On the validity of van der Waals theory of surface tension

O. Benois [1], T. Bodineau [2], P. Buttà [3] and E. Presutti [3]

École Normale Supérieure, Université de Rouen, Università di Roma Tor Vergata

Abstract. In this paper we prove a weak large deviation principle for the empirical distribution of Ising spins in $d \geq 2$ dimensions when the interaction is determined by a Kac potential and the temperature is below the critical value. We prove that the rate function is proportional to the area of the interface by a factor which identifies the surface tension. Its value is the same as that predicted by the van der Waals theory.

1. Introduction.

Lebowitz and Penrose have proved in [17] that in systems with Kac potentials the thermodynamic potentials, like the free energy and the pressure, converge, in an appropriate limit, to the values predicted by the van der Waals theory. In this paper we continue their program, shifting from the bulk to the surface properties of the system and proving the analogous statement for the surface tension. As in [2] we start by relating the surface tension to the rate function of large deviations and the main point will be the proof of a weak large deviation principle.

We consider in the beginning a general Ising system in $d \geq 2$ dimensions at inverse temperature β, supposing that it has two extremal, translationally invariant Gibbs measures, $\mu_\pm$ with magnetizations $\pm m_\beta$, $m_\beta > 0$. Let μ_Λ be the Gibbs measure on the torus Λ of $\mathbb{Z}^d$ of side L (for simplicity we take $L \in \{2^n,\, n \in \mathbb{N}\}$). Then, for large L, μ_Λ is well approximated by the 1/2-1/2 combination of μ_+ and μ_- so that we may suppose that with large μ_Λ-probability the configurations in Λ have empirical magnetizations either close to $+m_\beta$ or to $-m_\beta$. The probability of observing configurations where the magnetization varies across a surface Σ from $-m_\beta$ to $+m_\beta$, namely the probability of observing $m_\beta u$,

[1] URA 1378, Université de Rouen, 76821 Mont Saint Aignan, France

[2] DMI, École Normale Supérieure, 45 Rue d' Ulm, 75005 Paris, France

[3] Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy

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where u is the function equal to $+1$ and to -1 in the two regions, separated by Σ, where the phases are respectively $+m_\beta$ and $-m_\beta$, is then expected to be

$$\mu_\Lambda\Big(\{\sigma \approx m_\beta u\}\Big) \approx e^{-\beta\delta F} \tag{1.1}$$

where δF is the excess free energy due to the interface at Σ. In an isotropic case this would be

$$\delta F = |\Sigma|\tau_\beta \tag{1.2}$$

where τ_β is the surface tension (the dependence on β is made here explicit). In general $\tau_\beta = \tau_\beta(n)$, $n = n(r)$ the unit vector normal to Σ, and

$$\delta F = \int_\Sigma \lambda(dr)\tau_\beta\big(n(r)\big) \tag{1.3}$$

where $\lambda(dr)$ is the Hausdorff measure on Σ. If the diameter of Σ is proportional to L, the right hand side of (1.1) vanishes exponentially with exponent proportional to L^{d-1}, so that the analysis of the left hand side of (1.1) becomes a large deviation problem with the scaling factor L^{d-1} (and not the usual volume factor L^d, this is a feature related to the analysis of large deviations in the presence of a phase transition).

A mathematical definition based on the above ideas has been proposed in [2] and will be adopted here too. In macroscopic coordinates the torus of side L in $\mathbb{Z}^d$ becomes $\mathcal{T}$, the unit torus in $\mathbb{R}^d$ with center the origin. We identify a configuration σ in the original torus of $\mathbb{Z}^d$ of side L with a function $s \in L^\infty(\mathcal{T};\{\pm 1\})$ by setting $s(r) = \sigma(i)$, if Lr is in the unit cube of center i. In macroscopic coordinates a spin configuration is a very rapidly oscillating function, it is then natural to introduce a homogenization procedure which leads to the coarse grained configuration $s^{(\varepsilon)} \in L^\infty(\mathcal{T};[-1,1])$ (for simplicity we take $\varepsilon \in \{2^{-n},\, n \in \mathbb{N}\}$) defined by

$$s^{(\varepsilon)}(r) = \frac{1}{|C^{(\varepsilon)}(r)|}\int_{C^{(\varepsilon)}(r)} dr'\, s(r') \tag{1.4}$$

where $C^{(\varepsilon)}(r)$ is the atom containing r in the partition $\mathcal{D}^{(\varepsilon)}$ of $\mathbb{R}^d$ made of cubes of side ε and such that one of its atoms has center 0. $|C^{(\varepsilon)}(r)|$ is the volume of $C^{(\varepsilon)}(r)$. ε is a small parameter that eventually vanishes, but only after $L \to +\infty$, thus ε should be thought of as very large with respect to the lattice mesh L^{-1}.

The coarse grained configuration is used to locate the phases: we will say that with accuracy δ and coarse graining ε the cube $C^{(\varepsilon)}(r)$ is in the $+$ phase if $|s^{(\varepsilon)}(r) - m_\beta| < \delta$, it is in the $-$ phase if $|s^{(\varepsilon)}(r) + m_\beta| < \delta$. This is summarized by the variable $\eta_\delta^{(\varepsilon)} \in L^\infty(\mathcal{T};\{0,\pm 1\})$ defined as

$$\eta_\delta^{(\varepsilon)}(r) = \begin{cases} \pm 1 & \text{if } \big|s^{(\varepsilon)}(r') \mp m_\beta\big| < \delta \quad \text{for all } r' \in C^{(\varepsilon)}(r) \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

Then we will say that a macroscopic profile $u \in L^\infty(\mathcal{T};\{\pm 1\})$ is recognized with accuracy $\delta > 0$ and coarse graining ε by a spin configuration σ if

$$\|\eta_\delta^{(\varepsilon)} - u\| \le \delta \tag{1.6}$$

where $\|\cdot\|$ denotes hereafter the $L^1(\mathcal{T};dr)$-norm. We are using the same δ in (1.5) and (1.6) only for notational simplicity. (1.6) with δ small implies that for a large fraction of points σ recognizes the $\pm$ phase specified by u, namely the empirical average of σ that produces $s^{(\varepsilon)}$ is close to $\pm m_\beta$.

Recalling (1.1) and (1.2), the quantity

$$-\frac{1}{\beta L^{d-1}} \log \mu_\Lambda\Big(\{\|\eta_\delta^{(\varepsilon)} - u\| \le \delta\}\Big)$$

represents the excess free energy needed to create an interface described by u. To make precise this notion we restrict $u \in BV(\mathcal{T};\{\pm 1\})$ and let ∂u be the boundary of $\{u(r) = 1\}$. It is proved in [12] that there is a subset $\partial^\star u$ of ∂u, called the reduced boundary of u, which is a regular, C^1 surface with a well defined unit normal n and full Hausdorff measure λ. We can then pretend that, measure theoretically, ∂u is a regular surface with an area.

We next define the upper and lower functional for large deviations on $BV(\mathcal{T};\{\pm 1\})$ by setting

$$F'_\beta(u) := -\lim_{\delta\to 0^+} \liminf_{\varepsilon\to 0^+} \liminf_{L\to+\infty} \frac{1}{\beta L^{d-1}} \log \mu_\Lambda\Big(\{\|\eta_\delta^{(\varepsilon)} - u\| \le \delta\}\Big) \tag{1.7}$$

and

$$F''_\beta(u) := -\lim_{\delta\to 0^+} \limsup_{\varepsilon\to 0^+} \limsup_{L\to+\infty} \frac{1}{\beta L^{d-1}} \log \mu_\Lambda\Big(\{\|\eta_\delta^{(\varepsilon)} - u\| \le \delta\}\Big) \tag{1.8}$$

A weak large deviation principle holds if $F''_\beta(u) = F'_\beta(u) =: F_\beta(u)$ for all $u \in BV(\mathcal{T};\{\pm 1\})$. The surface tension $\tau_\beta(n)$ is well defined if for all such u

$$F_\beta(u) = \int_{\partial^\star u} \lambda(dr)\tau_\beta\big(n(r)\big) \tag{1.9}$$

If we understand correctly the literature, the existing results, [10], [18], [15] [16], [19], on the Ising model with nearest neighbor interactions in $d = 2$ dimensions imply the validity of (1.7), (1.8) and (1.9).

The above definitions need to be modified when studying the Kac potentials because the range of the interaction is long and eventually diverges, the interface being correspondingly thick. We first recall the basic definitions of Ising systems with Kac potentials. $\mathcal{X} := \{-1,1\}^{\mathbb{Z}^d}$ denotes the space of all the spin configurations σ and for any subset Λ of $\mathbb{Z}^d$, $\sigma_\Lambda \in \mathcal{X}_\Lambda = \{-1,1\}^\Lambda$ is the restriction of σ to Λ. $\gamma > 0$ is the Kac parameter (for simplicity, $\gamma \in \{2^{-n},\, n \in \mathbb{N}\}$), $\mu_{\gamma,\Lambda}(\cdot|\sigma_{\Lambda^c})$ is the Gibbs distribution in the finite region Λ with boundary conditions σ_{Λ^c}, Λ^c the complement of Λ:

$$\mu_{\gamma,\Lambda}(\sigma_\Lambda|\sigma_{\Lambda^c}) = Z_{\gamma,\Lambda}(\sigma_{\Lambda^c})^{-1} \exp\big[-\beta H_\gamma(\sigma_\Lambda|\sigma_{\Lambda^c})\big] \tag{1.10}$$

$Z_{\gamma,\Lambda}(\sigma_{\Lambda^c})$ is the partition function,

$$H_\gamma(\sigma_\Lambda|\sigma_{\Lambda^c}) = -\frac{1}{2}\sum_{i\neq j\in\Lambda} J_\gamma(i,j)\sigma(i)\sigma(j) - \sum_{i\in\Lambda, j\in\Lambda^c} J_\gamma(i,j)\sigma(i)\sigma(j) \tag{1.11}$$

$$J_\gamma(i,j) := \gamma^d J(\gamma|i-j|), \qquad \forall\, i,j\in\mathbb{Z}^d \tag{1.12}$$

J is a nonnegative, smooth function supported by $[0,1]$ and normalized so that

$$\int_{\mathbb{R}^d} dr\, J(|r|) = 1 \tag{1.13}$$

In [6] and [4] it is shown that if $\beta > 1$ there is $\gamma_\beta > 0$ so that for all $\gamma \le \gamma_\beta$ there are two distinct translationally invariant Gibbs states $\mu_\gamma^\pm$, limits of the finite volume Gibbs states with all $+1$ and, respectively, all -1 boundary conditions. In [5] it is shown that these are the only extremal, translationally invariant Gibbs states. Moreover their magnetizations, $\pm m_{\beta,\gamma}$, converge when $\gamma\to 0^+$ to the van der Waals values $\pm m_\beta$ (see (1.19) below).

As before we consider the Gibbs measure on a torus Λ of $\mathbb{Z}^d$ of side L and denote it for simplicity by $\mu_{\gamma,\Lambda}$. We define the lower functional for Kac potentials on $BV(\mathcal{T};\{\pm 1\})$ by setting

$$F'_{\beta,\gamma}(u) := -\lim_{\delta\to 0^+}\liminf_{\varepsilon\to 0^+}\liminf_{L\to+\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,\Lambda}\Big(\{\|\eta_\delta^{(\varepsilon)} - u\| \le \delta\}\Big) \tag{1.14}$$

The upper functional $F''_{\beta,\gamma}(u)$ is defined analogously. The factor $\gamma^{-1}L^{d-1}$ represents the scaling factor of the volume (in lattice units) of a neighborhood of the interface of thickness γ^{-1}, which by (1.12) is the range of the interaction.

In the simultaneous limit $L\to+\infty$ and $\gamma\to 0^+$ (actually $L=\gamma^{-1-a}$, $a>0$ and small) it is proved, [2], (see [1] for an extension to non isotropic interactions) that the limit upper and lower functionals, $F'_\beta(u)$ and $F''_\beta(u)$, are identical:

$$F'_\beta(u) = F''_\beta(u) = \tau_\beta P(u) \tag{1.15}$$

where $P(u)$ is the Hausdorff area of $\partial^\star u$ and the limit surface tension, τ_β, is found to have the value predicted by the van der Waals theory, as we briefly recall. We start from the excess free energy van der Waals functional $\mathcal{F}_\Lambda(m)$, Λ a measurable set in $\mathbb{R}^d$ (or in a torus), $m\in L^\infty(\Lambda,[-1,1])$,

$$\mathcal{F}_\Lambda(m) = \int_\Lambda dx\,\big[f(m(x)) - f(m_\beta)\big] + \frac{1}{4}\iint_{\Lambda\times\Lambda} dx\,dy\,J(|x-y|)\big[m(x)-m(y)\big]^2 \tag{1.16}$$

$$f(m) = -\frac{m^2}{2} - \beta^{-1} i(m) \tag{1.17}$$

$$i(m) = -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2} \tag{1.18}$$

$$m_\beta = \tanh\{\beta m_\beta\} \tag{1.19}$$

The convex envelope of $f(m)$ is the limit free energy of the Ising system after first the thermodynamic limit and then the scaling limit $\gamma \to 0^+$, [17].

Take now Λ in (1.16) to be the cylinder with axis the first coordinate r_1-axis and section the unit torus in $\mathbb{R}^{d-1}$. Denote by M_Λ the set of $m \in L^\infty(\Lambda, [-1,1])$ such that $\lim_{r_1\to\pm\infty} m(r) = \pm m_\beta$. Then, according to the van der Waals theory the surface tension is

$$\tau_\beta = \inf_{m\in M_\Lambda} \mathcal{F}_\Lambda(m) \tag{1.20}$$

τ_β in (1.20) is the same as τ_β in (1.15). In [9] it is also proved that the inf in (1.20) is a minimum attained on a unique (modulo translations) function $\bar{m}$ which depends only on the coordinate r_1 and which satisfies the mean field equation

$$\bar{m} = \tanh\{\beta J \star \bar{m}\} \tag{1.21}$$

where $J \star \bar{m}$ is the convolution of the two functions J and $\bar{m}$. Under the assumption $L = \gamma^{-1-a}$ of [2], one can exploit the structure of the Kac potentials and prove that the probability of a coarse grained configuration $s^{(\varepsilon)}$ is ($L_\gamma := \gamma L$ below)

$$\mu_{\gamma,\Lambda}(s^{(\varepsilon)}) \approx \exp\Big\{ -\gamma^{-d}\beta\mathcal{F}_{L_\gamma\mathcal{T}}\big(s^{(\varepsilon)}(L_\gamma\cdot)\big)\Big\} \tag{1.22}$$

The approximation is strong enough to yield

$$F'_\beta(u) := -\lim_{\delta\to 0^+}\limsup_{L_\gamma\to+\infty}\frac{1}{L_\gamma^{d-1}}\inf_{\{\|m-m_\beta u\|\le\delta\}}\mathcal{F}_{L_\gamma\mathcal{T}}(m(L_\gamma\cdot)) \tag{1.23}$$

where the inf is over all $m \in L^1(\mathcal{T};[-1,1])$. An analogous expression holds for the upper bound.

To make easier the comparison with [2], we call $L_\gamma = \varepsilon^{-1}$, ε here is not the same ε as in (1.22), observe also that γ in (1.23) appears only through L_γ. Then $-F'_\beta$ and $-F''_\beta$ are the Γ-upper and lower limits, [7], as $\varepsilon \to 0^+$, of the family of functionals $\{F^{(\varepsilon)}\}$ defined on $L^1(\mathcal{T},[-1,1])$ as

$$F^{(\varepsilon)}(u) = -\varepsilon^{d-1}\mathcal{F}_{\varepsilon^{-1}\mathcal{T}}\big(m_\beta u(\varepsilon^{-1}\cdot)\big) \tag{1.24}$$

In [2] and [3] it is proved that $\{F^{(\varepsilon)}\}$ Γ-converges to $\tau_\beta P(\cdot)$, where $P(u)$ is the perimeter of u if $u \in BV(\mathcal{T};\{\pm 1\})$ (more precisely the perimeter of the set $\{u=1\}$) and $P(u) = +\infty$ otherwise.

The fact that in (1.14) $L \to +\infty$ with $\gamma > 0$ fixed, prevents from using (1.22) which is no longer valid. But, as we shall see, after suitable conditioning we will reduce to computing probabilities of events in regions for which (1.22), or suitable modifications, may be applied. In this way we will recover parts of the proofs in [2] that will be used in the proof of the following main result of this paper:

Theorem 1.1

There are positive functions $\tau'_{\beta,\gamma}$ and $\tau''_{\beta,\gamma}$ such that for all $u \in BV(\mathcal{T};\{\pm 1\})$

$$F'_{\beta,\gamma}(u) \le \tau'_{\beta,\gamma} P(u), \qquad F''_{\beta,\gamma}(u) \ge \tau''_{\beta,\gamma} P(u) \tag{1.25}$$

and

$$\lim_{\gamma\to 0^+} \tau'_{\beta,\gamma} = \lim_{\gamma\to 0^+} \tau''_{\beta,\gamma} = \tau_\beta \tag{1.26}$$

The bounds in (1.25) must be strict as for $\gamma > 0$ the system is anisotropic and the surface tension depends on the orientation of the surface. The true theorem that we can only conjecture is that

$$F'_{\beta,\gamma}(u) = F''_{\beta,\gamma}(u) = \int_{\partial^\star u} \lambda(dr)\tau_{\beta,\gamma}\big(n(r)\big) \tag{1.27}$$

which, by our theorem, would imply that $\tau_{\beta,\gamma} \to \tau_\beta$ as $\gamma \to 0^+$.

In the next section we will introduce the main notation and define the different scales at which the system will be studied. Section 3 is devoted to the analysis of the typical configurations of the periodic Gibbs state. We prove that for any γ sufficiently small the macroscopic magnetization is close to the equilibrium value $m_{\beta,\gamma}$ (respectively $-m_{\beta,\gamma}$) with $\mu_{\gamma,\Lambda}$-probability 1/2.

Theorem 1.2

For any γ and $\delta > 0$ small enough the following holds. For any $\varepsilon > 0$ small enough,

$$\lim_{L\to+\infty} \mu_{\gamma,\Lambda}\Big(\|s^{(\varepsilon)} - m_{\beta,\gamma}\| \le \delta\Big) = \frac{1}{2}$$

Finally we derive Theorem 1.1 by computing the limit as γ goes to 0 of the lower and upper functionals $F'_{\beta,\gamma}(\cdot)$ and $F''_{\beta,\gamma}(\cdot)$, respectively in Sections 4 and 5. In an Appendix we report the statement on the Peierls estimates proved in [6] and the proofs of some technical lemmas that have been used earlier.

2. Basic notation and definitions.

Notations are particularly troublesome in this paper because we have three different scales and according to the case it is better to work with one or the other. We thus have basically the same object but with three different representations and this may be confusing, as it is even to these authors. For this reason we have decided to devote the present section to this issue.

The microscopic and the macroscopic levels

The basic space is the "microscopic space", i.e. the lattice $\mathbb{Z}^d$ whose elements are denoted by i, j and so on. We actually restrict to tori Λ of $\mathbb{Z}^d$, $d \geq 2$, of center 0 and side L and, for simplicity we take $L = 2^n$, $n \in \mathbb{N}$, we will prove Theorem 1.1 in this context.

The macroscopic region corresponding to any of the tori Λ is always the same unit torus $\mathcal{T}$ of $\mathbb{R}^d$ with center 0. We usually denote by r its points. The configuration σ_Λ in $\mathcal{X}_\Lambda$ is mapped into a function $s \in L^\infty(\mathcal{T}; \{\pm 1\})$, by

$$s(r) = \sigma(i), \qquad Lr \in C(i) \tag{2.1}$$

where $C(i)$ is the unit cube of $\mathbb{R}^d$ with center i.

Before going to the mesoscopic level it is convenient to introduce notation for the partitions of $\mathbb{R}^d$. We will denote by $\mathcal{D}^{(\ell)} = \{C^{(\ell)}\}$, $\ell > 0$, the partition of $\mathbb{R}^d$ into cubes of side ℓ with one of them having center 0. It is not important for the sequel to specify which one of the opposite faces of a cube should be taken off to make the different $C^{(\ell)}$ disjoint. We drop ℓ when $\ell = 1$ and we also use the convention that $C^{(\ell)}(r)$ is the cube of $\mathcal{D}^{(\ell)}$ that contains r.

The mesoscopic level: block spins and coarse graining

The mesoscopic level is determined by the range of the interaction. Points in $\mathbb{R}^d$ are here denoted by x, y and so on. At the mesoscopic level the spin configuration $S \in L^\infty(L_\gamma \mathcal{T}; \{\pm 1\})$ is

$$S(x) = s(L_\gamma^{-1} x), \quad L_\gamma = \gamma L, \qquad S(x) = \sigma(i), \quad \gamma^{-1} x \in C(i) \tag{2.2}$$

The block spins are better defined at this level. The coarse grain size is 2^{-k}, $k \in \mathbb{N}$, and the coarse grained spin configuration $S^{(2^{-k})} \in L^\infty(L_\gamma \mathcal{T}; [-1,1])$ is

$$S^{(2^{-k})}(x) = \frac{1}{|C^{(2^{-k})}(x)|} \int_{C^{(2^{-k})}(x)} dy\, S(y) \tag{2.3}$$

Given k and h in $\mathbb{N}$, $\zeta > 0$, we then define the block spin $\eta \in L^\infty(\gamma\Lambda; \{0, \pm 1\})$ as

$$\eta(x) = \begin{cases} \pm 1 & \text{if } \left| S^{(2^{-k})}(y) \mp m_\beta \right| < \zeta \quad \text{for all } y \in C^{(2^h)}(x) \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

This is the same definition as in [6] and [5] but employing mesoscopic coordinates. As in those papers k and h are "large" and ζ "small". We will also write, with an abuse of notation, $\eta_i = \eta(x)$ if $\gamma^{-1} x \in C(i)$. We also define the block spin $\eta(x)$ induced by a function $m \in L^\infty(L_\gamma \mathcal{T}; [-1,1])$ using the analogue of (2.4).

Observe that we are using the same symbol η (without subscript) with a different meaning than in the introduction. No ambiguity should arise if the reader recalls that in the introduction η had a subscript, which is here missing and this, if not the same context, will allow to distinguish the two. Using the same notation is not actually inconsistent since $\eta(x) = \pm 1$ means that the $\pm$ phase is recognized with accuracy ζ (in the sense of the introduction, but working on the mesoscopic scale) in each cube $C^{(2^{-k})}$ contained in $C^{(2^h)}(x)$.

Islands and contours

The point x is correct, or, equivalently, $\eta(x)$ is correct, if $\eta(x) \neq 0$ and $\eta(y) = \eta(x)$ on the cubes $C^{(2^h)}$ that are $\star$-connected to $C^{(2^h)}(x)$. x is incorrect if it is not correct.

The maximal connected components of the correct set are called islands. In an island $\eta(x)$ is constantly equal to 1 or to -1, accordingly the island is called a $\pm$ island. The boundary ∂I of an island I is the set of cubes $C^{(2^h)}$ not in I but at distance 0 from I. $\eta = \pm 1$ on the boundary ∂I of a $\pm$ island.

Each maximal $\star$-connected component of the incorrect set is the support of a contour, the contour Γ is defined by its support and by the values of the block spins on its support. When there is no risk of confusion, we may denote by Γ only its support. The boundary $\partial\Gamma$ of the contour Γ is the union of $\partial I \cap \Gamma$ over the islands, the $\pm$ boundary, $\partial\Gamma^{\pm}$, is the union of $\partial I \cap \Gamma$ over the $\pm$ islands I.

On the torus there is a special class of contours, the "winding" contours: a contour in the torus $L_\gamma \mathcal{T}$ is winding if its extension to $\mathbb{R}^d$ has an unbounded connected component.

A collection of contours $\{\Gamma_1, \dots, \Gamma_\kappa\}$ is *compatible* if it is produced by a spin configuration and we write $\sigma \Rightarrow \{\Gamma_1, \dots, \Gamma_\kappa\}$. We also write $m \Rightarrow \{\Gamma_1, \dots, \Gamma_\kappa\}$, $m \in L^\infty(L_\gamma \mathcal{T}; [-1, 1])$, if m induces $\{\Gamma_1, \dots, \Gamma_\kappa\}$. A collection $\{\Gamma_1, \dots, \Gamma_\kappa\}$ of compatible contours is *admissible* if there is no connected path of cubes from some $\partial\Gamma_i^+$ to some $\partial\Gamma_j^-$ in the complement of the supports of $\Gamma_1, \dots, \Gamma_\kappa$. The notion of an admissible collection of contours is relevant because in this case we can apply the Peierls estimates, see the Appendix and in particular Theorem 6.1. Notice that a non winding contour is admissible, which is not necessarily true for a winding one.

3. Typical configurations.

In this section we prove Theorem 1.2. We will be mainly working at the mesoscopic level, considering the torus $L_\gamma \mathcal{T}$ of side $L_\gamma = \gamma L$. We define the cornice

$$\Delta \doteq Q_{L_\gamma/3+(\log L_\gamma)^2} \setminus Q_{L_\gamma/3-(\log L_\gamma)^2} \tag{3.1}$$

where Q_R denotes the closed cube in $L_\gamma \mathcal{T}$ of side $2R$ and center in the origin. We define here a $\pm$ circuit $\mathcal{C}$ as a connected, $\mathcal{D}^{(2^h)}$-measurable set where η is constantly equal to ± 1. We also require that the complement of $\mathcal{C}$ has two connected components. We then call $K^{\pm}$ the set of configurations σ such that there is a $\pm$ circuit $\mathcal{C}^{\pm}$ inside Δ such that the two connected components of its complement contain $Q_{L_\gamma/3-(\log L_\gamma)^2}$ and $L_\gamma\mathcal{T}\setminus Q_{L_\gamma/3+(\log L_\gamma)^2}$ respectively.

If $\sigma \notin K^-\cup K^+$ there is a contour connecting the two boundaries of Δ, thus we can bound the probability of the complement of $K^-\cup K^+$ by computing the probability of a contour that connects the two boundaries. By the Peierls estimates proved in the Appendix, see Lemma 6.5 and Theorems 6.2 and 6.3, there is $c>0$ so that for all γ sufficiently small

$$\mu_{\gamma,\Lambda}\big(K^-\cup K^+\big)\ge 1-cL_\gamma^{d-1}e^{-c\gamma^{-d}(\log L_\gamma)^2} \tag{3.2}$$

The factor L_γ^{d-1} comes from the number of cubes $C^{(2^h)}$ on the boundary while $e^{-c\gamma^{-d}(\log L_\gamma)^2}$ bounds the probability of having a contour Γ that contains a given cube $C^{(2^h)}$ and such that $|\Gamma|>(\log L_\gamma)^2$.

By (3.2) and the spin flip symmetry of the model

$$\lim_{L\to+\infty}\mu_{\gamma,\Lambda}(K^\pm)=\frac{1}{2} \tag{3.3}$$

By (3.3) and using again symmetry it is then enough to prove that for γ sufficiently small

$$\lim_{L\to+\infty}\mu_{\gamma,\Lambda}\left(\|s^{(\varepsilon)}-m_{\beta,\gamma}\|\le\delta\,\Big|\,K^+\right)=1 \tag{3.4}$$

We first observe that given any $\delta>0$ for all $\varepsilon>0$ small enough and for all L large enough

$$\left\{\|s^{(\varepsilon)}-m_{\beta,\gamma}\|_{\Delta^0}\le\frac{\delta}{2}\right\}\subset\left\{\|s^{(\varepsilon)}-m_{\beta,\gamma}\|\le\delta\right\}$$

where Δ^0 is the union of all the cubes $C^{(\varepsilon L_\gamma)}$ that are in $L_\gamma\mathcal{T}\setminus\Delta$ which have distance greater than εL_γ from Δ. $\|\cdot\|_{\Delta^0}$ is the L^1-norm in $L_\gamma^{-1}\Delta^0$.

Let $\mathcal{C}^+$ be the first $+$ circuit in Δ coming from the outside and D the complement of $\mathcal{C}^+$. Then using the strong Markov property of the Gibbs measure $\mu_{\gamma,\Lambda}$ we have

$$\mu_{\gamma,\Lambda}\left(\|s^{(\varepsilon)}-m_{\beta,\gamma}\|_{\Delta^0}\le\frac{\delta}{2}\,\Big|\,K^+\right)=E^{\mu_{\gamma,\Lambda}(\cdot|K^+)}\left[\mu_{\gamma,\gamma^{-1}D}\left(\|s^{(\varepsilon)}-m_{\beta,\gamma}\|_{\Delta^0}\le\frac{\delta}{2}\Big|\sigma_{\gamma^{-1}\mathcal{C}^+}\right)\right] \tag{3.5}$$

Theorem 1.2 will follow from (3.5) and Lemma 3.1 below. $\square$

We denote in general by $G^+(\Lambda)$, Λ any bounded, $\mathcal{D}^{(2^h)}$-measurable subset of $\mathbb{Z}^d$, the set of configurations σ_{Λ^c} in $\mathcal{X}_{\Lambda^c}$ such that $\eta(x)=1$ for all $x\in\gamma\Lambda^c$ with $d(x,\gamma\Lambda)\le 1$.

Lemma 3.1

For all $\gamma>0$ small enough the following holds. With the above notation, for any $\sigma_{\gamma^{-1}D^c}\in G^+(\gamma^{-1}D)$,

$$\lim_{L\to\infty}\mu_{\gamma,\gamma^{-1}D}\left(\|s^{(\varepsilon)}-m_{\beta,\gamma}\|_{\Delta^0}\le\delta\,\Big|\,\sigma_{\gamma^{-1}D^c}\right)=1 \tag{3.6}$$

The same property holds for any arbitrary pair Δ^0 and D, with Δ^0 a $\mathcal{D}^{(\varepsilon L_\gamma)}$-measurable set at distance greater than εL_γ from D^c, D a $\mathcal{D}^{(2^h)}$-measurable set.

Proof.

Let $C_\ell^{(\varepsilon L_\gamma)}$, $\ell = 1, ..., N$, be the cubes that are in Δ^0, by the definition of Δ^0 they have distance greater than εL_γ from the complement of D. Then we need to prove

$$\lim_{L\to\infty} \mu_{\gamma,\gamma^{-1}D}\Big(\sum_{\ell=1}^N F_\ell > \delta N \,\Big|\, \sigma_{\gamma^{-1}D^c}\Big) = 0 \tag{3.7}$$

where,

$$F_\ell = \left| \frac{\gamma^d}{|C^{(\varepsilon L_\gamma)}|} \sum_{i\in\gamma^{-1}C_\ell^{(\varepsilon L_\gamma)}} \tilde\sigma(i) \right|, \qquad \tilde\sigma(i) = \sigma(i) - m_{\beta,\gamma} \tag{3.8}$$

By the Tchebyshev inequality,

$$\mu_{\gamma,\gamma^{-1}D}\Big(\sum_{\ell=1}^N F_\ell > \delta N \,\Big|\, \sigma_{\gamma^{-1}D^c}\Big) \le \frac{1}{\delta N}\sum_{\ell=1}^N E^{\mu_{\gamma,\gamma^{-1}D}(\cdot|\sigma_{\gamma^{-1}D^c})}[F_\ell] \tag{3.9}$$

Then

$$\mu_{\gamma,\gamma^{-1}D}\Big(\sum_{\ell=1}^N F_\ell > \delta N \,\Big|\, \sigma_{\gamma^{-1}D^c}\Big) \le \frac{1}{\delta N}\sum_{\ell=1}^N \Big\{E^{\mu_{\gamma,\gamma^{-1}D}(\cdot|\sigma_{\gamma^{-1}D^c})}[F_\ell^2]\Big\}^{1/2} \tag{3.10}$$

In [5] it is proved that there are c_1 and c_2 positive so that for all $\gamma > 0$ small enough, for any bounded, $\gamma^{-1}\mathcal{D}^{(2^h)}$-measurable region Λ and any $\sigma_{\Lambda^c} \in G^+(\Lambda)$ and for any i and j in Λ

$$\Big| E^{\mu_{\gamma,\Lambda}(\cdot|\sigma_{\Lambda^c})}\big(\tilde\sigma(i)\tilde\sigma(j)\big) - E^{\mu_\gamma^+}\big(\tilde\sigma(i)\tilde\sigma(j)\big)\Big| \le c_1 \exp(-c_2\gamma^2 \mathrm{dist}(\{i,j\},\Lambda^c)) \tag{3.11}$$

(recalling that μ_γ^+ is the extremal Gibbs measure with magnetization $m_{\beta,\gamma}$) and there is C_γ such that

$$\sum_{i\in\mathbb{Z}^d} E^{\mu_\gamma^+}\big(\tilde\sigma(0)\tilde\sigma(i)\big) \le C_\gamma \tag{3.12}$$

By (3.11) and (3.12) we obtain from (3.10)

$$\mu_{\gamma,\gamma^{-1}D}\Big(\sum_{\ell=1}^N F_\ell > \delta L^d \,\Big|\, \sigma_{\gamma^{-1}D^c}\Big) \le \frac{N}{\delta}\Big\{\frac{C_\gamma}{(\varepsilon L)^d} + c_1(\varepsilon L)^d e^{-c_2\gamma^2\varepsilon L}\Big\}^{1/2} \tag{3.13}$$

hence (3.6). This completes Lemma 3.1. $\square$

4. Lower bound.

In this section we will prove that for any $u \in BV(\mathcal{T};\{\pm 1\})$ and any $\omega > 0$

$$\liminf_{\gamma \to 0^+} -F'_{\beta,\gamma}(u) \geq -\tau_\beta P(u) - \omega \tag{4.1}$$

¿From (1.14) and Lemma 6.6 we have

$$-F'_{\beta,\gamma}(u) = \lim_{\delta \to 0^+} \liminf_{\varepsilon \to 0^+} \liminf_{L \to +\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,\Lambda}\Big(\{\|s^{(\varepsilon)} - m_{\beta,\gamma} u\| \leq \delta\}\Big) \tag{4.2}$$

By [13], for any $\delta > 0$ (and $\omega > 0$ as above) there is $u_\delta \in BV(\mathcal{T};\{\pm 1\})$ such that the boundary ∂u_δ is a C^∞ surface and

$$P(u_\delta) \leq P(u) + \omega', \qquad \|u_\delta - u\| \leq \frac{\delta}{2} \tag{4.3}$$

where ω' is a coefficient that will be specified later. Let Λ be the torus of $\mathbb{Z}^d$ of side L where the spin system lives and $L_\gamma = \gamma L$ its side in mesoscopic units. We set

$$\Sigma^\star = \Big\{x \in L_\gamma \mathcal{T} : L_\gamma^{-1} x \in \partial u_\delta\Big\} \tag{4.4}$$

Denoting by $d(x, \Sigma^\star)$ the signed distance from $\Sigma^\star$, positive where $u_\delta = 1$, we define

$$m^\star(x) = \bar{m}\big(d(x, \Sigma^\star)\big) \tag{4.5}$$

where $\bar{m}(s)$, $s \in \mathbb{R}$, is the instanton defined in (1.21). In [8] it is proved that $\bar{m}(s)$ is an increasing function that converges exponentially to $\pm m_\beta$ as $s \to \pm\infty$. In the proof of the Γ-convergence in [2], it is shown that

$$\lim_{L_\gamma \to +\infty} \frac{1}{L_\gamma^{d-1}} \mathcal{F}_{L_\gamma \mathcal{T}}(m^\star) = \tau_\beta P(u_\delta) \tag{4.6}$$

Given k, h and ζ, let $\eta^\star$ be the block spin configuration associated to $m^\star$, Γ the union of all its contours and $\mathcal{I}$ of its islands; $\delta\mathcal{I}$ is the union of all the cubes $C^{(2^h)}$ in $\mathcal{I}$ with 0-distance from Γ. We then define the following subsets of $\mathcal{X}_\Lambda$ (denoting by η the block spin configuration associated to σ_Λ)

$$B_{\delta\mathcal{I}} = \{\eta = \eta^\star \text{ on } \delta\mathcal{I}\}, \qquad B_\Gamma = \{\eta = \eta^\star \text{ on } \Gamma\} \tag{4.7}$$

We also denote by $\|\cdot\|_\mathcal{I}$ the L_1 norm on $L_\gamma^{-1}[\mathcal{I} \setminus \delta\mathcal{I}]$.

By exploiting the regularity of ∂u_δ, for any choice of k, h and ζ there is L' and for all $L \geq L'$ (calling $m = m_{\beta,\gamma} u$ and $m_\delta = m_{\beta,\gamma} u_\delta$)

$$\mu_{\gamma,\Lambda}\Big(\|s^{(\varepsilon)} - m\| \leq \delta\Big) \geq \mu_{\gamma,\Lambda}\Big(\{\|s^{(\varepsilon)} - m_\delta\|_\mathcal{I} \leq \delta/4\} \cap B_{\delta\mathcal{I}} \cap B_\Gamma\Big) \tag{4.8}$$

We are going to show that given ω' there is $\zeta > 0$ so that, for all $\gamma > 0$ small enough and all L large enough,

$$\mu_{\gamma,\Lambda}\Big(\|s^{(\varepsilon)} - m\| \le \delta\Big) \ge \mu_{\gamma,\gamma^{-1}\mathcal{I}}\Big(\{\|s^{(\varepsilon)} - m_\delta\|_{\mathcal{I}} \le \delta/4\} \cap B_{\delta\mathcal{I}}\Big)\, e^{-\beta(\tau_\beta P(u)+\omega')\gamma^{-1}L^{d-1}} \tag{4.9}$$

To prove (4.9) we use the continuum approximation in terms of the van der Waals functional, more precisely we condition on the complement of $\gamma^{-1}\Gamma$ and use (6.1). The conditional probability is then bounded from below by

$$\exp\Big(-\gamma^{-d}\beta \inf_{m\Rightarrow\Gamma} \mathcal{F}_\Gamma(m) - c\zeta^2\gamma^{-1}L^{d-1}P(u_\delta) - o(\gamma)\gamma^{-1}L^{d-1}P(u_\delta)\xi\Big) \tag{4.10}$$

The second term bounds

$$\int_\Gamma\int_{\Gamma^c} dx\, dy\, J(|x-y|)\big[m(x) - S^{(2^{-k})}(y)\big]^2$$

In fact in $B_{\delta\mathcal{I}} \cap B_\Gamma$ the square bracket is bounded by 2ζ and for L large, by the regularity of ∂u_δ, the integral becomes proportional to $L_\gamma^{d-1}P(u_\delta)4\zeta^2$. ξ in (4.10) is the smallest number of the form $n2^h$, $n \in \mathbb{N}$, such that $\bar{m}(n2^h) > m_\beta - \zeta$. Again for L large enough

$$|\Gamma| \le 2L_\gamma^{d-1}P(u_\delta)\xi$$

Recalling (4.6) and (4.3) we then obtain (4.9).

We will next prove that given $\delta > 0$ and $\omega' > 0$ there are $\varepsilon' > 0$, $L' > 0$ and $\gamma' > 0$ so that for all $L \ge L'$, $\gamma \le \gamma'$ and $\varepsilon \le \varepsilon'$:

$$\mu_{\gamma,\gamma^{-1}\mathcal{I}}\Big(B_{\delta\mathcal{I}} \cap \{\|s^{(\varepsilon)} - m_\delta\|_{\mathcal{I}} \le \delta/4\}\Big) \ge e^{-\beta\omega'\gamma^{-1}L^{d-1}}\mu_{\gamma,\gamma^{-1}\mathcal{I}}\big(B_{\delta\mathcal{I}}\big) \tag{4.11}$$

Since the region $\mathcal{I}$ is made of islands $\mathcal{I}_\ell$, $\ell = 1,..,\ell^\star$, then after conditioning on $\delta\mathcal{I}$ the measure $\mu_{\gamma,\gamma^{-1}\mathcal{I}}$ becomes a superposition of products of Gibbs measures over the spaces $\mathcal{X}_{\gamma^{-1}\mathcal{I}_\ell}$ with boundary conditions in $G^\pm$ and (4.11) follows from Lemma 3.1.

By using the spin flip symmetry we have

$$\mu_{\gamma,\gamma^{-1}\mathcal{I}}\big(B_{\delta\mathcal{I}}\big) = \mu_{\gamma,\gamma^{-1}\mathcal{I}}\big(B^+_{\delta\mathcal{I}}\big)$$

where $B^+_{\delta\mathcal{I}}$ is obtained from $B_{\delta\mathcal{I}}$ by changing all the minus constraints into plus constraints.

By repeating in reverse the argument that leads from (4.8) to (4.9) we then have

$$\mu_{\gamma,\gamma^{-1}\mathcal{I}}\big(B^+_{\delta\mathcal{I}}\big) \ge e^{-\beta\omega'\gamma^{-1}L^{d-1}}\mu_{\gamma,\Lambda}\big(B^+_{\delta\mathcal{I}}\big) \tag{4.12}$$

because now the function $m = m_\beta$ on Γ is compatible with $B^+_{\delta\mathcal{I}}$.

Calling $C_\ell^{(2^h)}$, $\ell = 1,...,N$, the cubes in $\delta\mathcal{I}$, we then have to estimate

$$\mu_{\gamma,\Lambda}\Big(\bigcap_{\ell=1}^N\{\eta = 1 \text{ on } C_\ell^{(2^h)}\}\Big)$$

We condition on the set

$$\Delta = \bigcap_{\ell=2}^{N} \{\eta = 1 \text{ on } C_\ell^{(2^h)}\}$$

and, by (6.3),

$$\mu_{\gamma,\Lambda\setminus\gamma^{-1}\Delta}\Big(\{\eta = 1 \text{ on } C_1^{(2^h)}\}\Big|\,\sigma_\Delta\Big) \geq 1 - e^{-c\gamma^{-d}} \geq \frac{1}{2}$$

for all γ small enough. Then

$$\mu_{\gamma,\Lambda}\Big(\bigcap_{\ell=1}^{N}\{\eta = 1 \text{ on } C_\ell^{(2^h)}\}\Big) \geq \frac{1}{2}\mu_{\gamma,\Lambda}\Big(\bigcap_{\ell=2}^{N}\{\eta = 1 \text{ on } C_\ell^{(2^h)}\}\Big)$$

By iteration we then get

$$\mu_{\gamma,\Lambda}\Big(\bigcap_{\ell=1}^{N}\{\eta = 1 \text{ on } C_\ell^{(2^h)}\}\Big) \geq \Big(\frac{1}{2}\Big)^N$$

Since $N \leq cL_\gamma^{d-1}$ we get for γ small enough

$$\mu_{\gamma,\Lambda}\Big(\bigcap_{\ell=1}^{N}\{\eta = 1 \text{ on } C_\ell^{(2^h)}\}\Big) \geq e^{-\beta\omega'\gamma^{-1}L^{d-1}}$$

By choosing $10\omega' < \omega$ we then obtain (4.1). $\square$

5. Upper bound.

In this section we prove that for any $u \in BV(\mathcal{T};\{\pm 1\})$ and any $\omega > 0$

$$\limsup_{\gamma\to 0^+} -F''_{\beta,\gamma}(u) \leq -\tau_\beta P(u) + \omega \tag{5.1}$$

¿From the analogous of (1.14) for the upper functional and Lemma 6.6 we have

$$-F''_{\beta,\gamma}(u) = \lim_{\delta\to 0^+}\limsup_{\varepsilon\to 0^+}\limsup_{L\to+\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,\Lambda}\Big(\{\|s^{(\varepsilon)} - m_{\beta,\gamma}u\| \leq \delta\}\Big) \tag{5.2}$$

We split the proof into several steps.

Reduction to rectangles

First we want to reduce (5.1) to the computation of the magnetization on "rectangles". Let $u \in \mathrm{BV}(\mathcal{T}_1,\{\pm 1\})$, we set $m = m_{\beta,\gamma}u$. We report below Theorem D.2 of [2] based on a theorem by Gromart, [14].

Theorem ([2])

For any $\delta > 0$, there is $\alpha > 0$ and an integer $\ell^\star$ such that the following holds. There are $\ell^\star$ disjoint parallelepipeds $\bar{R}_1, \dots, \bar{R}_{\ell^\star}$ with basis $\bar{B}_1, \dots, \bar{B}_{\ell^\star}$ respectively and equal height 2α such that

$$\frac{1}{\alpha}\sum_{\ell=1}^{\ell^\star}\int_{\bar{R}_\ell} dr\, |\chi_{\bar{R}_\ell}(r) - m(r)| \le \delta, \qquad \left|\sum_{\ell=1}^{\ell^\star}|\bar{B}_\ell| - P(u)\right| \le \delta \tag{5.3}$$

where $\chi_{\bar{R}_\ell} \doteq m_{\beta,\gamma}(\mathbf{1}_{\bar{R}_\ell^+} - \mathbf{1}_{\bar{R}_\ell^-})$ with $\{\bar{R}_\ell^+, \bar{R}_\ell^-\}$ the partition of $\bar{R}_\ell$ defined by $\bar{B}_\ell$.

As we will work only in the mesoscopic scale, we introduce R_ℓ, B_ℓ and $R_\ell^\pm$, the images in $L_\gamma \mathcal{T}$ of $\bar{R}_\ell$, $\bar{B}_\ell$ and $\bar{R}_\ell^\pm$ respectively. Define now the sets

$$E_\delta^\ell \doteq \left\{\sigma \,:\, \frac{1}{L_\gamma^d}\int_{R_\ell} dx\, \left|S^{(\varepsilon L_\gamma)}(x) - \chi_{R_\ell}(x)\right| \le \delta\right\}, \qquad E_\delta \doteq \bigcap_{\ell=1}^{\ell^\star} E_\delta^\ell \tag{5.4}$$

¿From the previous theorem we know that $\{\sigma : \|s^{(\varepsilon)} - m\| \le \delta\} \subset E_{2\delta}$ so that we have reduced the proof of (5.1) to

$$\limsup_{\varepsilon\to 0^+}\limsup_{L\to+\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,\Lambda}(E_\delta) \le -\tau_\beta \sum_{\ell=1}^{\ell^\star}|\bar{B}_\ell| + \omega \tag{5.5}$$

for δ and γ small enough.

Reduction to block spins

We introduce a piece of extra notation. Given a subset D of $L_\gamma\mathcal{T}$, we denote by N_D^+, N_D^-, N_D^0, the number of cubes $C^{(2^h)}$ in D where η is respectively equal to $+1$, -1, 0. We also call $N_D \doteq N_D^+ + N_D^- + N_D^0$, the total number of cubes in D and we denote by $S^{(D)}$ the average magnetization in D

$$S^{(D)} \doteq \frac{1}{|D|}\int_D dy\, S(y) \tag{5.6}$$

Lemma 5.1

Let $a > 0$, $\delta > 0$ and for any L let $\Delta_L \subset L_\gamma\mathcal{T}$ with $|\Delta_L| > (aL_\gamma)^d$. Then for any $\delta' > 0$ and $\gamma > 0$ small enough, we have the superexponential estimate

$$\lim_{L\to\infty}\frac{1}{L^{d-1}}\log\mu_{\gamma,\Lambda}\big(N_{\Delta_L}^{\mp} + N_{\Delta_L}^0 > \delta N_{\Delta_L}, |S^{(\Delta_L)} \mp m_{\beta,\gamma}| \le \delta'\big) = -\infty \tag{5.7}$$

Proof.

For simplicity we forget Δ_L in all the subscripts. We first estimate the probability of having more than δN incorrect blocks in Δ_L. Let $x_1,\dots,x_N$ be the centers of the spin blocks included in Δ_L. From Theorem 6.2 we know that there exists $c>0$ such that for γ small enough and for any $\{i_1,\dots,i_n\}\subset\{1,\dots,N\}$,

$$\mu_{\gamma,\Lambda}\big(\eta(x_{i_1})=0,\dots,\eta(x_{i_n})=0\big)\le \exp(-c\gamma^{-d}n) \tag{5.8}$$

so that, for any $M>0$,

$$\mu_{\gamma,\Lambda}\big(N^0\ge M\big)\le \exp(-c\gamma^{-d}M/2)\big(1+\exp(-c\gamma^{-d}/2)\big)^N \tag{5.9}$$

Taking $M=\delta^\star N$ and noticing that there is a constant $a'>0$ such that $N\ge (a'2^{-h}L_\gamma)^d$, we get

$$\lim_{L\to\infty}\frac{1}{L^{d-1}}\log\mu_{\gamma,\Lambda}\big(N^0>\delta^\star N\big)=-\infty \tag{5.10}$$

If $|S-m_{\beta,\gamma}|\le\delta'$, there is a positive constant c such that $N^-\le c\delta' N$. We choose δ' such that $c\delta'=\delta/2$, then the event in (5.7) is contained in the event $N^0>N\delta/2$, hence (5.7) follows from (5.10) So the lemma is proved. $\square$

Let now

$$M_\ell^\pm \doteq \#\Big\{C^{(2^h)}(x)\subseteq R_\ell^\pm\ ,\ \eta(x)\ne\pm1\Big\} \tag{5.11}$$

and consider the sets

$$F_\delta^\ell \doteq \big\{\sigma\ :\ M_\ell^+\le \delta 2^{-hd}L_\gamma^d,\ M_\ell^-\le\delta 2^{-hd}L_\gamma^d\big\}\quad\text{and}\quad F_\delta\doteq\bigcap_{\ell=1}^{\ell^\star}F_\delta^\ell \tag{5.12}$$

Then a straightforward consequence of the previous lemma is that the upper bound (5.5) will be established if for any δ and γ small enough

$$\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,\Lambda}(F_\delta)\le -\tau_\beta\sum_{\ell=1}^{\ell^\star}|\bar B_\ell|+\omega \tag{5.13}$$

In order to simplify notation, we consider for a while only one rectangle that we denote by R and write $R^\pm$, B and $M^\pm$.

Minimal section

Call $\hat n$ the normal to B directed toward R^+. Let $B^{(\kappa)}$ be the shift of B by the vector $2\kappa c(d)\hat n$, κ a positive integer and $c(d)$ a positive constant depending on the dimension d which will be fixed later. We define $\mathcal{H}^{(\kappa)}$ as the smallest connected set of cubes containing

$$\Big\{C^{(2^h)}\subseteq R^+\ :\ C^{(2^h)}\cap B^{(\kappa)}\ne\emptyset\Big\} \tag{5.14}$$

There is a choice of $c(d)$ such that

$$\mathcal{H}^{(\kappa)} \cap \mathcal{H}^{(\kappa')} = \emptyset \qquad \forall\, \kappa \neq \kappa'$$

Let $n_+(\kappa)$ be the number of cubes $C^{(2^h)}$ in $\mathcal{H}^{(\kappa)}$ where $\eta \leq 0$. We introduce also

$$n_+ = \min\bigl\{n_+(\kappa) \,:\, \kappa \in \mathbb{N} \text{ and } \kappa \leq \alpha 2^{-h} L_\gamma/(2c(d))\bigr\} \tag{5.15}$$

Call κ_+ the location where the minimum is achieved and define the minimal section in R^+ as

$$\mathcal{H}^+ \doteq \mathcal{H}^{(\kappa_+)} \tag{5.16}$$

In the same way we denote by $\mathcal{H}^-$ the minimal section in R^- and by n_- the number of cubes $C^{(2^h)}$ in $\mathcal{H}^-$ where $\eta \geq 0$. Then

Lemma 5.2

There is a positive constant c such that, for any $\delta > 0$ and for any $\sigma \in F_\delta$,

$$n_+ + n_- \leq c\delta 2^{-h(d-1)} L_\gamma^{d-1} \tag{5.17}$$

Proof. For any $\sigma \in F_\delta$,

$$\frac{2^{-h}L_\gamma}{2c(d)} n_+ \leq M^+ \leq \delta 2^{-hd} L_\gamma^d \tag{5.18}$$

¿From the analogous results for the $(-)$ case and the definition of $n_\pm$, we get (5.17). □

Upper and lower interfaces

We call here a + circuit in the rectangle R with basis B any connected set $\mathcal{C}$ of cubes in $\mathcal{D}^{(2^h)}$ where $\eta = 1$, such that $R \setminus \mathcal{C}$ has two connected components and each of them contains respectively the upper and lower faces of R parallel to B. A $-$ circuit is defined analogously.

Let $\bar{F}_\delta^{(\ell)}$ be the subset of $F_\delta^{(\ell)}$ which contains a + circuit and a $-$ circuit in R_ℓ. We want to reduce the estimate (5.13) to a computation of the probability of $\bar{F}_\delta \doteq \bigcap_{\ell=1}^{\ell^\star} \bar{F}_\delta^{(\ell)}$.

Lemma 5.3

There is a positive constant c_1 such that for any $\delta > 0$ and any γ sufficiently small the following holds

$$\mu_{\gamma,\Lambda}\bigl(F_\delta\bigr) \leq \exp\left[c_1 \gamma^{-1} \delta \log \delta L^{d-1}\right] \mu_{\gamma,\Lambda}\left(\bar{F}_\delta\right) \tag{5.19}$$

Proof.

Once again, to simplify notation we take $\ell^\star = 1$. Let F_δ^+ be the set of configurations containing a + circuit which separates the basis B from the top of R^+. First we want to change the configurations σ in F_δ into configurations $\bar{\sigma}$ of F_δ^+.

We denote by I the set of "bad" cubes $\{\eta \leq 0\}$ in the minimal section $\mathcal{H}^+$. ¿From Lemma 5.2, there is a positive constant c_2 such that

$$\#I = n_+ \leq c_2 \delta 2^{-h(d-1)} L_\gamma^{d-1} \tag{5.20}$$

We denote by $F(\kappa, I)$ the set of configurations in F_δ such that the minimal section is $\mathcal{H}^+ = \mathcal{H}^\kappa$ and such that the set of "bad" cubes in this minimal section is given by I.

With an abuse of notation, we denote by σ_I any fixed configuration of the spins in the set of cubes I such that $\eta = 1$ on I. We map every configuration σ in $F(\kappa, I)$ to $\bar{\sigma}$ by setting $\bar{\sigma}(i) = \sigma_I(i)$ for all $i \in \gamma^{-1} I$ and $\bar{\sigma}(i) = \sigma(i)$ elsewhere. Then there exists a positive constant c_3 such that we get for any $\{\kappa, I\}$

$$\mu_{\gamma,\Lambda}\left(F(\kappa, I)\right) \leq \sum_{\sigma \in F(\kappa, I)} \exp\left[c_3 n_+ 2^{hd} \gamma^{-d}\right] \mu_{\gamma,\Lambda}(\bar{\sigma}) \tag{5.21}$$

Now remark that for any $\sigma \in F(\kappa, I)$, $\bar{\sigma} \in \bar{F}_\delta^+$. Moreover, the map $\sigma_{I^c} \mapsto \bar{\sigma}$ is one to one. So, splitting the sum over $\sigma \in F(\kappa, I)$ into the sum over $\sigma' \in \bar{F}_\delta^+$ and the sum over $\sigma \in F(\kappa, I)$ such that $\bar{\sigma} = \sigma'$, we get, using also (5.20), that there is $c_4 > 0$ such that

$$\mu_{\gamma,\Lambda}\left(F(\kappa, I)\right) \leq \exp(\delta c_4 2^h \gamma^{-1} L^{d-1}) \mu_{\gamma,\Lambda}\left(\bar{F}_\delta^+\right) \tag{5.22}$$

For a given minimal section, the number of sets I is at most $\binom{c 2^{-h(d-1)} L^{d-1}}{c_3 \delta 2^{h(d-1)} L^{d-1}}$. We sum over all the different sets $F(\kappa, I)$ and we get for some constant $c_5 > 0$

$$\mu_{\gamma,\Lambda}\big(F_\delta\big) \leq \exp(c_5 \delta \log \delta 2^h \gamma^{-1} L^{d-1}) \mu_{\gamma,\Lambda}\left(\bar{F}_\delta^+\right) \tag{5.23}$$

By applying the same procedure in R^- we derive the lemma. $\square$

Energy estimates

Let σ be a configuration with a + circuit and a − circuit in the rectangle R. Then there is a + circuit $\mathcal{C}_+$ and a − circuit $\mathcal{C}_-$ such that there is no other circuit between them and such that all the block spins $C^{(2^h)}$ in $\mathcal{C}_+ \cup \mathcal{C}_-$ are incorrect. So one can find a contour Γ which contains both $\mathcal{C}_+$ and $\mathcal{C}_-$. Indeed there is a $\star$-connected path of incorrect spin blocks from $\mathcal{C}_+$ to $\mathcal{C}_-$ since in the opposite case there should be circuit between $\mathcal{C}_+$ and $\mathcal{C}_-$.

Therefore, from the previous step, we can associate a collection $\{\Gamma_1, \ldots, \Gamma_\kappa\}$ of compatible contours to any configuration in $\bar{F}_\delta$ so that in each rectangle R_ℓ there is a contour Γ_i containing a + circuit and a − circuit in R_ℓ.

Then using Lemma 6.5, we know that

$$\mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \{\Gamma_1, \ldots, \Gamma_\kappa\}\big) \leq \exp\left[-\beta \gamma^{-d} \left(\sum_{i=1}^{\kappa} \inf_{m \Rightarrow \Gamma_i} \mathcal{F}_{\Gamma_i}(m) + o(\gamma, \zeta, 2^{-h}) |\Gamma_i|\right)\right] \tag{5.24}$$

Lemma 5.4

Let $\{\Gamma_1, \ldots, \Gamma_\kappa\}$ be a collection of compatible contours such that for any rectangle R_ℓ there is a contour Γ_i which contains a $+$ circuit and a $-$ circuit in R_ℓ. Then, for L large enough

$$\sum_{i=1}^{\kappa} \inf_{m \Rightarrow \Gamma_i} \mathcal{F}_{\Gamma_i}(m) \geq \tau_\beta \sum_{\ell=1}^{\ell^\star} |B_\ell| - o(\zeta, 2^{-k}) \sum_{i=1}^{\kappa} |\Gamma_i| \tag{5.25}$$

Proof.

It is enough to consider one of the rectangles, denoted by R with basis B, and to argue with the restriction $\bar{\Gamma}$ to R of the contour. We will prove that

$$\inf_{m \Rightarrow \bar{\Gamma}} \mathcal{F}_{\bar{\Gamma}}(m) \geq \tau_\beta |B| - o(\zeta, 2^{-k}) |\bar{\Gamma}| \tag{5.26}$$

where B denotes the basis of R.

Any configuration m compatible with $\bar{\Gamma}$ takes values close to m_β in a circuit of R and close to $-m_\beta$ in another circuit. These two circuits make a partition of the rectangle into three regions that we call D_+, D_- and ΔD. We denote by $\mathcal{R}$ the set of all magnetizations m which satisfy $m = m_\beta$ in $D_+ \cap \bar{\Gamma}^c$, $m = -m_\beta$ in $D_- \cap \bar{\Gamma}^c$ and such that $m = \pm m_\beta$ in all the connected subsets of $\Delta D \cap \bar{\Gamma}^c$. Since we only have to pay for the interaction between these new conditions and the boundary of $\bar{\Gamma}$, we have

$$\inf_{m \Rightarrow \bar{\Gamma}} \mathcal{F}_{\bar{\Gamma}}(m) \geq \inf_{m \in \mathcal{R}} \mathcal{F}_R(m) - o(\zeta, 2^{-k}) |\bar{\Gamma}| \tag{5.27}$$

The inf of $\mathcal{F}_R$ when $m \in \mathcal{R}$ is studied in the section 4 of [2]. When proving the lower bound for the Γ-convergence, it is shown that

$$\lim_{L \to \infty} \frac{1}{L_\gamma^{d-1}} \inf_{m \Rightarrow \mathcal{R}} \mathcal{F}_R(m) \geq \tau_\beta |\bar{B}| \tag{5.28}$$

This proves the lemma. □

Conclusion

Let ε be a positive parameter. Combining estimates (5.23), (5.24), Lemma 5.3 and Lemma 5.4, we get for L large enough

$$\mu_{\gamma,\Lambda}(F_\delta) \leq \exp\left[-(1-\varepsilon)\beta\tau_\beta\gamma^{-1}L^{d-1}\sum_{i=1}^{\ell^\star}|\bar{B}_i|\right] \sum_{\kappa\geq 1}\sum_{\Gamma_1,\ldots,\Gamma_\kappa} \exp\left[c_1\delta\log\delta L^{d-1}\gamma^{-1}\right]$$
$$\times \exp\left[-\big(\varepsilon - o(\zeta, 2^{-k}) - o(\gamma, \zeta, 2^{-h})\big)c_2\gamma^{-d}\sum_{i=1}^{\kappa}|\Gamma_i|\right] \tag{5.29}$$

where the sum is taken over all the collections $\{\Gamma_1,\dots,\Gamma_\kappa\}$ which satisfy hypothesis of Lemma 5.4. We take ζ, γ small, k and h large so that $\varepsilon - o(\zeta, 2^{-k}) - o(\gamma,\zeta,2^{-h}) \geq \varepsilon/2$.

Now let A_ℓ be the set of the contours Γ satisfying $|\Gamma| \geq |B_\ell|$. Then

$$\sum_{\kappa\geq 1}\sum_{\Gamma_1,\dots,\Gamma_\kappa} \exp\left[-\varepsilon c_2/2\gamma^{-d}\sum_{i=1}^{\kappa}|\Gamma_i|\right] \leq \prod_{\ell=1}^{\ell^\star}\left(1+\sum_{\Gamma\in A_\ell}\exp\left[-\varepsilon c_3\gamma^{-d}|\Gamma|\right]\right) \tag{5.30}$$

for some constant $c_3>0$. Following Theorem 6.3, we count the number of contours which contain a given cube of R_ℓ so that there is a constant $c_4>0$ such that

$$\sum_{\kappa\geq 1}\sum_{\Gamma_1,\dots,\Gamma_\kappa} \exp\left[-\varepsilon c_2/2\gamma^{-d}\sum_{i=1}^{\kappa}|\Gamma_i|\right] \leq \left(1+L^d\exp\left[-\varepsilon c_4\gamma^{-1}L^{d-1}\right]\right)^{\ell^\star} \tag{5.31}$$

Combining the estimates (5.29), (5.31) and taking ε small enough, we derive the upper bound (5.5). $\square$

Appendix

In this Appendix we first recall some basic theorems on the Kac potentials and then prove some technical lemmas that have been used in the previous sections.

The first result, see for instance [6], is the basic estimate in the theory of Kac potentials ruling the transition to the continuum. Let Δ be a bounded, $\mathcal{D}^{(2^{-k})}$-measurable region, $k\in\mathbb{N}$. Then there is a function $o(\gamma)$ that vanishes as $\gamma\to 0^+$ ($o(\gamma)$ can be taken of the order of $\sqrt{\gamma}$) so that the following holds. For any $\gamma>0$ let $\Lambda=\{i\in\mathbb{Z}^d : \gamma i\in\Delta\}$, $\sigma_{\Lambda^c}\in\mathcal{X}_{\Lambda^c}$, and let Γ be a $\mathcal{D}^{(2^{-k})}$-measurable subset of configurations in Δ, then

$$\left|\log\mu_{\gamma,\Lambda}\big(\sigma\Rightarrow\Gamma\big|\sigma_{\Lambda^c}\big)+\beta\gamma^{-d}\inf_{m\Rightarrow\Gamma}\mathcal{F}_\Delta(m|S_{\Delta^c}^{(2^{-k})})\right|\leq|\Lambda|o(\gamma) \tag{6.1}$$

where for any measurable set Δ and any $m\in L^\infty(\Delta;[-1,1])$ and $m'\in L^\infty(\Delta^c;[-1,1])$

$$\mathcal{F}_\Delta(m|m')=\mathcal{F}_\Delta(m)+\frac{1}{2}\int_\Delta\int_{\Delta^c}dx\,dy\,J(|x-y|)\big[m(x)-m'(y)\big]^2 \tag{6.2}$$

Peierls estimates : known results

We first recall a result proved in [6] in the case of + boundary conditions. Let D be a $\mathcal{D}^{(2^h)}$-measurable region and $\sigma_{\gamma^{-1}D^c}$ a configuration with $\eta=1$ on $\{x\in D^c : d(x,\Lambda)\leq 1\}$. Let Δ be a $\mathcal{D}^{(2^h)}$-measurable region in D. Then

$$\mu_{\gamma,\gamma^{-1}D}\Big(\{\Delta\text{ is contained in a contour}\}\Big|\sigma_{\gamma^{-1}D^c}\Big)\leq e^{-c\gamma^{-d}} \tag{6.3}$$

where c is a constant (whose value depends on the choice of the parameters in the definition of the block spins). We will discuss here the extension of the result to periodic boundary conditions. We begin by some notation. Let Γ be an admissible contour then we can give a sign to each connected component of the complement of Γ according to the sign of its boundary. We define the map ϕ_Γ on the subsets of spin configurations compatible with Γ: for such a subset A, its image $\phi_\Gamma(A)$ consists in the spin configurations obtained from A by flipping the spins in all the $-$ islands $\star$-connected to Γ.

Theorem 6.1

Let $\underline{\Gamma} = \{\Gamma_1, \dots, \Gamma_\kappa\}$ be an admissible collection of contours in the torus $L_\gamma \mathcal{T}$. Then for any subset A of spin configurations measurable with respect to the complement of $\underline{\Gamma}$, we have

$$\mu_{\gamma,\Lambda}\Big(\{\sigma \Rightarrow \underline{\Gamma}\} \cap A\Big) \leq \mu_{\gamma,\Lambda}\big(\phi_{\underline{\Gamma}}(A)\big) \exp\left(-\beta\gamma^{-d}\sum_{i=1}^{\kappa}\left(\inf_{m \Rightarrow \Gamma_i} \mathcal{F}_{\Gamma_i}(m) - o(\gamma, \zeta, 2^{-h})|\Gamma_i|\right)\right) \tag{6.4}$$

where $o(\gamma, \zeta, 2^{-h}) = o(\gamma) + o(2^{-h})\zeta^2$ and $|\Gamma_i|$ is the volume of the support of Γ_i.

The assumption that the collection $\underline{\Gamma}$ is admissible is used to claim that if $\sigma \Rightarrow \underline{\Gamma}$ then we can divide the complement of the support of $\underline{\Gamma}$ into connected regions $\Delta_1, \dots, \Delta_n$ so that η is constant and non zero on $\partial\Delta_i$ (defined as the boundary of an island). Then the spin flip arguments of [6] in the Peierls estimates can be reproduced.

The Peierls condition can be obtained by an analysis of the functional $\mathcal{F}$ (see [6]). For Δ in $L_\gamma\mathcal{T}$, define $\mathcal{B}(\Delta)$ as the set of all $m \in L^\infty(L_\gamma\mathcal{T}; [-1,1])$ such that Δ is incorrect.

Theorem 6.2

There is $c > 0$ such that

$$\inf_{m \in \mathcal{B}(\Delta)} \mathcal{F}_\Delta(m) \geq c|\Delta| \tag{6.5}$$

A well known combinatorial argument (see [6], [11]) enables us to control the cost of large contours,

Theorem 6.3

For any $c > 0$ there is $\gamma' > 0$ so that for all $\gamma < \gamma'$,

$$\sum_{\Gamma \ni C, |\Gamma| \geq M} \exp\left(-c\gamma^{-d}|\Gamma|\right) \leq \exp\left(-c\gamma^{-d}M/2\right) \tag{6.6}$$

where the sum is taken over all the ($\star$-connected) contours of length greater than a constant M and which contain a given cube C.

The collection of all the contours produced by a spin configuration is admissible so that Theorem 6.1 applies. However, in the applications it often happens to have only a subset of contours, the next lemma deals with the case of all the winding contours.

Lemma 6.4

The collection of all the winding contours in Λ produced by a spin configuration σ is admissible.

Proof.

Let $\{\Gamma_1, \ldots, \Gamma_\kappa\}$ be the set of all the winding contours produced by a configuration σ. Let us suppose that there is a connected set in the complement of $\Gamma_1 \cup \ldots \cup \Gamma_\kappa$ which is connected to $\partial\Gamma_i^+$ and $\partial\Gamma_j^-$, we call it a path from Γ_i^+ to Γ_j^-. This path must cross a non winding contour $\Gamma^\star$, because it is included in the complement of all the winding contours.

Since $\Gamma^\star$ is non winding we can find a connected set of cubes where η is constant which goes around $\Gamma^\star$. Repeating the argument, we get a path where η is constant which connects $\partial\Gamma_i^+$ and $\partial\Gamma_j^-$, which is a contradiction. Lemma 6.4 is proved. $\square$

Lemma 6.5

We consider a collection of compatible contours $\underline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_\kappa\}$. For γ sufficiently small we have

$$\mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \underline{\Gamma}\big) \leq \exp\left[-\gamma^{-d}\beta \sum_{i=1}^{\kappa}\left(\inf_{m\Rightarrow\Gamma_i} \mathcal{F}_{\Gamma_i}(m) + o(\gamma)|\Gamma_i|\right)\right] \tag{6.7}$$

Proof.

We suppose that $\Gamma_1, \ldots, \Gamma_j$ are winding contours and $\Gamma_{j+1}, \ldots, \Gamma_\kappa$ are not winding contours. First we want to estimate the probability of the non winding contours. Since Γ_κ is non winding, it is admissible, then applying Theorem 6.1, we get

$$\begin{aligned}\mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \underline{\Gamma}\big) \leq &\mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \{\phi_{\Gamma_\kappa}(\Gamma_1), \ldots, \phi_{\Gamma_\kappa}(\Gamma_{\kappa-1})\}\big) \\ &\times \exp\left[-\gamma^{-d}\beta\left(\inf_{m\Rightarrow\Gamma_\kappa} \mathcal{F}_{\Gamma_\kappa}(m) + o(\gamma)|\Gamma_\kappa|\right)\right]\end{aligned} \tag{6.8}$$

with an abuse of notation on ϕ_{Γ_κ}.

As the contours are compatible, $\phi_{\Gamma_\kappa}(\Gamma_1), \ldots, \phi_{\Gamma_\kappa}(\Gamma_{\kappa-1})$ are still contours with the same support as $\Gamma_1, \ldots, \Gamma_{\kappa-1}$. Moreover the mapping ϕ_{Γ_κ} does not modify the value of the functional and we have $\inf_{m\Rightarrow\Gamma} \mathcal{F}_\Gamma(m) = \inf_{m\Rightarrow\phi_{\Gamma_\kappa}(\Gamma)} \mathcal{F}_{\phi_{\Gamma_\kappa}(\Gamma)}(m)$. ¿From this by iteration we get that

$$\mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \underline{\Gamma}\big) \leq \mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \underline{\Gamma}'\big) \exp\left[-\gamma^{-d}\beta\left(\sum_{i=j+1}^{\kappa} \inf_{m\Rightarrow\Gamma_i} \mathcal{F}_{\Gamma_i}(m) + o(\gamma)|\Gamma_i|\right)\right] \tag{6.9}$$

where we denote $\underline{\Gamma}' = \{\Gamma'_1 \cap \ldots \cap \Gamma'_j\}$ the images of the winding contours by all the previous mappings.

Now using Lemma 6.4, we introduce some other winding contours $\tilde\Gamma_1, \ldots, \tilde\Gamma_n$ such that $\Gamma'_1, \ldots, \Gamma'_j, \tilde\Gamma_1, \ldots, \tilde\Gamma'_n$ are admissible:

$$\mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \underline{\Gamma}'\big) \le \sum_{n\ge 0} \sum_{\tilde\Gamma_1,\ldots,\tilde\Gamma_n} \mu_{\gamma,\Lambda}\big(\sigma \Rightarrow \{\Gamma'_1, \ldots, \Gamma'_j, \tilde\Gamma_1, \ldots, \tilde\Gamma_n\}\big) \tag{6.10}$$

Since $\tilde\Gamma_1, \ldots, \tilde\Gamma_j, \Gamma'_1, \ldots, \Gamma'_n$ are admissible, we can use again Theorem 6.1 and get

$$\begin{aligned}\mu_{\gamma,\Lambda}(\sigma \Rightarrow \underline{\Gamma}') \le \sum_{n\ge 0} \exp\left[-\gamma^{-d}\beta \sum_{i=1}^{j} \left(\inf_{m\Rightarrow\Gamma'_i} \mathcal{F}_{\Gamma'_i}(m) + o(\gamma)|\Gamma_i|\right)\right] \\ \times \left(\sum_{\tilde\Gamma} \exp\left[-\gamma^{-d} c_1 \inf_{m\Rightarrow\tilde\Gamma} \mathcal{F}_{\tilde\Gamma}(m)\right]\right)^n\end{aligned} \tag{6.11}$$

where the last sum in the previous inequality is taken over all the winding contours $\tilde\Gamma$ and c is a constant. ¿From Theorem 6.2 we know that there is $c_2 > 0$ such that

$$\inf_{m\Rightarrow\tilde\Gamma} \mathcal{F}_{\tilde\Gamma}(m) \ge c_2|\tilde\Gamma| \tag{6.12}$$

So that it remains to get an upper bound for $\sum_{\tilde\Gamma} \exp\big(-\gamma^{-d}|\tilde\Gamma|\big)$. We note that any winding contour $\tilde\Gamma$ satisfies $|\tilde\Gamma| \ge L_\gamma$. So, using Theorem 6.3, there is a constant c_3 such that,

$$\sum_{\tilde\Gamma \ni C^{(2^h)}} \exp\big(-c_1 c_2 \gamma^{-d}|\tilde\Gamma|\big) \le \exp(-c_3\gamma^{1-d}L) \tag{6.13}$$

where the sum is taken over all the contours $\tilde\Gamma$ which contain the fixed cube $C^{(2^h)}$ and such that $|\tilde\Gamma| \ge L_\gamma$. Summing over all the cubes $C^{(2^h)}$ and taking L large enough, the statement will follow. $\square$

Lemma 6.6

For any $\varepsilon > 0$, $\delta > 0$ and $u \in BV(\mathcal{T}, \{\pm 1\})$ we have

$$\big\{\|s^{(\varepsilon)} - m_{\beta,\gamma}u\| \le \delta^2\big\} \subset \big\{\|\eta_\delta^{(\varepsilon)} - u\| \le 2\delta\big\} \quad \text{and} \quad \big\{\|\eta_\delta^{(\varepsilon)} - u\| \le \delta\big\} \subset \big\{\|s^{(\varepsilon)} - m_{\beta,\gamma}u\| \le 3\delta\big\} \tag{6.14}$$

Proof.

The first inclusion comes from

$$\|\eta_\delta^{(\varepsilon)} - u\| \leq 2 \int_{\mathcal{T}} dr\, \mathbf{1}_{\{|s^{(\varepsilon)}(r) - m_{\beta,\gamma} u(r)| \geq \delta\}} \tag{6.15}$$

and the Tchebychev inequality. The second one follows from

$$\|s^{(\varepsilon)} - m_{\beta,\gamma} u\| \leq \delta + 2 \int_{\mathcal{T}} dr\, \mathbf{1}_{\{|s^{(\varepsilon)}(r) - m_{\beta,\gamma} u(r)| \geq \delta\}} \leq \delta + 2 \int_{\mathcal{T}} dr\, \mathbf{1}_{\{|\eta_\delta^{(\varepsilon)}(r) - u(r)| \geq \delta\}} \tag{6.16}$$

and the lemma is proved. $\square$

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