Semi-oriented bootstrap percolation in three dimensions

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We consider the critical system size of a three dimensional semi-oriented bootstrap percolation model, constricted to a $3D$ cube wrapped to a torus, i.e. with periodical boundary conditions. We point out a possible form of the critical droplets for this model: occupied squares in a plane perpendicular to the primary direction of the dynamics behave as growing seeds when they are sufficiently large. Their critical linear size is of order $\exp O(\frac{1}{p}\log^2\frac{1}{p})$.

Also we prove that $\exp\exp O(\frac{1}{p}\log^2\frac{1}{p})$ is an upper bound on the critical system size. In the proof an element comes forward that is not contained in the two dimensional SBP model. Whereas in two dimensions a critical droplet can only either grow, survive or die, in three dimensions a growing critical droplet can also shrink.

It follows from this result that for the $\mathrm{SBP}^{2,1}$ model $p_c = 0$, i.e. on the infinite lattice this model almost surely fills the whole space for all nonzero p.

Key words: Bootstrap percolation, Critical lengths, exponential rates

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1 Introduction

Bootstrap percolation (BP) models are a special kind of cellular automata (CA), in which the rules of the dynamics can only create new occupied sites and cannot empty out an already existing occupied site. To be precise, BP models are CA such that the initial configurations are i.i.d. Bernoulli random variables with parameter p, and empty sites (state 0) that have at a given time one of some specific sets of nearest neighbours become occupied (state 1) at the next update. The configurations are updated in discrete time and occupied sites remain occupied for all time. BP models exist in different types according to the choice of nearest neighbours that cause an empty site to become occupied at the next update, for a survey see Adler (1991) or Schonmann (1992). The primary object of interest is the final configuration of occupied sites.

It is a fundamental observation, see Aizenman and Lebowitz (1988), Van Enter *et al* (1990) or Schonmann (1990, 1992), that for (nontrivial) BP models, when given some initial configuration with parameter p, a larger system has a higher probability of becoming completely occupied than a smaller one. The key to understanding this behaviour is in the following. Growing clusters of occupied sites can become isolated from other occupied sites when they are surrounded by layers of empty sites. Subsequently these isolated clusters will stop growing. However, if a cluster is so large that the probability of not

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finding an occupied site right next to its boundaries is vanishingly small, then this cluster will continue growing and in the long run engulf the entire lattice. The smallest cluster of occupied sites that behaves like this is called a critical droplet. A larger system will have a higher probability of containing a critical droplet. Therefore the following question arises. Given some initial density p, is there some critical system size, such that for larger sizes the system will, with high probability, be completely filled by the dynamics? In this paper we give a partial answer to this question for a three dimensional semi-oriented BP model.

The occurrence of such large occupied clusters in the random initial configuration is in general very improbable. The consequence of this is that it may take a very long time for the system to reach a fully occupied state, if it does so at all. This behaviour of BP models and especially the terms used to interpret this behaviour are reminiscent of metastability phenomena in the vicinity of first order phase transitions. Since bootstrap percolation models are relatively simple, the process in which the system leaves its initial metastable state and relaxes to its final state can be followed precisely. BP models are therefore particularly interesting for physicists. Analogies between BP models and physically more realistic models have been pointed out in, among others, Aizenman and Lebowitz (1988) and Dehghanpour and Schonmann (1996a, 1996b).

The name bootstrap percolation was apparently first mentioned by Chalupa *et al* (1979), although the idea of the models was independently proposed earlier by Pollak and Riesz and by Kopelmann. Many other authors have subsequently studied these type of models in a variety of contexts. In Van Enter (1987) one finds a result on a two dimensional $l = 2$ BP model, where contour arguments were used to prove that $p_c = 0$. In 1988 this result was generalized to arbitrary dimensions by Aizenman and Lebowitz (1988). Their rigorous results on the $l = 2$ basic BP model proved metastable behaviour and pointed out the finite volume effects. Since then variations of the model where treated in a similar way. Semi-oriented BP models were introduced by Duarte (1989). In his paper from 1990 Schonmann proved that $p_c = 0$ for this model in two dimensions. Also in 1990 a paper by Van Enter, Adler and Duarte appeared on the finite size effects of some variations of the BP model that were not covered in the article of Aizenman and Lebowitz. Among other things they used the results of Schonmann (1990) to obtain an upper bound on the finite size effects of $2D$ semi-oriented BP. In a private communication with these authors Schonmann pointed out that this bound was not optimal. A great step forward was made with Schonmann's next paper, Schonmann (1992). This contained several new results on the $l = d$ BP and the modified BP models. He pointed out some relations between the different models and described a useful induction on dimension argument. In 1995 a paper of Mountford was published, containing new rigorous results on the two dimensional semi-oriented bootstrap percolation model. He proved that the occurrence of critical droplets of a particular shape is a sufficient *and necessary* event for the system to be spanned by the dynamics.

In this paper we consider a three-dimensional semi-oriented BP model. In the remaining part of this section we will introduce this model. In section 2 we show a cluster of occupied sites that behaves as a growing seed and therefore is a candidate for being the critical droplet. In theorem 5 we state the main result of this paper: a bound on the critical system size. The remaining part of section 2 is devoted to a part of the proof: we consider the splitting probability of a growing critical droplet. This is an element that is not contained in $2D$ semi-oriented BP. In section 3 the bulk of the proof of theorem 5 is given. Section 4 contains a corollary and an outlook.

To a great length we will follow Schonmann (1992) in classifying the different models in terms of sets of nearest neighbours and we will adopt most of his notation. The states of a model are denoted by the variable $\eta_t(x)$ which in each x can only assume the values 0 and 1 and where t takes integer values.

Definition 1 *Let $\mathcal{N}_x = \{y \in \mathbb{Z}^d \mid \|y - x\|_1 = 1\}$ denote the set of nearest neighbours of a given site x. ($\|\cdot\|_1$ is the l_1-norm: $\|x\|_1 = \sum |x_i|, \quad i = 1, \ldots, d$.) If $\eta_t(x) = 0$ it will become 1 at the next update iff the set of occupied neighbours contains one of some specified subsets $A \subset \mathcal{N}_x$. We denote by $\mathcal{D} = \{A \subset \mathcal{N}_x \mid \eta_t(x) = 1,\ \forall x \in A \Rightarrow \eta_{t+1}(x) = 1\}$ the set of smallest subsets of $\mathcal{N}_x$ that cause the site x to become 1 if it were 0.*

Specifying $\mathcal{D}$ specifies the model. We discern basic, modified, semi-oriented and oriented BP models, see Schonmann (1992). Here we will only introduce those models that matter for this paper; these are modified BP and semi-oriented BP. In order to give the definitions of these models we define the unit vectors $e_i = (0, \ldots, 1, \ldots, 0), \quad i = 1, \ldots, d$. $|A|$ will denote the cardinality of the set A. By $\mathcal{N}$, without index, we mean a set of neighbours of an arbitrary site x. However, for convenience and clarity, we will talk about it as if it were $\mathcal{N}_0$.

Modified bootstrap percolation (MBP) In this model there must be at least one occupied nearest neighbour in every coordinate direction. So we have

$$\mathcal{D} = \{A \subset \mathcal{N} \mid A \cap \{-e_i, e_i\} \neq \emptyset, \quad \forall\, i = 1, \ldots, d\} \tag{1}$$

MBP and basic BP (see Schonmann (1992)) are both often called symmetric bootstrap percolation, in contrast with the following model.

Semi-oriented bootstrap percolation (SBP) We take a d_1-dimensional modified model and a d_2-dimensional oriented model and construct with these a d-dimensional semi-oriented model on the direct sum $\mathbb{Z}^{d_1} \oplus \mathbb{Z}^{d_2}$. We denote by $A = A_1 \cap A_2$ the bisection of A in separate sets of neighbours in the 'pure-model' sub-lattices. Let the vector $(a_1, \ldots, a_d) \in \{-1, +1\}^d$ be given. For notational convenience we define $b_i = a_{d_1+i}$ and $f_i = e_{d_1+i}$, for $i = 1, \ldots d_2$.

$$\begin{aligned} \mathcal{D} &= \\ &\{A \subset \mathcal{N} \mid A \cap \{-e_i, e_i\} \neq \emptyset,\ \forall i = 1, .., d_1 \wedge \{b_1 f_1, .., b_{d_2} f_{d_2}\} \subset A\} \end{aligned} \tag{2}$$

We will denote these systems as SBP^{d_1,d_2}. The directional vectors $e_1, \ldots, e_{d_1}$ are called secondary directions and the vectors $f_1, \ldots, f_{d_2}$ are called the primary directions. We add to this terminology the *primary* resp. *secondary sideplanes*, which are the sideplanes of a rectangular critical droplet perpendicular to the primary resp. secondary directions. In this paper we focus on the $\mathrm{SBP}^{2,1}$ model and without loss of generality we choose the $(1, 0, 0)$ direction as the primary direction.

Let us briefly list the properties that are made extensively use of.

First of all the process is attractive. Let $(\Omega, \mathcal{F})$ be defined as the set of configurations $\Omega = \{\eta \mid \eta(x) = 0, 1, \ \forall x \in \mathbb{Z}^d\} = \{0,1\}^{\mathbb{Z}^d}$, viewed as a compact metric space with a measurable structure given by the Borel σ-algebra $\mathcal{F}$ of subsets of Ω. If at $t = 0$ a state η_0' differs from another state η_0 only by the addition of some 1's, i.e. if at $t = 0$

$$\eta'(x) \geq \eta(x), \quad \forall\ x \in \mathbb{Z}^d \tag{3}$$

then this will remain true for all subsequent times. This property is called attractiveness. For a more extensive account the reader is referred to Liggett (1985). When equation (3) holds we write $\eta' \geq \eta$. This defines a partial order on the set $(\Omega, \mathcal{F})$.

Further it is easy to see from the rules of the dynamics that $\eta_\infty(n, x)$ is measurable with respect to $\sigma\{\eta_\infty(n-1, y), \eta_0(n, y) \mid \ y \in \{1, \ldots, N\}^2\}$. Therefore the $\{0,1\}^{\{1,\ldots,N\}^2}$ valued process $\eta_\infty(n, \cdot), \ n = 1, \ldots, N$ is a Markov chain.

Another property that can be derived directly from the rules of the dynamics is the existence of stationary planar interfaces. In $\mathrm{SBP}^{2,1}$, square planes of side N, parallel to one of the coordinate planes, will remain empty for all time if they are completely empty at $t = 0$. Other BP models have similar stationary planar interfaces.

We now define the exteriors of a three dimensional box $\{x \in \mathbb{Z}^3 \mid |x_i| \leq k_i\}$

with dimensions $2k_1+1$, $2k_2+1$ and $2k_3+1$ in resp. the e_1, e_2, and e_3 coordinate directions. We denote these boxes by Λ_k, where $k=\{k_1,k_2,k_3\}$. For convenience these boxes are defined as located at the origin, but in fact they can be anywhere in the lattice. This should not cause confusion since only their size is important for the growing process.

Definition 2 *Given $r\in\{1,\ldots 3\}$, a partition $1\le i_1<i_2<\cdots<i_r\le 3$, which we denote as a vector I, and a vector a, where $a_s\in\{-1,+1\}$, $\forall s=1,\ldots,r$ we define*

$$L^3_{I,a}(k)\equiv$$
$$\{x\in\mathbb{Z}^3\,|\,x_{i_s}=a_sk_{i_s},\ \forall s=1,\ldots,r\ \wedge\ |x_j|<k_j,\ \forall j\notin\{i_1,\ldots,i_r\}\}.$$

For each fixed r, let $\mathcal{F}_r$ denote the set of possible indices above, thus

$$\mathcal{F}_r=\{I,a\mid I=(i_1,\ldots,i_r),a=(a_1,\ldots,a_r)\}$$

and $\mathcal{F}\equiv\bigcup_{r=1}^3\mathcal{F}_r$.

The sets $L^3_{I,a}(k)$ denote a $(3-r)$-dimensional rectangular region with sides of length $2k_i-1$, where i is in $\{1,2,3\}$, the values of i specified by the set I. For $r=1,\ldots,3$ these sets respectively denote the interiors of the sideplanes, the ribs and the corner points; therefore we call them the exteriors of the region Λ_k. The motivation for this definition is clear. If a cluster of occupied sites is contained in a box Λ_{k-1}, where $k-1=\{k_1-1,k_2-1,k_3-1\}$, then growth of this cluster occurs via occupied sites in the sets $L^3_{I,a}(k)$ since the rules of the dynamics only involve nearest neighbour interaction.

2 Semi-oriented bootstrap percolation

We start by indicating a possible form for the critical droplets of the SBP^{2,1} model. Analogously to the critical droplets in SBP^{1,1}, see Mountford (1995), we could start with occupied squares in a plane of thickness 1, perpendicular to the primary direction of the system. As in all semi-oriented BP models we have extra requirements that determine the size of the critical droplets. These extra requirements are listed in the proof of the following lemma.

Lemma 3 *Squares with thickness 1, perpendicular to the primary direction of the system and having minimal sidelength* $k = \exp(1000\frac{1}{p}\log^2\frac{1}{p})$ *behave as critical droplets for the SBP*^{2,1} *system.*

PROOF. We denote a square region of occupied sites in a plane perpendicular to the primary direction as a box $\Lambda_{\{0,k_c,k_c\}}$, which has thickness 1 in the e_1 direction. We first ask how large k_c must be in order to let a new layer of occupied sites grow just next to this plane in the positive e_1 direction, thus in the set $L^3_{\{+1\}}(1, k_c+1, k_c+1)$. Since all members of this set already have an occupied neighbour in the negative e_1 direction this set forms a two dimensional modified BP system. We know from the results of Aizenman and Lebowitz (1988) that the critical size of such a system is of order $\exp O(\frac{1}{p}\log\frac{1}{p})$. So, a priori, k_c cannot be smaller than this value. Given this it is an easy calculation that in order to let $\Lambda_{\{0,k_c,k_c\}}$ have a high probability to grow $N-1$ new layers in the positive e_1 direction it is necessary that

$$k_c \geq \log N \cdot \exp O(\frac{1}{p}\log\frac{1}{p}). \tag{4}$$

Next, the planes $L^3_{\{-2\}}(N, k_c+1, k_c+1)$, $L^3_{\{+2\}}(N, k_c+1, k_c+1)$, $L^3_{\{-3\}}(N, k_c+1, k_c+1)$ and $L^3_{\{+3\}}(N, k_c+1, k_c+1)$, i.e. the secondary sideplanes of the grown droplet, have all neighbours in one of the secondary directions occupied. This means that each of these planes forms a $k_c \times N$ two dimensional SBP^{1,1} system. The results of Mountford (1995) tell us that this system has a critical size of order $\exp O(\frac{1}{p}\log^2\frac{1}{p})$. We see that both k_c and N must be larger than this value. This gives rise to two demands on k_c, the first being

$$k_c \geq k_{c,1} = \exp O(\frac{1}{p}\log^2\frac{1}{p}), \tag{5}$$

and the second one, using (4), being

$$k_c \geq k_{c,2} = O(\log N) \cdot \exp O(\frac{1}{p}\log\frac{1}{p}) = O(\frac{1}{p}\log^2\frac{1}{p}) \exp O(\frac{1}{p}\log\frac{1}{p}). \tag{6}$$

The minimal required sidelength is for any p the maximum of the bounds in eqs. (5) and (6). The reader can check that for very small p the former is the dominant one. The ratio $k_{c,1}/k_{c,2}$ even tends to infinity in limit $p \to 0$. This proves the lemma. $\square$

Lemma 3 suggests that, analogous to SBP^{1,1}, the critical system size N_c for the SBP^{2,1} system is of order $(1/p)^{k_c^2} = \exp\exp(2000\frac{1}{p}\log^2\frac{1}{p})$, where we have made the same choice for the prefactor as was done in Mountford (1995). In order to get a more rigorous confirmation of this we prove that $(1/p)^{k_c^2}$ is an upper bound on the critical system size. In order to do so we adapt a similar proof for the two dimensional SBP^{1,1} model in the supercritical regime, which can be found in Mountford (1995) and is also due to Schonmann. This is the main result of this paper.

Definition 4 (**Spanning probability**) *We say that an SBP system is (internally) spanned if in the end it is entirely covered by the dynamics. We denote the probability that the system is spanned by the variable* $R(N,p) \equiv \boldsymbol{P}[\eta_\infty(\cdot)=1]$*, where* N *denotes the linear size of the system.*

Theorem 5 *For an* $SBP^{2,1}$ *system on a three dimensional torus of linear size* N*, we have*

$$\inf_{N \geq \exp\exp(2000\frac{1}{p}\log^2\frac{1}{p})} \boldsymbol{P}[\eta_\infty(\cdot)=1] \to 1$$

when $p \to 0$*.*

We divide the square cross-section $[0,N]^2$, perpendicular to the primary direction of the system, up into $(10k_c)^2$ equal square regions Λ_{ij} and these in equal square regions Γ_{kl} of sidelength $\exp(1000\frac{1}{p}\log^2\frac{1}{p})$. It does not bother us if these numbers do not divide since it will become clear from the proof of the next lemma that this does not mean any loss of generality.

Lemma 6 *In order to prove theorem 5 it is sufficient to look at the dynamics restricted to* $[0,N]\times\Lambda$*, containing one* Γ *that is completely occupied.*

PROOF. In the decomposition that we have made we see that for an arbitrary Λ_{ij}, denoted Λ, the probability of not finding a completely occupied Γ is

$$(1-p^{|\Gamma|})^{|\Lambda|/|\Gamma|} = (1-p^{k_c^2})^{\frac{N^2}{100k_c{}^4}}.$$

$|\Lambda|$ denotes the cardinality of the set Λ. The reader can check for himself that this is vanishingly small compared to p, when $N \geq \exp\exp(2000\frac{1}{p}\log^2\frac{1}{p})$. We are thus justified in describing the behaviour of $\eta_\infty(n,\cdot)$ restricted to $[0,N]\times\Lambda$ containing some Γ that is completely occupied. Because of the attractiveness property of the model it is safe to set all sites outside of this restriction to zero. $\Box$

This decomposition is not strictly necessary for proving the theorem. Without it we could even find a smaller value for the prefactor in the doubly-exponential function. However, decomposition makes the proof easier and we are not primarily interested in the prefactor anyway.

Let Z_n be the smallest sidelength of the largest occupied rectangular region in $\eta^\Lambda(n,\cdot)$. This Z_n is the key variable in the proof. It is a Markov chain because, $\eta_\infty(n,\cdot)$, and therefore also $\eta_\infty^\Lambda(n,\cdot)$, is measurable with respect to $\sigma\{\eta_\infty(n-1,\cdot),\eta_0(n,\cdot)\}$. The transition probabilities of Z_n are complicated because of the many ways that a newly formed growing plane can split into smaller parts. Therefore we define another Markov chain X_n, in such a way that X_n will shrink faster and grow slower than Z_n. The behaviour of X_n can deduced. Let $P_\Box$ denote the probability that some site in a two dimensional lattice is the lower left corner point of an MBP^2 critical droplet. We know from Aizenman and Lebowitz (1988) that $P_\Box = \exp(-O(\frac{1}{p}\log\frac{1}{p}))$.

Definition 7 (The Markov chain X_n) *Let X_n be a Markov chain on $[0, N/10k_c]$ with transition probabilities*

$$
\begin{aligned}
p_{x,x+1} &= \frac{1}{k_c}\mathbf{I}_{\{x\geq k_c\}},\\
p_{xx} &= 1-5x^2(1-p)^x-(1-P_\square)^{x^2}-\frac{1}{k_c}\mathbf{I}_{\{x\geq k_c\}},\\
p_{xy} &= 5x(1-p)^x(1-(1-P_\square)^{xy}) \qquad 0<y<x,\\
p_{x0} &= (1-P_\square)^{x^2}.
\end{aligned} \tag{7}
$$

Further let $X_0 = \exp(1000\frac{1}{p}\log^2\frac{1}{p})$. $\mathbf{I}_{\{.\}}$ *denotes the indicator function.*

This definition needs some motivation. A growing critical droplet will grow an extra line in the secondary directions approximately every k_c times so we simply state that $p_{x,x+1} = 1/k_c$, given that the growing plane is supercritical. We see from Mountford (1995), that the critical droplet picture applies to SBP^{1,1}, i.e. that the occurrence of critical droplets is sufficient and necessary. Therefore we set the growth rate of X_n for sub-critical lengths to zero. It is now easy to see that X_n cannot grow faster than Z_n.

Establishing that X_n shrinks faster than Z_n is more difficult. In the definition of p_{xy}, where $0<y<x$, we see an element that is not contained in SBP^{1,1}. In 3 dimensions it is possible that the growing plane of the critical droplet splits and that only one part is internally spanned. For instance this happens when a complete line in a newly formed plane is empty while the part with smallest sidelength y is internally spanned and the other part is not. When y is sub-critical for the two dimensional MBP system it is highly probable that X_n will transit to state 0 the next time. These extra paths leading to state 0 account for a larger probability that the growth of the critical droplet stops. The essence of the proof of theorem 5 is therefore to show that this probability tends to zero faster than p, when $x\geq k_c$.

A single splitting, however, is not the only way for the growing plane to shrink. We must find a way to bound the probability of all possibilities in a term that can be handled and that shows that this probability is small enough. By reviewing systematically all possible contributions to p_{xy}, we see that p_{xy} is bounded by

$$p_{xy} \leq 5x(1-p)^x(1-(1-P_\Box)^{xy}). \tag{8}$$

Any possible way of shrinking needs some empty lines of thickness 1 that surround a growing MBP^2 critical droplet. It is easy to see that empty lines with an open ending in the system are no stable empty configurations. Therefore the only stable empty lines are (i) empty lines from one side to the opposite side and (ii) empty rectangular fringes with an exterior that is not internally spanned. For case (i) the possibilities are systematically generated by increasing the number of empty lines that create the splitting. It can easily be seen that these contributions to p_{xy} for any arbitrary number of empty lines n are bounded by

$$\delta p_{xy} \leq 2(1-p)^{nx}(1-(1-P_\Box)^{xy}). \tag{9}$$

For the case (ii), empty rectangular fringes, we must have an empty rectangle, where the interior is internally spanned while the exterior is not. It is an easy calculation to show that the probability that some site is the lower left corner of an empty rectangular fringe is equal to

$$P = (1-p)^{4y}(1-(1-P_\Box)^{y^2})(1-P_\Box)^{x^2-y^2}. \tag{10}$$

The probability that an empty rectangular fringe exists somewhere in the growing plane is another contribution to p_{xy}:

$$\begin{aligned}\delta p_{xy} &= 1-(1-P)^{x^2} \approx Px^2 \\ &= x^2(1-p)^{4y}(1-(1-P_\square)^{y^2})(1-P_\square)^{x^2-y^2}. \end{aligned} \tag{11}$$

For $y \geq \frac{1}{4}x$ it can be seen immediately that this is smaller than

$$x(1-p)^x(1-(1-P_\square)^{xy}). \tag{12}$$

For $y < \frac{1}{4}x$ we write (10) as

$$P \approx x(1-p)^x(1-(1-P_\square)^{xy})[xe^{p(x-y)}e^{-P_\square(x^2-y^2)}]. \tag{13}$$

The rightmost factors between square brackets are not smaller than 1 for all x, y but for large x they can be bounded by

$$\begin{aligned} & xe^{p(x-y)}e^{-P_\square(x^2-y^2)} \\ &= xe^{p(x-y)}e^{-p(x-y)\frac{P_\square}{p}(x+y)} \\ &= xe^{p(x-y)(\frac{P_\square}{p}(x+y)-1)} \\ &< xe^{-\frac{3}{4}px(\frac{P_\square}{p}x-1)}, \end{aligned}$$

which tends to zero if x is of large order, to be precise, if $\frac{1}{x} = o(\frac{P_\square}{p})$. If this is the case then we have also for $y < \frac{1}{4}x$, that P in (10) is smaller than

$$x(1-p)^x(1-(1-P_\square)^{xy}). \tag{14}$$

Now we gather all contributions to p_{xy}. By summing eq. (9) over all n, and adding the contributions from (12) and (14) we obtain (8). By summing the probabilities p_{xy} for all $y < x$ we obtain

$$\sum_{y=0}^{x} p_{xy} \leq e^{-(p-P_\square)x} \frac{1 - e^{-P_\square x^2} - x e^{-P_\square x^2} + x e^{-P_\square x(x+1)}}{1 - e^{-P_\square x}}. \tag{15}$$

There are now three cases to consider, corresponding to the scales of x for which the products px and px^2, either tend to 0 or to ∞, with $p \to 0$. The reader can work out himself what happens with (15) in these three cases, and we obtain for arbitrary x

$$\sum_{y} p_{xy} \leq 5x^2(1-p)^x. \tag{16}$$

3 Bounding of the spanning probability

The rest of the proof consists of crude labour. In order to prove the following lemma we must show that at time N the Markov chain X_n will with high probability be in state $N/10k_c$.

Lemma 8 *The probability that the Markov chain X_n is not in state $N/10k_c$ at time N is vanishingly small compared to p, to be precise, we have*

$$\inf_{N \geq \exp\exp(2000\frac{1}{p}\log^2\frac{1}{p})} \boldsymbol{P}[X_N = \frac{N}{10k_c}] \to 1,$$

when $p \to 0$.

PROOF. We observe that the probability $Q_{x,x+1}$ that X_n never grows to state $x+1$, given that it is in state x, is equal to the probability of all paths

$\{x, k_1, k_2, \ldots\}$ with all $k_i \leq x$, which eventually end in state 0. We have

$$Q_{x,x+1} = \sum_{n=0}^{\infty} (p_{xx})^n Q_{xx} \tag{17}$$

where $Q_{xx} = \sum_{m=0}^{\infty} p_{x0}(m)$ is the probability of all paths from state x to state 0 with $k_1 < x$. $p_{x0}(m)$ is the transition probability from x to 0 in m steps. Using the Chapman-Kolmogorov equations we get

$$p_{x0}(m+2) = \sum_{k=1}^{x-1} p_{xk}(m) p_{k0}(2). \tag{18}$$

These transition probabilities can be calculated directly. We get

$$p_{k0}(2) \leq \frac{2}{P_\square}(1-p)^k \tag{19}$$

and

$$p_{xk}(m) \leq \frac{1}{m! p^m} e^{-px}(1 - e^{-px}) \quad \forall k, 0 < k < x. \tag{20}$$

When this is substituted in eq. (18) we get

$$p_{x0}(m+2) \leq \frac{2}{P_\square} \frac{1}{m! p^{m+1}} e^{-px}(1 - e^{-px})^2. \tag{21}$$

This is a very enjoyable result since it tends to zero both when $px \to 0$ and when $px \to \infty$. Also it can be summed over m, which yields Q_{xx}:

$$\begin{aligned} Q_{xx} &= p_{x0} + p_{x0}(2) + \sum_{m=1}^{\infty} p_{x0}(m+2) \\ &\leq (1 - P_\square)^{x^2} \\ &+ \frac{2}{P_\square} e^{-px} \\ &+ \frac{2}{p P_\square} e^{\frac{1}{p}} e^{-px}(1 - e^{-px})^2. \end{aligned} \tag{22}$$

The reader can now easily check, that when p is taken to zero while it is assumed that $x \geq k_c = \exp(1000\frac{1}{p}\log^2\frac{1}{p})$, the quantity $Q_{xx} = o(\frac{1}{x^n})$, for any $n \in \mathbb{N}$. This is vanishingly small compared to p. Now we calculate $Q_{x,x+1}$ according to eq. (17).

$$\begin{aligned}
Q_{x,x+1} &= Q_{xx}\sum_{m=0}^{\infty} p_{xx}^m \\
&= Q_{xx}\sum_{m=0}^{\infty}\left[1 - o(e^{-(p-P_\Box)x}\frac{1}{P_\Box}) - e^{-P_\Box x^2} - \frac{1}{k_c}\right]^m \\
&\leq Q_{xx}\sum_{m=0}^{\infty}(1-\frac{1}{k_c})^m \\
&= Q_{xx}\frac{1}{1-1+\frac{1}{k_c}} \\
&= k_c Q_{xx}. \qquad (23)
\end{aligned}$$

Since we have $Q_{xx} = o(1/x^n)$, $\forall n \in \mathbb{N}$ we also have $Q_{x,x+1} = o(1/x^{n-1})$, $\forall n \geq 2$. We see immediately that $Q_{x,x+1}$ is also vanishingly small compared to p.

The next question we ask ourselves is what is the probability that X_n will never reach the state $\frac{N}{10k_c}$, given that it is in a state k_c. This probability is equal to

$$\begin{aligned}
&\sum_{k=x}^{\frac{N}{10k_c}}\sum_{m}^{\infty} p_{xk}(m) Q_{k,k+1} \\
&< \sum_{k=x}^{\frac{N}{10k_c}} Q_{k,k+1} \quad \approx \quad \int_x^{\frac{N}{10k_c}} o(\frac{1}{x^n})dx \\
&= o(\frac{1}{k_c^{n-2}}), \quad \forall n \geq 3. \qquad (24)
\end{aligned}$$

We now come closer to the end. We have seen from lemma 3 how rectangular columns grow in the SBP^{2,1} model. If the probability that X_n shrinks is set to zero for the moment then X_n will transit from x to $x+1$ approximately once in every k_c times. So if W is a binomial random variable with parameters N and $\frac{1}{k_c}$ then the probability $P\{X_n$ does not reach $\frac{N}{10k_c}$ by time $N\}$ is equal to

$$\begin{aligned}
P[W < \frac{N}{10k_c}] &= \frac{1}{2}P[|W - EW| > \frac{N}{k_c} - \frac{N}{10k_c}] \\
&\leq \frac{1}{2}(\frac{10k_c}{9N})^2 \mathrm{Var}(W) \\
&= \frac{1}{2}(\frac{10k_c}{9N})^2 \frac{N}{k_c}(1 - \frac{1}{k_c}) \\
&\leq \frac{50}{81}\frac{k_c}{N} = o(\frac{1}{k_c^n}), \quad \forall n \in \mathbb{N}. \qquad (25)
\end{aligned}$$

Note that using a quite rough Chebychev-inequality in the second line is sufficient to come to this result. Combining this with the former result we see that the probability that X_n does not reach state $\frac{N}{10k_c}$ by time N is of smaller order than $o(\frac{1}{k_c^n})$ for any $n \in \mathbb{N}$.

The last possibility we have to exclude is the chance that X_n does reach state $N/10k_c$, but starts shrinking again before time N. It can easily be seen that this probability is smaller than:

$$\begin{aligned}
&\sum_{m=0}^{N}(p(\frac{N}{10k_c}, \frac{N}{10k_c}))^m Q(\frac{N}{10k_c}, \frac{N}{10k_c}) \\
&= o((\frac{10k_c}{N})^n)\sum_{m=0}^{N}(1 - \frac{1}{k_c^n})^m \\
&= o(\frac{k_c^{2n}}{N^n}), \quad \forall n \in \mathbb{N}, \qquad (26)
\end{aligned}$$

where we wrote $p(x,x)$ and $Q(x,x)$ instead of p_{xx} and Q_{xx} for typographical convenience. We conclude that this contribution to the probability that $X_N \neq$

$N/10k_c$ is also vanishingly small compared to p.

Gathering all the results we see that the probability that X_n is not in state $\frac{N}{10k_c}$ at time N is of smaller order than $o(\frac{1}{k_c^n}), \forall n \in \mathbb{N}$. This proves the lemma. □

4 Proof of the theorem, corollary and outlook

Proof of theorem 5 All that is left now is gathering the results. We see from lemma 8 that X_n will almost surely be in state $N/10k_c$ at "time" N. Because X_n can be coupled to Z_n, this means that the dynamics restricted to $[0, N] \times \Lambda$ will fully occupy the rightmost side Λ_N of the restricted system. Lemma 6 says that this will be true for any arbitrary Λ and therefore the rightmost side of the entire system will almost surely be completely occupied by the dynamics when $N \geq \exp\exp(2000\frac{1}{p}\log^2\frac{1}{p})$. Because of the periodical boundary conditions this means that the entire system will become occupied. This proves the theorem. □

Since we have no proof that $\sup_{N \leq (1/p)^{k_c}} \mathbf{P}[\eta_\infty(N, \cdot) = 1] \to 0$, this result must be considered as an upper bound on the critical system size.

We give a corollary that follows immediately from the theorem. On the infinite lattice the dynamics of the SBP^{2,1} model will almost surely fill up the whole space, for all nonzero p, i.e. we have:

Corollary 9 $p_c = 0$ *for the SBP^{2,1} model.*

The proof of this corollary is left to the reader.

Finally, we remark that the results in this paper can be transfered to a $\mathrm{SBP}^{2,1}$ model in a cube with free boundary conditions, if we alter the definition of internal spanning. This definition only involves the right hand side wall of the system, in the sense that we say that the system is spanned if the $\eta_\infty(N,\cdot)=1$. The equivalent of theorem 5 then says that for an $\mathrm{SBP}^{2,1}$ system with free boundary conditions, we have in the supercritical regime

$$\inf_{N\geq\exp\exp(2000\frac{1}{p}\log^2\frac{1}{p})}\mathbf{P}[\eta_\infty(N,\cdot)=1]\to 1,$$

when $p\to 0$. Only minor adaptations of the proof are needed to make this clear, see also Mountford (1995).

Outlook We have not been able to prove that the spanning probability of the $\mathrm{SBP}^{2,1}$ model tends to zero for scales smaller than $\exp\exp O(\frac{1}{p}\log^2\frac{1}{p})$. This would be the first thing to do in order to extend the results on this model.

Also for other BP models, the proofs for the sub-critical regimes are very difficult to achieve. The basic BP model is the only one known where this has been done for arbitrary dimensions. For the other variants of the BP model no proofs exist for three or more dimensions.

Another possibility to extend the results in this paper might be a generalization of the proof of the supercritical regime to $\mathrm{SBP}^{n,1}$ models, where $n\geq 3$. Whether this is possible or not depends on how the higher dimensional versions of the splitting probability behave. It does not seem to be too difficult to deduce this.

Our last remark here concerns the regime of p that is covered. We derived our results for values of p which are asymptotically close to zero. For other models, e.g. the $l = 2$ BP it has been observed numerically that the slope of the graph of $p_c(k)$ against $1/\log k$ changes for larger values of p (see, for instance, remarks about this in Van Enter *et al* (1990)). This suggests that at those values of p other spanning processes can prevail. Such dual regimes, with a corresponding crossover point, may exist too in semi-oriented BP models. To the best of our knowledge nothing is known about this.

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References

Adler, J. Bootstrap percolation, Physica A **171** (1991) 453–470.

Aizenman, M. and Lebowitz, J.L. Metastability effects in bootstrap percolation, J. Phys. A **21** (1988) 3801–3813.

Chalupa, J., Leath, P.L. and Reich, G.R. Bootstrap percolation on a Bethe lattice, J. Phys. C **12** (1979) L31–35.

Dehghanpour, P. and Schonmann, R.H. A nucleation-and-growth model, (1996a) preprint.

Dehghanpour, P. and Schonmann, R.H. Metropolis dynamics relaxation via nucleation and growth, (1996b) preprint.

Duarte, J.A.M.S. Simulation of a cellular automat with an oriented bootstrap rule, Physica A **157** (1989) 1075.

Van Enter, A.C.D. Proof of Straley's argument for bootstrap percolation, J. Stat. Phys. **48** (1987) 943–945.

Van Enter, A.C.D., Adler, J. and Duarte, J.A.M.S. Finite size effects for some bootstrap percolation models, J. Stat. Phys. **60** (1990) 323–332.
+ addendum, J. Stat. Phys. **62** (1991) 505–506.

Liggett, T. Interacting particle systems, Grundlehren der mathematischen wissenschaften 276, Springer Verlag (1985).

MOUNTFORD, T.S. Critical length for semi-oriented bootstrap percolation, Stochastic processes and their applications, **56(2)** (1995) 185–206.

SCHONMANN, R.H. Critical points of two-dimensional bootstrap percolation-like cellular automata, J. Stat. Phys. **58** (1990) 1239–1244.

SCHONMANN, R.H. On the behaviour of some cellular automata related to bootstrap percolation, Ann. of Prob. **20** (1992) 174–193.