytic in $\theta$ for $\theta$ close to the minima of these eigenvalues; this will be done only when the periodic integrated density of states $n(E)$ is non-degenerate at energy 0 that is

$$\limsup_{E\to 0^+} \left|\frac{\log(n(E) - n(0))}{\log E}\right| = d/2$$

We still denote the Floquet eigenvalues of $H$ that vanish at some point in $\mathbb{T}^*$ by $E_1(\theta) \leq \cdots \leq E_n(\theta)$. By Proposition 1.1, we know that, if $\theta^0 \in \mathbb{T}^*$ is such that, for some $j \in J$, one has $E_j(\theta^0) = 0$, then $\theta^0$ is non-degenerate minimum of $E_j$ that is, there exist $C > 1$ and $\delta > 0$ such that

$$\forall |\theta - \theta^0| \leq \delta, \quad \frac{1}{C}|\theta - \theta^0|^2 \leq E_j(\theta) \leq C|\theta - \theta^0|^2. \tag{13}$$

Let $Z$ be the set of $\theta$ in $\mathbb{T}^*$ such that for some $1 \leq j \leq n$, $E_j(\theta) = 0$. $Z$ is a set of isolated points by (13).

As the Floquet eigenvalues have been ordered increasingly, if for $1 \leq j \leq n$, we define $Z_j = \{\theta \in \mathbb{T}^*;\ E_j(\theta) = 0\}$, the sequence $(Z_j)_{1\leq j\leq n}$ is decreasing and $Z_1 = Z$. For $\theta^0 \in Z$, let $N_{\theta^0}$ be the set of indices $1 \leq j \leq n$ such that $E_j(\theta^0) = 0$. Finally we recall that $u_j(\cdot, \theta)$ is the Floquet eigenfunction associated to the eigenvalue $E_j(\theta)$. Then, we have the

**Lemma 3.1.** *There exists $(v_j(\cdot, \theta))_{1\leq j\leq n}$ functions in $L^2(\mathbb{T}^*, \mathcal{H}^2_\theta)$ such that:*

1. *for $\theta^0 \in Z$ and $j \in N_{\theta^0}$, there exists $V_{\theta^0}$, a neighbourhood of $\theta^0$ (in $\mathbb{T}^*$) such that,*
   - *the mapping $\theta \in V_{\theta^0} \mapsto v_j(\cdot, \theta) \in L^2(\mathbb{T}, \mathcal{H}_\theta)$ is real analytic (i.e $\theta \mapsto T_{\theta\mapsto\theta_0} v_j(\cdot, \theta)$ is real analytic from $V_{\theta^0}$ to $L^2(\mathbb{T}, \mathcal{H}_{\theta_0})$).*
   - *$span((v_j(\cdot, \theta))_{j\in N_{\theta^0}}) = span((u_j(\cdot, \theta))_{j\in N_{\theta^0}})$ in $\mathcal{H}_\theta$.*
2. *for $\theta \in \mathbb{T}^*$, we have*
   - *The system $(v_j(\cdot, \theta))_{1\leq j\leq n}$ is orthonormal in $\mathcal{H}_\theta$.*

- *span*$((v_j(\cdot,\theta))_{1\leq j\leq n}) =$ *span*$((u_j(\cdot,\theta))_{1\leq j\leq n})$ *in* $\mathcal{H}_\theta$.

**Proof.** Pick $\theta^0 \in Z$. For $\theta$ close to $\theta^0$, consider the Riesz integral

$$\Pi_{\theta,N_{\theta^0}} = \int_{\mathcal{C}_0} (z - H_\theta)^{-1} dz. \tag{14}$$

(where $\mathcal{C}_0$ is a closed path around 0 that does not enclose any other eigenvalue of $H_{\theta^0}$). It is well known (see e.g. [29]) that for $\theta \in V_{\theta^0}$ a sufficiently small neighbourhood of $\theta^0$, this integral defines the orthonormal projector onto the vector space spanned by $(u_j(\cdot,\theta))_{j\in N_{\theta^0}}$; moreover, it is analytic in $\theta$. Then for $j \in N_{\theta^0}$ and $\theta \in V_{\theta^0}$, we set $\tilde{v}_j(\cdot,\theta) = \Pi_{\theta,N_{\theta^0}}(T_{\theta^0\to\theta}u_j(\cdot,\theta^0))$. Using the Gram-Schmidt algorithm, we orthonormalize the system $(\tilde{v}_j(\cdot,\theta))_{j\in N_{\theta^0}}$ into a system $(v_j(\cdot,\theta))_{j\in N_{\theta^0}}$. As the system $(u_j(\cdot,\theta^0))_{j\in N_{\theta^0}}$ is orthonormal (if necessary reducing $V_{\theta^0}$ somewhat) we see that the function $(\theta \mapsto v_j(\cdot,\theta))_{j\in N_{\theta^0}}$ are real analytic in $V_{\theta^0}$.

Now, as $Z$ is a finite set, we may reduce the above constructed neighbourhoods $(V_{\theta^0})_{\theta^0\in Z}$ so that, two by two, they do not intersect. For $\theta \notin \cup_{\theta^0\in Z} V_{\theta^0}$, we set $v_j(\cdot,\theta) = u_j(\cdot,\theta)$. The thus defined functions satisfy Lemma 3.1.

□

Remark 3.2. The construction done above can always be carried out locally near a given point in $\mathbb{T}^*$. We globalised it using the fact that the points where we want the construction to be analytic are isolated.

## 4. A reduction procedure

To study the behaviour of $N(E)$ at the edge of a gap, our main tool will be a reduction theorem.

### 4.1. Energy localization for the density of states.

The aim of this section is to give an estimation on $N(E)$ (for an energy near the edge of a gap) by means of the integrated density of states of a reference operator (that is localized in energy near the edge of the gap). In general we will need to use different reference operators to estimate $N(E)$ from above and below. We first define the reference operators.

4.1.1. *The reference operators.* Let $\mathcal{E}$ be a closed $\Gamma$-translation invariant subspace of finite energy for $H$. We have the

**Proposition 4.1.** *Assume that $H$ and $H_\omega$ are constructed as in sections 1 and 2 and that assumptions (H.1)-(H.2) are satisfied. Define $H^{\mathcal{E}}_\omega = \Pi^{\mathcal{E}} H_\omega \Pi^{\mathcal{E}}$. $H^{\mathcal{E}}_\omega$ is bounded random ergodic operator.*

*We may define its integrated density of states in the following way. Let $(u_{j,0})_{1\leq j\leq n}$ be a set of Wannier generators for $\mathcal{E}$ and for $k \in \mathbb{N}$, let $\Lambda_k \subset \Gamma$ be the cube of centre 0 and sidelength $k$. Define $\Pi^{\mathcal{E}}_k$ to be the orthogonal projector on the vector space spanned by $(u_{j,\gamma})_{1\leq j\leq n;\ \gamma\in\Lambda_k}$ (here $u_{j,\gamma}(x) = u_{j,0}(x-\gamma)$) and*

$$H^{\mathcal{E}}_{\omega,k} = \Pi^{\mathcal{E}}_k H_\omega \Pi^{\mathcal{E}}_k.$$

*Let u denote by $N_{\mathcal{E},\Lambda_k}$ the normalized counting function of $H^{\mathcal{E}}_{\omega,k}$ that is*

$$N_{\mathcal{E},\Lambda_k} = \frac{1}{Vol(C_0)\cdot \sharp\Lambda_k}\sharp\{\textit{eigenvalues of } H^{\mathcal{E}}_{\omega,k} \leq \lambda\} \textit{ for } \lambda\in\mathbb{R}$$

*and $dN_{\mathcal{E},\Lambda_k}$ the associated counting measure (i.e its derivative).*

*Then, almost surely, the sequence of positive measures $(dN_{\mathcal{E},\Lambda_k})_{k\in\mathbb{N}}$ converges vaguely to $dN_{\mathcal{E}}$ a positive measure that is non random. Moreover $dN_{\mathcal{E}}$ is independent of the Wannier generators chosen for the construction of the approximation measures. It is the density of states measure of $H^{\mathcal{E}}_{\omega}$. For $\varphi\in\mathcal{C}^\infty_0$, one has*

$$\int_{\mathbb{R}}\varphi(\lambda)dN_{\mathcal{E}}(\lambda) = \frac{1}{Vol(C_0)}\sum_{j=1}^{n}\mathbb{E}\left(\left\langle u_{j,0}, \varphi\left(H^{\mathcal{E}}_{\omega}\right)u_{j,0}\right\rangle\right).$$

*Furthermore, if we define $N_{\mathcal{E}}(\lambda) = \displaystyle\int_{(-\infty,\lambda]} dN_{\mathcal{E}}(\lambda)$ then this function is non-decreasing right continuous. Almost surely, for $\lambda\in\mathbb{R}$ a continuity point of $N_{\mathcal{E}}$, we have that*

$$N_{\mathcal{E}}(\lambda) = \lim_{k\to+\infty}\frac{1}{Vol(C_0)\cdot\sharp\Lambda_k}\sharp\{\textit{eigenvalues of } H^{\mathcal{E}}_{\omega,k}\leq\lambda\}.$$

Remark 4.1. As will be seen from its proof, the result on the density of states given above holds for a general bounded random ergodic operator acting on $l^2(\Gamma)$, under no further decrease assumptions on the matrix coefficients of the operator. This improves on previously known results [23].

Notice that the factor $\mathrm{Vol}(C_0)\cdot\sharp\Lambda_k$ is nothing but the volume of $\Lambda_k$ seen as a parallelogram in $\mathbb{R}^d$.

4.1.2. *The reduction to a discrete problem.* To perform this reduction, we are going to decompose our random hamiltonian on different translation invariant subspaces; the thus obtained random operators will be the reference operators used to estimate the density of states. We will need two decompositions depending on which bound (lower or upper) we want to get.
Let us first describe the decomposition needed for the upper bound. We partition $\mathbb{N}$ in $\mathbb{N} = J_-\cup J_0\cup J_+$ where

- $J_0$ is the set of indices of the Floquet eigenvalues of $H$ that vanish at some point in $\mathbb{T}^*$. It is a finite set of consecutive integers.
- $J_-$ is the set of indices less than $\inf J_0$.
- $J_+$ is the set of indices larger than $\sup J_0$.

Denote by $\Pi_0(\theta)$, $\Pi_-(\theta)$ and $\Pi_+(\theta)$ the orthogonal projections in $\mathcal{H}_\theta$ on the vector spaces spanned by respectively $(u_j(\cdot,\theta))_{j\in J_0}$, $(u_j(\cdot,\theta))_{j\in J_-}$ and $(u_j(\cdot,\theta))_{j\in J_+}$ (we recall that the $(u_j(\cdot,\theta))_{j\in\mathbb{N}}$ are the Floquet eigenfunctions for $H$). Obviously, these projectors are mutually orthogonal and their sum is the identity for any $\theta\in\mathbb{T}^*$.

Define $\Pi_\cdot = U^*\Pi_\cdot(\theta)U$ where $\cdot\in\{-,0,+\}$. Then, $\Pi_\cdot$ is an orthogonal projector on $L^2(\mathbb{R}^d)$ and for $\gamma\in\Gamma$, we have $\tau^*_\gamma\Pi_\cdot\tau_\gamma = \Pi_\cdot$. It is clear that $\Pi_-$, $\Pi_0$ and $\Pi_+$ are mutually orthogonal and that $\Pi_-+\Pi_0+\Pi_+ = Id_{L^2(\mathbb{R}^d)}$. For $\cdot\in\{-,0,+\}$, we set $\mathcal{E}_\cdot = \Pi_\cdot(L^2(\mathbb{R}^d))$. These spaces are translation invariant. Moreover, $\mathcal{E}_-$ and $\mathcal{E}_0$ are of finite energy.

The decomposition needed for the lower bound is the following. For $l \in \mathbb{N}$, we define $\Pi_{l,-}$ (resp. $\Pi_{l,+}$) to be the orthogonal projection in $L^2(\mathbb{R}^d)$ on the vector space spanned by the Wannier generators $(u_j(\cdot,\theta))_{0\leq j\leq l}$ (resp. by the Wannier generators $(u_j(\cdot,\theta))_{l+1\leq j}$). As above, we define $\mathcal{E}_{l,-} = \Pi_{l,-}(L^2(\mathbb{R}^d))$ and $\mathcal{E}_{l,+} = \Pi_{l,+}(L^2(\mathbb{R}^d))$. $\mathcal{E}_{l,-}$ is translation invariant and of finite energy. Then we get

**Theorem 4.1.** *Assume that $H$ and $H_\omega$ are as above and satisfy assumptions (H.1), (H.2) and (H.3). There exists $l_0 \in \mathbb{N}$, $E_0 > 0$ and $C > 1$ such that, for $0 \leq E \leq E_0$ and $l \leq l_0$,*

- *if (H.4) holds, one has*
$$N_{\mathcal{E}_{l,-}}(E) - N_{\mathcal{E}_{l,-}}(0^-) \leq N(E) - N(0^-)$$
*where $N_{\mathcal{E}_{l,-}}$ is the integrated density of states of $\Pi_{l,-}H_\omega\Pi_{l,-}$.*
- *if (H.4s) holds, one has*
$$N_{\mathcal{E}_{l,-}}(E) - N_{\mathcal{E}_{l,-}}(0^-) \leq N(E) - N(0^-) \leq N_{\mathcal{E}_0}(C\cdot E)$$
*where $N_{\mathcal{E}^0}$ is the integrated density of states of $\Pi_0 H_\omega \Pi_0$.*

Remark 4.2. If we replace (H.4s) by the stronger assumption given in the Remark 2.2, then in Theorem 4.1, we can replace $N_{\mathcal{E}_{l,-}}$ in the lower bound by $N_{\mathcal{E}_0}$ (modulo a dilation in energy). If we accept to replace $N_{\mathcal{E}_0}$ by $N_{\mathcal{E}_{l,-}}$ in the upper bound, we can choose $C$ as close to 1 as we want if we increase $l$.
Let us point out here that while proving Theorem 4.1, we will also prove that $\Pi_{l,-}H_\omega\Pi_{l,-}$ has a gap of size at least $\delta'/2$ below 0.
When 0 is the upper edge of a gap ((H.1), (H.2), (H.3), (H.4) and (H.4s) being replaced by their counterparts), one shows that an analogue of Theorem 4.1 still holds.

Theorem 4.1 has an immediate corollary

**Corollary 4.1.** *Under assumption (H.4s), 0 is continuity point for $N$. More precisely, for $C$ and $E_0$ as in Theorem 4.1 and for $0 \leq E \leq E_0$, one has*
$$0 \leq N(E) - N(0^-) \leq C\cdot(n(C\cdot E) - n(0)).$$

**Proof.** We notice that, as $V_\omega$ is non-negative, $\Pi_0 H_\omega \Pi_0 \geq \Pi_0 H \Pi_0$; so that if $n_{\mathcal{E}_0}$ denotes the density of states of $\Pi_0 H \Pi_0$, we have
$$N_{\mathcal{E}_0}(E) \leq n_{\mathcal{E}_0}(E).$$
By construction on $\Pi_0$, we have
$$0 \leq n_{\mathcal{E}_0}(E) \leq n(E) - n(0).$$
The continuity of $N$ at 0 is then immediate as $n(E)$ is continuous.

□

Then to study the behaviour of $N(E)$ for $E$ small, we just need to study the behaviour of $N_{\mathcal{E}_0}(E)$ and $N_{\mathcal{E}_{l,-}}(E)$. But this now presents two advantages. First, $\Pi_0 H_\omega \Pi_0$ is a discrete operator that is, via a Wannier basis for $\mathcal{E}_0$, $\Pi_0 H_\omega \Pi_0$ is in a very natural way unitarily equivalent to a random Jacobi matrix acting on $\ell^2(\Gamma)\otimes\mathbb{C}^n$. The same holds for $\Pi_{l,-}H_\omega\Pi_{l,-}$.

Moreover, for $H_\omega$, 0 was at an interior edge of the spectrum. For $\Pi_0 H_\omega \Pi_0$ it is an exterior edge, the infimum of the spectrum. In this case more techniques are available.

The proof of Theorem 4.1 is the object of the next section. In section 6 we show the required lower bound for $N_{\mathcal{E}_{l,-}}(E)$. And in section 7, we prove the upper bound on $N_{\mathcal{E}_0}(E)$. Putting this together we get Theorem 2.1.

## 5. The proof of Theorem 4.1

Our strategy will be first to approach the density of states of $H_\omega$ by the density of states of periodic approximations. Then, we will show that, for those periodic approximations, we may localize the density of states in energy, i.e control the behaviour of the density of states near 0 by the behaviour of the density of states of some reduced operator (that are bounded $d$-dimensional Jacobi matrix). We then compute the limit of the density of states of this reduced operator to get the convergence to the density of states of $\Pi_0 H_\omega \Pi_0$ and $\Pi_{l,-} H_\omega \Pi_{l,-}$, and thus get the announced control on the density of states of $H_\omega$.
The *density of states* of $H_\omega$ is the distributional derivative of the integrated density of states $n(E)$; it defines a Stieltjes measure denoted by $dn$. It satisfies, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\langle \varphi, dn \rangle = \frac{1}{(2\pi)^d} \int_{\theta \in T^*} \mathrm{tr}_{\mathcal{H}_\theta}(\varphi(H_\theta)) d\theta. \tag{15}$$

To begin the proof of this reduction procedure, we will express the density of states $dn$ in a different way. This form will be of use later on.

**Proposition 5.1.** *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and let $\chi_{C_0}$ be the characteristic function of $C_0$, the fundamental cell of $\Gamma$. Then $\chi_{C_0}\varphi(H)$ is trace-class (cf [26]) and we have*

$$\langle \varphi, dn \rangle = \frac{1}{Vol(C_0)} tr\left(\chi_{C_0}\varphi(H)\chi_{C_0}\right). \tag{16}$$

**Proof.** Let $(u_n(\cdot, \theta))_{n\in\mathbb{N}}$ be the Floquet eigenvectors of $H$. Then, for $\gamma \in \Gamma$ and $n \in \mathbb{N}$, define

$$v_{n,\gamma} = \frac{1}{\sqrt{\mathrm{Vol}(C_0^*)}} \tau_\gamma U^* u_n.$$

The vectors $(v_{n,\gamma})_{\gamma\in\Gamma;\ n\in\mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{R}^d)$. Hence, as $H$ is periodic,

$$\begin{aligned}
\frac{1}{\mathrm{Vol}(C_0)} \mathrm{tr}\left(\chi_{C_0}\varphi(H)\chi_{C_0}\right) &= \frac{1}{\mathrm{Vol}(C_0)} \mathrm{tr}\left(\chi_{C_0}\varphi(H)\right) \\
&= \frac{1}{\mathrm{Vol}(C_0)\mathrm{Vol}(C_0^*)} \sum_{n\in\mathbb{N}} \sum_{\gamma\in\Gamma} \langle v_{n,\gamma}, \chi_{C_0}\varphi(H) v_{n,\gamma}\rangle \\
&= \frac{1}{\mathrm{Vol}(C_0)\mathrm{Vol}(C_0^*)} \sum_{n\in\mathbb{N}} \sum_{\gamma\in\Gamma} \langle U^* u_n, \tau_{-\gamma}(\chi_{C_0})\varphi(H) U^* u_n\rangle.
\end{aligned}$$

As $\sum_\gamma \tau_\gamma(\chi_{C_0}) = 1$ , we get

$$\begin{aligned}
\frac{1}{\mathrm{Vol}(C_0)}\mathrm{tr}\,(\chi_{C_0}\varphi(H)\chi_{C_0}) &= \frac{1}{\mathrm{Vol}(C_0)\mathrm{Vol}(C_0^*)}\sum_{n\in\mathbb{N}}\langle U^*u_n, \varphi(H)U^*u_n\rangle \\
&= \frac{1}{\mathrm{Vol}(C_0)\mathrm{Vol}(C_0^*)}\sum_{n\in\mathbb{N}}\int_{T^*}\langle u_n(\cdot,\theta), \varphi(H_\theta)u_n(\cdot,\theta)\rangle_\theta d\theta \\
&= \frac{1}{(2\pi)^d}\int_{\theta\in T^*}\mathrm{tr}_\theta(\varphi(H_\theta))d\theta \\
&= \langle\varphi, dn\rangle.
\end{aligned}$$

using the unitarity of $U$ and the fact that $\mathrm{Vol}(C_0)\mathrm{Vol}(C_0^*) = (2\pi)^d$.

□

5.1. **The periodic approximation.** Let $k \in \mathbb{N}^*$ and define the following periodic Schrödinger operator

$$H_{\omega,k} = H + \sum_{\gamma\in C_k\cap\Gamma}\omega_\gamma\sum_{\beta\in(2k+1)\Gamma}V(x-\gamma-\beta) = H + V_{\omega,k} \tag{17}$$

where $C_k = \{x\in\mathbb{R}^d;\ \text{for } j=1,\dots,d,\ -k-1/2 < x_j \le k+1/2\}$.
Then, for any $\omega\in\Omega$ and $k\in\mathbb{N}^*$, $H_{\omega,k}$ is a $(2k+1)\Gamma$-periodic essentially self-adjoint Schrödinger operator. It is a $H$-bounded perturbation of $H$ with relative bound 0. We notice that the boundedness assumption on the $(\omega_\gamma)$ implies that the family $(H_{\omega,k})_{\omega,k}$ is uniformly $H$-bounded. $H_{\omega,k}$ will also denote its self-adjoint extension.

According to [24], [25] or [29], $H_{\omega,k}$ admits an integrated density of states denoted by $N_{\omega,k}$ that is given by (2), the torus $\mathbb{T}^*$ being replaced by the torus $\mathbb{T}_n^* = \mathbb{R}^d/((2k+1)\Gamma)^*$. Then we prove

**Theorem 5.1.** *For any $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$, we have that*

$$\lim_{k\to+\infty}\mathbb{E}\left(\int_{\mathbb{R}}\varphi(\lambda)dN_{\omega,k}\right) = \int_{\mathbb{R}}\varphi(\lambda)dN. \tag{18}$$

*Moreover, one has, for $\lambda\in\mathbb{R}$ a continuity point of $N(\lambda)$,*

$$\lim_{k\to+\infty}\mathbb{E}\left(N_{\omega,k}(\lambda)\right) = N(\lambda). \tag{19}$$

**Proof.** Let $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$. It is well known (see [23] or [4]) that

$$\int_{\mathbb{R}}\varphi(\lambda)dN = \mathbb{E}\left(\frac{1}{\mathrm{Vol}(C_0)}\mathrm{tr}\,(\chi_{C_0}\varphi(H_\omega)\chi_{C_0})\right).$$

By Proposition 5.1, we know that

$$\int_{\mathbb{R}}\varphi(\lambda)dN_{\omega,k} = \frac{1}{\mathrm{Vol}(C_k)}\mathrm{tr}\,(\chi_{C_k}\varphi(H_{\omega,k})\chi_{C_k}).$$

We decompose $\chi_{C_k} = \sum_{\gamma \in C_k \cap \Gamma} \tau_\gamma(\chi_{C_0})$ to get

$$\begin{aligned}\mathrm{tr}\left(\chi_{C_k}\varphi(H_{\omega,k})\chi_{C_k}\right) &= \sum_{\gamma \in C_k \cap \Gamma} \mathrm{tr}\left(\tau_\gamma(\chi_{C_0})\varphi(H_{\omega,k})\right) \\ &= \sum_{\gamma \in C_k \cap \Gamma} \mathrm{tr}\left(\chi_{C_0}\varphi(H_{\tau_\gamma(\omega),k})\right)\end{aligned}$$

as $H$ is $\Gamma$-periodic and $C_k$ is symmetric with respect to 0.
Hence, as the random variables $(\omega_\gamma)_\gamma$ are independent identically distributed, we get

$$\mathbb{E}\left(\mathrm{tr}\left(\chi_{C_k}\varphi(H_{\omega,k})\chi_{C_k}\right)\right) = \sharp(C_k \cap \Gamma) \cdot \mathbb{E}\left(\mathrm{tr}\left(\chi_{C_0}\varphi(H_{\omega,k})\right)\right).$$

But $\sharp(C_k \cap \Gamma) \cdot \mathrm{Vol}(C_0) = \mathrm{Vol}(C_k)$ hence

$$\mathbb{E}\left(\int_{\mathbb{R}} \varphi(\lambda) dN_{\omega,k}\right) = \frac{1}{\mathrm{Vol}(C_0)} \mathbb{E}\left(\mathrm{tr}\left(\chi_{C_0}\varphi(H_{\omega,k})\chi_{C_0}\right)\right). \tag{20}$$

Now, as the random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ are bounded, we know that $H_{\omega,k}$ is lower-bounded uniformly in $\omega$ and $k$. Pick $\lambda_0 \in \mathbb{R}$ be such that $I \leq \lambda_0 + H_{\omega,k}$ and $I \leq \lambda_0 + H_\omega$ for any $\omega \in \Omega$ and $k \in \mathbb{N}$.

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\tilde\varphi$ be an almost analytic extension of $(\lambda_0 + x)^q \varphi(x)$ for some $q > d/2$. Then, by [8] and [17], we know that, for any $n$ and $\omega \in \Omega$, the following formula holds,

$$\varphi(H_{\omega,k}) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial\tilde\varphi}{\partial\overline{z}}(z) \cdot (\lambda_0 + H_{\omega,k})^{-q}(z - H_{\omega,k})^{-1} dz \wedge d\overline{z}. \tag{21}$$

As $q > d/2$, $\chi_{C_0}(\lambda_0 + H_{\omega,k})^{-q}(z - H_{\omega,k})^{-1}$ is trace-class and we have

$$\mathrm{tr}\left(\chi_{C_0}\varphi(H_{\omega,k})\chi_{C_0}\right) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial\tilde\varphi}{\partial\overline{z}}(z) \cdot \mathrm{tr}\left(\chi_{C_0}(\lambda_0 + H_{\omega,k})^{-q}(z - H_{\omega,k})^{-1}\chi_{C_0}\right) dz \wedge d\overline{z}. \tag{22}$$

Let us now estimate

$$\left\|\chi_{C_0}(\lambda_0 + H_{\omega,k})^{-q}(z - H_{\omega,k})^{-1}\chi_{C_0} - \chi_{C_0}(\lambda_0 + H_\omega)^{-q}(z - H_\omega)^{-1}\chi_{C_0}\right\|_{\mathrm{tr}} \leq A + B \tag{23}$$

where

$$\begin{aligned}A &= \left\|\chi_{C_0}\left((z - H_{\omega,k})^{-1} - (z - H_\omega)^{-1}\right)(\lambda_0 + H_\omega)^{-q}\chi_{C_0}\right\|_{\mathrm{tr}} \\ &= \left\|\chi_{C_0}\left((z - H_{\omega,k})^{-1}\left(\sum_{\substack{|\gamma|>k \\ \gamma\in\Gamma}} (\omega_{\tilde\gamma} - \omega_\gamma) V(x-\gamma)\right)(z - H_\omega)^{-1}\right)(\lambda_0 + H_\omega)^{-q}\chi_{C_0}\right\|_{\mathrm{tr}}\end{aligned}$$

and

$$\begin{aligned}B &= \left\|\chi_{C_0}(z - H_{\omega,k})^{-1}\left((\lambda_0 + H_{\omega,k})^{-q} - (\lambda_0 + H_\omega)^{-q}\right)\chi_{C_0}\right\|_{\mathrm{tr}} \\ &= \left\|\chi_{C_0}(z - H_{\omega,k})^{-1}\sum_{l=1}^{q-1}(\lambda_0 + H_{\omega,k})^{l-q}\left(\sum_{\substack{|\gamma|>k \\ \gamma\in\Gamma}} (\omega_{\tilde\gamma} - \omega_\gamma) V(x-\gamma)\right)(\lambda_0 + H_\omega)^{-l}\chi_{C_0}\right\|_{\mathrm{tr}}\end{aligned}$$

(here $\|\cdot\|_{\mathrm{tr}}$ denotes the trace-class norm, and $\gamma = \tilde\gamma \mod ((2k+1)\cdot\Gamma)$.)

We separately control $A$ and $B$; for $A$, as the $(\omega_\gamma)_\gamma$ are bounded, we get, for some $M>0$,

$$\begin{aligned}
A &\leq \sum_{\beta\in\Gamma} \left\| \chi_{C_0}(z-H_{\omega,k})^{-1} \left( \sum_{\substack{|\gamma|>k\\ \gamma\in\Gamma}} (\omega_{\tilde\gamma}-\omega_\gamma)\, V(x-\gamma)\right) \chi_{\beta+C_0} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \times \\
&\qquad\qquad \times \left\| \chi_{\beta+C_0}(z-H_\omega)^{-1}(\lambda_0+H_\omega)^{-q}\chi_{C_0}\right\|_{\mathrm{tr}} \\
&\leq M \sum_{\substack{\beta\in\Gamma\\ |\gamma|>k\\ \gamma\in\Gamma}} \left\| \chi_{C_0}(z-H_{\omega,k})^{-1} V \chi_{\beta+\gamma+C_0}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \cdot \left\| \chi_{\beta+C_0}(z-H_\omega)^{-1}(\lambda_0+H_\omega)^{-q}\chi_{C_0}\right\|_{\mathrm{tr}}.
\end{aligned}$$

By the appendix of [17] (Proposition 4.3) and by the proof of Theorem XIII.96 of [24], we know that, for some $K>1$, we have, uniformly in $\omega$,

$$\left\| \chi_{\beta+C_0}(z-H_\omega)^{-1}(\lambda_0+H_\omega)^{-q}\chi_{C_0}\right\|_{\mathrm{tr}} \leq \frac{K}{|\mathrm{Im} z|} e^{-|\mathrm{Im} z||\beta|_\Gamma/K}$$

and

$$\left\| \chi_{C_0}(z-H_{\omega,k})^{-1} V \chi_{\beta+\gamma+C_0}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \frac{K}{|\mathrm{Im} z|} \|\tau_{\beta+\gamma} V\|_{L^p(C_0)}.$$

(where $\tau_\gamma$ is the translation by $\gamma$).
Hence, we get that

$$A \leq \sum_{\beta\in\Gamma} \sum_{\substack{|\gamma|>k\\ \gamma\in\Gamma}} \frac{K}{|\mathrm{Im} z|^2} \|\tau_{\beta+\gamma}V\|_{L^p(C_0)} \cdot e^{-|\mathrm{Im} z||\beta|_\Gamma/K}.$$

Doing the same computation for $B$, we get the same estimate.

Hence, integrating these estimates, as $\partial_{\overline{z}}\tilde\varphi$ vanishes to infinite order on the real axis and using (22) and (23), we get that,

$$\begin{aligned}
&\mathrm{Vol}(C_0) \cdot \left| \mathbb{E}\left( \int_{\mathbb{R}} \varphi(\lambda) dN_{\omega,k}\right) - \langle \varphi, dN\rangle \right| \\
&= \left| \mathbb{E}\left(\mathrm{tr}\left(\chi_{C_0}\varphi(H_{\omega,k})\chi_{C_0}\right)\right) - \mathbb{E}\left(\mathrm{tr}\left(\chi_{C_0}\varphi(H_\omega)\chi_{C_0}\right)\right)\right| \\
&\leq \mathbb{E}\left( \frac{1}{\pi} \int_{\mathbb{C}} \left| \frac{\partial\tilde\varphi}{\partial\overline{z}}(z)\right| \cdot \left\| \chi_{C_0}(\lambda_0+H_{\omega,k})^{-q}(z-H_{\omega,k})^{-1}\chi_{C_0} - \right.\right. \\
&\qquad\qquad \left.\left. -\chi_{C_0}(\lambda_0+H_\omega)^{-q}(z-H_\omega)^{-1}\chi_{C_0}\right\|_{\mathrm{tr}} dxdy\right) \qquad (24) \\
&\leq \frac{1}{\pi}\int_{\mathbb{C}} \frac{C}{|\mathrm{Im} z|^2} \cdot \left| \frac{\partial\tilde\varphi}{\partial\overline{z}}(z)\right| \cdot \left( \sum_{\substack{|\gamma|>k\\ \gamma\in\Gamma\\ \beta\in\Gamma}} \|\tau_{\beta+\gamma}V\|_{L^p(C_0)} e^{-|\mathrm{Im} z||\beta|_\Gamma/C}\right) dxdy
\end{aligned}$$

Now as $\sum_{\gamma\in\Gamma}\|\tau_\gamma V\|_{L^p(C_0)} < +\infty$, we apply Lebesgue's dominated convergence Theorem to right hand side of (24) to get that

$$\left|\mathbb{E}\left(\int_{\mathbb{R}}\varphi(\lambda)dN_{\omega,k}\right) - \langle\varphi, dN\rangle\right| \to 0 \text{ as } k \to +\infty.$$

This gives equation (18) of Theorem 5.1. To get (19), we just have to notice that, if $\lambda$ is a continuity point of $N$ then

$$N(\lambda) = \inf_{\varphi\in\mathcal{E}_\lambda^+}\langle\varphi, dN_{\omega,k}\rangle = \sup_{\varphi\in\mathcal{E}_\lambda^-}\langle\varphi, dN_{\omega,k}\rangle \tag{25}$$

where

$$\mathcal{E}_\lambda^+ = \{\varphi\in\mathcal{C}_0^\infty(\mathbb{R});\ 0\le\varphi\le 1 \text{ and } \varphi\equiv 1 \text{ on } [-\lambda_0,\lambda]\}$$
$$\text{and}$$
$$\mathcal{E}_\lambda^+ = \{\varphi\in\mathcal{C}_0^\infty(\mathbb{R});\ 0\le\varphi\le 1 \text{ and } \varphi\equiv 0 \text{ on } [\lambda,+\infty)\}.$$

As $N_{\omega,k}$ is continuous, (25) is valid for $N_{\omega,k}$ at any point $\lambda$. Then, for $\varphi\in\mathcal{E}_\lambda^+$, we have $N_{\omega,k}(\lambda)\le\langle\varphi, dN_{\omega,k}\rangle$; hence $\mathbb{E}(N_{\omega,k}(\lambda))\le\mathbb{E}(\langle\varphi, dN_{\omega,k}\rangle)$. Using (18), we get that $\limsup_n \mathbb{E}(N_{\omega,k}(\lambda))\le\langle\varphi, dN\rangle$. Hence taking the infimum over $\mathcal{E}_\lambda^+$, we get

$$\limsup_n \mathbb{E}(N_{\omega,k}(\lambda))\le N(\lambda).$$

The same way, using the other characterization of $N(\lambda)$ given by (25), we get

$$\liminf_n \mathbb{E}(N_{\omega,k}(\lambda))\ge N(\lambda).$$

This completes the proof of (19), thus of Theorem 5.1.

□

REMARK 5.1. In this proof we need a slight extension of Proposition 4.3 of [17] as in our case $V_\omega$ is not bounded but only $H$-relatively bounded. This extension is easily derived from the computations done in [17].

To define the integrated density of states, there are many approximation schemes available. Some have also been used very efficiently to estimate the integrated density of states (e.g. Dirichlet-Neumann bracketing). The periodic approximation scheme developed here could also proved quite useful as the approximation can be controlled quite precisely. Indeed, if we assume that $V$ satisfies (H.2bis) one sees that

$$\begin{aligned}\left\|\chi_{C_0}(\lambda_0+H_{\omega,k})^{-q}(z-H_{\omega,k})^{-1}\chi_{C_0}-\chi_{C_0}(\lambda_0+H_\omega)^{-q}(z-H_\omega)^{-1}\chi_{C_0}\right\|_{\mathrm{tr}}&\\ \le\frac{K}{|\mathrm{Im}z|^2}\left(e^{-k\cdot|\mathrm{Im}z|/K}+k^{-(\nu-d)}\right).&\end{aligned}$$

Integrating this, (24) tells us that, for $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$, we have

$$\langle\varphi, dN\rangle = \mathbb{E}\left(\langle\varphi, dN_{\omega,k}\rangle\right)+O(k^{-(\nu-d)}). \tag{26}$$

If we now assume that $V$ is exponentially decreasing at infinity, e.g. for some $0 \leq g_+ \in L^p(C_0)$ and $\kappa > 0$, we have, for $x \in C$ and $\gamma \in \Gamma$,

$$V(x+\gamma) \cdot e^{\kappa \cdot |\gamma|_\Gamma} \leq g_+(x),$$

then, we get that

$$\begin{aligned}\left\|\chi_{C_0}(\lambda_0 + H_{\omega,k})^{-q}(z-H_{\omega,k})^{-1}\chi_{C_0} - \chi_{C_0}(\lambda_0 + H_\omega)^{-q}(z-H_\omega)^{-1}\chi_{C_0}\right\|_{\mathrm{tr}} \\ \leq \frac{K}{|\mathrm{Im} z|^2} e^{-k\cdot|\mathrm{Im} z|/K}.\end{aligned}$$

Hence, (24) tells us that, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have

$$\langle \varphi, dN \rangle = \mathbb{E}\left(\langle \varphi, dN_{\omega,k} \rangle\right) + O(k^{-\infty}). \tag{27}$$

5.2. **Existence of the density of states for the references operators.** We will now prove Proposition 4.1. Let $\mathcal{E}$ be a $\Gamma$-translation invariant subspace of $L^2(\mathbb{R}^d)$ of finite energy for $H$. Define $H_\omega^{\mathcal{E}} = \Pi^{\mathcal{E}} H_\omega \Pi^{\mathcal{E}}$. Then, as $\mathcal{E}$ is of finite energy, $H_\omega^{\mathcal{E}}$ is a bounded operator on $\mathcal{E}$ for any $\omega \in \Omega$. It is even uniformly bounded on $\Omega$ by (H.2) and (H.3). Moreover it is clearly weakly measurable. Its ergodicity (with respect to the translations $(\tau_\gamma)_{\gamma\in\Gamma}$) is immediately deduced from the fact that $\Pi^{\mathcal{E}}$ is $\Gamma$-periodic. Let $(u_{j,0})_{1\leq j\leq n}$ and $\Lambda_k$ be as in Proposition 4.1. We recall that $\Pi_k^{\mathcal{E}}$ is the orthogonal projector on the vector space spanned by $(u_{j,\gamma})_{1\leq j\leq n;\ \gamma\in\Lambda_k}$ and

$$H_{\omega,k}^{\mathcal{E}} = \Pi_k^{\mathcal{E}} H_\omega \Pi_k^{\mathcal{E}}.$$

For $O$, an operator with discrete spectrum, we define its counting function by

$$\vartheta(O,E) = \sharp\{\text{ eigenvalue of } O \text{ less than } E\}.$$

Let us denote by $\vartheta'(O,E)$ the distributional derivative of $\vartheta(O,E)$. We will prove the

**Lemma 5.1.** *For almost every $\omega$, we have, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,*

$$\lim_{k\to+\infty} \frac{tr\left(\varphi(H_{\omega,k}^{\mathcal{E}})\right)}{\sharp\Lambda_k} = \lim_{k\to+\infty} \frac{\langle \varphi, \vartheta'(H_{\omega,k}^{\mathcal{E}})\rangle}{\sharp\Lambda_k} = \sum_{j=1}^n \mathbb{E}\left(\langle u_{j,0}, \varphi(H_\omega^{\mathcal{E}})u_{j,0}\rangle\right). \tag{28}$$

**Proof.** By (21), for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have

$$\mathrm{tr}\left(\varphi(H_{\omega,k}^{\mathcal{E}})\right) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\varphi}}{\partial\overline{z}}(z) \cdot \mathrm{tr}\left((z-H_{\omega,k}^{\mathcal{E}})^{-1}\right) dz \wedge d\overline{z}, \tag{29}$$

(where $\tilde{\varphi}$ is an almost analytic extension of $\varphi$).
We will show that, $\omega$-almost surely,

$$\frac{1}{\sharp\Lambda_k}\mathrm{tr}((z-H_{\omega,k})^{-1}) \to \sum_{j=1}^n \mathbb{E}(\langle u_{j,0}, (z-H_\omega^{\mathcal{E}})^{-1}u_{j,0}\rangle) \text{ as } k\to+\infty$$

Then, Lemma 5.1 will be an immediate consequence of the infinite vanishing of analytic extensions on the real axis, of the fact that we are dealing with operators of finite rank and Lebesgue's dominated convergence theorem.

Obviously, by the shift invariance property of the probability measure, we have

$$\sum_{j=1}^{n}\mathbb{E}(\langle u_{j,0},(z-H_\omega^{\mathcal{E}})^{-1}u_{j,0}\rangle)=\mathbb{E}\left(\frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k}\langle u_{j,\gamma},(z-H_\omega^{\mathcal{E}})^{-1}u_{j,\gamma}\rangle\right).$$

Then

$$\begin{aligned}
&\frac{1}{\sharp\Lambda_k}\mathrm{tr}((z-H_{\omega,k}^{\mathcal{E}})^{-1})-\frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k}\langle u_{j,\gamma},(z-H_\omega^{\mathcal{E}})^{-1}u_{j,\gamma}\rangle\\
&=\frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k}\langle u_{j,\gamma},(z-H_{\omega,k}^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}H_\omega(\Pi^{\mathcal{E}}-\Pi_k^{\mathcal{E}})(z-H_\omega^{\mathcal{E}})^{-1}u_{j,\gamma}\rangle.\\
&=\frac{1}{\sharp\Lambda_k}\mathrm{tr}\left(\Pi_{\mathcal{E},k}(z-H_{\omega,k}^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}H_\omega(\Pi^{\mathcal{E}}-\Pi_k^{\mathcal{E}})(z-H_\omega^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}\right)
\end{aligned}$$

Now, let $0<\varepsilon<1$, we can write

$$\Pi^{\mathcal{E}}-\Pi_k^{\mathcal{E}}=\Pi_{(1+\varepsilon)k}^{\mathcal{E},c}+\Pi_k^{\mathcal{E},\partial\varepsilon}$$

where

$$\begin{aligned}
\Pi_k^{\mathcal{E},\partial\varepsilon}&=\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_{(1+\varepsilon)k}\setminus\Lambda_k}u_{j,\gamma}\otimes u_{j,\gamma},\\
\Pi_{(1+\varepsilon)k}^{\mathcal{E},c}&=\sum_{j=1}^{n}\sum_{\gamma\notin\Lambda_{(1+\varepsilon)k}}u_{j,\gamma}\otimes u_{j,\gamma}.
\end{aligned}$$

$\Pi_k^{\mathcal{E},\partial\varepsilon}$ is of rank less than $C\varepsilon k^d$ hence, for some $C>0$, we have

$$\frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k}\left|\langle u_{j,\gamma},(z-H_{\omega,k}^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}H_\omega\Pi_k^{\mathcal{E},\partial\varepsilon}(z-H_\omega^{\mathcal{E}})^{-1}u_{j,\gamma}\rangle\right|\le C\varepsilon(\mathrm{Im}z)^{-2}.$$

Let us now estimate

$$\Pi_k^{\mathcal{E}}(z-H_{\omega,k}^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}H_\omega\Pi_{(1+\varepsilon)k}^{\mathcal{E},c}(z-H_\omega^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}=A\cdot B\cdot C$$

where

$$\begin{aligned}
A&=\Pi_k^{\mathcal{E}}(z-H_{\omega,k}^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}\\
B&=\Pi_k^{\mathcal{E}}H_\omega\Pi_{(1+\varepsilon)k}^{\mathcal{E},c}\\
C&=\Pi_{(1+\varepsilon)k}^{\mathcal{E},c}(z-H_\omega^{\mathcal{E}})^{-1}\Pi_k^{\mathcal{E}}.
\end{aligned}$$

Then

$$|\mathrm{tr}(ABC)|^2=|\mathrm{tr}(CAB)|^2\le n\cdot\sharp\Lambda_k\cdot\mathrm{tr}((CA)BB^*(CA)^*)\le n\cdot\sharp\Lambda_k\cdot\|CA\|^2\mathrm{tr}(BB^*).$$

Hence, for some $K>0$, we have

$$\frac{1}{\sharp\Lambda_k}|\mathrm{tr}(ABC)|\le\frac{K}{(\mathrm{Im}(z))^2}\sqrt{\frac{\mathrm{tr}\left(\Pi_k^{\mathcal{E}}H_\omega\Pi_{(1+\varepsilon)k}^{\mathcal{E},c}H_\omega\Pi_k^{\mathcal{E}}\right)}{\sharp\Lambda_k}}.$$

Now we compute

$$\begin{aligned}
\alpha_k &= \frac{1}{\sharp\Lambda_k}\mathrm{tr}(\Pi_k^{\mathcal{E}} H_\omega \Pi_{(1+\varepsilon)k}^{\mathcal{E},c} H_\omega \Pi_k^{\mathcal{E}}) \\
&= \frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k} \|\Pi_{(1+\varepsilon)k}^{\mathcal{E},c} H_\omega u_{j,\gamma}\|^2 \\
&= \frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k} \|\left(\tau_\gamma\circ\Pi_{(1+\varepsilon)k}^{\mathcal{E},c}\circ\tau_\gamma^*\right) H_{\tau_\gamma(\omega)} u_{j,0}\|^2.
\end{aligned}$$

By construction, for any $k_0\in\mathbb{N}$ and $\varepsilon>0$ fixed, if $k$ is large enough then, for $\gamma\in\Lambda_k$, $(\gamma+\Gamma\setminus\Lambda_{(1+\varepsilon)k})\cap\Lambda_{k_0}=\emptyset$. Hence, we get

$$\alpha_k \le \frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k} \|\Pi_{k_0}^{\mathcal{E},c} H_{\tau_\gamma(\omega)} u_{j,0}\|^2.$$

But $H_\omega$ is ergodic and, as $\mathcal{E}$ is of finite energy, the function $\omega\mapsto\|\Pi_{k_0}^{\mathcal{E},c}H_\omega u_{j,\gamma}\|^2$ is a bounded random variable. Hence, by the Birkhoff-Khintchine ergodic theorem (cf [23]), we know that, $\omega$-almost surely,

$$\lim_{k\to+\infty}\frac{1}{\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k} \|\Pi_{k_0}^{\mathcal{E},c} H_{\tau_\gamma(\omega)} u_{j,0}\|^2 = \sum_{j=1}^{n}\mathbb{E}\left(\|\Pi_{k_0}^{\mathcal{E},c}H_\omega u_{j,0}\|^2\right).$$

Then, for any $k_0\in\mathbb{N}$, we have

$$0\le\limsup_{k\to+\infty}\alpha_k\le\sum_{j=1}^{n}\mathbb{E}\left(\|\Pi_{k_0}^{\mathcal{E},c}H_\omega u_{j,0}\|^2\right).$$

But, we know that for some $C>0$, for any $\omega$,

$$\|\Pi_{k_0}^{\mathcal{E},c}H_\omega u_{j,\gamma}\|\le C \text{ and } \lim_{k_0\to+\infty}\|\Pi_{k_0}^{\mathcal{E},c}H_\omega u_{j,0}\|=0.$$

So, by the Lebesgue's dominated convergence Theorem, we get that

$$\lim_{k_0\to+\infty}\sum_{j=1}^{n}\mathbb{E}\left(\|\Pi_{k_0}^{\mathcal{E},c}H_\omega u_{j,0}\|^2\right)=0.$$

Hence $\lim\limits_{k\to+\infty}\alpha_k=0$. This proves that, for $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$ fixed, $\omega$-almost surely, one has (28). Notice that by (29), $\varphi\mapsto\dfrac{\mathrm{tr}\left(\varphi(H_{\omega,k}^{\mathcal{E}})\right)}{\sharp\Lambda_k}$ is continuous in $\mathcal{C}_0^\infty(\mathbb{R})$ (uniformly in $\omega$ and $k$) and that we can find a countable dense set of functions in $\mathcal{C}_0^\infty(\mathbb{R})$. This ends the proof of Lemma 5.1.

□

Then $\omega$ almost surely, $\dfrac{1}{\sharp\Lambda_k}\vartheta'(H_{\omega,k}^{\mathcal{E}},\lambda)$ converges in the distributional sense. Its limit is obviously non random. To get the announced vague convergence, we just have to notice that the expression in (28) are well defined for continuous $\varphi$, are bounded in $\varphi$ in the $\|\cdot\|_\infty$ (the supremum norm for continuous functions) and that the supports of the measures $\vartheta'(H_{\omega,k}^{\mathcal{E}},\lambda)$ are uniformly contained in some bounded set.

The pointwise convergence of the integrated density of states at the continuity points is achieved by the same arguments as in the proof of (19).

We just have to check that the result is independent of the Wannier generators. To prove this let us use the Floquet reduction of $\Pi^{\mathcal{E}}$. Using the same notations as in the subsection 1.2, we know that

$$\begin{aligned}\sum_{j=1}^{n}\mathbb{E}\left(\langle u_{j,0},\varphi(H^{\mathcal{E}}_{\omega})u_{j,0}\rangle\right) &= \sum_{j=1}^{n}\mathbb{E}\left(\langle u_{j,0},\Pi^{\mathcal{E}}\varphi(H^{\mathcal{E}}_{\omega})\Pi^{\mathcal{E}}u_{j,0}\rangle\right)\\ &= \sum_{j=1}^{n}\mathbb{E}\left(\langle U^*(u_j(\cdot,\theta)),\Pi^{\mathcal{E}}\varphi(H^{\mathcal{E}}_{\omega})\Pi^{\mathcal{E}}U^*u_j(\cdot,\theta)\rangle\right)\end{aligned}$$

hence

$$\begin{aligned}\sum_{j=1}^{n}\mathbb{E}\left(\langle u_{j,0},\varphi(H^{\mathcal{E}}_{\omega})u_{j,0}\rangle\right) &= \sum_{j=1}^{n}\mathbb{E}\left(\int_{\mathbb{T}^*}\langle u_j(\cdot,\theta),\Pi^{\mathcal{E}}_{\theta}U\varphi(H^{\mathcal{E}}_{\omega})U^*\Pi^{\mathcal{E}}_{\theta}u_j(\cdot,\theta)\rangle_{\theta}d\theta\right)\\ &= \sum_{j=1}^{n}\mathbb{E}\left(\int_{\mathbb{T}^*}\mathrm{tr}_{\mathcal{H}_\theta}\left[\Pi^{\mathcal{E}}_{\theta}U\varphi(H^{\mathcal{E}}_{\omega})U^*\Pi^{\mathcal{E}}_{\theta}\right]d\theta\right),\end{aligned}$$

and this last expression is clearly independent of the choice of Wannier generators. This ends the proof of Proposition 4.1.

□

REMARK 5.2. Proposition 4.1 can obviously be generalized to any ergodic self-adjoint operator acting on $\ell^2(\Gamma)\otimes\mathbb{C}^n$ that is almost surely bounded (in a uniform way). Hence, it extends the results of [23] in the sense that we do not need any a priori decrease assumption on the matrix coefficients of the random operator.

5.3. **Localization in energy for the periodic approximations.** The notations being the same as in sections 5.1 and 4, we define, for $l\in\mathbb{N}$,

$$H_{\omega,k,-}=\Pi_-H_{\omega,k}\Pi_-=\Pi_-H\Pi_-+\Pi_-V_{\omega,k}\Pi_-=H_-+V_{\omega,k,-}$$

and

$$H_{\omega,k,l,-}=\Pi_{l,-}H_{\omega,k}\Pi_{l,-}=\Pi_{l,+}H\Pi_{l,-}+\Pi_{l,-}V_{\omega,k}\Pi_{l,-}=H_{l,-}+V_{\omega,k,l,-}.$$

For $l\in\mathbb{N}$, the same way we define $H_0$, $V_{\omega,k,0}$, $H_{\omega,k,0}$, $H_+$, $V_{\omega,k,+}$, $H_{\omega,k,+}$, $H_{l,+}$, $V_{\omega,k,l,+}$ and $H_{\omega,k,l,+}$. By assumptions (H.1), (H.2) and (H.3) know that, for some $a>0$, $\omega$-almost surely, for any $k\in\mathbb{N}$, we have

$$H_{\omega,k,+}\geq H_+\geq a\Pi_+ \text{ and } H_{\omega,k,l,+}\geq H_{l,+}\geq a\Pi_{l,+} \tag{30}$$

We define $H_{\omega,k,0}=\Pi_0H_{\omega,k}\Pi_0$.

Let us now fix $k\in\mathbb{N}$ and $l\in\mathbb{N}$. The projector $\Pi_-$ is $\Gamma$-periodic, hence it is $(2k+1)\cdot\Gamma$-periodic. So we may Floquet decompose it jointly with $H_{\omega,k}$. More precisely

$$\Pi_-=\int_{\mathbb{T}^*_k}^{\oplus}\Pi_{-,k}(\theta)d\theta.$$

For $\theta\in\mathbb{T}^*_k$, $\Pi_{-,k}(\theta)$ is an orthogonal projector. The same statements hold for $\Pi_{+,k}(\theta)$, $\Pi_{0,k}(\theta)$, $\Pi_{l,-,k}(\theta)$ and $\Pi_{l,+,k}(\theta)$.

At last we define

$$H_{\omega,k,-}(\theta) = \Pi_{-,k}(\theta) H_{\omega,k}(\theta) \Pi_{-,k}(\theta),$$

and in an analogous way $H_{\omega,k,+}(\theta)$, $H_{\omega,k,0}(\theta)$, $H_{\omega,k,l,-}(\theta)$ and $H_{\omega,k,l,-}(\theta)$. By (30) we have

$$H_{\omega,k,+}(\theta) \geq a\Pi_{+,k}(\theta). \tag{31}$$

We then get

**Lemma 5.2.** *There exists $l_0 \in \mathbb{N}$, $E_0 > 0$ and $C > 0$ such that for any $l \geq l_0$, there exists and for $0 \leq E \leq E_0$,*

- *if (H.4) holds, we have*

$$\vartheta\left(H_{\omega,k,l,-}(\theta), E\right) - \vartheta(H_{\omega,k,l,-}(\theta), 0) \leq \vartheta(H_{\omega,k}(\theta), E) - \vartheta(H_{\omega,k}(\theta), 0^-). \tag{32}$$

- *if (H.4s) holds, we have*

$$\begin{aligned} \vartheta\left(H_{\omega,k,l,-}(\theta), E\right) &- \vartheta(H_{\omega,k,l,-}(\theta), 0) \\ &\leq \vartheta(H_{\omega,k}(\theta), E) - \vartheta(H_{\omega,k}(\theta), 0^-) \leq \vartheta(H_{\omega,k,0}(\theta), C \cdot E). \end{aligned} \tag{33}$$

**Proof.** Before starting the proof, let us notice that as the random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ are bounded, $V_\omega$ is relatively bounded with respect to $H$ with bound 0 uniformly in $\omega$ i.e, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ (independent of $\omega$) such that, for $\varphi \in H^2(\mathbb{R}^d)$,

$$\|V_\omega \varphi\| \leq \varepsilon \|H\varphi\| + C_\varepsilon \|\varphi\|.$$

This immediately tell us that, uniformly in $\omega$, $k$, $l$ and $\theta$, $V_{\omega,k,0}$ (resp. $V_{\omega,k,+}$, $V_{\omega,k,-}$, $V_{\omega,k,l,+}$ and $V_{\omega,k,l,+}$) is relatively bounded with respect to $H_0$ (resp. $H_+$, $H_-$, $H_{l,+}$ and $H_{l,-}$) with bound 0. Moreover the operators $H_{\omega,k,0}$, $H_{\omega,k,+}$, $H_{\omega,k,-}$, $H_{\omega,k,l,+}$ and $H_{\omega,k,l,+}$ admit the same lower bound (uniform in $\omega$, $k$, $l$ and $\theta$). In the sequel of this proof, to simplify the notations, we will omit all references to $\theta$, $k$ and $\omega$. So for instance, $\mathcal{H}$ will denote $\mathcal{H}_\theta$, and $V$ will denote $V_{\omega,k}$.

This lemma will be proved in two distinct steps. Let us assume that (H.4) holds and prove the lower bound. By the definition of $\Pi_{l,+}$, we know that, as $l \to +\infty$, the spectrum of $H_{l,+}$ (acting on $\mathcal{H}_{l,+}$) tends to $+\infty$. Let us pick $l$ be such that $H_{l,+}+V_{l,+} \geq \Pi_{l,+}$. Hence, for $E < 1$, the following operator is a well defined relatively bounded perturbation (with relative bound 0) of $H_{l,-}$ (acting on $\mathcal{H}_{l,-}$)

$$H_l^-(E) = H_{l,-} + V_{l,-} + V_{-,l,+}\left(E - H_{l,+} - V_{l,+}\right)^{-1} V_{+,l,-}. \tag{34}$$

where $V_{-,l,+} = \Pi_{l,-} V \Pi_{l,+}$ and $V_{+,l,-}$ is its adjoint. Let us denote the positive eigenvalues (ordered increasingly) of $H+V$, $H_{l,-}+V_{l,-}$ and $H_l^-(E)$ respectively by $(E_n)_{n\geq 0}$, $(E_n^-)_{n\geq 0}$ and $(E_n(E))_{n\geq 0}$. We decompose $H+V$ over $\mathcal{H}_{l,-} \oplus \mathcal{H}_{l,+}$ as the matrix

$$H + V = \begin{pmatrix} H_{l,-} + V_{l,-} & V_{-,l,+} \\ V_{+,l,-} & H_{l,+} + V_{l,+} \end{pmatrix}.$$

From this form, one sees that $0 \leq E < 1$ is an eigenvalue of $H+V$ if and only if $E$ is an eigenvalue of $H_l^-(E)$.

Obviously, for $n \geq 0$ and $E < 1$, $E_n(E)$ is a decreasing function of $E$ and one has $E_n(E) \leq E_n^-$. This immediately tells us that $E_n \leq E_n^-$. This gives us, for $E < 1$,

$$\vartheta(H_{\omega,k,l,-}(\theta), E) \leq \vartheta(H_{\omega,k}(\theta), E).$$

On the other hand, we know as $V_{l,-}$ is relatively bounded with respect to $H_{l,-}$ with bound 0, it is relatively form bounded with the same bound. Hence, we can pick $\lambda > 0$ such that, uniformly in $l$ (and $k$, $\omega$ and $\theta$), we have

$$\|\sqrt{V}\Pi_{l,-}(\lambda + H_{l,-} + V_{l,-})^{-1/2}\| \leq 1.$$

Analogously $V_{l,+}$ is relatively form bounded with respect to $H_{l,+}$ with bound 0 uniformly in $l$; hence, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for any $l \geq l_0$ and $\varphi \in \mathcal{H}$, we have

$$\|\sqrt{V}\Pi_{l,+}\varphi\|^2 \leq \varepsilon\|(H_{l,+} + V_{l,+})^{1/2}\varphi\|^2 + C_\varepsilon\|\varphi\|^2.$$

Now replacing $\varphi$ by $(H_{l,+} + V_{l,+} - E)^{-1/2}\varphi$, we get

$$\|\sqrt{V}\Pi_{l,+}(H_{l,+} + V_{l,+} - E)^{-1/2}\varphi\|^2 \leq \varepsilon\|\varphi\|^2 + (C_\varepsilon + 1)\|(H_{l,+} + V_{l,+} - E)^{-1/2}\varphi\|^2.$$

Then for any $\varepsilon > 0$, we can find $l_\varepsilon \in \mathbb{N}$ such that, for $l \geq l_\varepsilon$, we have

$$\|\sqrt{V}\Pi_{l,+}(H_{l,+} + V_{l,+} - E)^{-1/2}\|^2 \leq \varepsilon$$

If we use this and (34), we get, for $\varphi \in \mathcal{H}$,

$$\begin{aligned}
\langle H_l^-(E)\varphi, \varphi\rangle &\geq \langle (H_{l,-} + V_{l,-})\varphi, \varphi\rangle \\
&\quad - \Big\| \Big((E - H_{l,+} - V_{l,+})^{-1/2}\Pi_{l,+}\sqrt{V}\Big) \cdot \\
&\qquad \Big(\sqrt{V}\Pi_{l,-}(\lambda + H_{l,-} + V_{l,-})^{-1/2}\Big) \cdot \\
&\qquad\quad \Big((\lambda + H_{l,-} + V_{l,-})^{1/2}\varphi\Big)\Big\|^2 \\
&\geq \langle (H_{l,-} + V_{l,-})\varphi, \varphi\rangle - \varepsilon\left\|(\lambda + H_{l,-} + V_{l,-})^{1/2}\varphi\right\|^2 \\
&= (1 - \varepsilon)\langle (H_{l,-} + V_{l,-})\varphi, \varphi\rangle - \varepsilon\lambda.
\end{aligned}$$

By the minimax principle, this immediately gives us that, for any $\varepsilon > 0$, we can find $l_\varepsilon \in \mathbb{N}$ such that, for $l \geq l_\varepsilon$, we have for $n \in \mathbb{N}$ and $E < 1$,

$$E_n(E) \geq (1 - \varepsilon)E_n^- \varepsilon\lambda$$

hence for $n \in \mathbb{N}$ such that $E_n < 1$, we have

$$E_n \geq (1 - \varepsilon)E_n^- \varepsilon\lambda$$

Choose now $0 < \varepsilon < 1/3$ small such that $\varepsilon\lambda < \delta'/4$, then for $l$ large enough and $n$ such that $E_n \leq 1$, we have

$$\frac{2}{3}(E_n + \delta'/4) \geq E_n^-. \tag{35}$$

So as $H + V$ has a gap of length at least $\delta'$ below 0, if $E_n < 0$ then $E_n \leq -\delta'$. In this case, by (35), we have $E_n^- \leq -\delta'/2$. Hence we get that

$$\vartheta(H_{\omega,k,l,-}(\theta), 0^-) \geq \vartheta(H_{\omega,k}(\theta), 0^-).$$

This ends the proof of the lower bound.

REMARK 5.3. We notice that, incidentally, we proved that $H_{\omega,k,l,-}(\theta)$ has a gap of length at least $\delta'/2$ below 0 (for $l$ large enough). This is uniform in $\theta$. Then, if we take $k \to +\infty$, we get that $H_{\omega,l,-}$ has a gap of length at least $\delta'/2$ below 0 (for $l$ large enough). One is easily convinced that the length of this gap tends to the length of the gap for $H_\omega$ when $l$ tends to $+\infty$.

Let us now assume that (H.4s) holds. Of course we already have proved the announced lower bound as (H.4s) implies (H.4). Under assumption (H.4s), by [13], we know that, for any $t \in [0,1]$ and any $k \in \mathbb{N}^*$, the interval $[-\delta', 0)$ does not intersect the spectrum of $H_{\omega,k,t} = H + tV_{\omega,k}$. Hence, for $t \in [0,1]$, $k \in \mathbb{N}^*$ and $E \in [-\delta', 0)$, we know that

$$\vartheta(H_{\omega,k,t}(\theta), E) = \vartheta(H_{\omega,k,t}(\theta), 0^-).$$

But $t \mapsto \vartheta(H_{\omega,k,t}(\theta), E)$ is continuous for $t \in [0,1]$ and $E \in [-\delta', 0)$ and it is integer valued hence it is constant. So we get that, for any $t \in [0,1]$ and $E \in [-\delta', 0]$,

$$\begin{aligned} \vartheta(H_{\omega,k,t}(\theta), E) &= \vartheta(H_{k,0}(\theta), E) \\ &= (2k+1)^d \cdot \sharp\{\text{Floquet eigenvalues of } H \text{ less than } E\} \\ &= (2k+1)^d \sharp J_-. \end{aligned} \tag{36}$$

We recall that $J_-$ is the set of indices of the Floquet eigenvalues of $H$ smaller than 0. Let us again omit the reference to $k$, $\omega$ and $\theta$. Consider the following orthogonal decomposition of decomposition of $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_- \overset{\perp}{\oplus} \mathcal{H}^+ \text{ where } \mathcal{H}_- = \Pi_- \mathcal{H} \text{ and } \mathcal{H}^+ = \Pi^+ \mathcal{H} = (\Pi_0 + \Pi_+)\,\mathcal{H}.$$

Using this decomposition, we can write $H$ in the following matrix form

$$\begin{pmatrix} H_- + V_{--} & V_-{}^+ \\ V^+{}_- & H^+ + V^{++} \end{pmatrix} \text{ where } V_-{}^+ = \Pi_- V \Pi^+,\ V^+{}_- = \Pi^+ V \Pi_-,\ H^+ = \Pi^+ H,$$
$$H^- = \Pi^- H \text{ and } V^{++} = \Pi^+ V \Pi^+.$$

Notice that by (36), the number of eigenvalues of $H + V$ less than 0 is exactly equal to the rank of $\Pi_-$. We get the

**Lemma 5.3.** *For $0 \leq E$, we have*

$$0 \leq \vartheta(H, E) - \vartheta(H, 0^-) \leq \vartheta(H^+, E). \tag{37}$$

**Proof.** This lemma is a consequence of the minimax principle. Indeed, by what has been said above, the number or eigenvalues of $H + V$ lying between 0 and $E$ is equal to the number of eigenvalues of $H + V$ (ordered increasingly) less than $E$ that have an index larger than the rank of $\Pi_-$. Let us denote the rank of $\Pi_-$ by $r$. If we denote the eigenvalues of $H + V$ by $E_n$, the minimax formulae tells us

$$E_n = \sup_{\substack{\mathcal{E} \subset \mathcal{H} \\ \dim(\mathcal{E}) = n-1}} \inf_{\substack{\varphi \perp \mathcal{E} \\ \|\varphi\| = 1 \\ \varphi \in \mathcal{D}(H)}} \langle (H + V)\varphi, \varphi \rangle.$$

So if $n > r$, in the variational formula, we may choose $\mathcal{E}$ so that it contains $\Pi_-\mathcal{H}$; this gives us that, for $n \geq r+1$,

$$E_n \geq \sup_{\substack{\mathcal{E}\subset\mathcal{H} \\ \dim(\mathcal{E})=n-1-r}} \inf_{\substack{\varphi\perp\mathcal{E}\oplus\Pi_-\mathcal{H} \\ \|\varphi\|=1 \\ \varphi\in\mathcal{D}(H)}} \langle (H+V)\varphi,\varphi\rangle = E^+_{n-r}$$

where $E^+_n$ are the eigenvalues of $H^+ + V^{++}$ (acting on $\Pi^+\mathcal{H}$). This ends the proof of Lemma 5.3.

□

Now we decompose $\mathcal{H}^+$ into $\mathcal{H}^+ = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_+$. This induces the following matrix form for $H^+ + V^{++}$

$$H^+ + V^{++} = \begin{pmatrix} H_0 + V_{00} & V_{0+} \\ V_{+0} & H_+ + V_{++} \end{pmatrix}.$$

To get Lemma 5.2, as $H_+ \geq a\Pi_+$ and $V \geq 0$, it is then enough to prove

**Lemma 5.4.** *There exists $C > 0$ such that*

$$\begin{pmatrix} H_0 + V_{00} & V_{0+} \\ V_{+0} & H_+ + V_{++} \end{pmatrix} \geq \frac{1}{C}\begin{pmatrix} H_0 + V_{00} & 0 \\ 0 & H_+ + V_{++} \end{pmatrix}.$$

Before proving Lemma 5.4, let us just show how it implies (33). We know that $H_+ \geq a\Pi_+$ by (31). Hence, the small non-negative eigenvalues of $H^+ + V^{++}$ are lower bounded by the eigenvalues of $1/C(H_0 + V_{00})$. This ends the proof of Lemma 5.2.

**Proof of Lemma 5.4.** We know that $V$ is $H$-bounded and that $H_+ \geq a\Pi_+$ for some $a > 0$. Hence for some $C > 0$, we have $\|H_+^{-1/2}V_{++}H_+^{-1/2}\| \leq C$. For any $\varphi_+ \in \mathcal{H}_+\cap\mathcal{D}(H)$, this immediately gives

$$|\langle V_{++}\varphi_+,\varphi_+\rangle| \leq \langle H_+\varphi_+,\varphi_+\rangle \leq \langle H\varphi,\varphi\rangle \text{ if } \varphi = \varphi_+ + \varphi_0.$$

Then as

$$\begin{aligned} |\langle V_{+0}\varphi_0,\varphi_+\rangle| &\leq |\langle V^{1/2}\varphi, V^{1/2}\varphi_+\rangle| + |\langle V_{++}\varphi_+,\varphi_+\rangle| \\ &\leq \|V^{1/2}\varphi\|\cdot\|V^{1/2}\varphi_+\| + |\langle V_{++}\varphi_+,\varphi_+\rangle|, \end{aligned}$$

for some $C > 0$, we get

$$|\langle V_{+0}\varphi_0,\varphi_+\rangle| \leq C\langle H^+\varphi,\varphi\rangle \text{ if } \varphi = \varphi_+ + \varphi_0.$$

The proof of Lemma 5.4 is then ended by noticing that

$$\langle (H_0+V_{00})\varphi_0,\varphi_0\rangle + \langle (H_+ + V_{++})\varphi_+,\varphi_+\rangle \leq \langle (H^+ + V^{++})\varphi,\varphi\rangle + 2|\left(\langle V_{0+}\varphi_+,\varphi_0\rangle\right)|$$

and using the non-negativity of $V$. □

To end the proof of Theorem 4.1, we will first prove that, for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\lim_{k\to+\infty} \mathbb{E}\left(\int_{\mathbb{R}} \varphi(\lambda)dN(H_{\omega,k,0})\right) = \int_{\mathbb{R}} \varphi(\lambda)dN_{\mathcal{E}_0}. \tag{38}$$

Therefore let us notice that $H_{\omega,k,0}$ is $(2k+1)\cdot\Gamma$-periodic operator acting on $\mathcal{E}_0$ ($\mathcal{E}_0$ is $\Gamma$-translation invariant). Let $(u_{j,0})_{1\leq j\leq n}$ be Wannier generators for $\mathcal{E}_0$. We may then compute the density of states of $H_{\omega,k,0}$ in the same way as we did it in Proposition 5.1 (just replace $\chi_{C_0}$ by the projector $\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k} u_{j,\gamma}\otimes u_{j,\gamma}$) and get, for $\varphi\in\mathcal{C}_0^\infty$,

$$\int_{\mathbb{R}}\varphi(\lambda)dN(H_{\omega,k,0})=\frac{1}{\mathrm{Vol}(C_0)\cdot\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k}\langle u_{j,\gamma},\varphi(H_{\omega,k,0})u_{j,\gamma}\rangle.$$

Hence, using the translation invariance of the random variables, we get

$$\begin{aligned}\mathbb{E}\left(\int_{\mathbb{R}}\varphi(\lambda)dN(H_{\omega,k,0})\right)&=\frac{1}{\mathrm{Vol}(C_0)\cdot\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k}\mathbb{E}\left(\langle u_{j,\gamma},\varphi(H_{\omega,k,0})u_{j,\gamma}\rangle\right)\\&=\frac{1}{\mathrm{Vol}(C_0)\cdot\sharp\Lambda_k}\sum_{j=1}^{n}\sum_{\gamma\in\Lambda_k}\mathbb{E}\left(\langle u_{j,0},\varphi(H_{\tau_\gamma(\omega),n,0})u_{j,0}\rangle\right)\\&=\frac{1}{\mathrm{Vol}(C_0)}\sum_{j=1}^{n}\mathbb{E}\left(\langle u_{j,0},\varphi(H_{\omega,k,0})u_{j,0}\rangle\right).\end{aligned}$$

So we just need to show that, for $1\leq j\leq n$,

$$\mathbb{E}\left(\langle u_{j,0},\varphi(H_{\omega,k,0})u_{j,0}\rangle\right)\underset{k\to+\infty}{\longrightarrow}\mathbb{E}\left(\langle u_{j,0},\varphi(H_{\omega,0})u_{j,0}\rangle\right).$$

This is clear as $H_{\omega,k}$ converges to $H_\omega$ in strong resolvent sense. This ends the proof of (38). The same way one proves that, for any $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$,

$$\lim_{k\to+\infty}\mathbb{E}\left(\int_{\mathbb{R}}\varphi(\lambda)dN(H_{\omega,k,l,-})\right)=\int_{\mathbb{R}}\varphi(\lambda)dN_{\mathcal{E}_{l,-}}.$$

We now notice that

$$N(H_{\omega,k},E)=\int_{T_k^*}\vartheta(H_{\omega,k}(\theta),E)d\theta,\ N(H_{\omega,k,l,-},E)=\int_{T_k^*}\vartheta(H_{\omega,k,l,-}(\theta),E)d\theta$$
$$\text{and } N(H_{\omega,k,0},E)=\int_{T_k^*}\vartheta(H_{\omega,k,0}(\theta),E)d\theta$$

Integrating (32) if (H.4) holds (resp. (33) if (H.4s) holds), using Theorem 5.1 and Proposition 4.1, for any $0\leq E$ small enough that is continuity point for $N(E)$, $N_{\mathcal{E}_{l,-}}(E/C)$ and $N_{\mathcal{E}_0}(C\cdot E)$, we get

$$N_{\mathcal{E}_{l,-}}(E)-N_{\mathcal{E}_{l,-}}(0^-)\leq N(E)-N(0^-)$$
$$N_{\mathcal{E}_{l,-}}(E)-N_{\mathcal{E}_{l,-}}(0^-)\leq N(E)-N(0^-)\leq N_{\mathcal{E}_0}(CE).$$

But $N$, $N_{\mathcal{E}_{l,-}}$ and $N_{\mathcal{E}_0}$ are non-decreasing; so they have at most countably many jump points. Furthermore they are right continuous; so any $0\leq E$ is the right-limit of a sequence of continuity points. This proves Theorem 4.1.

$\square$

To end this section, we will give a more explicit form for the reference operators that will be used later on. Let $\mathcal{E}$ be a translation invariant subspace of $L^2(\mathbb{R}^d)$ of finite energy. Let $(v_j)_{1\leq j\leq n}$ be a set of Wannier generators for $\mathcal{E}$. Define $(v_j(\cdot,\theta))_{1\leq j\leq n}=(Uv_jU^*)_{1\leq j\leq n}$

and $\Pi^{\mathcal{E}}$ the orthogonal projector on $\mathcal{E}$. We are now interested in the density of states of the random ergodic operator $H_\omega^{\mathcal{E}} = \Pi^{\mathcal{E}} H_\omega \Pi^{\mathcal{E}}$ acting on $\mathcal{E}$.

For $u = (u_j)_{j\in J_0} \in L^2(\mathbb{T}^*) \otimes \mathbb{C}^n$, we define

$$W(u) = \sum_{j\in J_0} U^*(u_j(\theta)v_j(\cdot,\theta)) \tag{39}$$

The mapping $W:\ L^2(\mathbb{T}^*) \otimes \mathbb{C}^n \to \mathcal{E}$ defines a unitary equivalence. For $u \in \mathcal{E}$, its inverse is given by

$$W^*(u) = \langle (Uu)(\cdot,\theta), v_j(\cdot,\theta))\rangle_{L^2(C_0)}.$$

Hence, $H_\omega^{\mathcal{E}}$ is unitarily equivalent to the operator $H^{\mathcal{E}} + V_\omega^{\mathcal{E}}$ acting on $L^2(\mathbb{T}^*) \otimes \mathbb{C}^n$ where $H^{\mathcal{E}}$ is the multiplication by the matrix

$$H^{\mathcal{E}}(\theta) = \begin{pmatrix} \langle v_1(\cdot,\theta)), Hv_1(\cdot,\theta))\rangle_{L^2(C_0)} & \cdots & \langle v_1(\cdot,\theta)), Hv_n(\cdot,\theta))\rangle_{L^2(C_0)} \\ \vdots & \ddots & \vdots \\ \langle v_n(\cdot,\theta)), Hv_1(\cdot,\theta))\rangle_{L^2(C_0)} & \cdots & \langle v_n(\cdot,\theta)), Hv_n(\cdot,\theta))\rangle_{L^2(C_0)} \end{pmatrix}$$

and $V_\omega^{\mathcal{E}}$ has the matrix kernel

$$\begin{aligned} V_\omega^{\mathcal{E}}(\theta,\theta') &= \begin{pmatrix} v_{\omega,11}(\theta,\theta') & \cdots & v_{\omega,1n}(\theta,\theta') \\ \vdots & \ddots & \vdots \\ v_{\omega,n1}(\theta,\theta') & \cdots & v_{\omega,nn}(\theta,\theta') \end{pmatrix} \\ &= \sum_{\gamma\in\Gamma} \omega_\gamma \begin{pmatrix} v_{11}(\gamma;\theta,\theta') & \cdots & v_{1n}(\gamma;\theta,\theta') \\ \vdots & \ddots & \vdots \\ v_{n1}(\gamma;\theta,\theta') & \cdots & v_{nn}(\gamma;\theta,\theta') \end{pmatrix}, \end{aligned} \tag{40}$$

where, for $1 \le i,j \le n$,

$$v_{\omega,ij}(\theta,\theta') = \int_{\mathbb{R}^d} V_\omega(x) v_i(x,\theta)\overline{v_j(x,\theta')}dx$$

and

$$v_{ij}(\gamma;\theta,\theta') = \int_{\mathbb{R}^d} V(x-\gamma) v_i(x,\theta)\overline{v_j(x,\theta')}dx.$$

Remark 5.4. The sum in equation (40) is convergent as the kernel of a series of operators converging in the strong operator topology.

The following lemma will be of use

**Lemma 5.5.** *Let $v \in L^2(\mathbb{T}^*, \mathcal{H}_\theta^2)$. Define the two norms*

$$\operatorname*{ess\text{-}sup}_{\theta\in\mathbb{T}^*} \left( |\langle v(\cdot,\theta), H_\theta v(\cdot,\theta)\rangle_{L^2(C_0)}| + \|v(\cdot,\theta)\|^2_{L^2(C_0)} \right) = \|v\|^2_{1,H,\infty}$$

$$\operatorname*{ess\text{-}sup}_{\theta\in\mathbb{T}^*} \left( \|H_\theta v(\cdot,\theta)\|^2_{L^2(C_0)} + \|v(\cdot,\theta)\|^2_{L^2(C_0)} \right) = \|v\|^2_{2,H,\infty}.$$

*Let $0 \le V \in L^p_{loc,unif}(\mathbb{R}^d)$ for some $p$ chosen as in assumption (H.1).*

*We define the operators $OP_{1,V,v}: L^2(\mathbb{T}^*) \to L^2(\mathbb{R}^d)$ and $OP_{2,V,v}: L^2(\mathbb{T}^*) \to L^2(\mathbb{R}^d)$ by*

$$\forall u \in L^2(\mathbb{T}^*), \quad OP_{1,V,v}(u) = \sqrt{V(x)} \int_{\mathbb{T}^*} v(x,\theta) u(\theta) d\theta$$
$$OP_{2,V,v}(u) = V(x) \int_{\mathbb{T}^*} v(x,\theta) u(\theta) d\theta.$$

*Then, if $\|v\|^2_{1,H,\infty} < +\infty$ (resp. $\|v\|^2_{2,H,\infty} < +\infty$) then $OP_{1,V,v}$ (resp. $OP_{2,V,v}$) is bounded; more precisely, there exists $C > 0$ such that for any $(v, V)$ as above, one has*

$$\|OP_{1,V,v}\|_{\mathcal{L}(L^2(\mathbb{T}^*),L^2(\mathbb{R}^d))} \leq C\|v\|_{1,H,\infty} \sqrt{\|V\|_{L^p_{loc,unif}}}$$
$$\|OP_{2,V,v}\|_{\mathcal{L}(L^2(\mathbb{T}^*),L^2(\mathbb{R}^d))} \leq C\|v\|_{2,H,\infty} \|V\|_{L^p_{loc,unif}}.$$

**Proof.** We will only give the proof for $\mathrm{OP}_{1,V,v}$ as it is the same for $\mathrm{OP}_{2,V,v}$. Let us compute

$$\begin{aligned}\|\mathrm{Op}_{1,V,v}(u)\|^2_{L^2(\mathbb{R}^d)} &= \int_{R^d} V(x) \left| \int_{\mathbb{T}^*} v(x,\theta) u(\theta) d\theta \right|^2 dx \\ &= \langle V u_v, u_v \rangle \text{ where } u_v(x) = \int_{\mathbb{T}^*} v(x,\theta) u(\theta) d\theta \in H^2(\mathbb{R}^d),\end{aligned}$$

hence, by [24] Th. XIII.96 and [10] Th. VI.1.38, there exists $C > 1$ such that

$$\begin{aligned}\|\mathrm{Op}_{1,V,v}(u)\|^2_{L^2(\mathbb{R}^d)} &\leq C\|V\|_{L^p_{\mathrm{loc,unif}}} \left( |\langle H u_v, u_v \rangle| + \|u_v\|^2_{L^2(\mathbb{R}^d)} \right) \\ &\leq C\|V\|_{L^p_{\mathrm{loc,unif}}} \int_{\mathbb{T}^*} \left( |\langle v(\cdot,\theta), H_\theta v(\cdot,\theta) \rangle_{L^2(C_0)}| + \|v(\cdot,\theta)\|^2_{L^2(C_0)} \right) |u(\theta)|^2 d\theta \\ &\leq C\|v\|^2_{H,\infty} \|V\|_{L^p_{\mathrm{loc,unif}}} \|u\|_{L^2(\mathbb{T}^*)}.\end{aligned}$$

□

Notice that the functions $(v_j(\cdot,\theta))_{1 \leq j \leq n}$ constructed in Lemma 3.1 satisfy the assumptions of Lemma 5.5.

## 6. The lower bound

To lower bound the integrated density of states of $H_\omega$ near 0, we will use Theorem 4.1. Let $\mathcal{E}_{l,-} \subset L^2(\mathbb{R}^d)$ be defined as in Theorem 4.1 that is $\mathcal{E}_{l,-}$ is the vector space spanned by the Wannier generators $(u_j(\cdot,\theta))_{1 \leq j \leq l}$ for some $l$ large enough (we recall that the $(u_j(\cdot,\theta))_{j \in \mathbb{N}}$ are the Floquet eigenfunctions for $H$). $\Pi_{l,-}$ denotes the orthogonal projection on $\mathcal{E}_{l,-}$. By Theorem 4.1, to lower bound the integrated density of states $N(E) - N(0^-)$, we just need to lower bound $N_{\mathcal{E}_{l,-}}(E) - N_{\mathcal{E}_{l,-}}(0^-)$ where $N_{\mathcal{E}_{l,-}}$ is the integrated density of states of the reference operator $\Pi_{l,-} H_\omega \Pi_{l,-} = H_{\omega,l,-}$. Now in Remark 5.3, we noticed that $H_{\omega,l,-}$ has a gap of length at least $-\delta'/2$ below 0. Hence

$$N_{\mathcal{E}_{l,-}}(E) - N_{\mathcal{E}_{l,-}}(0^-) = N_{\mathcal{E}_{l,-}}(E) - N_{\mathcal{E}_{l,-}}(-E)$$

For the rest of this section, to simplify the notations, we reorder the Floquet eigenvalues $(E_j)_{1 \leq j \leq l}$ and the associated eigenvectors. We will denote the Floquet eigenvalues generating the band at 0 (ordered increasingly) by $(E_j)_{1 \leq j \leq n}$ and the corresponding eigenvectors

will be enumerated correspondingly. The rest of the eigenvalues will have indices going from $n+1$ to $l$ (their precise ordering will not be of importance).
Let us now pick a new set of Wannier generators for $\mathcal{E}_{l,-}$. Therefore we will distinguish between the two possible behaviours of the periodic integrated density of states $n$:

- if $n$ is non-degenerate i.e $\limsup_{E\to 0^+}(\log(n(E)-n(0)))/(\log E) = d/2$, then we replace $(u_j)_{1\leq j\leq n}$ by $(v_j)_{1\leq j\leq n}$ given by Lemma 3.1. These are now analytic in $\theta$ near any point where the Floquet eigenvalues $(E_j)_{1\leq j\leq n}$ vanish.
- if $n$ is degenerate i.e $\limsup_{E\to 0^+}(\log(n(E)-n(0)))/(\log E) < d/2$, then by Remark 3.1, near some $\theta^0$, the behaviour of $E_1(\theta)$ is degenerate. Let $(E_j)_{1\leq j\leq q}$ be the Floquet eigenvalues vanishing at $\theta^0$. Following the proof of Lemma 3.1, we construct $(v_j)_{1\leq j\leq q}$ analytic in $\theta$ near $\theta^0$ such that, for $\theta\in\mathbb{T}^*$, we have
  - The system $(v_j(\cdot,\theta))_{1\leq j\leq q})$ is orthonormal.
  - $\mathrm{span}((v_j(\cdot,\theta))_{1\leq j\leq q}) = \mathrm{span}((u_j(\cdot,\theta))_{1\leq j\leq q})$.

  Then we replace $(u_j)_{1\leq j\leq q}$ by $(v_j)_{1\leq j\leq q}$.

Now let $\theta^0$ be a point where $E_1(\theta)$ vanishes in a degenerate (resp. non-degenerate) way if $n$ is degenerate (resp. non-degenerate) and assume $\theta^0=0$. Then, by the proof of Proposition 1.1 and Remark 3.1, we know that, for some $C>0$,

- if $n$ is non-degenerate then $v_1(\cdot,\theta)$ satisfies
$$\|H_\theta v_1(\cdot,\theta)\|_{L^2(C_0)} \leq C|\theta|^2.$$
We set $f(\cdot,\theta)=v_1(\cdot,\theta)$.
- if $n$ is degenerate then, modulo a linear transformation in $\theta$, by Remark 3.1, we know that there exists a vector $f(\theta)=(f_1(\theta),\dots,f_q(\theta))$ linear in $\theta$ for $\theta$ close to 0 such that, if we define $\tilde f(\cdot,\theta)=\sum_{1\leq j\leq q} f_j(\theta)v_j(\cdot,\theta)$ then
$$|\theta_1| \leq \|\tilde f(\cdot,\theta)\|_{L^2(C_0)} = \|f(\theta)\|_{\mathbb{C}^q}.$$
and
$$\|H_\theta \tilde f(\cdot,\theta)\|_{L^2(C_0)} \leq C(|\theta_1|\cdot|\theta'|^2+|\theta'|^3+|\theta|^4).$$
(here $\theta=(\theta_1,\theta')$). We the set $f(\cdot,\theta)=\tilde f(\cdot,\theta)/\theta_1$.

To simplify the notations, we now set $H_{l,-}=\Pi_{l,-}H\Pi_{l,-}$. We also identify $\mathcal{E}_{l,-}$ with $L^2(\mathbb{T}^*)\otimes\mathbb{C}^l$ by use of the Wannier basis described above.

To lower bound $N_{\mathcal{E}_{l,-}}(E)-N_{\mathcal{E}_{l,-}}(-E)$, we will use the definition of the integrated density of states given in Proposition 4.1. More precisely, for any $N$ large enough, we will restrict $H_\omega$ to the cube $\Lambda_N$ and show that, with a good probability, $(H_\omega)_{|\Lambda_N}$ has many eigenvalues in the interval $(-E,E)$. This will be done by explicitly constructing orthogonal families of approximate eigenfunctions. We recall that $\Lambda_N=\{\gamma\in\Gamma;\ |\gamma|\leq N\}$, $\Pi_N$ is the orthogonal projector on the vector space spanned by $(\delta_{\gamma,j})_{\gamma\in\Lambda_N;\ 1\leq j\leq l}$ (here $\delta_{\gamma,j}=e^{i\gamma\cdot\theta}\otimes\delta_j\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^l$ and $(\delta_j)_{1\leq j\leq l}$ is the canonical basis in $\mathbb{C}^l$).

Let $\alpha>0$ and $1>\zeta>0$ be small constants to be chosen later on. Pick $\varepsilon>0$. Let $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$ be such that $0\leq\chi\leq 1$, $\chi$ is supported in $[\zeta/2,\zeta]$ ($K$ being the constant defined above) and $\int_{\mathbb{R}} \chi(t)^2 dt = 2$. Then we define

$$w_\varepsilon(\theta) = \varepsilon^{-\frac{d-1+2\kappa}{4}} \chi(\varepsilon^{-\kappa}\theta_1) \prod_{j=2}^{d} \chi(\varepsilon^{-1/2}\theta_j) \in L^2(\mathbb{T}^*)$$

where $\kappa$ is defined by

- $\kappa = 1/2$ if $n$ has a non-degenerate behaviour.
- $\kappa = 1/3$ if $n$ has a degenerate behaviour.

We also define $w_\varepsilon^f(\cdot,\theta) = w_\varepsilon(\theta)\cdot f(\cdot,\theta)$. One checks that

$$\|w_\varepsilon^f\|^2 = \int_{\mathbb{T}^*} \|f(\cdot,\theta))\|^2_{L^2(C_0)} |w_\varepsilon(\theta)|^2 d\theta \geq 2^d \tag{41}$$

and that

(42)

$$\begin{aligned}
\|H_{l,-}w_\varepsilon^f\|^2 &= \int_{\mathbb{T}^*} \|H_\theta f(\cdot,\theta))\|^2_{L^2(C_0)} |w_\varepsilon(\theta)|^2 d\theta \\
&\leq \begin{cases} C^2 \int_{\mathbb{T}^*} |\theta|^4 |w_\varepsilon(\theta)|^2 d\theta & \text{if } n \text{ is non-degenerate} \\ C^2 \int_{\mathbb{T}^*} (|\theta'|^2 + |\theta'|^3/|\theta_1| + |\theta|^4/|\theta_1|)^2 |w_\varepsilon(\theta)|^2 d\theta & \text{if } n \text{ is degenerate} \end{cases} \\
&\leq \begin{cases} C^2\varepsilon^2 \int_{[\zeta/2,\zeta]^d} |\theta|^4 \prod_{j=1}^{d} \chi^2(\theta_j) d\theta \text{ if } n \text{ is non-degenerate} \\ C^2\varepsilon^2 \int_{[\zeta/2,\zeta]^d} (|\theta'|^2 + \varepsilon^{1/6}|\theta'|^3/|\theta_1| + |\theta|^4/|\theta_1|)^2 \prod_{j=1}^{d} \chi^2(\theta_j) d\theta & \text{if not} \end{cases} \\
&\leq \varepsilon^2/16.
\end{aligned}$$

if we pick $\zeta$ small enough (independent of $\varepsilon$).
Now, for $\beta \in \Gamma$, we define

$$\Lambda_\alpha(\varepsilon) = \left\{\gamma \in \Gamma;\ |\gamma_1| \leq \varepsilon^{-(\kappa+\alpha)},\ \text{for } 2 \leq j \leq d,\ |\gamma_j| \leq \varepsilon^{-(1/2+\alpha)}\right\},$$
$$w^f_{\varepsilon,\beta}(\cdot,\theta) = e^{i\beta\cdot\theta} w_\varepsilon^f(\cdot,\theta) \text{ and } w^f_{\alpha,\varepsilon,\beta}(\cdot,\theta) = e^{i\beta\cdot\theta}(\Pi_{\Lambda_\alpha(\varepsilon)} w_\varepsilon^f)(\cdot,\theta).$$

For $N$ large enough and $\beta$ well chosen, $w^f_{\alpha,\varepsilon,\beta}$ will be an approximate eigenfunction associated to an eigenvalue in $[-\varepsilon,\varepsilon]$.

Let us first notice that $w^f_{\alpha,\varepsilon,\beta}$ and $w^f_{\varepsilon,\beta}$ are close to each other; more precisely, for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that, for any $\beta \in \Gamma$, we have

$$\|w^f_{\alpha,\varepsilon,\beta} - w^f_{\varepsilon,\beta}\| \leq C_k \varepsilon^{\alpha k}. \tag{43}$$

Indeed $\|w_{\alpha,\varepsilon,\beta} - w_{\varepsilon,\beta}\|$ is obviously independent of $\beta$ so let us take $\beta = 0$. Then, to estimate this norm, we only need to estimate the $\gamma$'s Fourier coefficient of $w_\varepsilon^f$ for $\gamma$ outside of $\Lambda_\alpha(\varepsilon)$;

for any $k \in \mathbb{N}$, for some $C_k > 0$, we get

$$\left| \int_{\mathbb{T}^*} e^{-i\gamma\theta} w_\varepsilon f(x,\theta) d\theta \right| = \left| \varepsilon^{\frac{d-1+2\kappa}{4}} \int_{[\zeta/2,\zeta]^d} e^{-i\varepsilon^\kappa \gamma_1 \theta_1 + \varepsilon^{\frac{1}{2}} \gamma' \theta'} \prod_{j=1}^d \chi(\theta_j) f(x, \varepsilon^\kappa \theta_1, \varepsilon^{\frac{1}{2}} \theta') d\theta \right|$$
$$\leq C_k (|\varepsilon^\kappa \gamma_1|^2 + |\varepsilon^{\frac{1}{2}} \gamma'|^2)^{-k/2}$$

using the smoothness of $\theta \mapsto f(x,\theta)$ and the non-stationary phase. If we sum this estimate over $\gamma$ outside of $\Lambda_\alpha(\varepsilon)$, we get (43). Notice that, combining (43) with (41), we get that, for $\varepsilon$ small enough, $\|w_{\alpha,\varepsilon,\beta}\| \geq 1$.
Obviously,

$$\|H_{\omega,l,-} w^f_{\alpha,\varepsilon,\beta}\|^2 \leq 2\|H_{l,-} w^f_{\alpha,\varepsilon,\beta}\|^2 + 2\|V_{\omega,l,-} w^f_{\alpha,\varepsilon,\beta}\|^2. \tag{44}$$

As the $(u_j(\cdot,\theta))_{j\in\mathbb{N}}$ form a Hilbert basis of $L^2(\mathbb{T}^*, \mathcal{H}_\theta)$, we have

$$\begin{aligned}
\|V_{\omega,l,-} w^f_{\alpha,\varepsilon,\beta}\|^2 &= \sum_{j=1}^l \int_{\mathbb{T}^*} \left| \int_{\mathbb{R}^d} V_\omega(x) u_j(x,\theta') \left( \int_{\mathbb{T}^*} f(x,\theta) w_{\alpha,\varepsilon,\beta}(\theta) d\theta \right) dx \right|^2 d\theta' \\
&\leq \sum_{j\in\mathbb{N}} \int_{\mathbb{T}^*} \left| \int_{\mathbb{R}^d} V_\omega(x) u_j(x,\theta') \left( \int_{\mathbb{T}^*} f(x,\theta) w_{\alpha,\varepsilon,\beta}(\theta) d\theta \right) dx \right|^2 d\theta' \\
&= \int_{\mathbb{R}^d} V^2_\omega(x) \left| \int_{\mathbb{T}^*} f(x,\theta) w_{\alpha,\varepsilon,\beta}(\theta) d\theta \right|^2 dx.
\end{aligned}$$

Hence using Lemma 5.5, (42), (43) and (44), for $\varepsilon$ small enough, we get

$$\begin{aligned}
\|H_{\omega,l,-} w^f_{\alpha,\varepsilon,\beta}\|^2 &\leq \varepsilon^2/8 + 2\int_{\mathbb{R}^d} V^2_\omega(x) \left| \int_{\mathbb{T}^*} f(x,\theta) w_{\alpha,\varepsilon,\beta}(\theta) d\theta \right|^2 dx \\
&\leq \varepsilon^2/6 + \int_{\mathbb{R}^d} V^2_\omega(x) \left| \int_{\mathbb{T}^*} f(x,\theta) w_{\varepsilon,\beta}(\theta) d\theta \right|^2 dx
\end{aligned} \tag{45}$$

Then, expanding $V^2_\omega$ and using assumption (H.2bis), we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} V^2_\omega(x) \left| \int_{\mathbb{T}^*} f(x,\theta) w_{\varepsilon,\beta}(\theta) d\theta \right|^2 dx \\
&= \sum_{\substack{\eta\in\Gamma \\ \gamma\in\Gamma \\ \gamma'\in\Gamma}} \omega_\gamma \omega_{\gamma'} \int_{C_0} V(x+\eta-\gamma) V(x+\eta-\gamma') \left| \int_{\mathbb{T}^*} f(x,\theta) w_{\varepsilon,\beta}(\theta) e^{-i\eta\cdot\theta} d\theta \right|^2 dx \\
&\leq \sum_{\eta\in\Gamma} \left( \sum_{\gamma\in\Gamma} (1+|\eta-\gamma|)^{-\nu} \omega_\gamma \right)^2 \int_{C_0} g^2_+(x) \left| \int_{\mathbb{T}^*} f(x,\theta) w_\varepsilon(\theta) e^{-i(\eta-\beta)\cdot\theta} d\theta \right|^2 dx.
\end{aligned}$$

One easily checks that, for some $C > 0$ (depending only on $\alpha$),

$$\sum_{\eta\notin\Lambda_\alpha(\varepsilon)} (1+|\eta|)^{-\nu} \leq C\varepsilon^{(\nu-d)/2+\alpha(\nu-d)}. \tag{46}$$

We set $s=1$ if $V$ satisfies (H.2bis.1) and $s=2/(\nu-d)$ if $V$ satisfies (H.2bis.1)). Then using the boundedness of the random variables $(\omega_\gamma)_{\gamma\in\Gamma}$, we get that, for some $C>0$,

$$\begin{aligned}\int_{\mathbb{R}^d} V_\omega^2(x)\left|\int_{\mathbb{T}^*} f(x,\theta)w_{\varepsilon,\beta}(\theta)d\theta\right|^2 dx \\ \leq \sum_{\eta\in\Gamma}\left(C\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\eta+\Lambda_\alpha(\varepsilon^s)}(1+|\eta-\gamma|)^{-\nu}\omega_\gamma\right)^2\cdot \\ \cdot\int_{C_0} g_+^2(x)\left|\int_{\mathbb{T}^*} f(x,\theta)w_\varepsilon(\theta)e^{-i(\eta-\beta)\cdot\theta}d\theta\right|^2 dx.\end{aligned}$$

Moreover, for some $C>0$, we have

$$\begin{aligned}&\sum_{\eta\notin\beta+\Lambda_\alpha(\varepsilon^s)}\left(C\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\eta+\Lambda_\alpha(\varepsilon^s)}(1+|\eta-\gamma|)^{-\nu}\omega_\gamma\right)^2\cdot \\ &\qquad\cdot\int_{C_0} g_+^2(x)\left|\int_{\mathbb{T}^*} f(x,\theta)w_\varepsilon(\theta)e^{-i(\eta-\beta)\cdot\theta}d\theta\right|^2 dx \\ &\leq C\sum_{\eta\notin\Lambda_\alpha(\varepsilon^s)}\int_{C_0} g_+^2(x)\left|\int_{\mathbb{T}^*} f(x,\theta)w_\varepsilon(\theta)e^{-i\eta\cdot\theta}d\theta\right|^2 dx \\ &=\int_{C_0} g_+^2(x)\|f(x,\cdot)w_\varepsilon-\Pi_{\Lambda_\alpha(\varepsilon)}f(x,\cdot)w_\varepsilon\|_{L^2(\mathbb{T}^*)}dx.\end{aligned}$$

Using the same computations as in the proofs of Lemma 5.5 and of (43), we get that, for any $k\in\mathbb{N}$, there exists $C_k>0$ such that, for any $\beta\in\Gamma$, we have

$$\int_{C_0} g_+^2(x)\|f(x,\cdot)w_\varepsilon-\Pi_{\Lambda_\alpha(\varepsilon)}f(x,\cdot)w_\varepsilon\|_{L^2(\mathbb{T}^*)}dx\leq C_k\varepsilon^{s\alpha k}.$$

Combining this computation with (45), we get that, for $\varepsilon>0$ and some $C>0$

$$\begin{aligned}\|H_{\omega,l,-}w^f_{\alpha,\varepsilon,\beta}\|^2\leq\varepsilon^2/2 \\ +\sum_{\eta\in\beta+\Lambda_\alpha(\varepsilon^s)}\left(C\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)}(1+|\gamma|)^{-\nu}\omega_{\gamma+\eta}\right)^2\cdot \\ \cdot\int_{C_0} g_+^2(x)\left|\int_{\mathbb{T}^*} f(x,\theta)w_\varepsilon(\theta)e^{-i(\eta-\beta)\cdot\theta}d\theta\right|^2 dx.\end{aligned}$$

Now, as $f(x,\theta)$ satisfies the assumption of Lemma 5.5 and as $g_+$ is $H$-relatively bounded, using Bessel's inequality, for $\varepsilon > 0$ small enough and some $K > 1$, we have

(47)

$$\begin{aligned}\|H_{\omega,l,-}w^f_{\alpha,\varepsilon,\beta}\|^2 &\leq \varepsilon^2/2 + \left(C\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)}(1+|\gamma|)^{-\nu}\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega_\gamma\right)^2\cdot \\ &\qquad\cdot\sum_{\eta\in\Lambda_\alpha(\varepsilon^s)}\int_{C_0}g_+^2(x)\left|\int_{\mathbb{T}^*}f(x,\theta)w_\varepsilon(\theta)e^{-i\eta\cdot\theta}d\theta\right|^2 dx \\ &\leq \varepsilon^2/2 + K\left(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega_\gamma\right)^2\end{aligned}$$

Now, for $N\in\mathbb{N}$ very large, we may divide $\Lambda_N$ into $N(\varepsilon)$ disjoint sets of size $2\Lambda_\alpha(\varepsilon^s)$. More precisely, we may write

$$\bigcup_{j=1}^{N(\varepsilon)}(\beta_j + 2\Lambda_\alpha(\varepsilon^s)) \subset \Lambda_N \text{ and}$$
$$\text{for } j\neq j',\quad (\beta_j + 2\Lambda_\alpha(\varepsilon^s))\cup(\beta_{j'} + 2\Lambda_\alpha(\varepsilon^s)) = \emptyset,$$

(hence, for some $C>0$, we have $N(\varepsilon)\geq (N\varepsilon^s)^d/C$).
This implies that, for $j\neq j'$, $w^f_{\alpha,\varepsilon,\beta_j}$ and $w^f_{\alpha,\varepsilon,\beta_{j'}}$ are orthogonal. Hence, by (46) and (47), we get that

(48)

$$\begin{aligned}\mathbb{E}(\sharp\{-\varepsilon\leq \text{eigenvalues of }\Pi_N H^0_\omega\Pi_N\leq\varepsilon\}) &\geq \mathbb{E}(\sharp\{1\leq j\leq N(\varepsilon);\ \|H_{\omega,l,-}w^f_{\alpha,\varepsilon,\beta_j}\|\leq\varepsilon\}) \\ &\geq \mathbb{E}\left(\sum_{j=1}^{N(\varepsilon)}B_j(\omega)\right),\end{aligned}$$

where

$$B_j(\omega) = \begin{cases} 1 & \text{if } K\left(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega_\gamma\right)^2 \leq \varepsilon^2/2 \text{ where } K \text{ is given in (47)} \\ 0 & \text{if not.}\end{cases}$$

The $(B_j)_{1\leq j\leq N(\varepsilon)}$ are independent identically distributed Bernoulli random variables. So by (48), we get that, for some $C>0$

$$\begin{aligned}\frac{1}{(2N+1)^d}\mathbb{E}(\sharp\{-\varepsilon\leq \text{eigenvalues of }\Pi_N H^0_\omega\Pi_N\leq\varepsilon\}) &\geq \frac{N(\varepsilon)}{(2N+1)^d}\mathbb{P}(B_1=1) \\ &\geq \frac{1}{C}\varepsilon^{sd}\mathbb{P}(B_j=1)\end{aligned}$$

Hence, taking the limit $N \to +\infty$, we get that, for $\varepsilon > 0$ small enough,

$$N(\varepsilon) - N(0^-) \geq N_{\mathcal{E}_{l,-}}(\varepsilon) - N_{\mathcal{E}_{l,-}}(-\varepsilon) \geq \frac{1}{C}\varepsilon^{sd}\mathbb{P}(B_j = 1). \tag{49}$$

We are only left with estimating $\mathbb{P}(B_j = 1)$. It is now obvious that there exists $K > 0$ such that, for any $\varepsilon$ small enough and any $1 \leq j \leq N(\varepsilon)$, if for all $\gamma \in \beta_j + 2\Lambda_\alpha(\varepsilon^s)$, we have $\omega_\gamma \leq \varepsilon/(2K)$ then

$$K\left(\varepsilon^{s\alpha(\nu-d)} \cdot \varepsilon + \sum_{\gamma\in\beta_j+2\Lambda_\alpha(\varepsilon^s)} (1+|\gamma|)^{-\nu}\omega_\gamma\right)^2 \leq \varepsilon^2 \cdot K(\varepsilon^{s\alpha(\nu-d)} + 1/(2K))^2 \leq \varepsilon^2/2$$

if $\varepsilon$ is small enough (depending on $\alpha$). This tells us that, for some $C > 1$,

$$\begin{aligned} \log \mathbb{P}(B_j = 1) &\geq \sharp\Lambda_\alpha(\varepsilon^s) \cdot \log \mathbb{P}(\omega_0 \leq \varepsilon/2K) \\ &\geq \varepsilon^{-s\kappa - s\alpha - s(d-1)/2 - s(d-1)\alpha} \cdot \log \mathbb{P}(\omega_0 \leq \varepsilon/2K)/C. \end{aligned} \tag{50}$$

Hence taking the double logarithm of (49), using assumption (H.3) and (50), we get that,

$$\begin{aligned} \liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\log|\log(N(\varepsilon) - N(0^-))|}{\log \varepsilon} &\geq -s(\kappa + \alpha + (d-1)/2 + (d-1)\alpha) \\ &= -s((d-1)/2 + \kappa) - sd\alpha. \end{aligned}$$

Now, we may choose $\alpha > 0$ as small as we like; hence, we conclude that

$$\begin{aligned} \liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\log|\log(N(\varepsilon) - N(0^-))|}{\log \varepsilon} &\geq -\frac{s(d-1+2\kappa)}{2} \\ &\geq -\frac{sd}{2} = \begin{cases} -\dfrac{d}{2} & \text{if } V \text{ satisfies (H.2bis.2)} \\ -\dfrac{d}{\nu - d} & \text{if } V \text{ satisfies (H.2bis.1)}. \end{cases} \end{aligned}$$

This ends the proof of Theorem 2.1. More precisely, we get that, if $n$ is degenerate then

$$\liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\log|\log(N(\varepsilon) - N(0^-))|}{\log \varepsilon} \geq \begin{cases} -\dfrac{d}{2} + \dfrac{1}{6} & \text{if } V \text{ satisfies (H.2bis.2)} \\ -\dfrac{d}{\nu - d} + \dfrac{1}{3(\nu - d)} & \text{if } V \text{ satisfies (H.2bis.1)}. \end{cases}$$

## 7. The upper bound

In this section, we will assume that $H$ satisfies assumption (H.1) and that, if we denote its density of states by $n(E)$ then, at the edge 0, we have

$$\limsup_{E\to 0^+} \left|\frac{\log(n(E) - n(0))}{\log E}\right| = d/2$$

Then to prove the upper bound the strategy will be the following: by Theorem 4.1, we know that we just need to upper bound $N_{\mathcal{E}_0}(E)$ (defined in Theorem 4.1). Therefore, we will use (13) and Lemma 3.1 to compare $H^0_\omega = \Pi_0 H_\omega \Pi_0$ with a discrete Anderson model.

For this model, the Lifshits tail behaviour hence the upper bound on the integrated density of states has already been proved in [27] or [21].
The idea of the reduction is simple: $N_{\mathcal{E}_0}(E)$ is the density of states for the reference operator $H^0_\omega = \Pi_0 H_\omega \Pi_0$ where $\Pi_0$ is the orthogonal projection on the eigenspace associated to the eigenvalues generating the band at 0. Let $V^0_\omega = \Pi_0 V_\omega \Pi_0$ and$H^0_\omega = \Pi_0 H \Pi_0$. As $V^0_\omega$ is non-negative, if a vector minimizes $H^0_\omega$, it has to minimize $H^0$ hence to be concentrated near the zeros of the $(E_j)_{j\in N}$. But these zeros are non-degenerate and isolated; hence near these zeros, $H^0$ resembles the usual discrete Laplace operator. On the other hand, if the vector concentrates near a point $\theta^0$ then, in the kernel $V_\omega(\theta,\theta')$, we may freeze $\theta=\theta'=\theta^0$. A close look at the $V_\omega(\theta^0,\theta^0)$ shows that it resembles the usual Anderson potential. A fact that is crucially used is that, as the zeros of the $(E_j)_{j\in N}$ are non-degenerate, it is possible to bound the difference $V_\omega(\theta,\theta') - V_\omega(\theta^0,\theta^0)$ by $H^0_\omega$.

The above heuristics tells us that one of the problems we will encounter while proving the upper bound is that we will have to take into account all the points where the Floquet eigenvalues vanish. So we will begin with a lemma that shows that enables us to isolate the contributions of the differents zeos of the Floquet eigenvalues. Let us just shortly recall the structure of the zeros of the Floquet eigenvalues. For $1\leq j\leq n$, we define $Z_j = \{\theta\in\mathbb{T}^*;\ E_j(\theta)=0\}$; the sequence $(Z_j)_{1\leq j\leq n}$ is decreasing and $Z_1 = Z$, the set of all points where some Floquet eigenvalue vanishes. Let $m=\sharp Z$. In the sequel, the vectors $(v_j(\cdot,\theta))_{1\leq j\leq n}$ will be the vector generating the band at 0 defined in Lemma 3.1. Then we prove the

**Lemma 7.1.** *There exists $c_0>0$ such that, for $0\leq E$, we have*

$$N_{\mathcal{E}_0}(E) \leq \sum_{\theta^0\in Z} N_{\theta^0}(m\cdot E),$$

*where*

- *$N_{\theta^0}$ is the density of states of a random Schrödinger operator acting on $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$ of the form $\varpi_{\theta^0}\otimes I_n + V^0_\omega$.*
- *$\varpi_{\theta^0}$ denotes the multiplication by the function $\varpi_{\theta^0}(\theta) = c_0\sum_{j=1}^d (1-\cos(\theta_j-\theta^0_j))$ living on $\mathbb{T}^*$.*
- *$I_n$ is the $n\times n$ identity matrix.*
- *$V^0_\omega$ is the random operator with matrix kernel*

$$V^0_\omega(\theta,\theta') = \begin{pmatrix} v_{\omega,11}(\theta,\theta') & \cdots & v_{\omega,1n}(\theta,\theta') \\ \vdots & \ddots & \vdots \\ v_{\omega,n1}(\theta,\theta') & \cdots & v_{\omega,nn}(\theta,\theta') \end{pmatrix} \tag{51}$$

*where, for $1\leq i,j\leq n$,*

$$v_{\omega,ij}(\theta,\theta') = \int_{\mathbb{R}^d} V_\omega(x) v_i(x,\theta)\overline{v_j(x,\theta')}dx$$

**Proof.** For $\theta^0\in Z$, let $j(\theta^0) = \sup N_{\theta^0}$. By the construction made in Lemma 3.1, for $\theta$ close to $\theta^0$, the operator $H^0$ (acting on $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$) is a multiplication by a matrix; the matrix takes the following block diagonal form

$$H^0(\theta) = \begin{pmatrix} B_{j(\theta^0)}(\theta) & 0 & 0 & \cdots & 0 \\ 0 & E_{j(\theta^0)+1}(\theta) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & E_n(\theta) \end{pmatrix}$$

where $B_{j(\theta^0)}(\theta)$ is $j(\theta^0) \times j(\theta^0)$ block given by

$$B_{j(\theta^0)}(\theta) = \begin{pmatrix} \langle v_1(\cdot,\theta)), Hv_1(\cdot,\theta))\rangle_{L^2(C_0)} & \cdots & \langle v_1(\cdot,\theta)), Hv_{j(\theta^0)}(\cdot,\theta))\rangle_{L^2(C_0)} \\ \vdots & \ddots & \vdots \\ \langle v_{j(\theta^0)}(\cdot,\theta)), Hv_1(\cdot,\theta))\rangle_{L^2(C_0)} & \cdots & \langle v_{j(\theta^0)}(\cdot,\theta)), Hv_{j(\theta^0)}(\cdot,\theta))\rangle_{L^2(C_0)} \end{pmatrix}.$$

For $\theta^0 \in Z$, define $\varpi_{\theta^0}(\theta) = c\sum_{j=1}^{d}(1-\cos(\theta_j - \theta^0_j))$. Here the constant $c$ is small and will be chosen later on. As the zeros of the Floquet eigenvalues are non-degenerate, there exists $c > 0$ and $V_{\theta^0}$ an open neighbourhood of $\theta^0$ such that, for $\theta \in V_{\theta^0}$

- for $1 \le j \le j(\theta^0)$, $E_j(\theta) \ge \varpi_{\theta^0}(\theta)$,
- for $j \ge j(\theta^0)$, $E_j(\theta) \ge 3c$.

Hence, for $\theta$ close to $\theta^0$,

$$H^b_{\theta^0}(\theta) \le H^0(\theta) \tag{52}$$

where $H^b_{\theta^0}$ is the diagonal matrix with entries $\varpi_{\theta^0}(\theta)$ for $1 \le j \le j(\theta^0)$ and $c$ for $j \ge j(\theta^0)$. We then extend $H^b_{\theta^0}$ to the whole $\mathbb{T}^*$ by setting $H^b_{\theta^0}(\theta) = 2c$ outside $V_{\theta^0}$.
Finally, let us notice that the $(V_{\theta^0})_{\theta^0\in Z}$ may be chosen so that they cover $\mathbb{T}^*$ i.e $\cup_{\theta^0\in Z} V_{\theta^0} = \mathbb{T}^*$ and such that each of them contains exactly one point of $Z$ i.e if $\theta \in Z$, $\theta' \in Z$ and $\theta \ne \theta'$ then $\theta' \notin \overline{V_\theta}$.

We order the points in $Z = \{\theta^k;\ 1 \le k \le m\}$. Let $(\chi_k)_{1\le k\le m}$ be a $\mathcal{C}^\infty_0$-partition of unity in $\mathbb{T}^*$ such that

- for $1 \le k \le m$, $\mathrm{supp}(\chi_k) \subset V_{\theta^k}$,
- for $1 \le k \le m$, $0 \le \chi_k \le 1$,
- for $1 \le k \le m$, $\chi_k \equiv 1$ in some neighbourhood of $\theta^k$,

Then we have $1/m \le \sum_{1\le k\le m} \chi_k^2$.
For $u \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$, we will write $u = ((w_{j,k}))_{\substack{1\le k\le m \\ 1\le j\le n}}$. We will consider $u$ as a system of $m$ columns denoted by $w_{\cdot,k}$ for $1 \le k \le m$, each of them belonging to $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$. We endow $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$ with the scalar product generating the following euclidean norm

$$\|u\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m} = \sum_{k=1}^{m} \|w_{\cdot,k}\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n}.$$

We define the mapping $S:\ L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\to L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$ by

$$S(u)=(\chi_k u)_{1\leq k\leq m}=((\chi_k u_j))_{\substack{1\leq k\leq m\\ 1\leq j\leq n}}\text{ if }u=(u_j)_{1\leq j\leq n}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n.$$

The adjoint of $S$, $S^*:\ L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m\to L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$ is defined by

$$S^*(u)=\left(\sum_{1\leq k\leq m}\chi_k u_{j,k}\right)_{1\leq j\leq n}\quad\text{if }u=((u_{j,k}))_{\substack{1\leq k\leq m\\ 1\leq j\leq n}}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m.$$

One easily checks that $\frac{1}{m}I\leq S^*\circ S\leq I$ (here $I$ is the identity of $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$) so that $S$ is one-to-one.
We will now define a random ergodic operator on $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$ that will be used to lower bound $H^0_\omega$. Let $u=((u_{j,k}))_{\substack{1\leq k\leq m\\ 1\leq j\leq n}}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$ and define

$$H^a_\omega u=H^a u+V^a_\omega u=(H^b_{\theta k}u_{\cdot,k}+V^0_\omega u_{\cdot,k})_{1\leq k\leq m}$$

where $H^b_{\theta_k}$ has been defined in (52) and $V^0_\omega$ in (51).
As $(\chi_k)_{1\leq k\leq m}$ is a partition of unity, one checks that, for $u\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$,

$$\langle V^0_\omega u,u\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n}=\langle V^a_\omega Su,Su\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m}.$$

As $H^b_{\theta^k}\leq H^0$ and $\sum_{k=1}^m\chi_k^2\leq 1$, we immediately get that, for $u\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$, we have

$$\begin{aligned}\langle H^a Su,Su\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m}&=\sum_{k=1}^m\langle H^b_{\theta^k}\chi_k u,\chi_k u\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n}\\&\leq\langle(\sum_{k=1}^m H^b_{\theta^k}\chi_k^2)u,u\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n}\\&\leq\langle H^0u,u\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n}.\end{aligned}$$

hence

$$\langle H^a_\omega Su,Su\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m}\leq\langle H^0_\omega u,u\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n}.\tag{53}$$

By Proposition 4.1 and Remark 5.2, we know that the operators $H^0_\omega$ and $H^a_\omega$ both admit a density of states denoted respectively by $N^0(E)$ and $N^a(E)$. Moreover, for $E\in\mathbb{R}$ such that $E$ is continuity point of $N^0$ and $N^a$, we know that

- $N^0(E)$ is given by

  $$N^0(E)=\lim_{q\to+\infty}\frac{1}{\sharp\Lambda_q}\sharp\{\text{eigenvalues of }H^0_{\omega,q}\text{ less than }E\}$$

  where

  $$H^0_{\omega,q}=\Pi^0_q H^0_\omega\Pi^0_q$$

  and $\forall u=(u_j)_{1\leq j\leq n}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$,

  $$\Pi^0_q u=\left(\sum_{\gamma\in\Lambda_q}e^{i\gamma\theta}\int_{\mathbb{T}^*}e^{-i\gamma\theta}u_j(\theta)d\theta\right)_{1\leq j\leq n},$$

  where $\Lambda_q=\{\gamma\in\Gamma;\ |\gamma|_\Gamma\leq q\}$.

- $N^a(E)$ is given by

$$N^a(E) = \lim_{q\to+\infty} \frac{1}{\sharp\Lambda_q} \sharp\{\text{eigenvalues of } H^a_{\omega,q} \text{ less than } E\}$$

where

$$H^a_{\omega,q} = \Pi^a_q H^a_\omega \Pi^a_q$$

and $\forall u = (u_{j,k})_{\substack{1\le j\le n\\ 1\le k\le m}} \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$,

$$\Pi^a_q u = \left(\sum_{\gamma\in\Lambda_q} e^{i\gamma\theta}\int_{\mathbb{T}^*} e^{-i\gamma\theta} u_{j,k}(\theta)d\theta\right)_{\substack{1\le j\le n\\ 1\le k\le m}}.$$

Let us compare $N^0$ and $N^a$. Therefore we will need the

**Lemma 7.2.** *For any $\delta>0$ and any $l\in\mathbb{N}$, there exists $C_l(\delta)>0$ such that, for $q\ge 1$,*

$$\left\|\Pi^a_{(1+\delta)q}\circ S\circ\Pi^0_q - S\circ\Pi^0_q\right\|_{\mathcal{L}(L^2(\mathbb{T}^*)\otimes\mathbb{C}^n, L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m)} \le C_l(\delta)q^{-l}.$$

**Proof.** The operator $(I-\Pi^a_{(1+\delta)q})\circ S\circ\Pi^0_q$ is a $n\times(m\cdot n)$ matrix of operators; each component is acting on $L^2(\mathbb{T}^*)$. The components are of the form $(I-\Pi_{(1+\delta)q})\chi_j\Pi_q$ where $\Pi_q = \sum_{\gamma\in\Lambda_q} e^{i\gamma\theta}\otimes e^{-i\gamma\theta}$ is acting on $L^2(\mathbb{T}^*)$. Then the operator $\Pi^0_q\chi_j(I-\Pi^a_{(1+\delta)q})\chi_j\Pi^0_q$ has the following kernels

$$[\Pi^0_q\chi_j(I-\Pi^a_{(1+\delta)q})\chi_j\Pi^0_q](\theta,\theta') = \sum_{\substack{|\beta|_\Gamma\le q\\ |\beta'|_\Gamma\le q}} e^{i\beta\theta-i\beta'\theta'} \sum_{|\gamma|_\Gamma\ge(1+\delta)q}\int_{\mathbb{T}^*}\chi_j e^{i(\beta-\gamma)\theta}d\theta\int_{\mathbb{T}^*}\chi_j e^{i(\gamma-\beta')\theta}d\theta$$

Hence

$$\begin{aligned}\|(I-\Pi^a_{(1+\delta)q})\chi_j\Pi^0_q\|_{\mathcal{L}(L^2(\mathbb{T}^*))} &\le \sum_{\substack{|\beta|_\Gamma\le q\\ |\beta'|_\Gamma\le q}}\sum_{|\gamma|_\Gamma\ge(1+\delta)q}|\hat\chi_j(\beta-\gamma)\hat\chi_j(\gamma-\beta')|\\ &\le \sum_{\substack{|\beta|_\Gamma\le q\\ |\beta'|_\Gamma\le q}}\sum_{|\gamma|_\Gamma\ge\delta q}|\hat\chi_j(\beta-\beta'-\gamma)\chi_j(\gamma)|\\ &\le Cq^{2d}\sum_{|\gamma|_\Gamma\ge\delta q}|\hat\chi_j(\gamma)|\\ &\le C_k(\delta)q^{-k}\quad \text{as } \chi_j\in\mathcal{C}^\infty_0(\mathbb{T}^*).\end{aligned}$$

This ends the proof of Lemma 7.2.

□

Fix $\delta>0$ and $E>0$. Pick $u\in\Pi^0_q(L^2(\mathbb{T}^*)\otimes\mathbb{C}^n)$ such that $\langle H^0_\omega u,u\rangle\le E\|u\|^2$. Then

$$\langle H^a_\omega Su, Su\rangle \le E\|u\|^2$$

Hence, by Lemma 7.2,

$$\begin{aligned}\langle H^a_\omega(\Pi^a_{(1+\delta)q}S)u,(\Pi^a_{(1+\delta)q}S)u\rangle &\leq E\|u\|^2+2|\langle H^a_\omega(1-\Pi^a_{(1+\delta)q}S)u,(\Pi^a_{(1+\delta)q})Su\rangle|+\\ &\qquad +|\langle H^a_\omega(1-\Pi^a_{(1+\delta)q})Su,(1-\Pi^a_{(1+\delta)q})Su\rangle|\\ &\leq (E+C_\delta q^{-1})\|u\|^2\end{aligned}$$

Moreover, by Lemma 7.2, as $S^*\circ S\geq 1/m$,

$$\|u\|^2\leq\frac{m}{1-C_\delta q^{-1}}\|(\Pi^a_{(1+\delta)q}S)u\|^2. \tag{54}$$

Hence, for $q$ large enough (depending on $\delta$),

$$\langle H^a_\omega(\Pi^a_{(1+\delta)q}S)u,(\Pi^a_{(1+\delta)q}S)u\rangle\leq(1+\delta)m\cdot E\|(\Pi^a_{(1+\delta)q}S)u\|^2$$

By (54), we see that, for large enough $q$, $\Pi^a_{(1+\delta)q}S\Pi^0_q$ is one-to-one from $\Pi^0_q(L^2(\mathbb{T}^*)\otimes\mathbb{C}^n)$ to $\Pi^a_q(L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m)$. On the other hand, we know that

$$\begin{aligned}&\sharp\{\text{eigenvalues of } H^0_{\omega,q}\text{ less than } E\}=\\ &\quad=\sup\{\dim(\mathcal{E});\ \mathcal{E}\text{ linear subspace of }\Pi^0_q(L^2(\mathbb{T}^*)\otimes\mathbb{C}^n)\text{ such that}\\ &\qquad\qquad\forall u\in\mathcal{E},\ \langle H^0_\omega u,u\rangle\leq E\|u\|^2\}\end{aligned}$$

and, mutandi mutandis, the same holds for $H^a_\omega$; by what has be done above, for $q$ large enough, we get that

$$\sharp\{\text{eigenvalues of } H^0_{\omega,q}\text{ less than } E\}\leq\sharp\{\text{eigenvalues of } H^a_{\omega,(1+\delta)q}\text{ less than }(1+\delta)m\cdot E\}.$$

Dividing by the volume factor and letting $q\to+\infty$, for $E$ small enough, $E$ a continuity point of $N^0(E)$ and $N^a((1+\delta)E)$, we obtain

$$N^0(E)\leq(1+\delta)^dN^a((1+\delta)m\cdot E). \tag{55}$$

Using the right continuity of the integrated density of states and the fact that there is at most an enumerable number of discontinuity points, we extend (55) immediately to some interval of the form $[0,E_0]$. Now we can let $\delta$ go to 0 and get

$$N^0(E)\leq N^a(m\cdot E).$$

To end the proof of the upper bound, we notice that $H^a$ is a $m$-fold tensor product of operators of the form $H^b_{\theta^0}+V^0_\omega$ where $\theta^0$ runs through $Z$ (all these operators are acting on $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$ where $H^b_\theta$ was defined in (52) and $V^0_\omega$ in (51). But by its definition, one sees that

$$H^b_{\theta^0}\geq\varpi_{\theta^0}\otimes I_n.$$

Hence, $H^a$ is lower bounded by the $m$-fold tensor product of operators of the form $\varpi_{\theta^0}\otimes I_n+V^0_\omega$ where $\theta^0$ runs through $Z$. This ends the proof of Lemma 7.1.

$\square$

Now we need to understand the behaviour of the density of states $N_{\theta^0}$ of the operators $\varpi_{\theta^0}\otimes I_n+V^0_\omega$ constructed in Lemma 7.1. Here $\varpi_{\theta^0}$ has a very simple form: up to a unitary transformation and a dilation, it is nothing but the usual $d$-dimensional discrete Laplace operator (seen on the Fourier side). We will now use this to compare $\varpi_{\theta^0} \otimes I_n + V^0_\omega$ with the usual discrete Anderson model. We have the

**Lemma 7.3.** *There exists $C > 1$ such that*

$$h^a_\omega \otimes I_n \leq C \cdot \left( \varpi_{\theta^0} \otimes I_n + V^0_\omega \right)$$

*as operators acting on $L^2(\mathbb{T}^*) \otimes \mathbb{C}^n$ where*

- *if $V$ satisfies assumption (H.2bis.2), $h^a_\omega$ is unitarily equivalent to the usual Anderson model acting on $\ell^2(\Gamma)$ defined by*

$$-\Delta_\Gamma + \sum_{\gamma \in \Gamma} \omega_\gamma \delta_\gamma \otimes \delta_\gamma.$$

- *if $V$ satisfies assumption (H.2bis.1), $h^a_\omega$ is unitarily equivalent to a long range Anderson model acting on $\ell^2(\Gamma)$ defined by*

$$-\Delta_\Gamma + \sum_{\gamma \in \Gamma} \omega_\gamma \sum_{\beta \in \Gamma} (1 + |\beta|)^{-\nu} \delta_{\beta+\gamma} \otimes \delta_{\beta+\gamma}.$$

- *$\delta_\gamma$ is the vector $(\delta_{\beta\gamma})_{\gamma \in \Gamma}$ and $\Delta_\Gamma$ is the discrete Laplacian on $\ell^2(\Gamma)$ defined by*

$$(\Delta_\Gamma u)_\gamma = \sum_{|\beta - \gamma|_\Gamma = 1} (u_\beta - 2u_\gamma).$$

From this lemma, Lemma 7.1 and Theorem 4.1, one easily deduces the upper bound in Theorem 2.1 from what is known for the usual Anderson model ([27] or [21]). In the case of the long range Anderson model, it is not difficult to extend the results of [27] or [21] to suit our needs.

Remark 7.1. Notice that our assumption on the behaviour of the common probability measure of the $(\omega_\gamma)_{\gamma \in \Gamma}$ is not exactly the same as in [27] or [21]; these can easily be handled by replacing the $(\omega_\gamma)_{\gamma \in \Gamma}$ by a set of new i.i.d random variables (in the Anderson model) defined as follows:

- if $\omega_\gamma \in [0, \text{ess-sup}(\omega_0)/2]$, we set $\tilde{\omega}_\gamma = 0$;
- if $\omega_\gamma \in (\text{ess-sup}(\omega_0)/2, \text{ess-sup}(\omega_0)]$, we set $\tilde{\omega}_\gamma = \text{ess-sup}(\omega_0)/2$.

Then we get that

$$\tilde{A}_\omega = -\Delta_\Gamma + \sum_{\gamma \in \Gamma} \tilde{\omega}_\gamma \delta_\gamma \otimes \delta_\gamma \leq -\Delta_\Gamma + \sum_{\gamma \in \Gamma} \omega_\gamma \delta_\gamma \otimes \delta_\gamma$$

and to $\tilde{A}_\omega$ we may directly apply the results of [27] or [21].
The same idea may be used to remove the assumption on the not to fast decay of the common probability measure of the $(\omega_\gamma)_{\gamma \in \Gamma}$ at the edge of its support. The other assumptions remaining the same, we then get that

$$\limsup_{E \to 0^+} \frac{\log |\log(N(E) - N(0))|}{\log E} \leq \begin{cases} -\dfrac{d}{\nu - d} & \text{if } V \text{ satisfies (H.2bis.1)} \\ -\dfrac{d}{2} & \text{if } V \text{ satisfies (H.2bis.2)} \end{cases}$$

**Proof.** For the sake of simplicity let us forget about the subscript $\theta^0$ and define $\varpi = \varpi_{\theta^0}$ and $H^0_\omega = \varpi \otimes I_n + V^0_\omega$. Pick $1 \leq k \leq n$. Then, consider the function $\tilde{v}_k(x,\theta) = v_k(x,\theta^0)e^{i(\theta-\theta^0)x}$. Clearly $\tilde{v}_k$ belongs to $L^2(\mathbb{T}^*, \mathcal{H}^2_\theta)$ and $\|\tilde{v}_k\|_{1,H,\infty} < +\infty$. We define

$$\delta v_k(x,\theta) = \frac{1}{\sqrt{\varpi(\theta)}} \left(\tilde{v}_k(x,\theta) - v_k(x,\theta)\right).$$

As $v_k(x,\theta)$ is analytic in $\theta$ for $\theta$ close to $\theta^0$, we see that $\theta \mapsto \|\delta v_k(x,\theta)\|_{L^2(C_0)}$ satisfies the assumption of Lemma 5.5. As the family $(V_\omega)_\omega$ is bounded in $L^p_{\text{loc,unif}}$ (for some $p$ chosen as in (H.1)), we use Lemma 5.5 to get that the family of operators $(\mathrm{Op}_{1,\delta v_k(\cdot,\theta),V_\omega})_\omega$ is bounded. Now we compute for $u = (u_j)_{1\leq j\leq n} \in L^2(\mathbb{T}^*) \otimes C^n$,

$$\begin{aligned}
\langle V^0_\omega u_j, u_k\rangle &= \sum_{1\leq j,k\leq n} \langle v_{\omega,jk} u_j, u_k\rangle \\
&= \| \sum_{1\leq j\leq n} \mathrm{Op}_{1,v_j(\cdot,\theta),V_\omega}(u_j)\|^2_{L^2(\mathbb{R}^d)} \\
&\geq \frac{1}{2}\| \sum_{1\leq j\leq n} \mathrm{Op}_{1,\tilde{v}_j(\cdot,\theta),V_\omega}(u_j)\|^2_{L^2(\mathbb{R}^d)} - \| \sum_{1\leq j\leq n} \mathrm{Op}_{1,\delta v_j(\cdot,\theta),V_\omega}(\sqrt{\varpi} u_j)\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Hence, for some $C > 0$ we get, uniformly in $\omega$

$$\begin{aligned}
\| \sum_{1\leq j\leq n} \mathrm{Op}_{1,\tilde{v}_j(\cdot,\theta),V_\omega}(u_j)\|^2_{L^2(\mathbb{R}^d)} &\leq C\left(\langle V^0_\omega u_j, u_k\rangle + \sum_{1\leq j\leq n} \|\sqrt{\varpi} u_j\|^2\right) \\
&= C\langle H^0_\omega u_j, u_k\rangle
\end{aligned}$$

This immediately tell us that, for some $C > 0$

$$\langle H^b_\omega u, u\rangle \leq C\langle H^0_\omega u, u\rangle \tag{56}$$

where

- $H^b_\omega = \varpi \otimes I_n + V^b_\omega$
- $V^b_\omega$ is the operator matrix with entries the operators with the following kernels

$$v^b_{\omega,jk}(\theta,\theta') = \int_{\mathbb{R}^d} V_\omega(x) v_j(x,\theta^0)\overline{v_k(x,\theta^0)} e^{i(\theta-\theta')x} dx.$$

Let us study these kernels more precisely; one has

$$\int_{\mathbb{R}^d} V_\omega(x) v_j(x,\theta^0)\overline{v_k(x,\theta^0)} e^{i(\theta-\theta')x} dx = \sum_{\gamma\in\Gamma} \omega_\gamma e^{i(\theta-\theta')\gamma} v^b_{jk}(\theta,\theta') \tag{57}$$

where

$$v^b_{jk}(\theta,\theta') = \int_{\mathbb{R}^d} V(x) v_j(x,\theta^0)\overline{v_k(x,\theta^0)} e^{i(\theta-\theta')x} dx.$$

Remark 7.2. In (57), the sum converges in the strong operator topology.

So we get that

$$V^b_\omega = \sum_{\gamma\in\Gamma} \omega_\gamma V^b_\gamma$$

where $V_\gamma^b$ is an operator matrix with entries the $((v_{jk}^b))_{1\leq j,k\leq n}$. By definition, for $u = (u_j)_{1\leq j\leq n} \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$, we have

$$\langle V_\gamma^b u, u\rangle = \int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{C_0^*} e^{i\theta x}e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx. \tag{58}$$

To lower bound the operator $V_\omega^b$ we will need to distinguish the two cases in assumption (H.2.bis).

7.0.1. *In case 2) of assumption (H.2bis).* In this case, $|\, x\, |^2\, V(x)$ is in $\ell^1(L^p(\mathbb{R}^d))$ hence for $\gamma\in\Gamma$ and $u = (u_j)_{1\leq j\leq n}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$, expanding (58), we get

(59)

$$\begin{aligned}\langle V_\gamma^b u, u\rangle = &\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx\\
&+2\mathrm{Re}\left(\int_{\mathbb{R}^d} V(x)\left(\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right)\cdot\right.\\
&\qquad\left.\cdot\overline{\left(\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right)}dx\right)\\
&+\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2 dx\end{aligned}$$

where we defined

$$g(x,\theta) = \frac{e^{i(\theta-\theta^0)x}-1}{\sqrt{\varpi(\theta)}} \leq C(1+|x|) \text{ for some } C>0. \tag{60}$$

By (59), using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\langle V_\gamma^b u, u\rangle \geq &\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx\\
&-2\sqrt{\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2 dx}\,\cdot\\
&\qquad\cdot\sqrt{\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx}\\
&+\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2 dx\end{aligned}$$

Pick $0<\alpha<1$, using the inequality $2ab \geq -\alpha a^2 - b^2/\alpha$ (for $a,b>0$), we get

$$\begin{aligned}\langle V_\gamma^b u,u\rangle \geq (1-\alpha)\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx \\ +(1-1/\alpha)\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2 dx\end{aligned}$$

So, we get that, for some $C>0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx \\ &\leq C\left(\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2 dx + \langle V_\gamma^b u,u\rangle\right).\end{aligned}\tag{61}$$

Expanding the square in the left hand side term of (61), we see that

$$\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx = \sum_{1\leq j,k\leq n} v_{jk}\hat{u}_j(\gamma)\overline{\hat{u}_k(\gamma)}$$

where

- $\hat{u}_j(\gamma)$ is the $\gamma$-th Fourier coefficient of $u_j$.
- $v_{jk} = \int_{\mathbb{R}^d} V(x)v_j(x,\theta^0)\overline{v_k(x,\theta^0)}dx$.

Now we claim that the matrix $((v_{jk}))_{1\leq j,k\leq n}$ is positive definite. The positivity is obvious. Assume it is not definite; then its has 0 as an eigenvalue. In other words there exists $(\lambda_j)_{1\leq j\leq n}$ such that

$$\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n \lambda_j v_j(x,\theta^0)\right|^2 dx = 0.$$

Hence as $V(x)$ does not vanish almost everywhere, the continuous function $v(\cdot,\theta_0) = \sum_{j=1}^n \lambda_j v_j(\cdot,\theta^0)$ must vanish on some open set. But, by construction (see Lemma 3.1) all the $(v_j(\cdot,\theta^0))_{1\leq j\leq n}$ are linearly independent solutions of the elliptic equation $H_{\theta^0}u=0$; hence $v(\cdot,\theta^0)$ is also a solution of $H_{\theta^0}u=0$. By the unique continuation principle (e.g. see [9]), it has to vanish everywhere. This contradicts the fact that the $(v_j(\cdot,\theta^0))_{1\leq j\leq n}$ are linearly independent.
This now tells us that, for some $C>0$, we have

$$\sum_{j=1}^n\left|\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 \leq C\cdot\int_{\mathbb{R}^d} V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2 dx\tag{62}$$

If we sum (61) over $\gamma$, using (62) and the definition of $V^b_\omega$, we get

$$\sum_{\gamma\in\Gamma}\omega_\gamma\sum_{j=1}^n\left|\int_{\mathbb{T}^*}e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2$$
$$\leq C\left(\sum_{\gamma\in\Gamma}\omega_\gamma\int_{\mathbb{R}^d}V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*}g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2dx+\langle V^b_\omega u,u\rangle\right)$$

As the random variables $(\omega_\gamma)_\gamma$ are bounded, using Parseval's identity, (60) and assumption (H.2bis.1), we obtain, for some $C>0$,

(63)
$$\sum_{\gamma\in\Gamma}\omega_\gamma\sum_{j=1}^n\left|\int_{\mathbb{T}^*}e^{-i\theta\gamma}u_j(\theta)d\theta\right|^2$$
$$\leq\sqrt{C}\left(\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d}V(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*}g(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2dx+\langle V^b_\omega u,u\rangle\right)$$
$$\leq\sqrt{C}\left(\int_{\mathbb{R}^d}V(x)\int_{\mathbb{T}^*}g(x,\theta)\left|\sum_{j=1}^n v_j(x,\theta^0)\sqrt{\varpi(\theta)}u_j(\theta)\right|^2d\theta dx+\langle V^b_\omega u,u\rangle\right)$$
$$\leq C\left(\sum_{j=1}^n\int_{\mathbb{R}^d}V(x)|v_j(x,\theta^0)|^2(1+|x|)^2dx\int_{\mathbb{T}^*}\varpi(\theta)|u_j(\theta)|^2d\theta+\langle V^b_\omega u,u\rangle\right)$$
$$\leq C\left(\sum_{j=1}^n\int_{\mathbb{T}^*}\varpi(\theta)|u_j(\theta)|^2d\theta+\langle V^b_\omega u,u\rangle\right),$$

Now using (56) and (63), we get that there exists $C>0$ such that for $u\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$,

$$\langle H^a_\omega u,u\rangle\leq C\cdot\langle H^0_\omega u,u\rangle,$$

where $H^a_\omega=h^a_\omega\otimes I_n$ and $h^a_\omega$ is unitarily equivalent to the $d$-dimensional discrete Anderson model (written in Fourier variables); indeed we have

$$h^a_\omega=\varpi_{\theta^0}+\sum_{\gamma\in\Gamma}\omega_\gamma\pi_\gamma\text{ where }\pi_\gamma\text{ is the orthogonal projector on }e^{i\theta\cdot\gamma}.$$

Written in Fourier variables, the discrete Laplace operator on $\Gamma$ is nothing but $\varpi_0$ (i.e. for $\theta^0=0$) that is the conjugate of $\varpi_{\theta^0}$ by a translation by $\theta^0$; but the projections $(\pi_\gamma)_{\gamma\in\Gamma}$ commute with translations. This ends the proof of Lemma 7.3 in the case 2 of assumption (H.2bis).

7.0.2. *In case 1) of assumption (H.2bis).* As $\mid x\mid^2 V(x)$ is not in $\ell^1(L^p(\mathbb{R}^d))$ in this case, using (H.2bis.1), for some $C>0$, we get

$$\begin{aligned}\langle V_\gamma^b u,u\rangle &= \sum_{\beta\in\Gamma}\int_{C_0} V(x-\beta)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{i\theta x}e^{-i\theta(\gamma+\beta)}u_j(\theta)d\theta\right|^2 dx\\ &\geq \frac{1}{C}\sum_{\beta\in\Gamma}(1+|\beta|)^{-\nu}\int_{C_0} g_-(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} e^{i\theta x}e^{-i\theta(\gamma+\beta)}u_j(\theta)d\theta\right|^2 dx\\ &= \frac{1}{C}\sum_{\beta\in\Gamma}(1+|\beta|)^{-\nu}s_{\gamma,\beta}\end{aligned}$$

We can then analyse each of the $s_{\gamma,\beta}$ in the same way as in in the previous case to get, for some $C>0$

$$\begin{aligned}&\sum_{j=1}^n\left|\int_{\mathbb{T}^*} e^{-i\theta(\gamma+\beta)}u_j(\theta)d\theta\right|^2\\ &\leq C\left(s_{\gamma,\beta}+\int_{C_0} g_-(x)\left|\sum_{j=1}^n v_j(x,\theta^0)\int_{\mathbb{T}^*} g(x,\theta)e^{-i\theta(\gamma+\beta)}\sqrt{\varpi(\theta)}u_j(\theta)d\theta\right|^2 dx\right).\end{aligned}$$

If we now multiply this by $\omega_\gamma\cdot(1+|\beta|)^{-\nu}$ and sum it over $\gamma$ and $\beta$, using the same arguments as before, we get, for some $C>0$,

$$\sum_{j=1}^n\sum_{\substack{\gamma\in\Gamma\\ \beta\in\Gamma}}(1+|\beta|)^{-\nu}\omega_\gamma\left|\int_{\mathbb{T}^*} e^{-i\theta(\gamma+\beta)}u_j(\theta)d\theta\right|^2 \leq C\left(\sum_{j=1}^n\int_{\mathbb{T}^*}\varpi(\theta)|u_j(\theta)|^2 d\theta+\langle V_\omega^b u,u\rangle\right).$$

This immediately implies that there exists $C>0$ such that for $u\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$,

$$\langle H_\omega^a u,u\rangle \leq C\langle H_\omega^0 u,u\rangle$$

where $H_\omega^a = h_\omega^a\otimes I_n$ and $h_\omega^a$ is a $d$-dimensional long range discrete Anderson model (written in Fourier variable) i.e.

$$h_\omega^a = \varpi+\sum_{\gamma\in\Gamma}\omega_\gamma\sum_{\beta\in\Gamma}(1+\mid\beta\mid)^{-\nu}\pi_{\gamma+\beta}.$$

This ends the proof of Lemma 7.3.

$\square$

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