Anderson Localization for Random Schrödinger Operators with Long Range Interactions

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Abstract

We prove pure point spectrum with power-decaying eigenfunctions for Schrödinger Operators with a periodic potential plus a random potential of the form $V_\omega(x) = \sum q_i(\omega) f(x-i)$ where f decays at infinity like $|x|^{-m}$ for $m > 3d$ resp. $m > 2d$ depending on the regularity of f. The random variables q_i are supposed to be independent and identically distributed. We suppose that their distribution has a bounded density of compact support.

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1 Introduction

At least since the groundbreaking work of Lifshitz in the early 60s it is widely accepted among physicists that random models of solid state physics should exhibit pure point spectrum near fluctuation boundaries. The latter are those parts of the spectrum which are determined by rather rare events. To present a more concrete picture consider a Schrödinger operator of the form $-\Delta + V_{per} + V_\omega$ describing a periodic solid with additional impurities given by the random perturbation. The spectrum will typically consist of a union of closed intervals, bands, whose edges correspond to the unlikely events that the random perturbation takes on its maximal respectively minimal value. Thus these band edges form the fluctuation boundaries.

Mathematical rigorous proofs of localization in multidimensional models have so far been restricted to Anderson type models with additional technical restrictions. They go back to the pioneering works for the discrete case ([11, 10, 4, 17]) with $V_{per} = 0$ which found considerable simplification in [5]. The first paper which treated the continuum case was [12] which was extended and simplified substantially in [2, 13, 16]. In the latter paper the case of a non-trivial periodic background potential was considered for the first time; however, as in the other articles mentioned so far, the results showed localization near the bottom of the spectrum, only.

Localization near arbitrary band edges was investigated in a series of papers [7, 8, 1, 15, 18]: In [8] a discrete Schrödinger operator was considered, while [7] treats a divergence form model for acoustic waves and contains most of the technique necessary for the Schrödinger case (in [18] an extension of the results for anisotropic models is given). In [1] a fairly general situation is considered, including Schrödinger operators of the above type under some mild assumptions on the coefficients. In our recent paper [15] we also studied the latter case (allowing more general

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V_{per} and f) using among other ingredients a multi-scale analysis which is based on a variation of ideas from [2, 7].

To point out the progress achieved in the present note we recall that alloy type models have the form

$$V_\omega(x) = \sum q_i(\omega) f(x-i) \tag{1.1}$$

where we will assume that the q_i are independent identically distributed random variables which have a bounded density g. We suppose that supp(g) is an interval and denote by q_- and q_+ its infimum and supremum respectively. The function $f \geq 0$,($f \geq c > 0$ on an open set,) is assumed to belong to $l^1(L^p)$ with $p = 2$ for $d \geq 3$, $\quad p > \frac{d}{2}$ for $d \geq 4$.

Also let $H_0 = -\Delta + V_0$, where $V_0 \in L^p_{loc}$ is a periodic potential with p as above. We will study the random Schrödinger operator $H_\omega = H_0 + V_\omega$.

Despite some discussion in [2, 13], localization had so far been settled for compactly supported f only, even in the case $V_{per} = 0$. Here we show how the technique introduced in [15] can be used to obtain localization for long range interaction f with

$$|f(x)| \leq C\langle x\rangle^{-m} \quad \text{for } |x| \text{ large,}$$

where $\langle x\rangle := (1+|x|^2)^{1/2}$. Peter Hislop recently announced results (joined work with Combes and Mourre in preparation) on the problem of localization for long range f as well.

The difficulties one has to overcome are caused by the fact that $V_\omega(x)$ and $V_\omega(y)$ are statistically correlated even if the distance between x and y is large. For other results on localization for correlated potentials, see e.g. [6], [9].

A simple extension of the argument given in [15] shows that $\sigma(H_\omega) = \Sigma$ for almost every ω, where

$$\Sigma = \bigcup_{q\in[q_-,q_+]} \sigma(H_0 + q\cdot\sum_k f(\cdot - k)).$$

If $m > 3d$ we will prove pure point spectrum with power decaying eigenfunctions near the bottom $\inf\Sigma$ of the spectrum. If we impose the conditions of [1] or [2] we may even have $m > 2d$ since their Wegner estimate is stronger. Note that [1] [2] in particular assume boundedness of f. Under slightly stronger assumptions on the density g we can prove localization near all band edges of the spectrum.

Let us formulate the main results of this paper.

Theorem 1.1 *If $m > 3d$ then for almost every ω the spectrum of H_ω is pure point in a neighborhood of* $\inf\Sigma$ *with eigenfunctions decaying more rapidly than every inverse polynomial.*

Theorem 1.2 *Assume that $m > 3d$ and that there exists $\tau > d/2$ such that g satisfies*

$$\int_{q_-}^{q_-+h} g(s)\, ds + \int_{q_+-h}^{q_+} g(s)\, ds \leq h^\tau \quad \textit{for small } h. \tag{1.2}$$

Then for almost every ω the spectrum of H_ω is pure point in a neighborhood of $\partial\Sigma$ with eigenfunctions decaying more rapidly than every inverse polynomial.

In our proof the particular form of the Wegner bound turns out to be crucial. In the Wegner estimate we take from [13] and [15] the Wegner bound behaves quadratic in the volume term. The above results rely on this type of Wegner estimate. Barbaroux, Combes, Hislop and Mourre ([1], [2] and [3]) prove a Wegner estimate that is linear in the volume. In their case our proof below requires only $m > 2d$ instead of $m > 3d$.

2 Probabilistic Preliminaries and Strategy of the Proof

In this paper we use the structure of the probability space Ω of the random variables which still has product structure although the random potential itself might be (and typically will be) deterministic in the sense of the technical definition of this term (see [14]). The strategy of our proof is to make probabilistic estimates for the box-hamiltonians $H_{\Lambda_l(x)}$ (see below) which are valid *uniformly* in q_i for those i with $\mathrm{dist}(i, \Lambda_l(x)) \geq r_l$ for some "security distance" r_l.

To define this procedure more precisely note that our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has a product structure, i.e. $\Omega = \otimes_{i\in\mathbb{Z}^d} S, \quad S \subset \mathbb{R}, \quad \mathcal{F} = \otimes_{\rangle\in\mathbb{Z}^\lceil} \mathcal{B}(\mathcal{S}), \quad \mathbb{P} = \otimes_{\rangle\in\mathbb{Z}^\lceil} \mathbb{P}_\prime$ where $\mathbb{P}_0$ is the distribution of q_0 and S its (compact) support for any subset Λ of the lattice $\mathbb{Z}^d$. We define the projection $\Pi_\Lambda : \Omega \longrightarrow \otimes_{i\in\Lambda} S$ in the obvious way $(\Pi_\Lambda \omega)_i = \omega_i$ for $i \in \Lambda$.

Definition 2.1 *If $A \in \mathcal{F}$ is an event and $\Lambda \subset \mathbb{Z}^d$ we define*

$$A^*_\Lambda = \{\omega \,|\exists \omega' \in A \quad \Pi_\Lambda \omega' = \Pi_\Lambda \omega\} = \Pi_\Lambda^{-1}(\Pi_\Lambda A). \tag{2.1}$$

Proposition 2.2 *If $\Lambda_1 \cap \Lambda_2 = \emptyset, \quad A, B \in \mathcal{F}$ then $A^*_{\Lambda_1}$ and $B^*_{\Lambda_2}$ are independent events.*

Proof. The characteristic function of the event $A^*_{\Lambda_1}$ (resp. $B^*_{\Lambda_2}$) is a function of the ω_i for $i \in \Lambda_1$ (resp. Λ_2) only. Hence these random variables are independent if $\Lambda_1 \cap \Lambda_2 = \emptyset$. □

We denote by H_Λ the operator $H_\omega = H_0 + V_\omega$ restricted to the cube Λ with periodic boundary conditions and by $R_\Lambda(z)$ its resolvent. By $\Lambda_l(x)$ we denote the cube of side length l around the point $x \in \mathbb{R}$. We will drop the letter "x" whenever it is understood which center is meant or if x is the origin. For shortness of notation we write $R_{l,x}$ for $R_{\Lambda_l(x)}$, $\chi_{l,x}$ for $\chi_{\Lambda_l(x)}$ and so on. $\varphi_{l,x}$ is a $C_0^\infty-$function which vanishes near the boundary of $\Lambda_l(x)$ and is one on that cube away from the boundary. By $W(\varphi)$ we denote the commutator $[-\Delta, \varphi]$.

Given an energy E and a configuration ω we call $\Lambda_l(x)$ γ-*good* if

$$\left\| W(\varphi_{l,x}) R_{l,x}(E + i\varepsilon) \chi_{l/3,x} \right\| \leq e^{-\gamma l} \tag{2.2}$$

independent of $\varepsilon \neq 0$.

The goal of multiscale analysis is to prove that

$$\mathbb{P}\left(\{\omega \,|\Lambda_l \text{ is not } \gamma\text{-good}\}\right) < l^{-\xi} \tag{2.3}$$

for some ξ and all l from a scale l_n with $l_{n+1} = l_n^\alpha$ for some $\alpha \in (1, 2)$. This is done by an induction procedure, i.e. the estimate (2.3) is first proven for an initial length scale l_0 and one deduces (2.3) for scale $l_{n+1} = l_n^\alpha$ from the corresponding estimate for l_n. In our case we have to replace the estimate (2.3) by a "uniform" version of it. This means that we prove the stronger estimate:

$$\mathbb{P}\left(\{\omega \,|\Lambda_l \text{ is not } \gamma\text{-good}\}^*_{4l}\right) < l^{-\xi} \tag{2.4}$$

This estimate allows us to use the independence stated in Proposition 2.2 during the induction step.

Once (2.4) is done for the scale l_n the localization proof can be completed by a standard method, e.g. as in [15]. This will be done in Section 6 with many references to [15].

We note that we can do the initial scale estimate precisely as in [15]. Before we do the induction step in section 4 and 5 we will need a "uniform" version of the Wegner estimate. This is the content of the following section.

3 The Wegner estimates

Let us set $A = A(l,\varepsilon) = \{\omega \,|dist(E, \sigma(H_{\Lambda_l})) < \varepsilon\}$. The Wegner Lemma tells us that this event has small probability, more precisely

Proposition 3.1 *([13],[15]) Under the assumptions of Theorem 1.1 there exists a neighborhood U of $\partial\Sigma$ such that for $E \in U$*

$$\mathbb{P}(A(l,\varepsilon)) \leq C\varepsilon \,|\Lambda_l|^2 \tag{3.1}$$

As usual $|M|$ denotes the Lebesgue-measure of the set M.

Combes-Hislop [2] and Barbaraux-Combes-Hislop [1] have a better Wegner estimate which gives $|\Lambda_l|$ instead of $|\Lambda_l|^2$ on the right hand side of 3.1:

Proposition 3.2 *([2],[1]) Under the assumptions in [2] resp. [1] we have for E in a neighborhood of $\partial\Sigma$*

$$\mathbb{P}(A(l,\varepsilon)) \leq C\varepsilon \,|\Lambda_l| \tag{3.2}$$

We will need a "uniform" Wegner-Lemma, i.e. an estimate of A^*_{l+r} (where $A = A(l,\varepsilon)$)

Proposition 3.3 *(uniform Wegner-estimate):*

$$\mathbb{P}(A^*_{l+r}) \leq C(\varepsilon + r^{-(m-d)}) \,|\Lambda_l|^2 \tag{3.3}$$

Moreover in the case of Proposition 3.2 we have

$$\mathbb{P}(A^*_{l+r}) \leq C(\varepsilon + r^{-(m-d)}) \,|\Lambda_l| \tag{3.4}$$

Proof. Suppose $\omega \in A^*_{l+r}$ then there exists an $\omega' \in A$ such that

$$\Pi_{\Lambda_{l+r}}\omega' = \Pi_{\Lambda_{l+r}}\omega.$$

Consequently, for any $x \in \Lambda_l$ we have

$$\begin{aligned} |V_\omega(x) - V_{\omega'}(x)| &\leq \sum_{i \notin \Lambda_{l+r}} \left(|q_i(\omega)| + |q_i(\omega')|\right) f(x-i) \\ &\leq C \sum_{i \notin \Lambda_{l+r}} |x-i|^{-m} \leq C' r^{-(m-d)}. \end{aligned}$$

Thus $\mathrm{dist}(E, \sigma(H_{\Lambda_l}(\omega'))) < \varepsilon$ implies $\mathrm{dist}(E, \sigma(H_{\Lambda_l}(\omega))) < \varepsilon + c' r^{-(m-d)}$ and hence

$$\mathbb{P}(A^*_{l+r}) \leq \mathbb{P}(A(l, \varepsilon + c' r^{-(m-d)})).$$

The latter probability can be estimated by (3.1) und (3.2) respectively. □

4 Multiscale Analysis (Deterministic Part)

Here we collect some notations and results from [15] and add a few new ones.

Fix two scales l and l' with $l' \gg l$ and both l and l' multiples of 3.

By $\Gamma_b = \Gamma_b^{l,l'}(\omega)$ we denote the union of all γ-bad cubes in $\Lambda_{3l'}$, more precisely

$$\Gamma_b := \cup \left\{ \Lambda_l(x) \,\middle|\, \Lambda_l(x) \text{ is not } \gamma\text{-good and } x \in \frac{l}{3}\mathbb{Z}^d \cap \Lambda_{3l'} \right\} \tag{4.1}$$

By $\kappa = \kappa_l$ we denote the diameter of Γ_b. The following result is contained in [15].

Proposition 4.1

$$\left\|W(\varphi_{l'})R_{l'}(E+i\varepsilon)\chi_{l/3}\right\| \leq C\left(\frac{l'}{l}\right)^{2d}\|R_{3l'}\|\left(1+\|R_{l'}\|\right)\left(3^d e^{-\gamma l}\right)^{\left(\frac{l'}{l}-\frac{3\kappa_l}{l}\right)} \tag{4.2}$$

Proof. Proposition 4.1 is essentially proved during the proof of Lemma 5.3 in [15], see in particular equation (5.7) there. We also observe that Lemma 5.2 (ii) in [15] extends to the case considered here with (5.1) of [15] replaced with

$$\left\|\chi_{l/3,x}R_{3l'}\chi_{l/3,y}\right\| \leq \left(3^d e^{-\gamma l}\right)^{\frac{3|x-y|}{l}-\frac{3\kappa_l}{l}}\|R_{3l'}\|. \tag{4.3}$$

□

As a Corollary we note:

Corollary 4.2 *If* $dist(E,\sigma(H_{\Lambda_{l'}})) \geq c(l')^{-\beta}$ *and* $dist(E,\sigma(H_{\Lambda_{3l'}})) \geq c(l')^{-\beta}$ *for a* $\beta \geq 0$ *then*

$$\left\|W(\varphi_{l'})R_{l'}(E+i\varepsilon)\chi_{l/3}\right\| \leq c\frac{(l')^{2d+2\beta}}{l^{2d}}(3^d e^{-\gamma l})^{\frac{l'}{l}-\frac{3\kappa_l}{l}}. \tag{4.4}$$

5 Multiscale Analysis (Probabilistic Part)

In this section we do the induction step crucial to multiscale analysis, i.e. we show: The probability, that $\Lambda_{l'}$ is not γ-super-good, is very small provided we know the same for the length scale l. For our purpose it will not be enough to require that Λ_l is γ-good. Because of the long range tails of f we have to require this property "uniformly" in the sense explained in Section 2.

Definition 5.1

$$G_l(\gamma) := \{\omega \,|\Lambda_l \textit{ is } \gamma\textit{-good}\}$$

We call Λ_l *"*γ*-super-good" for a certain* ω*, if* $\omega' \in G_l(\gamma)$ *for all* ω' *with* $\Pi_M\omega' = \Pi_M\omega$ *with* $M = \Lambda_{4l}$.

In essence, Λ_l is γ-super-good if Λ_l is γ-good independently of the values of the q_i with $\mathrm{dist}(i,\Lambda_l) > 3l$.

We start with:

Proposition 5.2 *Suppose that* $\mathbb{P}\left(\Lambda_l \textit{ is not } \gamma\textit{-super-good}\right) \leq \eta$*, then*

$$\mathbb{P}\left(\left\{\omega \,\middle|\, diam\ \Gamma_b^{l,l'} \geq 8l\right\}^*_{4l'}\right) \leq C\left(\frac{l'}{l}\right)^{2d}\eta^2.$$

Proof. Take $x, y \in \frac{l}{3}\mathbb{Z}^d \cap \Lambda_{l'}$ with $|x-y| > 4l$, i.e. such that $\Lambda_{4l}(x)\cap\Lambda_{4l}(y) = \emptyset$. Then the events $\{\Lambda_l(x) \text{ is not super-good}\}$ and $\{\Lambda_l(y) \text{ is not super-good}\}$ are independent due to Proposition 2.2.

Hence

$$\mathbb{P}\left(\exists\, \Lambda_l(x) \text{ and } \Lambda_l(y),\ |x-y| > 4l \text{ which both are not } \gamma\text{-super-good}\right)$$

$$\leq \quad C\left(\frac{l'}{l}\right)^{2d}\eta^2.$$

Consequently

$$\mathbb{P}\left(\left\{\omega \,\middle|\, \mathrm{diam}\ \Gamma_b^{l,l'} \geq 8l\right\}^*_{4l'}\right) \leq C\left(\frac{l'}{l}\right)^{2d}\eta^2$$

□

Proposition 5.3 *Let $E \in U$ as in Proposition 3.1 and $\beta > 0$. Then there exist constants c_1, c_2 and c such that if*

$$\mathbb{P}\left(\Lambda_l(x) \text{ is not } \gamma\text{-super-good}\right) \leq \eta$$

for some $\gamma > 0$, $\eta > 0$ and a multiple l of 3, then

$$\mathbb{P}\left(\Lambda_{l'}(x) \text{ is not } \gamma'\text{-super-good}\right) \leq \eta'$$

for every multiple l' of 3 with $l' > 24l$ and

$$\gamma' = \gamma(1 - \frac{24l}{l'}) - \frac{c_1}{l} - \frac{c_2 \log l'}{l'}, \tag{5.1}$$

$$\eta' = c\left((\frac{l'}{l})^{2d}\eta^2 + (l')^{2d-\beta} + (l')^{3d-m}\right). \tag{5.2}$$

Proof. Suppose that

$$\operatorname{diam} \Gamma_b^{l,l'} < 8l \tag{5.3}$$

and

$$\operatorname{dist}(E, \sigma(H_{\Lambda_{l'}})) \geq C(l')^{-\beta}, \quad \operatorname{dist}(E, \sigma(H_{\Lambda_{3l'}}) \geq c(l')^{-\beta} \tag{5.4}$$

then by Corollary 4.2

$$\begin{aligned}
&\left\|W(\varphi_{l'})R_{l'}\chi_{l/3}\right\| \\
&\leq C\left(\frac{l'}{l}\right)^{2d}(3^d e^{-\gamma l})^{\frac{l'}{l}-24}(l')^{2\beta} \\
&= C(l')^{2d(1-1/\alpha)+2\beta}(3^d)^{l'/l-24}e^{24\gamma l}e^{-\gamma l'} \\
&\leq \exp\left(-l'\left[\gamma(1-\frac{24l}{l'}) - \frac{d\log 3}{l} - \frac{(2d(1-1/\alpha)+2\beta)\log l'}{l'} - \frac{\log C}{l'}\right]\right).
\end{aligned}$$

We have

$$\mathbb{P}\left(\{\omega \,|\, (5.3) \text{ is not satisfied}\}^*_{4l'}\right) \leq c\left(\frac{l'}{l}\right)^{2d}\eta^2 \tag{5.5}$$

by Proposition 5.2 and

$$\begin{aligned}
&\mathbb{P}\left(\{\omega \,|\, (5.4) \text{ is wrong}\}^*_{4l'}\right) \\
&\leq \mathbb{P}\left(\left\{\omega \,\middle|\, \operatorname{dist}(E, \sigma(H_{\Lambda_{l'}})) < C(l')^{\beta}\right\}^*\right) + \mathbb{P}\left(\left\{\omega \,\middle|\, \operatorname{dist}(E, \sigma(H_{\Lambda_{3l'}})) < C(l')^{\beta}\right\}^*\right) \\
&\leq C\left((l')^{-\beta} + (l')^{-(m-d)}\right)(l')^{2d}
\end{aligned} \tag{5.6}$$

by Proposition 3.3. □

Below $[x]_3$ denotes the largest multiple of 3 which is smaller or equal to x.

Proposition 5.4 *Under the assumptions of Theorem 1.1 respectively Theorem 1.2 the following holds for small $\xi > 0$, $\alpha \in \left(1, \frac{2d+2\xi}{2d+\xi}\right)$, and E in a neighborhood U of $\inf\Sigma$ respectively $\partial\Sigma$: There exist $\ell^* > 0$ and $\gamma > 0$ such that*

$$\mathbb{P}\left(\Lambda_{l_n} \text{ is not } \gamma\text{-super-good}\right) \leq l_n^{-\xi} \quad \text{for } n = 0, 1, 2, \ldots,$$

where $l_0 \geq l^*$ is an arbitrary multiple of 3 and $l_n = \left[l_0^{(\alpha^n)}\right]_3$.

Proof. Exactly as in [15] one proves an initial scale estimate and uses a Combes-Thomas argument to get that for every $\xi > 0$ (respectively every $\xi \in (0, 2\tau - d)$ in the situation of Theorem 1.2)

$$\mathbb{P}\left(\Lambda_{l_0} \text{ is not } \gamma_0\text{-super-good}\right) \leq l_0^{-\xi} \tag{5.7}$$

for $l_0 \geq l^*$ and some $\gamma_0 > 0$. Note that here one directly gets "super-good" instead of merely "good". Applying Proposition 5.3 to $l_n := \left[l_0^{(\alpha^n)}\right]_3$ (with l^* sufficiently large) yields

$$\mathbb{P}\left(\Lambda_{l_n} \text{ is not } \gamma_n\text{-super-good}\right) \leq \eta_n$$

with

$$\begin{aligned} \gamma_{n+1} &= \gamma_n \left(1 - \frac{24 l_n}{l_{n+1}}\right) - \frac{c_1}{l_n} - \frac{c_2 \log l_{n+1}}{l_{n+1}}, \\ \eta_{n+1} &= c\left((\frac{l_{n+1}}{l_n})^{2d} \eta_n^2 + l_{n+1}^{2d-\beta} + l_{n+1}^{3d-m}\right). \end{aligned}$$

The first equation implies that $\gamma_n \geq \gamma > 0$ for all n, where $\sum l_n / l_{n+1} < \infty$ is used. From the second equation it is found inductively that $\eta_n \leq l_n^{-\xi}$ for all n for l^* sufficiently large:

(i) If $\xi > 0$ is sufficiently small then

$$c l_{n+1}^{3d-m} < \frac{1}{3} l_{n+1}^{-\xi}$$

since $m > 3d$,

(ii)

$$c\left(\frac{l_{n+1}}{l_n}\right)^{2d} \eta_n^2 \leq c l_{n+1}^{2d(1-1/\alpha) - 2\xi/\alpha} < \frac{1}{3} l_{n+1}^{-\xi}$$

since $\alpha < (2\xi + 2d)/(\xi + 2d)$,

(iii)

$$c l_{n+1}^{2d-\beta} \leq \frac{1}{3} l_{n+1}^{-\xi}$$

if the choice $\beta > 2d + \xi$ is made.

□

Under the assumptions of [2], [1] we can use the uniform Wegner estimate in the form (3.4) in the proofs of Propositions 5.3 and 5.4. The second and third term on the r.h.s. of (5.2) are replaced by $(l')^{d-\beta}$ resp. $(l')^{2d-m}$. From this it follows that Proposition 5.4 does now hold under the weaker assumption $m > 2d$.

6 Conclusion of the Proofs

By establishing Propositions 5.3 and 5.4 we have overcome the main obstacles in extending the arguments from [15] to the long range case considered here. Theorems 1.1 and 1.2 can now be proven by modifying Sections 6 and 8 of [15]. Thus we only sketch the main steps.

Localization will be a consequence of the following Simon-Wolff-type result:

Theorem 6.1 *Let $r > 2d$. Under the assumptions of Theorem 1.1 there exists a neighborhood U of $\inf \Sigma$ and under the assumptions of Theorem 1.2 there exists a neighborhood U of $\partial\Sigma$ such that the following holds with probability one:*

(a) For almost every $E \in U$

$$\sup_{\varepsilon \neq 0} \|R(E + i\varepsilon)\langle \cdot \rangle^{-r}\| < \infty.$$

(b) For every $D > 2d$ and almost every $E \in U$ there exists C such that

$$\sup_{\varepsilon \neq 0} \|\chi_x R(E + i\varepsilon)\langle \cdot \rangle^{-r}\| \leq C\langle x \rangle^{D-r}$$

for every $x \in \mathbb{Z}^d$. Here $\chi_x := \chi_{\Lambda_1(x)}$.

In Section 6 of [15] a corresponding result is proven for a compactly supported function replacing $\langle \cdot \rangle^{-r}$. One ingredient to extend the argument in [15] to the above situation is

Proposition 6.2 *Let $H = -\Delta + V$ where V_- is in the Kato class and V_+ is locally in the Kato class, and let M_g be multiplication by $g \in L^2(\mathbb{R}^d)$. Then there exists a subset $S_0 \subset \mathbb{R}$ with $|\mathbb{R} \setminus S_0| = 0$ such that*

$$\sup_{\varepsilon \neq 0} \|\chi_A (H - E - i\varepsilon)^{-1} M_g\| < \infty$$

for every $E \in S_0$ and every compact subset of $\mathbb{R}^d$.

The proof of Proposition 6.2 is a direct extension of the proof of Lemma 3 in [19] where g is assumed to be a characteristic function of a compact set.

We will also use the following estimates on localized resolvents of finite box operators:

Proposition 6.3 *Let U be the neighborhood of $\inf \Sigma$ respectively $\partial\Sigma$ found in Proposition 5.4 and $E \in U$ be fixed. With l^*, l_k, $\gamma > 0$ and $\xi > 0$ as in Proposition 5.4 let $\Psi_k := \chi_{3l_k} - \chi_{3l_{k-1}}$. Then, if l^* is sufficiently large, with probability $\mathbb{P} \geq 1 - l_k^{-\xi}$ it holds that*

$$\sup_{\varepsilon \neq 0} \|\Psi_k R_{9l_k}(E + i\varepsilon) W(\varphi_{9l_k})\| \leq e^{-\frac{9}{2}\gamma l_k}, \tag{6.1}$$

and

$$\sup_{\varepsilon \neq 0} \|\Psi_k R_{9l_k}(E + i\varepsilon) \chi_{\Lambda_{k-1}/3}\| \leq C l_k^{3d-d/\alpha+\xi} e^{-5\gamma l_{k-1}}. \tag{6.2}$$

Proposition 6.3 and its proof are almost identical to Lemma 6.3 of [15]. The proof uses Propositions 5.3 and 5.4 above as well as Lemma 5.2 of [15] (where the latter remains true in the long range case).

We can now prove Theorem 6.1. The estimates (6.1) and (6.2) hold with $\mathbb{P} \geq 1 - l_k^{-\xi}$. In addition, the Wegner estimate (3.1) implies that

$$\|R_{9l_k}\| \leq C l_k^{\xi+2d} \tag{6.3}$$

with $\mathbb{P} \geq 1 - c l_k^{-\xi}$. Since $\sum_k l_k^{-\xi} < \infty$ we conclude from the Borel Cantelli Lemma that for every fixed $E \in U$ (U as in Proposition 6.3) and almost every ω there exists $k_0 = k_0(\omega)$ such that

(6.1), (6.2) and (6.3) hold for every $k \geq k_0$. We now estimate as in the proof of Theorem 6.1(a) of [15] (with $Q(x) := \langle x \rangle^{-r}$)

$$\begin{aligned} & \|(1-\chi_{3l_{k_0}})R(E+i\varepsilon)Qf\|^2 \\ & \leq \; 2\sum_{k\geq k_0}\Big(\|\Psi_k R_{9l_k}\varphi_{9l_k}Q\|^2\|f\|^2 \\ & \qquad + \|\Psi_k R_{9l_k}(E+i\varepsilon)W(\varphi_{9l_k})\|^2\,\|\Psi_{k+1}R(E+i\varepsilon)Qf\|^2\Big). \end{aligned} \tag{6.4}$$

With $Q_k := Q|_{\Lambda_{l_{k-1}/3}}$ we estimate

$$\begin{aligned} \|\Psi_k R_{9l_k}\varphi_{9l_k}Q\| &\leq \|\Psi_k R_{9l_k}Q_k\| + \|R_{9l_k}\|\|Q-Q_k\| \\ &\leq Cl_k^{3d-d/\alpha+\xi}e^{-5\gamma l_{k-1}} + Cl_{k-1}^{\alpha(\xi+2d)-r}, \end{aligned} \tag{6.5}$$

where (6.2) and (6.3) are used. With $\xi > 0$ sufficiently close to 0 and $\alpha \in \left(1, \frac{2d+2\xi}{2d+\xi}\right)$ sufficiently close to 1 the r.h.s. of (6.4) is summable in k. Using this observation and (6.4), the proof of Theorem 6.1(a) can now be completed as in [15]. (Note that Proposition 6.2 applies with $g = Q$.)

For an $E \in U$ sucht (a) holds let $x \in \mathbb{Z}^d$ and choose k such that $x \in \Lambda_{3l_k \setminus 3l_{k-1}}$. Then

$$\|\chi_x RQ\| \leq \|\chi_x R_{9l_k}\varphi_{9l_k}Q\| + \|\chi_x R_{9l_k}W(\varphi_{9l_k})RQ\|. \tag{6.6}$$

Using (6.1) and part (a) the second term on the r.h.s. of (6.6) can be estimated by $e^{-\frac{3}{2}\gamma|x|}$ with $\mathbb{P} \geq 1 - l_k^{-\xi}$. For the first term on the r.h.s. of (6.6) one has

$$\begin{aligned} \|\chi_x R_{9l_k}\varphi_{9l_k}Q\| &\leq \|\chi_x R_{9l_k}Q_k\| + \|R_{9l_k}\|\,\|Q-Q_k\| \\ &\leq e^{-\gamma'|x|} + Cl_k^{(\xi+2d)-r/\alpha} \end{aligned}$$

for $0 < \gamma' < \gamma/3$ and $|x|$ sufficiently large with $\mathbb{P} \geq 1 - cl_k^{-\xi}$. In the last step $\|\chi_x R_{9l_k}Q_k\|$ was estimated as in (6.9) of [15] and $\|R_{9l_k}\|\,\|Q-Q_k\|$ was estimated as in (6.5) above. Since $D > 2d$ one has $\xi + 2d - r/\alpha < D - r$ if ξ is sufficiently close to 0 and α sufficiently close to 1. It follows that $Cl_k^{(\xi+2d)-r/\alpha} \leq C\langle x \rangle^{D-r}$. Inserting in (6.6) we arrive at

$$\|\chi_x RQ\| \leq C\langle x \rangle^{D-r}$$

for x sufficiently large with $\mathbb{P} \geq 1 - Cl_k^{-\xi}$. From here the proof of (b) is completed as the proof of Theorem 6.1(b) in [15]. □

The above proof of Theorem 6.1 also shows that one may replace $\langle \cdot \rangle^{-r}$ by a compactly supported function Q and then gets for almost every $E \in U$ the existence of C, $\gamma > 0$ such that

$$\sup_{\varepsilon \neq 0}\|\chi_x R(E+i\varepsilon)Q\| \leq Ce^{-\gamma|x|} \tag{6.7}$$

for all $x \in \mathbb{Z}^d$.

The proof of Theorems 1.1 and 1.2 can now be completed by arguments similar to those in Section 8 of [15]. For U as in Theorem 6.1 and $\omega \in \Omega$ let

$$\begin{aligned} \tilde{S_\omega} \;:=\; & \{E \in U : \text{ for all } y \in \mathbb{Z}^d \text{ there exist } C, \gamma > 0 \text{ such that} \\ & \sup_{\varepsilon \neq 0}\|\chi_x R(E+i\varepsilon,\omega)\chi_y\| \leq Ce^{-\gamma|x|} \text{ for all } x \in \mathbb{Z}^d\} \end{aligned}$$

and for $s > d/2$ such that $m - s > 2d$ and $n = 0, 1, 2, \ldots$ let

$$\begin{aligned} S_\omega^n &:= \{E \in U : \text{ there exists } C \text{ such that} \\ &\quad \sup_{\varepsilon \neq 0} \|\chi_x R(E + i\varepsilon, \omega) \langle \cdot \rangle^{-m+s-n}\| \leq C \langle x \rangle^{s-n-1} \text{ for all } x \in \mathbb{Z}^d\}. \end{aligned}$$

Finally, let $S_\omega := (\tilde{S}_\omega \cap \bigcap_{n=0}^\infty S_\omega^n) \setminus \{\text{eigenvalues of } H(\omega)\}$. If follows from Theorem 6.1 (with $D = m - 1$ and (6.7) (compare [15]) that $|U \setminus S_\omega| = 0$ with probability one. If $E \in S_\omega$ and $\varphi \in H^1_{-s}$ is a weak solution of $(H(\omega) + \lambda f)\varphi = E\varphi$, then the arguments in the proof of Lemma 8.1 of [15] can be repeated to yield

$$\|\chi_y \varphi\| \leq |\lambda| \sup_{\varepsilon \neq 0} \|\chi_y R(E + i\varepsilon) f \varphi\|. \tag{6.8}$$

If B is a ball such that $f(x) \leq C\langle x \rangle^{-m}$ for $x \notin B$, then we can estimate (6.8) further by

$$\begin{aligned} & C \sup_{\varepsilon \neq 0} \|\chi_y R(E + i\varepsilon) \langle \cdot \rangle^{-m+s}\| \, \|\langle \cdot \rangle^{-s} \varphi\| \\ & \quad + C \sup_{\varepsilon \neq 0} \|\chi_y R(E + i\varepsilon) \chi_B\| \, \|\chi_B f \varphi\| \\ \leq \; & C \langle y \rangle^{s-1}, \end{aligned}$$

using that $E \in \tilde{S}_\omega \cap S_\omega^0$ and that $\varphi \in L^\infty_{loc}$ by a subsolution estimate. The latter now also yields $|\varphi(y)| \leq C\langle y \rangle^{s-1}$. Now we can apply a boot strap procedure by inserting this estimate into the r.h.s. of (6.8) and successively winning an additional factor $\langle y \rangle^{-1}$ in the estimate for $|\varphi(y)|$. Thus φ decays more rapidly than every inverse polynomial. ¿From here on the proof of Theorems 1.1 and 1.2 is completed as in Section 8 of [15].

We finally note that Theorem 6.1 can be shown to hold with $r > d$ and $D > d$ if the improved Wegner estimate from [2, 1] is used in its proof. Only $m > 2d$ has to be assumed in this case since Proposition 5.4 now holds under this assumption. This allows to complete the proof of Theorems 1.1 and 1.2 under the weaker assumption $m > 2d$.

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