Asymptotic of the Density of States for the Schrödinger Operator with Periodic Electro-Magnetic Potential

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Abstract

For the Schrödinger operator in $L^2(\mathbf{R}^n)$, $n > 1$, with C^∞ periodic electro-magnetic potential, we give an asymptotic formula of the integrate density of states of the form

$$N(\mu) = a_n \mu^{n/2} + \mathbf{O}(\mu^{(n-2+\epsilon)/2}), \quad \forall\, \epsilon > 0.$$

When $n = 2$, this estimate enables us to prove the finiteness of gaps in the spectrum.

I Introduction

This paper is the continuation of[1] in which a rigorous proof of the Bethe-Sommerfeld conjecture was given for the Schrödinger operator with a periodic electrical potential in the low dimensions.

Here we are interested in the study of the Schrödinger operator with periodic electro-magnetic potential, on $L^2(\mathbf{R}^n)$, with $n \geq 2$,

$$P_{V,A} = \sum_{j=1}^{n} (D_{x_j} - A_j(x))^2 + V(x); \tag{I.1}$$

$D_s = -i\partial_s$ where $\partial_s = \dfrac{\partial}{\partial s}$, if s is a real variable.
$V(x)$ and $A(x) := (A_1(x), \ldots, A_n(x))$ are real and C^∞ :

$$V(x),\ A_1(x),\ \ldots,\ A_n(x)\ \in C^\infty(\mathbf{R}^n; \mathbf{R}). \tag{I.2}$$

Let Γ be a lattice on $\mathbf{R}^n$:

$$\Gamma = \{\sum_{j=1}^{n} k_j e_j;\ k = (k_1, \ldots, k_n) \in \mathbf{Z}^n\}, \tag{I.3}$$

where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbf{R}^n$.

We assume that the electrical potential V and the magnetic one A are periodic:

$$V(x + a) = V(x) \ \text{ and } \ A(x + a) = A(x), \quad \forall a \in \Gamma. \tag{I.4}$$

It is well known that $P_{V,A}$ is essentially self-adjoint on $L^2(\mathbf{R}^n)$, from $C_0^\infty(\mathbf{R}^n)$, the space of C^∞ functions with compact support.

For any continuous function of compact support $f \in C_0^0(\mathbf{R})$, the bounded operator on $L^2(\mathbf{R}^n)$, $f(P_{V,A})$ has a C^∞ kernel $K_{f(P_{V,A})}(x,y)$.

The periodicity of V and A implied that the kernel is also periodic on the diagonal:

$$K_{f(P_{V,A})}(x+a,x+a) = K_{f(P_{V,A})}(x,x), \quad \forall a \in \Gamma \text{ and } x \in \mathbf{R}^n. \tag{I.5}$$

See[2] and[3] for more general situations.

If $\varphi(x) \in C^\infty(\mathbf{R}^n;\mathbf{R})$, then the two operators $P_{V,A}$ and $P_{V,A+\nabla\varphi}$ are gauge equivalent and then unitarily equivalent. Due to the formula

$$K_{f(P_{V,A})}(x,x) = K_{f(P_{V,A+\nabla\varphi})}(x,x), \quad \forall f \in C_0^0(\mathbf{R}),$$

the spectral results on $P_{V,A}$ we will make out will be gauge-invariant. We need only V and the magnetic field B to be C^∞ and periodic, and the two-form b associted to B on the torus $\mathbf{T}^n := \mathbf{R}^n/\Gamma$ to be exact:
$B(x) = (B_{j,k}(x))_{1\le j,k\le n},\ B_{j,k}(x) = \partial_{x_j}A_k(x) - \partial_{x_k}A_j(x),\ b := \sum_{1\le j<k\le n} B_{j,k}(x)dx_j \wedge dx_k.$

We may change gauge, so it is easy to see that we can choose A such that

$$\int_{\mathcal{K}} A(x)dx = 0 \quad \text{and} \quad div(A) := \sum_{j=1}^n \partial_{x_j}A_j = 0. \tag{I.6}$$

$\mathcal{K}$ denotes the periodic cell:

$$\mathcal{K} = \{\sum_{j=1}^n t_j e_j \ ;\ t_j \in [0,1]\}. \tag{I.7}$$

A such periodic magnetic potential is unique and is given by

$$A(x) = \sum_{\gamma\in\Gamma^\star} A_\gamma e^{ix\gamma} \ \text{ with } \ A_0 = (0,\dots,0) \ \text{ and } A_\gamma = \frac{i}{|\gamma|^2}B_\gamma\gamma \text{ if } \gamma \neq 0, \tag{I.8}$$

if $B(x) = \sum_{\gamma\in\Gamma^\star} e^{ix\gamma}B_\gamma$ and $\Gamma^\star$ denotes the dual lattice:

$$\Gamma^\star := \{\gamma \in \mathbf{R}^n;\ \gamma a \in 2\pi\mathbf{Z},\ \forall a \in \Gamma\} = \{\sum_{j=1}^n k_j e_j^\star;\ k_j \in \mathbf{Z}\}, \tag{I.9}$$

$(e_j^\star)_{1\le j\le n}$ is the dual basis of $(e_j)_{1\le j\le n}$ in (I.3): $e_j e_k^\star = 2\pi\delta_{jk}$.

An important measure associated to $P_{V,A}$ and which is gauge-invariant is the density of states of $P_{V,A}$. It is the Borel measure $\rho_{V,A}$ defined from (I.5) and (I.7) by

$$\rho_{V,A}(f) := \int_{\mathbf{R}} f(\tau)d\rho_{V,A}(\tau) = |\mathcal{K}|^{-1}\int_{\mathcal{K}} K_{f(P_{V,A})}(x,x)dx, \quad \forall f \in C_0^0(\mathbf{R}). \tag{I.10}$$

As $P_{V,A}$ is an elliptic and bounded below differential operator, for every caracteristic function $f = \chi_I$ of bounded above interval I, the kernel $K_{\chi_I(P_{V,A}}(x,y)$ is also C^∞, so $\rho_{V,A}(\chi_I)$ in (I.10) is well defined. The integrated density of state is the real function

$$N(\mu;P_{V,A}) = \int_{-\infty}^{\mu} d\rho_{V,A}(\tau) = \rho_{V,A}(\chi_{]-\infty,\mu[}). \tag{I.11}$$

This increasing function is defined also by

$$\int_{\mathbf{R}} f(\tau)d\rho_{V,A}(\tau) = -\int_{\mathbf{R}} \partial_\tau f(\tau)N(\tau; P_{V,A})d\tau, \quad \forall f \in C_0^\infty(\mathbf{R}), \tag{I.12}$$

and

$$N(\tau; P_{V,A}) = 0, \quad \forall \tau \in]-\infty, -\parallel V \parallel_{L^\infty(\mathbf{R}^n)}[. \tag{I.13}$$

In the case without magnetic field, it is well known by the Floquet theory, see[4], that the spectrum of $P_{V,0}$, $\sigma(P_{V,0})$, is purely absolutely continuous. It is formed by bands seperated possibly by open intervals called gaps. In[1] it was proved that

$$N(\mu; P_{V,0}) = a_n\mu^{\frac{n}{2}} + a_{n-2}\mu^{\frac{n-2}{2}} + \mathbf{O}(1 + \mu^{(n-3+\epsilon)/2}); \tag{I.14}$$

with $a_n = (2\pi)^{-n}|\mathbf{S}^{n-1}|/n$, and $a_{n-2} = -\frac{na_n}{2}|\mathcal{K}|^{-1}\int_{\mathcal{K}} V(x)dx$.

The estimate (I.14) enables their authors to give an alternative proof of the Bethe-Sommerfeld conjecture, the finiteness of gaps in the spectrum of $P_{V,0}$, for the low dimensions, $2 \leq n \leq 4$.

The Floquet theory works also for $P_{V,A}$, (the magnetic potential A is periodic). If $\mathcal{K}^\star$ is the basic periodic cell of the dual lattice $\Gamma^\star$, then $P_{V,A}$ is unitarily equivalent to $\int_{\mathcal{K}^\star}^{\oplus} P_{V,A}^\theta \frac{d\theta}{|\mathcal{K}^\star|}$; $P_{V,A}^\theta$ is the self-adjoint operator on $L^2(\mathbf{T}^n)$ defined by

$$P_{V,A}^\theta := \sum_{j=1}^n (D_{x_j} - \theta_j - A_j(x))^2 + V(x), \quad \forall \theta \in \mathcal{K}^\star.$$

So the spectrum of $P_{V,A}$, $\sigma(P_{V,A})$, is formed by bands $b_j = \{\lambda_j(\theta);\ \theta \in \mathcal{K}^\star\}$, seperated possibly by gaps, we denoted $(\lambda_j(\theta))_j$ the increasing sequence of eigenvalues of the operator $P_{V,A}^\theta$.

Following the proof of[5] reproduced in[4], on the absolutely continuity of the spectrum of $P_{V,0}$, it is easy to see that that the singular spectrum of $P_{V,A}$, $\sigma_s(P_{V,A})$, is reduced to the closure of its point spectrum $\sigma_p(P_{V,A})$ and that

$$\sigma_p(P_{V,A}) \subset \sigma(P_{V,A}^\theta), \quad \forall \theta \in \mathcal{K}^\star.$$

So, if $\sigma_s(P_{V,A})$ is not empty, then it is discrete and $+\infty$ is its only accumulation point.

In[6], it was proved the $\sigma(P_{V,A})$ is still absolutely continuous provided the $\parallel A \parallel_{L^\infty(\mathbf{R}^n)}$ is small enough. The proof of[7] which shows that there is no eigenvalue with finite multiplicity for $P_{V,0}$ is still valid for $P_{V,A}$.

Let us recall some useful formulas coming from the Floquet theory:

$$N(\mu; P_{V,A}) = |\mathcal{K}^\star|^{-1}\int_{\mathcal{K}^\star}\Big(\sum_{\lambda_j(\theta)<\mu} 1\Big)d\theta, \tag{I.15}$$

and more generally

$$\int f(\tau)d\rho_{V,A}(\tau) = |\mathcal{K}^\star|^{-1}\int_{\mathcal{K}^\star}\sum_j f(\lambda_j(\theta))d\theta, \quad \forall f \in C_0^0(\mathbf{R}).$$

In this paper we generalize the estimate (I.14) in the following way.

Theorem I.1 : *Under the assumptions on V and A (I.2) and (I.4), if $n \geq 2$, then for any fixed $\epsilon > 0$, we have the following asymptotic expansion as $\mu \to +\infty$,*

$$N(\mu; P_{V,A}) = a_n \mu^{\frac{n}{2}} \; + \; \mathbf{O}(\mu^{(n-2+\epsilon)/2}); \tag{I.16}$$

with $a_n = (2\pi)^{-n} |\mathbf{S}^{n-1}|/n$.

Using (I.15) to get an asymtotic formula for $N(\mu; P_{V,A})$, the modern techniques of[10] to study the counting function of the eigenvalues of elliptic differential operator would give

$$N(\mu; P_V) = a_n \mu^{\frac{n}{2}} + \circ(\mu^{\frac{n-1}{2}}),$$

when they are applied to the $P^{\theta}_{V,A}$.

The remainder estimate (I.16) is sharp, if we compare it to the well known one above.

When V takes its minimum only on Γ and when the associated hessian is non degenerated then, if $B(0) = 0$, using semi-classical analysis, (see for exemple[8]), it is easy to see that there exists $\eta_N > 0$ such that the N first bands of the spectrum of $P_{\lambda^2 V, \lambda A}$ are localised in the intervals $I_k =] - \eta_N + \lambda a_k, \lambda a_k + \eta_N[$, for $k = 1, \ldots, N$. (a_k), is the increasing sequence of eigenvalues of an harmonic oscillator with constant magnetic field. So, for large $\lambda > 0$, there are many gaps in the spectrum of $P_{\lambda^2 V, \lambda A}$.

The same is true for $P_{V, \lambda A}$ if the only zero of $B(x)$ are Γ and if they are zero of some order m, one can take $I_k =] - \eta_N + \lambda^{2/(m+2)} a_k, \lambda^{2/(m+2)} a_k + \eta_N[$, see[9]. (a_k), is the sequence of eigenvalues of a Schrödinger operator with polynomial magnetic field.

The crutial question is to know if the gaps in the spectrum are finite or no. As in the case without magnetic field in[1], the estimate (I.16) enables us to give an answer for this question for some dimension n. Here we can just say that the equivalent of Bethe-Sommerfeld conjecture is true when $n = 2$: the gaps in the spectrum of $P_{V,A}$ are finite if $n = 2$. (see Corollary (II.2)).

Remind that the condition zero flux for the magnetic field, (i.e. the two-form b is exact), is necessary for the finiteness of the gaps; for $n = 2$ and for a constant magnetic field B_0, $A(x) = (b_0 x_2, -b_0 x_1)$ with $b_0 \neq 0$,
the gaps in the spectrum of $P_{\epsilon V, A}$ are infinite if $\epsilon > 0$ is small enough.

Our method is the same as in[1]. For $\lambda_0 > 0$ such that

$$P_{V,A} + \lambda_0 > 0, \tag{I.17}$$

we have to construct the half-wave operator e^{itQ},

$$Q := [P_{V,A} + \lambda_0]^{\frac{1}{2}}, \tag{I.18}$$

as a Fourier integral operator. For small $|t|$ the calculus is well known, (see[11] or[10]). For large $|t|$, we need just to control the symbol of the pseudo-differential operator, (p.d.o.), $e^{itQ} e^{-itQ_0}$,

$$Q_0 := [P_{0,0} + \lambda_0]^{\frac{1}{2}} = (-\Delta + \lambda_0)^{\frac{1}{2}}. \tag{I.19}$$

For an operator which is a periodic p.d.o. one, it was proved in[1] that its symbol can be estimated in term of $L^2(\mathbf{T}^n)$ norm of the related Floquet operator. Here, as $Q - Q_0$ is a p.d.o. of order zero, (instead of order -1 for the case without magnetic field), the estimates of the derivatives of the symbol of $e^{itQ} e^{-itQ_0}$ are less fine for large $|t|$.

If $w_{V,A}(t,x,\xi)$ is the symbol of $e^{itQ}e^{-itQ_0}$, $w_{V,A}(x,D_x) = e^{itQ}e^{-itQ_0}$,
and if $r_{V,A}(t,\xi) = \int_{\mathcal{K}}(w_{V,A}(t,x,\xi)-1)dx$,
then, when V is normalized, it was proved in[1] that, for any $\alpha \in \mathbf{N}^n$,

$$< \xi >^{1+2|\alpha|} [< t > (< \xi > + < t >)]^{-|\alpha|} \partial_\xi^\alpha r_{V,0}(t,\xi) \in L^\infty(\mathbf{R}\times\mathbf{R}^n),$$

$(< z >= (|z|^2+1)^{1/2}$ if $z \in \mathbf{R}^m)$. Here with a magnetic potential, we have only

$$< \xi >^{|\alpha|} < t >^{-2|\alpha|} \partial_\xi^\alpha r_{V,A}(t,\xi) \in L^\infty(\mathbf{R}\times\mathbf{R}^n).$$

This explains the roughness of the estimate (I.16) compared to the one (I.14).

Later on, we will assume the assumptions of Theorem (I.1), the magnetic potential A will satisfy (I.6) and we will normalize $V(x)$ by

$$\int_{\mathcal{K}} V(x)dx = 0. \tag{I.20}$$

If E is a Hilbert space, $\mathcal{L}(E)$ will denote the space of bounded linear operator of E.

For any periodic, (Γ-periodic), function g on $\mathbf{R}^n$, (g_γ) will be its Fourier coefficients:

$$g(x) = \sum_{\gamma\in\Gamma^\star} g_\gamma e^{ix\gamma}, \text{ if } g \in L^2_{loc}(\mathbf{R}^n) \text{ and } g(x+a) = g(x), \ \forall a \in \Gamma. \tag{I.21}$$

For any operator $T \in \mathcal{L}(\mathbf{R}^n)$, $K_T(x,y)$ will be its Schwartz kernel:

$$< Tf|g >:= \int_{\mathbf{R}^n} Tf(x)\overline{g}(x)dx = \int_{\mathbf{R}^n\times\mathbf{R}^n} K_T(x,y)f(y)\overline{g}(x)dydx, \ \forall f,g \in L^2(\mathbf{R}^n). \tag{I.22}$$

For any integer m and any $f \in L^2(\mathbf{R}^m)$, $\hat{f}$ will denote the Fourier transform of f,

$$\hat{f}(\xi) = \int_{\mathbf{R}^m} e^{-ix\xi} f(x)dx, \tag{I.23}$$

and for any $x \in \mathbf{R}^m$, $< x >= (|x|^2+1)^{\frac{1}{2}}$.

II The main result for the Fourier coefficients of the number of Floquet eigenvalues

Let $\lambda_0 > 0$ be fixed such that $(P_{V,A}+\lambda_0)^{-1}$ exists, for exemple we take

$$\lambda_0 = 1 + \| V \|_{L^\infty(\mathbf{R}^n)} . \tag{II.1}$$

We define the operators Q and Q_0 by (I.18) and (I.19) .

For a periodic operator $L \in \mathcal{L}(\mathbf{L}^2(\mathbf{R}^n))$, $LT_a = T_aL$, $\forall a \in \Gamma$,
with $T_a(f)(x) = f(x+a)$, $\forall f \in L^2(\mathbf{R}^n)$,
such that L has a smooth kernel $K_L(x,y)$, we denote the average trace of L by $\widetilde{\mathrm{Tr}}(\mathrm{L})$,

$$\widetilde{Tr}(L) := \frac{1}{|\mathcal{K}|}\int_{\mathcal{K}} K_L(x,x)dx. \tag{II.2}$$

Then we get

$$\widetilde{Tr} f(Q) = \rho_{V,A}(F), \ \forall f \in C_0^\infty(\mathbf{R}), \tag{II.3}$$

with $F(\tau) = f((\tau+\lambda_0)^{\frac{1}{2}})$, if $\tau \in [-\lambda_0 + \mathrm{Inf}\{\sigma(P_{V,A})\}, +\infty[$,
the value of $F(\tau)$, when $\tau < -\lambda_0 + \mathrm{Inf}\{\sigma(P_{V,A})\}$, does not matter. ($\rho_{V,A}$ is defined in (I.10)).

Let $b \in \Gamma$ and L a periodic operator with continuous kernel, then the function $x \to K_L(x+b,x)$, is periodic. We define its average

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}(\mathrm{L}) := \widetilde{\mathrm{Tr}}(\mathrm{T_b L}) = |\mathcal{K}|^{-1}\int_{\mathcal{K}} \mathrm{K_L(x+b,x)dx}. \tag{II.4}$$

Then we define the complex measure $\varrho^b_{V,A}$,

$$\varrho^b_{V,A}(f) = \int_{\mathbf{R}} f(\tau) d\varrho^b_{V,A}(\tau) := \widetilde{\mathrm{Tr}}_{\mathrm{b}}(\mathrm{f(P_{V,A})}), \quad \forall \mathrm{f} \in \mathrm{C}^0_0(\mathbf{R}). \tag{II.5}$$

Remark that $\varrho_{V,A} = \varrho^0_{V,A}$.

The Floquet theory gives

$$\varrho^b_{V,A}(f) = -\int_{\mathbf{R}} \partial_\tau f(\tau) N_b(\tau, P_{V,A}) d\tau, \quad \forall f \in C^\infty_0(\mathbf{R}), \tag{II.6}$$

with $N_b(\tau, P_{V,A})$ defined as in (I.15) by

$$N_b(\tau, P_{V,A}) := |\mathcal{K}^\star|^{-1} \int_{\mathcal{K}^\star} e^{ib\theta} (\sum_{\lambda_j(\theta)<\tau} 1) d\theta. \tag{II.7}$$

We get a similar formula to (II.3)

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}(\mathrm{f(Q)}) = \varrho^{\mathrm{b}}_{\mathrm{V,A}}(\mathrm{F}), \quad \forall \mathrm{f} \in \mathrm{C}^\infty_0(\mathbf{R}), \tag{II.8}$$

(F is related to f as in (II.3)).

The formulas (II.6) and (II.7) show that $\varrho^b_{V,A}$ is a Borel measure and

$$|\varrho^b_{V,A}(f)| \le \varrho_{V,A}(|f|), \quad \forall f \in C^\infty_0(\mathbf{R}). \tag{II.9}$$

The main result of this chapter is the following theorem.

Theorem II.1 *Assume the assumptions of Theorem (I.1), (I.20) and* $n = 2$.
Then there exists $b \in \Gamma$, $b \neq 0$, *and* $\eta_b > 0$, *depending only on b, such that*

$$|N_b(\mu; P_{V,A})| + |N_{2b}(\mu; P_{V,A})| \ge \eta_b \mu^{\frac{n-1}{4}}, \quad \forall \mu > 1 \tag{II.10}$$

We define now

$$N^+(\mu; P_{V,A}) = \sup_\theta \{ \sum_{\lambda_j(\theta)<\mu} 1\} \text{ and } N^-(\mu; P_{V,A}) = \inf_\theta \{ \sum_{\lambda_j(\theta)<\mu} 1\}. \tag{II.11}$$

Following an idea of Dahlberg and Trubowitz[12], as in[1], we get the following Corollary from (I.1) and (II.1).

Corollary II.2 *If* $n = 2$, *then there exists a non negative constant* $\eta_0 > 0$ *such that*

$$N^+(\mu; P_{V,A}) - N^-(\mu; P_{V,A}) \ge \eta_0 \mu^{(n-1)/4}, \ \forall \mu > 1. \tag{II.12}$$

So there exists $\Lambda > 0$, *such that*

$$[\Lambda, +\infty[\subset \sigma(P_{V,A}). \tag{II.13}$$

The inclusion (II.13) proves a Bethe-Sommerfeld conjecture for $P_{V,A}$, the finiteness of gaps in the spectrum of $P_{V,A}$, when $n = 2$.

One can find the history of the proof of the original Bethe-Sommerfeld conjecture for $P_{V,0}$ in[1].

The theorems (I.1) and (II.1) will be proved in section 6.

Remark II.3 *The estimate (II.10) is always satisfied for any non zero b, if the magnetic field is small enough, i.e. when $n = 2$, there exits $\epsilon_0 > 0$ such that, if $0 < \epsilon < \epsilon_0$, then*

$$|N_b(\mu; P_{V,\epsilon A})| + |N_{2b}(\mu; P_{V,\epsilon A})| \geq \eta_b \mu^{\frac{n-1}{4}}, \quad \forall \mu > 1 \tag{II.14}$$

η_b is as in (II.10) and can be chosen independent of ϵ.

III Periodic pseudo-differential calculus

We adopt the notations of[1]. Let $d > 0$ be an integer and g_z a Riemannian diagonal metric on $\mathbf{R}^d$,

$$g_z = \sum_{j=1}^{d} q_j^2(z) dz_j^2, \quad (\text{with } q_j(z) > 0). \tag{III.1}$$

For a non negative continuous function, $M(z) \in C(\mathbf{R}^d; \mathbf{R}_+)$,
and for a regular function $p(z)$, we write as in[10]

$$p(z) \in S(M(z); g_z) \quad iff \quad \forall \alpha \in \mathbf{N}^d, \ \exists C_\alpha > 0, \ s.t. \ |\partial_z^\alpha p(z)| \leq C_\alpha M(z) \prod_{j=1}^{d} q_j^{\alpha_j}(z). \tag{III.2}$$

We will denote by g_z^0 the usual Euclidian metric on $\mathbf{R}^d$:

$$g_z^0 = \sum_{j=1}^{d} dz_j^2. \tag{III.3}$$

$\mathcal{S}(\mathbf{R}^d)$ will denote the Schwartz space of C^∞ fast-decreasing functions and $\mathcal{S}'(\mathbf{R}^d)$ its dual space. For any real m, $H^m(\mathbf{R}^d)$ will denote the usual Sobolev space of order m on $\mathbf{R}^d$.

We consider on $\mathbf{R}^{2n}$ the metric

$$g_{x,\xi} = g_x^0 + <\xi>^{-2} g_\xi^0, \quad \forall (x,\xi) \in \mathbf{R}^{2n}. \tag{III.4}$$

A symbol $p(x,\xi) \in S(<\xi>^m; g_{x,\xi})$ is said periodic iff

$$p(x+a,\xi) = p(x,\xi), \ \forall a \in \Gamma, \tag{III.5}$$

in this case, we will write $\quad p(x,\xi) = \sum_{\gamma \in \Gamma^\star} e^{ix\gamma} p_\gamma(\xi)$.

Then, for any $(j,\alpha) \in \mathbf{N} \times \mathbf{N}^n$, $\exists \ C_{j,\alpha} > 0$ s.t.

$$|\partial_\xi^\alpha p_\gamma(\xi)| \leq C_{j,\alpha} <\gamma>^{-j} <\xi>^{m-|\alpha|} \quad \forall (\gamma,\xi) \in \Gamma^\star \times \mathbf{R}^n. \tag{III.6}$$

For a periodic symbol $p(x,\xi) \in S(<\xi>^m; g_{x,\xi})$,
and for any fixed $\theta \in \mathbf{R}^n$, the p.d.o. $p(x,\theta + D)$, defined by the symbol $p(x,\theta+\xi)$, leads to

an operator $p^\theta(x,D) : C^\infty(\mathbf{T}^n) \longrightarrow (C^\infty(\mathbf{T}^n))'$,
$p^\theta(x,D)v(x) = p(x,\theta+D)\tilde{v}(x)$, ($\tilde{v}(x)$ is the extension of $v(x)$ on $\mathbf{R}^n$ by Γ-periodicity), and, if $v(x) = \sum_{\gamma\in\Gamma^\star} v_\gamma e^{ix\gamma} \in C^\infty(\mathbf{T}^n)$,

$$p^\theta(x,D)v(x) = \sum_{\gamma\in\Gamma^\star} p(x,\theta+\gamma)v_\gamma e^{ix\gamma} = \sum_{\gamma\in\Gamma^\star} e^{ix\gamma} \sum_{\varrho\in\Gamma^\star} v_\varrho p_{\gamma-\varrho}(\theta+\varrho). \tag{III.7}$$

$p^\theta(x,D)$ operates continuously on the Sobolev spaces in the following way,

$$p^\theta(x,D) : H^s(\mathbf{T}^n) \longrightarrow H^{s-m}(\mathbf{T}^n), \quad \forall s \in \mathbf{R}.$$

If $q(x,\xi)$ is an other periodic symbol,

$$q(x,\xi) = \sum_{\gamma\in\Gamma^\star} q_\gamma(\xi)e^{ix\gamma} \in S(<\xi>^l; g^\delta_{x,\xi}),$$

then

$$s(x,D) = p(x,D)q(x,D) \iff s^\theta(x,D) = p^\theta(x,D)q^\theta(x,D), \ \forall\theta\in\mathcal{K}^\star, \tag{III.8}$$

and obviously, in this case $s(x,\xi)$ is a periodic symbol

$$s(x,\xi) = \sum_{\gamma\in\Gamma^\star} s_\gamma(\xi)e^{ix\gamma}, \ s_\gamma(\xi) = \sum_{\varrho\in\Gamma^\star} p_{\gamma-\varrho}(\xi+\varrho)q_\varrho(\xi). \tag{III.9}$$

We mention that $s \in S(<\xi>^{m+l}; g_{x,\xi})$, and for any integer N,

$$s(x,\xi) - \sum_{|\alpha|<N} \frac{1}{i^{|\alpha|}\alpha!}\partial^\alpha_\xi p(x,\xi)\partial^\alpha_x q(x,\xi) \ \in S(<\xi>^{m+l-N}\ g_{x,\xi}). \tag{III.10}$$

Let us recall the Floquet theory for p.d.o.. Let T be an operator defined on $L^2(\mathbf{R}^n)$, which is a periodic p.d.o. of order m,

$$Tu(x) = \sigma(x,D)u(x), \quad \forall u \in D(T), \tag{III.11}$$

with $\quad \sigma(x,\xi) = \sum_{\gamma\in\Gamma^\star} \sigma_\gamma(\xi)e^{ix\gamma} \in S(<\xi>^m; g_{x,\xi})$.

If $m>0$, we assume T elliptic, i.e. there exists $a(x,\xi) \in S(<\xi>^{-m}; g_{x,\xi})$, such that

$$a(x,D)\sigma(x,D) - I_d \in OpS(<\xi>^{-1}; g_{x,\xi}). \tag{III.12}$$

Then

$$T = \int^\oplus_{\mathcal{K}^\star} T^\theta \frac{d\theta}{|\mathcal{K}^\star|}, \quad \text{with} \ \ T^\theta := \sigma^\theta(x,D). \tag{III.13}$$

More over, if $m \le 0$, then

$$\sum_{\gamma\in\Gamma^\star} |\sigma_\gamma(\xi)|^2 \le \| \sigma(x,D) \|^2_{\mathcal{L}(L^2(\mathbf{R}^n))} . \tag{III.14}$$

This estimate comes from (III.13) and

$$\sum_{\gamma\in\Gamma^\star} \sigma_{\gamma-\rho}(\xi)e^{ix\gamma} = \sigma^\theta(x,D)(e_\rho), \quad \text{with} \ e_\rho(x) := e^{ix\rho},$$

if $\rho\in\Gamma^\star$ and $\xi-\rho = \theta\in\mathcal{K}^\star$; (see[1] for more details).

IV The half wave unitary group

Let us assume the hypotheses of Theorem (I.1) with electro-magnetic potentials V and A such that (I.6) and (I.20) are satisfied. If Q is the operator defined by (I.17) and (I.18) , then by[13] and [14], (see also[15]), Q is an p.d.o.:

$$Q \in OpS(<\xi>; g_{x,\xi}), \quad Qu(x) = q(x,D)u(x), \quad \forall u \in \mathcal{S}(\mathbf{R}^n). \tag{IV.1}$$

It is obvious from (I.4) that Q is a periodic p.d.o.,

$$q(x,\xi) = \sum_{\gamma\in\Gamma^\star} q_\gamma(\xi)e^{ix\gamma} \ \in S(<\xi>; g_{x,\xi}). \tag{IV.2}$$

The Seeley calculus[13], (see also[10]), shows that

$$q(x,\xi) - (|\xi - A(x)|^2 + \lambda_0)^{1/2} - \frac{1}{2}V(x)(|\xi - A(x)|^2 + \lambda_0)^{-1/2} \tag{IV.3}$$

$$+\frac{1}{2i}(|\xi - A(x)|^2 + \lambda_0)^{-3/2}\xi\partial_x(\xi A(x)) \ \in S(<\xi>^{-2}; g_{x,\xi}).$$

So, if

$$\sigma^0(\xi) = (|\xi|^2 + \lambda_0)^{\frac{1}{2}}, \quad (Q_0 = \sigma^0(D)), \tag{IV.4}$$

then $\quad q(x,\xi) \sim \sigma^0(\xi) + \sum_{j\geq 0} q_{-j}(x,\xi)$,

in the sense that $\quad q_{-j}(x,\xi) \in S(<\xi>^{-j}; g_{x\xi}), \ \forall j \geq 0$,
and for any $N > 0$,

$$q(x,\xi) - \sigma^0(\xi) - \sum_{j=0}^{N-1} q_{-j}(x,\xi) \ \in S(<\xi>^{-N}; g_{x,\xi}); \tag{IV.5}$$

¿From (IV.3) and (IV.4), we may take

$$q_0(x,\xi) = -\frac{\xi A(x)}{\sigma^0(\xi)} \tag{IV.6}$$

and

$$q_{-1}(x,\xi) = \frac{|A(x)|^2}{2\sigma^0(\xi)} - \frac{(\xi A(x))^2}{2(\sigma^0(\xi))^3} + \frac{V(x)}{2\sigma^0(\xi)} - \frac{\xi\partial_x(\xi A(x))}{2i(\sigma^0(\xi))^{-3}}. \tag{IV.7}$$

The main point we will need is that (I.6) and (I.20) imply that

$$\int_{\mathcal{K}} q_0(x,\xi)dx = 0 \ \text{ and } \int_{\mathcal{K}} q_{-1}(x,\xi)dx = \int_{\mathcal{K}} (\frac{|A(x)|^2}{2\sigma^0(\xi)} - \frac{(\xi A(x))^2}{2(\sigma^0(\xi))^3})dx. \tag{IV.8}$$

Theorem IV.1 *Assume the assumptions of Theorem (II.1) .*
Let $c(x,D)$ be a given periodic p.d.o.:

$$c(x,\xi) = \sum_{\gamma\in\Gamma^\star} c_\gamma(\xi)e^{ix\gamma} \ \in S(<\xi>^{-m}; g_{x,\xi}), \quad \text{with } m \geq 0. \tag{IV.9}$$

Then for any real fixed t_0,

$$W_c(t_0) := e^{it_0Q}c(x,D)e^{-it_0Q_0} \in OpS(<\xi>^{-m}; g_{x,\xi}), \tag{IV.10}$$

and it is a periodic p.d.o.. More precisely there exists a symbol
$w_c(t,x,\xi) \in C^\infty(\mathbf{R}\times\mathbf{R}^{2n}), \quad w_c(t,x+a,\xi) = w_c(t,x,\xi), \quad \forall a \in \Gamma,$
such that

$$W_c(t) = w_c(t,x,D_x), \quad \forall t \in \mathbf{R}. \tag{IV.11}$$

More over, if

$$w_c(t,x,\xi) = \sum_{\gamma\in\Gamma^\star} w_{c,\gamma}(t,\xi)e^{ix\gamma}, \tag{IV.12}$$

then for any $(j,k,\alpha) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}^n$, there exists a contant $C_{j,k,\alpha}$ such that

$$(\sum_{\gamma\in\Gamma^\star} |\gamma|^{2j}|\partial_t^k\partial_\xi^\alpha w_{c,\gamma}(t,\xi)|^2)^{1/2} \leq C_{j,k,\alpha} < t >^{j+2|\alpha|}< \xi >^{-m-|\alpha|}, \tag{IV.13}$$

$\forall (t,\xi) \in \mathbf{R}\times\mathbf{R}^n$, *. So*

$$w_c(t,x,\xi) \in S(< t >^{[\frac{n}{2}]+1}< \xi >^{-m}; dt^2 + < t >^2\ g_x^0 + \frac{< t >^4}{< \xi >^2} g_\xi^0). \tag{IV.14}$$

(For a real x, $[x]$ is the integer part of x, $[x] \leq x < [x]+1$ and $[x] \in \mathbf{Z}$.)

<u>Proof</u> It is well known that $W_c(t)$ is p.d.o. and as Q and Q_0 are periodic, $W_c(t)$ is also periodic. We have just to prove (IV.13) and as in[1], we will follow the proof of (IV.14) using the theorem of characterization of p.d.o. of Beals[14], (see also[15]).

For operators L, M and T,

$$ad_L(T) = [L;T] = LT - TL \quad \text{and} \quad ad_{L,M}(T) = LT - TM. \tag{IV.15}$$

Let $k \in \mathbf{N}$ be given and let

$$W_c^{(k)}(t) = \partial_t^k W_c(t) = i^k e^{itQ} ad_{Q,Q_0}^k(c(x,D))e^{-itQ_0}. \tag{IV.16}$$

As $ad_{Q,Q_0}^k(c(x,D)) \in OpS(< \xi >^{-m}; g_{x,\xi})$,
there exists a constant $C_k > 0$ such that

$$\parallel W_c^{(k)}(t) \parallel_{\mathcal{L}(L^2(\mathbf{R}^n))} \leq C_k, \quad \forall t \in \mathbf{R}, \tag{IV.17}$$

and more precisely, for any real s, there exists a constant $C_{k,s} > 0$ such that

$$\parallel W_c^{(k)}(t) \parallel_{\mathcal{L}(H^s(\mathbf{R}^n);H^{m+s}(\mathbf{R}^n))} \leq C_{k,s}, \quad \forall t \in \mathbf{R}. \tag{IV.18}$$

The estimate (IV.17) is obvious and so is this one

$$\parallel Q^{m+s}W_c^{(k)}(t)Q_0^{-s} \parallel_{\mathcal{L}(L^2(\mathbf{R}^n))} = \parallel i^k e^{itQ}Q^{m+s}ad_{Q,Q_0}^k(c(x,D))e^{-itQ_0} \parallel_{\mathcal{L}(L^2(\mathbf{R}^n))} \leq C_{k,s},$$

which is equivalent to (IV.18), ($C_{k,s}$ is as in (IV.18)).

Let L be an operator, then

$$\partial_t(ad_L(W_c^{(k)}(t)) = iad_{Q,Q_0}(ad_L(W_c^{(k)}(t)) + i[ad_L(Q)W_c^{(k)}(t) - W_c^{(k)}(t)ad_L(Q_0)]. \tag{IV.19}$$

This differential equation, in the t variable, satisfied by $ad_L(W_c^{(k)}(t)$ implies

$$ad_L(W_c^{(k)}(t)) = e^{itQ}ad_L(W_c^{(k)}(0))e^{-itQ_0} + \tag{IV.20}$$

$$ie^{itQ}\{\int_0^t e^{-i\tau Q}[ad_L(Q)W_c^{(k)}(\tau) - W_c^{(k)}(\tau)ad_L(Q_0)]e^{i\tau Q_0}d\tau\}e^{-itQ_0}.$$

Now if $T(t)$ is an operator depending smoothly on $t \in \mathbf{R}$, then

$$ad_L(e^{itQ}T(t)e^{-itQ_0}) = e^{itQ}ad_L(T(0))e^{-itQ_0}+ \tag{IV.21}$$

$$ie^{itQ}(\int_0^t [e^{-i\tau Q}(ad_L(Q-Q_0)+ad_L(Q_0))e^{i\tau Q}T(\tau) - T(\tau)ad_L(Q_0)]d\tau)e^{-itQ_0}$$

$$+e^{itQ}(\int_0^t [e^{-i\tau Q}ad_L(e^{i\tau Q}\partial_t T(\tau)e^{-i\tau Q_0})e^{i\tau Q_0}d\tau)e^{-itQ_0}.$$

We recall that (IV.4) and (IV.5) imply

$$ad_{D_{x_j}}(Q_0) = 0 \quad \text{and} \quad ad_{D_{x_j}}(Q-Q_0) \in OpS(1; g_{x,\xi}).$$

We need a technical lemma.

Lemma IV.2 *Let $j > 0$ be an integer and for $l = 0, 1 \ldots, j$ let $a_l(x, D)$ be periodic p.d.o., $a_l(x,\xi) \in S(<\xi>^{-m_l}; g_{x,\xi})$, with $m_l \geq 0$. If*

$$S(t) = \int_0^t e^{-i\tau_j Q}a_j(x,D)e^{i\tau_j Q}\int_0^{\tau_j} e^{-i\tau_{j-1}Q}a_{j-1}(x,D)e^{i\tau_{j-1}Q}\ldots \tag{IV.22}$$

$$\ldots\int_0^{\tau_2} e^{-i\tau_1 Q}a_1(x,D)e^{i\tau_1 Q}a_0(x,D)d\tau_1\ldots d\tau_j,$$

then there exist an integer $M(j)$, depending only on j,
$M(j)\times(j+1)$ periodic p.d.o.

$$b_{d,l}(x,D),\ l = 0,1\ldots j,\ 1 \leq d \leq M(j),\ b_{d,l}(x,\xi) \in S(<\xi>^{-p_l}; g_{x,\xi}),\ (p_l \geq 0),$$

and $M(j)\times(j+2)$ periodic p.d.o.

$$b'_{d,l}(x,D),\ l = 0,1\ldots j+1,\ 1 \leq d \leq M(j),\ b'_{d,l}(x,\xi) \in S(<\xi>^{-p'_l}; g_{x,\xi}),\ (p'_l \geq 0),$$

such that, if $A = D_{x_k}$, then

$$e^{-itQ}ad_A(e^{itQ}S(t)e^{-itQ_0})e^{itQ_0} = \tag{IV.23}$$

$$\sum_{d=1}^{M(j)}\int_0^t e^{-i\tau_j Q}b_{d,j}(x,D)e^{i\tau_j Q}\int_0^{\tau_j} e^{-i\tau_{j-1}Q}b_{d,j-1}(x,D)e^{i\tau_{j-1}Q}\ldots$$

$$\ldots\int_0^{\tau_2} e^{-i\tau_1 Q}b_{d,1}(x,D)e^{i\tau_1 Q}b_{d,0}(x,D)d\tau_1\ldots d\tau_j$$

$$+\sum_{d=1}^{M(j)}\int_0^t e^{-i\tau_{j+1}Q}b'_{d,j+1}(x,D)e^{i\tau_{j+1}Q}\int_0^{\tau_{j+1}} e^{-i\tau_j Q}b'_{d,j}(x,D)e^{i\tau_j Q}\ldots$$

$$\ldots\int_0^{\tau_2} e^{-i\tau_1 Q}b'_{d,1}(x,D)e^{i\tau_1 Q}b'_{d,0}(x,D)d\tau_1\ldots d\tau_{j+1}.$$

Proof of Lemma (IV.2) We do it by induction. For $j=1$, (IV.23) comes from (IV.21) and (IV.22). If the lemma is valid for any $j,\ j<N$, and if $j=N$, then

$$e^{itQ}\partial_t S(t)e^{-itQ_0} = a_N(x,D)e^{itQ}S_{N-1}(t)e^{-itQ_0}$$

with $S_{N-1}(t)$ satisfying (IV.22) with $j=N-1$. We write that

$$ad_A(e^{itQ}\partial_t S(t)e^{-itQ_0}) = \tag{IV.24}$$

$$ad_A(a_N(x,D))e^{itQ}S_{N-1}(t)e^{-itQ_0} + a_N(x,D)ad_A(e^{itQ}S_{N-1}(t)e^{-itQ_0}).$$

Then applying (IV.23) to $ad_A(e^{itQ}S_{N-1}(t)e^{-itQ_0})$ and taking into account (IV.21) and (IV.24), we found that $ad_A(e^{itQ}S(t)e^{-itQ_0})$ satifies also (IV.23). The Lemma is proved.

Now if L and $T(t)$ are as in (IV.21) and if $ad_{ad_L(Q_0)}(Q_0)=0$, then we have also the formula

$$e^{-itQ}ad_L(e^{itQ}T(t)e^{-itQ_0})e^{itQ_0} = ad_L(T(0)) + itad_{ad_L(Q_0)}(T(0)) \tag{IV.25}$$

$$+\int_0^t [ie^{-i\tau Q}ad_L(Q-Q_0)e^{i\tau Q}T(\tau) + e^{-i\tau Q}ad_L(e^{i\tau Q}\partial_t T(\tau)e^{-i\tau Q_0})e^{i\tau Q_0}]d\tau$$
$$+\int_0^t\int_0^\rho e^{-i\tau Q}[i^2 ad_{ad_L(Q_0)}(Q-Q_0)e^{i\tau Q}T(\tau) + iad_{ad_L(Q_0)}(e^{i\tau Q}\partial_t T(\tau)e^{-i\tau Q_0})e^{i\tau Q_0}]d\tau d\rho.$$

We need an other technical lemma as (IV.2).

Lemma IV.3 *Let $j>0$ be an integer and for $l=0,1\dots,j$ let $a_l(x,D)$ be periodic p.d.o.,*

$$a_l(x,\xi)\in S(<\xi>^{-m_l};g_{x,\xi}), \quad \text{with} \quad m_l\geq 0.$$

If

$$S(t) = \int_0^t e^{-i\tau_j Q}a_j(x,D)e^{i\tau_j Q}\int_0^{\tau_j} e^{-i\tau_{j-1}Q}a_{j-1}(x,D)e^{i\tau_{j-1}Q}\dots \tag{IV.26}$$
$$\dots\int_0^{\tau_2} e^{-i\tau_1 Q}a_1(x,D)e^{i\tau_1 Q}a_0(x,D)d\tau_1\dots d\tau_j,$$

then there exist an integer $M(j)$, depending only on j,
$M(j)\times(j+1)$ periodic p.d.o.

$$b_{d,l}(x,D),\ l=0,1\dots j,\ 1\leq d\leq M(j),\ b_{d,l}(x,\xi)\in S(<\xi>^{-p_l};g_{x,\xi}),\ (p_l\geq 0),$$

$M(j)\times(j+2)$ periodic p.d.o.

$$b'_{d,l}(x,D),\ l=0,1\dots j+1,\ 1\leq d\leq M(j),\ b'_{d,l}(x,\xi)\in S(<\xi>^{-p'_l};g_{x,\xi}),\ (p'_l\geq 0),$$

and $M(j)\times(j+3)$ periodic p.d.o.

$$b''_{d,l}(x,D),\ l=0,1\dots j+2,\ 1\leq d\leq M(j),\ b''_{d,l}(x,\xi)\in S(<\xi>^{-p''_l};g_{x,\xi}),\ (p''_l\geq 0),$$

such that

$$e^{-itQ}ad_A(e^{itQ}S(t)e^{-itQ_0})e^{itQ_0} = \sum_{d=1}^{M(j)}\int_0^t e^{-i\tau_j Q}b_{d,j}(x,D)e^{i\tau_j Q}\int_0^{\tau_j} e^{-i\tau_{j-1}Q}b_{d,j-1}(x,D)e^{i\tau_{j-1}Q}\dots \tag{IV.27}$$

$$\dots \int_0^{\tau_2} e^{-i\tau_1 Q} b_{d,1}(x,D) e^{i\tau_1 Q} b_{d,0}(x,D) d\tau_1 \dots d\tau_j$$

$$+ \sum_{d=1}^{M(j)} \int_0^t e^{-i\tau_{j+1}Q} b'_{d,j+1}(x,D) e^{i\tau_{j+1}Q} \int_0^{\tau_{j+1}} e^{-i\tau_j Q} b'_{d,j}(x,D) e^{i\tau_j Q} \dots$$

$$\dots \int_0^{\tau_2} e^{-i\tau_1 Q} b'_{d,1}(x,D) e^{i\tau_1 Q} b'_{d,0}(x,D) d\tau_1 \dots d\tau_{j+1}$$

$$+ \sum_{d=1}^{M(j)} \int_0^t e^{-i\tau_{j+1}Q} b"_{d,j+2}(x,D) e^{i\tau_{j+2}Q} \int_0^{\tau_{j+2}} e^{-i\tau_{j+1} Q} b"_{d,j+1}(x,D) e^{i\tau_{j+1} Q} \dots$$

$$\dots \int_0^{\tau_2} e^{-i\tau_1 Q} b"_{d,1}(x,D) e^{i\tau_1 Q} b"_{d,0}(x,D) d\tau_1 \dots d\tau_{j+2},$$

for $A = x_k$ *or* $A = s_0(D)$ *with* $s(\xi) \in S(1; <\xi>^{-2} g^0_\xi)$.

We have the following properties

$$1 + \sum_{l=0}^{j} m_l \le \sum_{l=0}^{j} p_l,\ 1 + \sum_{l=0}^{j} m_l \le \sum_{l=0}^{j+1} p'_l \ \text{ and } 1 + \sum_{l=0}^{j} m_l \le \sum_{l=0}^{j+2} p"_l, \tag{IV.28}$$

and for $l = 0, \dots, j+2$,

$$p"_l(x,\xi) = 0, \quad \text{if} \quad A = s_0(D) \in OpS(1; <\xi>^{-2} g^0_\xi). \tag{IV.29}$$

Proof of Lemma (IV.3) We do it also by induction. For $j = 1$, (IV.27) and (IV.28) come from (IV.25) and (IV.26). We use (IV.21), (instead of (IV.25)), to get (IV.29).

If the lemma is valid for any j, $j < N$, and if $j = N$,
then, when $A = s_0(D) \in OpS(1; <\xi>^{-2} g^0_\xi)$,
we do as for Lemma (IV.2). We get

$$e^{itQ} \partial_t S(t) e^{-itQ_0} = a_N(x,D) e^{itQ} S_{N-1}(t) e^{-itQ_0}$$

with $S_{N-1}(t)$ satisfying (IV.26) with $j = N-1$. So

$$ad_A(e^{itQ}\partial_t S(t) e^{-itQ_0}) = \tag{IV.30}$$

$$ad_A(a_N(x,D)) e^{itQ} S_{N-1}(t) e^{-itQ_0} + a_N(x,D) ad_A(e^{itQ} S_{N-1}(t) e^{-itQ_0}).$$

Then from (IV.21), (IV.26) with $j = N$, and from (IV.30), when we apply the hypothesis of the recurrence to $S_{N-1}(t)$, we get (IV.27) and (IV.28) with $b"_{d,l}(x,D) = 0$.

If $A = x_k$, then we write that

$$ad_{ad_A(Q_0)}(e^{itQ}\partial_t S(t) e^{-itQ_0}) = \tag{IV.31}$$

$$ad_{ad_A(Q_0)}(a_N(x,D)) e^{itQ} S_{N-1}(t) e^{-itQ_0} + a_N(x,D) ad_{ad_A(Q_0)}(e^{itQ} S_{N-1}(t) e^{-itQ_0}).$$

As $ad_A(Q_0) = s_{A,0}(D) \in OpS(1; <\xi>^{-2} g^0_\xi)$,
then from (IV.25), (instead of (IV.21)), from (IV.26) with $j = N$, and from (IV.31), when we apply the hypothesis of the recurrence to $S_{N-1}(t)$, we get (IV.27) and (IV.28).
The Lemma (IV.3) is proved.

Continuation of the proof of Theorem (IV.1)

Let $L_1,\dots L_j$ be j operators such that for every $k\in\{1,\dots,j\}$, $L_k=x_{l(k)}$ or $L_k=D_{x_{l(k)}}$ for some $l(k)\in\{1,\dots,n\}$. Using Lemma (IV.3) when $A=x_p$, and Lemma (IV.2) when $A=D_{x_p}$, we get

$$ad_{L_j}\circ\dots\circ ad_{L_1}(W_c^{(k)}(t))=e^{itQ}\{ad_{L_j}\circ\dots\circ ad_{L_1}(W_c^{(k)}(0))+ \tag{IV.32}$$

$$\sum_{d=1}^{M(j)}\sum_{l=1}^{j+N(j)}\int_0^t e^{-i\tau_l Q}a_{d,j,l,l}(x,D)e^{i\tau_l Q}\int_0^{\tau_l}e^{-i\tau_{l_1}Q}a_{d,j,l,l-1}(x,D)e^{i\tau_{l-1}Q}\dots\dots$$

$$\int_0^{\tau_2}e^{-i\tau_1 Q}a_{d,j,l,1}(x,D)e^{i\tau_1 Q}a_{d,j,l,0}(x,D)d\tau_1\dots d\tau_l\}e^{-itQ_0},$$

and

$$a_{d,j,l,p}(x,\xi)\in S(<\xi>^{-m_{j,l,p}};g_{x,\xi}),\quad \text{with}\quad m_{j,l,p}\geq 0, \tag{IV.33}$$

and, if $N(j)$ is the number of L_l such that $L_l=x_{d(l)}$, then

$$\sum_{p=0}^{l}m_{j,l,p}\geq m_j:=m+N(j). \tag{IV.34}$$

So, for any real s, there exists $C_{j,s}>0$ such that, $\forall t\in\mathbf{R}$,

$$\|\, ad_{L_j}\circ\dots\circ ad_{L_1}(W_c^{(k)}(t))\,\|_{\mathcal{L}(H^s(\mathbf{R}^n);H^{m_j+N(j)+s}(\mathbf{R}^n))}\leq C_{j,s}<t>^{j+N(j)}. \tag{IV.35}$$

The Beals characterization theorem proves that, for any fixed $t_0\in\mathbf{R}$,

$$W_c^{(k)}(t_0)\in OpS(<\xi>^{-m};g_{x,\xi}).$$

We get that the periodic symbol, (in x), $w_{c,t}(x,\xi)=w_c(t,x,\xi)$ of $W_c(t)$, is regular in (t,x,ξ). The estimate (IV.13) comes from (III.14) and (IV.35). We recall that (IV.14) comes from (IV.13) and the continuous injection

$$H^{[\frac{n}{2}]+1}(\mathbf{T}^n)\ \hookrightarrow\ C^0(\mathbf{T}^n).$$

The Theorem (IV.1) is proved •

We let

$$W(t):=e^{itQ}e^{-itQ_0}=w(t,x,D_x). \tag{IV.36}$$

We know from Theorem (IV.1) that $W(t)$ is a periodic p.d.o.:

$$w(t,x,\xi)=\sum_{\gamma\in\Gamma^\star}w_\gamma(t,\xi)e^{ix\gamma}, \tag{IV.37}$$

and

$$w(t,x,\xi)\ \in S(<t>^{[\frac{n}{2}]+1};g_t^0+<t>^2\,g_x^0+\frac{<t>^4}{<\xi>^2}g_\xi^0). \tag{IV.38}$$

From the definition of $w_0(t,\xi)$ in (IV.37) and from the estimate (IV.13) we get

$$w_0(t,\xi)\in S(1;g_t^0+\frac{<t>^4}{<\xi>^2}g_\xi^0). \tag{IV.39}$$

We recall the Taylor expansion of $W(t)$.

Theorem IV.4 *The symbol $w(t,x,\xi)$ of $W(t)$ has the asymptotic expansion*

$$w(t,x,\xi) \sim \sum_{j\geq 0} \frac{t^j}{j!} w^{(j)}(x,\xi), \tag{IV.40}$$

with $w^{(j)}(x,\xi) = \partial_t^j w(t,x,\xi)_{|t=0}$,

$$w^{(j)}(x,D) = i^j ad^j_{Q,Q_0}(I_d). \tag{IV.41}$$

The asymptotic expansion has the following meaning.

For any $N \in \mathbf{N}$ and for any regular function with compact support on $\mathbf{R}$, $\varphi(t) \in C_0^\infty(\mathbf{R})$,

$$t^{-N-1}\varphi(t)[w(t,x,\xi) - \sum_{j=0}^{N} \frac{t^j}{j!} w^{(j)}(x,\xi)] \in S(1; g_t^0 + g_{x,\xi}). \tag{IV.42}$$

The proof comes from Theorem (IV.1) and the well known equality

$$W(t) = \sum_{j=0}^{N} \frac{(it)^j}{j!} ad^j_{Q,Q_0}(I_d) + ie^{itQ}\{\int_0^t e^{-i\tau Q}\frac{(i\tau)^N}{N!} ad^N_{Q,Q_0}(I_d)e^{i\tau Q_0}d\tau\}e^{-i\tau Q_0}.$$

We have also an asymptotic expansion of $w(t,x,\xi)$ in the classical meaning of p.d.o. as in[1]. Instead of the case without magnetic field, here we have a real phase.

Let $q_0(x,\xi)$ be the symbol defined in (IV.6) and satifying (IV.5). We write its Fourier expansion

$$q_0(x,\xi) = \sum_{\gamma\in\Gamma^\star} \frac{\xi}{\sigma^0(\xi)} A_\gamma e^{ix\gamma}, \tag{IV.43}$$

and for every $N > 0$, there exists $C_N > 0$, such that

$$< \gamma >^N |A_\gamma| \leq C_N, \quad \forall\gamma \in \Gamma^\star. \tag{IV.44}$$

Let $\psi(t,x,\xi)$ be the real phase satisfying

$$\partial_t\psi(t,x,\xi) = \frac{\xi}{\sigma^0(\xi)}\partial_x\psi(t,x,\xi) + q_0(x,\xi) \text{ and } \psi(0,x,\xi) = 0. \tag{IV.45}$$

Then

$$\psi(t,x,\xi) \in S(< t >; g_t^0 + g_x^0 + < t >^2 < \xi >^{-2} g_\xi^0), \tag{IV.46}$$
$$\psi(t,x+a,\xi) = \psi(t,x,\xi), \forall a \in \Gamma,$$

and we shall see below that

$$\psi(t,x,\xi) = t\sum_{\gamma\in\Gamma^\star} \frac{\xi}{\sigma^0(\xi)} A_\gamma \zeta(it\frac{\xi}{\sigma^0(\xi)}\gamma)e^{ix\gamma} \tag{IV.47}$$

with the function on $\mathbf{C}$, $\zeta(z) = (e^z - 1)/z$.

Theorem IV.5 *The symbol* $w(t,x,\xi)$ *of* $W(t)$ *has the asymptotic expansion*

$$w(t,x,\xi) \sim e^{i\psi(t,x,\xi)} \sum_{j\geq 0} p_{-j}(t,x,\xi), \tag{IV.48}$$

with $p_0(t,x,\xi)=1$, $p_{-j}(0,x,\xi)=0$, *and for* $j \geq 1$,

$$p_{-j}(t,x,\xi) \ \in S(<t>^{3j}<\xi>^{-j}; g_t^0+g_x^0+<t>^2<\xi>^{-2}\, g_\xi^0), \tag{IV.49}$$

$$p_{-j}(t,x+a,\xi) = p_{-j}(t,x,\xi), \quad \forall a \in \Gamma.$$

The asymptotic expansion has the following meaning.
For any $\varphi(t) \in C_0^\infty(\mathbf{R})$, *and for any* $N \in \mathbf{N}$,

$$\varphi(t)[w(t,x,\xi) - e^{i\psi(t,x,\xi)} \sum_{j=0}^{N} p_{-j}(t,x,\xi)] \in S(<\xi>^{-N-1}; g_t^0 + g_{x,\xi}). \tag{IV.50}$$

<u>Proof</u> We recall that $W(t)$ is the solution of $\partial_t W(t) = i[QW(t) - W(t)Q_0]$.
We know that for any function $\varphi(t)$ as in (IV.50),

$$\varphi(t)w(t,x,\xi) \in S(1; g_t^0 + g_{x,\xi}).$$

So, for any positive integer N,

$$\varphi(t)\{i\partial_t w(t,x,\xi) + [q(x,\xi) - \sigma^0(\xi)]w(t,x,\xi) + \tag{IV.51}$$

$$\sum_{1\leq|\alpha|\leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha q(x,\xi)\partial_x^\alpha w(t,x,\xi)]\} \ \in S(<\xi>^{-N-1}; g_t^0 + g_{x,\xi}).$$

Looking for $w(t,x,\xi)$ satifying (IV.48) with (IV.49), the relations (IV.5) and (IV.51) lead to the phase equation (IV.45) and the transport equations, for $j>0$,

$$\partial_t p_{-j}(t,x,\xi) - \frac{\xi}{\sigma^0(\xi)}\partial_x p_{-j}(t,x,\xi) = i\sum_{k=1}^{j} q_{-k}(x,\xi)p_{-j+k}(t,x,\xi) \tag{IV.52}$$

$$+ie^{-i\psi(t,x,\xi)} \sum_{2\leq|\alpha|\leq j+1} \frac{(-i)^{|\alpha|}}{\alpha!} \sum_{k=-1}^{j-|\alpha|} \partial_\xi^\alpha q_{-k}(x,\xi)\partial_x^\alpha [e^{i\psi(t,x,\xi)} p_{-j+k+|\alpha|}(t,x,\xi)],$$

with $q_1(x,\xi) = \sigma^0(\xi)$.

Lemma IV.6 *If* $c(t,x,\xi) = \sum_{\gamma\in\Gamma^\star} c_\gamma(t,\xi)e^{ix\gamma}$ *is a periodic symbol,*

$$c(t,x,\xi) \ \in S(<t>^l<\xi>^m; g_t^0+g_x^0+<t>^2<\xi>^{-2}\, g_\xi^0) \tag{IV.53}$$

$c(t,x+a,\xi) = c(t,x,\xi), \forall a \in \Gamma$.
Then there exists a symbol $a(t,x,\xi)$ *solution of the equation*

$$\partial_t a(t,x,\xi) - \frac{\xi}{\sigma^0(\xi)}\partial_x a(t,x,\xi) = c(t,x,\xi) \text{ and } a(0,x,\xi)=0, \tag{IV.54}$$

which is periodic

$$a(t,x+a,\xi)=a(t,x,\xi),\forall a\in\Gamma,\quad a(t,x,\xi)=\sum_{\gamma\in\Gamma^\star}a_\gamma(t,\xi)e^{ix\gamma},$$

and

$$a(t,x,\xi)\ \in S(<t>^{l+1}<\xi>^m;g_t^0+g_x^0+<t>^2<\xi>^{-2}\,g_\xi^0) \tag{IV.55}$$

i.e. $\forall N,\ k\ \in\mathbf{N}$ *and* $\forall\alpha\in\mathbf{N}^n,\ \ \exists\ C_{N,\alpha}>0$ *such that*

$$|\partial_t^k\partial_\xi^\alpha a_\gamma(t,\xi)|\le C_{N,k,\alpha}<\gamma>^{-N}<t>^{l+1+|\alpha|}<\xi>^{m-|\alpha|}, \tag{IV.56}$$

Proof of the lemma The assumption (IV.53) means that, for all $N,\ k\in\mathbf{N}$ and $\alpha\in\mathbf{N}^n$, there exists $C_{N,\alpha}>0$ such that

$$|\partial_t^k\partial_\xi^\alpha c_\gamma(t,\xi)|\le C_{N,k,\alpha}<\gamma>^{-N}<t>^{l+|\alpha|}<\xi>^{m-|\alpha|}\ . \tag{IV.57}$$

Equation (IV.54) is equivalent to

$$\partial_t a_\gamma(t,\xi)-i\frac{\xi\gamma}{\sigma^0(\xi)}a_\gamma(t,\xi)=c_\gamma(t,\xi),\ \ \text{and}\ \ a_\gamma(0,\xi)=0,\ \forall\gamma\in\Gamma^\star. \tag{IV.58}$$

So, for all $\gamma\in\Gamma^\star$,

$$a_\gamma(t,\xi)=e^{it\frac{\xi\gamma}{\sigma^0(\xi)}}\int_0^t e^{-i\tau\frac{\xi\gamma}{\sigma^0(\xi)}}c_\gamma(\tau,\xi)d\tau. \tag{IV.59}$$

The estimate (IV.56) comes from (IV.57) and (IV.59).

In fact we have a global formula for the symbol $a(t,x,\xi)$

$$a(t,x,\xi)=\int_{-\frac{1}{2}(t+\kappa(x,\xi))}^{\frac{1}{2}(t-\kappa(x,\xi))}\tilde{c}(t,x,\xi,\tau)d\tau, \tag{IV.60}$$

if $\kappa(x,\xi)=\frac{\sigma^0(\xi)}{|\xi|^2}x\xi$, then

$$\tilde{c}(t,x,\xi,\tau):=c(\frac{1}{2}(t+\kappa(x,\xi))+\tau,[\frac{1}{2}(t+\kappa(x,\xi))-\tau]\frac{\xi}{\sigma^0(\xi)}+[x-x\xi\frac{\xi}{|\xi|^2}],\xi).$$

We get (IV.47) from (IV.45) and (IV.59).

Now Theorem (IV.5) follows from (IV.45), (IV.46), (IV.51), (IV.52) and the Lemma (IV.6).

V The asymptotic of the density of state

As in the case of without magnetic field, we have a good functional calculus .

Theorem V.1 *Let $f(\tau)$ be a regular real function on* $\mathbf{R}$ *such that*

$$f(\tau)\in S(<\tau>^m;<\tau>^{-2}\,g_\tau^0). \tag{V.1}$$

Then the operator $f(Q)$ is a periodic p.d.o.. There exists a periodic symbol

$$\sigma_{f(Q)}(x,\xi) = \sum_{\gamma\in\Gamma^\star} \sigma_{f(Q),\gamma}(\xi) e^{ix\gamma}, \tag{V.2}$$

such that

$$\sigma_{f(Q)}(x,\xi) \in S(<\xi>^m; g_{x,\xi}), \tag{V.3}$$

and $\quad f(Q) = \sigma_{f(Q)}(x,D)$.

More over, $\sigma_{f(Q)}(x,\xi)$ has an asymptotic expansion of the form

$$\sigma_{f(Q)}(x,\xi) \sim \sum_{j\geq 0} \sigma_{f(Q),m-j}(x,\xi), \tag{V.4}$$

with

$$\sigma_{f(Q),m-j}(x,\xi) \in S(<\xi>^{m-j}; g_{x,\xi}) \tag{V.5}$$

and for every postive integer N,

$$\sigma_{f(Q)}(x,\xi) - \sum_{j=0}^{N} \sigma_{f(Q),m-j}(x,\xi) \in S(<\xi>^{m-N-1}; g_{x,\xi}). \tag{V.6}$$

Proof Here we reproduce briefly the proof of[1] in the case without magnetic field.

We may assume $m<-1$, even taking $g(\tau) :=<\tau>^{-2[|m|]-2} f(\tau)$ instead of f, ($f(Q) = (Q^2+1)^{[|m|]+1} g(Q)$).

In this case $f(\tau) \in H^s(\mathbf{R}^n)$, $\forall s\in\mathbf{R}$,
and we can write $\quad f(Q) = (2\pi)^{-1}\int_{\mathbf{R}} \hat f(t) e^{itQ} dt = \sigma_{f(Q)}(x,D)$,

$$\sigma_{f(Q)}(x,\xi) = (2\pi)^{-1}\int_{\mathbf{R}} \hat f(t) w(t,x,\xi) e^{it\sigma^0(\xi)} dt. \tag{V.7}$$

Let N be a fixed positive integer. We choose an other integer M larger then $N:\quad M<N+n$. Then by Taylor formula, (see Theorem (IV.4)), we may write

$$w(t,x,\xi) = \sum_{j=0}^{M} \frac{t^j}{j!} w^{(j)}(x,\xi) \ + \ t^{M+1} r_{M+1}(w)(t,x,\xi), \tag{V.8}$$

so, if for any positive integer j, $\quad f^{(j)}(\tau) = \partial_\tau^j f(\tau)$, then, (with $\sigma^0(\xi)$ defined in (IV.4)),

$$\sigma_{f(Q)}(x,\xi) = \sum_{j=0}^{M} \frac{(-i)^j}{j!} f^{(j)}(\sigma^0(\xi)) w^{(j)}(x,\xi) + \tag{V.9}$$

$$(2\pi)^{-1}\int_{\mathbf{R}} \hat f(t) t^{M+1} r_{M+1}(w)(t,x,\xi) e^{it\sigma^0(\xi)} dt.$$

As there exists some constant $\eta_0>0$ such that $\sigma^0(\xi) > \eta_0 <\xi>$,
integrating by parts, we get

$$\int_{\mathbf{R}} \hat f(t) t^{M+1} r_{M+1}(w)(t,x,\xi) e^{it\sigma^0(\xi)} dt = \tag{V.10}$$

$$\int_{\mathbf{R}} (\sigma^0(\xi))^{-M-1} e^{it\sigma^0(\xi)} (-i\partial_t)^{M+1} \{\hat f(t) t^{M+1} r_{M+1}(w)(t,x,\xi)\} dt.$$

(V.1) with $m < -1$ shows that for any three integers $j,\ k$ and $l,\ 0 \leq k \leq l$,

$$< t >^j \partial_t^k (\hat{f}(t) t^l) \in L^2(\mathbf{R}). \tag{V.11}$$

Recalling that $w^{(j)}(x, D) = i^j ad^j_{Q,Q_0}(I_d)$
and $r_{M+1} = \frac{i}{t^{M+1}} e^{itQ} \{ \int_0^t e^{-i\tau Q} \frac{(i\tau)^N}{M!} ad^M_{Q,Q_0}(I_d) e^{i\tau Q_0} d\tau \} e^{-i\tau Q_0}$,
then Theorem (V.1) comes from Theorem (IV.1), (V.1), and (V.8)–(V.11) •

We are able to explicit (V.4) in the following way.

Remark V.2 *In the asymptotic expansion (V.4), we can take*

$$\sigma_{f(Q),m}(x, \xi) = f(\sigma^0(\xi)), \tag{V.12}$$

$$\sigma_{f(Q),m-1}(x, \xi) - [-i(q(x, \xi) - \sigma^0(\xi)) \times f'(\sigma^0(\xi))] \in S(< \xi >^{m-2}; g_{x,\xi}),$$

so by (IV.5) and (IV.6), we may assume

$$\sigma_{f(Q),m-1}(x, \xi) = i \frac{\xi A(x)}{\sigma^0(\xi)} \times f'(\sigma^0(\xi)). \tag{V.13}$$

Notice also that $\widetilde{\mathrm{Tr}}(\mathrm{f(Q)})$ is well defined whenever
$f(\tau) \in S(< \tau >^{-m}; < \tau >^{-2} g^0_\tau)$, with $m > n$. ($\widetilde{\mathrm{Tr}}(\mathrm{A})$ is defined by (II.2)).

Theorem V.3 *Let* $f(\tau) \in C_0^\infty(\mathbf{R})$
and let $\lambda >> 1$ be a large non negative real number.
Then, as $\lambda \to +\infty$, $\widetilde{\mathrm{Tr}}(\mathrm{f(Q - \lambda I_d)})$ *has the following asymptotic expansion*

$$\widetilde{\mathrm{Tr}}(\mathrm{f(Q - \lambda I_d)}) \sim \sum_{\mathrm{j \geq 0}} \int_0^{+\infty} \mathrm{a_{n-1-j}(r) f((r^2 + \lambda_0)^{1/2} - \lambda) dr}, \tag{V.14}$$

with

$$a_{n-1}(r) = (2\pi)^{-n} |S_n| r^{n-1}, \quad a_{n-2}(r) = 0, \tag{V.15}$$

and

$$a_{n-3}(r) = \frac{r^{n-1}}{2|\mathcal{K}|} \times \{\partial_r + \frac{n-1}{r}\} \{ \int_{\mathcal{K} \times \mathbf{S}^{n-1}} [\frac{r(\omega A(x))^2}{r^2 + \lambda_0} - \frac{|A(x)|^2}{r}] dx d\Omega(\omega) \},$$

($\mathbf{S}^{n-1}$ is the unit sphere and $d\Omega(\omega)$ its usual measure),
and more generally

$$a_k(r) \in S(< r >^k; < r >^{-2} g^0_r). \tag{V.16}$$

The asymptotic expansion is to be understood in the following way: for any integer $N > n$, there exists a constant $C_N > 0$, depending only on N, (and not on λ nor on $f(\tau)$), such that

$$|\widetilde{\mathrm{Tr}}\mathrm{f((Q - \lambda I_d))} - \sum_{\mathrm{j=0}}^{\mathrm{N-1}} \int_0^{+\infty} \mathrm{a_{n-1-j}(r) f((r^2 + \lambda_0)^{1/2} - \lambda) dr}| \tag{V.17}$$

$$\leq C_N \lambda^{n-N} \{ \| t^N \hat{f}(t) \|_{L^1(\mathbf{R})} + \sum_{j=0}^{N} \| \partial_t^j \hat{f}(t) \|_{L^1(\mathbf{R})} \}.$$

Proof The proof is the same as in the one in[1] for case without magnetic field. Let us sketch it. Let $\chi(\tau)$ be a cut off function satifying

$$\chi(\tau) \in C_0^\infty(\mathbf{R}),\ \chi(\tau) = \chi(-\tau),\ 0 \le \chi(\tau) \le 1, \tag{V.18}$$

$$\chi(\tau) = 1,\ \text{if } |\tau| < \frac{1}{2},\quad \text{and}\quad \chi(\tau) = 0,\ \text{if } |\tau| > 1.$$

We denote by χ_λ the function

$$\chi_\lambda(\tau) = \chi(2\lambda^{-1}(\tau - \lambda)). \tag{V.19}$$

Let $w_0(t,\xi)$ be the symbol defined by (IV.37) $\quad w_0(t,\xi) = |\mathcal{K}|^{-1}\int_{\mathcal{K}} w(t,x,\xi)dx.$
By (II.2) and (V.7),

$$\widetilde{\mathrm{Tr}}(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})(\mathrm{I_d} - \chi_\lambda(\mathrm{Q_0}))) = \tag{V.20}$$

$$(2\pi)^{-n-1}\int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}\hat{f}(t)w_0(t,\xi)(1-\chi_\lambda(\sigma^0(\xi)))dtd\xi.$$

As $|\sigma^0(\xi) - \lambda| > \lambda/4$ on the support of $\ 1 - \chi_\lambda(\sigma^0(\xi))$,
we may integrate by parts in t and then, for any integer M,
$\widetilde{\mathrm{Tr}}(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})(\mathrm{I_d} - \chi_\lambda(\mathrm{Q_0}))) =$

$$i^M(2\pi)^{-n-1}\int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}(\sigma^0(\xi)-\lambda)^{-M}(1-\chi_\lambda(\sigma^0(\xi)))\partial_t^M[\hat{f}(t)w_0(t,\xi)]dtd\xi.$$

So taking into a count (IV.39) and (IV.4), if $M > n$, we get the estimate

$$|\widetilde{\mathrm{Tr}}(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})(\mathrm{I_d} - \chi_\lambda(\mathrm{Q_0})))| \le \lambda^{\mathrm{n-M}}\mathrm{C_M}\sum_{\mathrm{j=0}}^{\mathrm{M}} \|\, \partial_t^{\mathrm{j}}\hat{\mathrm{f}}(\mathrm{t})\,\|_{\mathrm{L^1}(\mathbf{R})} \tag{V.21}$$

with C_M depending only on M, (and not on f).
Now writing that

$$\widetilde{\mathrm{Tr}}(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})\chi_\lambda(\mathrm{Q_0})) = (2\pi)^{-\mathrm{n}-1}\int_{\mathbf{R}\times\mathbf{R}^{\mathrm{n}}} \hat{\mathrm{f}}(\mathrm{t})\mathrm{w_0}(\mathrm{t},\xi)\chi_\lambda(\sigma^0(\xi))\mathrm{e}^{\mathrm{it}(\sigma^0(\xi)-\lambda)}\mathrm{dtd}\xi,$$

we integrate by parts in $|\xi|$, using

$$t^j e^{it(\sigma^0(\xi)} = (-i)^j\{\sigma^0(\xi)\frac{\xi}{|\xi|^2}\partial_\xi\}^j e^{it(\sigma^0(\xi)}.$$

In the vicinity of infinity in t, we found that for any integer M,

$$\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(t))\hat{f}(t)w_0(t,\xi)\chi_\lambda(\sigma^0(\xi))e^{it(\sigma^0(\xi)-\lambda)}dtd\xi =$$

$$i^M\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(t))t^{-M}\hat{f}(t)e^{it(\sigma^0(\xi)-\lambda)}\{\partial_\xi\frac{\xi}{|\xi|^2}\sigma^0(\xi)\}^M\{w_0(t,\xi)\chi_\lambda(\sigma^0(\xi))\}dtd\xi.$$

Using (IV.39), we found that there exists a contant C_M, as in (V.21), such that

$$|(2\pi)^{-n-1}\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(t))\hat{f}(t)w_0(t,\xi)\chi_\lambda(\sigma^0(\xi))e^{it(\sigma^0(\xi)-\lambda)}dtd\xi| \le$$

$$C_M(\int_{\lambda/2}^{3\lambda/2} r^{n-1-M}dr) \times \int_{\mathbf{R}} |t|^M(1-\chi(t))|\hat{f}(t)|dt.$$

So there exists a C_M as in (V.21), such that

$$|(2\pi)^{-n-1}\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(t))\hat{f}(t)w_0(t,\xi)\chi_\lambda(\sigma^0(\xi))e^{it(\sigma^0(\xi)-\lambda)}dtd\xi| \leq \tag{V.22}$$

$$\lambda^{n-M}C_M\int_{\mathbf{R}}|t|^M(1-\chi(t))|\hat{f}(t)|dt.$$

Now we use the Taylor formula, (cf. Theorem (IV.4)), and we integrate by parts as above, then by (IV.42), for any integer M, there exists a C_M as in (V.21), such that

$$|\int_{\mathbf{R}\times\mathbf{R}^n}\chi(t)\hat{f}(t)w_0(t,\xi)\chi_\lambda(\sigma^0(\xi))e^{it(\sigma^0(\xi)-\lambda)}dtd\xi- \tag{V.23}$$

$$\sum_{j=0}^{M-1}\frac{i^j}{j!}\int_{\mathbf{R}\times\mathbf{R}^n}\chi(t)\hat{f}(t)e^{it(\sigma^0(\xi)-\lambda)}\{\partial_\xi\frac{\xi}{|\xi|^2}\sigma^0(\xi)\}^j\{w_0^{(j)}(\xi)\chi_\lambda(\sigma^0(\xi))\}dtd\xi|$$

$$\leq \lambda^{n-M}C_M\int_{\mathbf{R}}\chi(t)|\hat{f}(t)|dt;$$

$(w_0^{(j)}(\xi) := \partial_t^j w_0(t,\xi)_{|t=0} = \frac{1}{|\mathcal{K}|}\int_{\mathcal{K}} w^{(j)}(x,\xi)dx)$.

But for any integers $j,\ 0 \leq j < M$,

$$|\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(t))\hat{f}(t)e^{it(\sigma^0(\xi)-\lambda)}\{(\partial_\xi\frac{\xi}{|\xi|^2}\sigma^0(\xi)\}^j\{w_0^{(j)}(\xi)\chi_\lambda(\sigma^0(\xi))\}dtd\xi| =$$

$$|i^{M-j}\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(t))\hat{f}(t)t^{-(M-j)}e^{it(\sigma^0(\xi)-\lambda)}\{(\partial_\xi\frac{\xi}{|\xi|^2}\sigma^0(\xi)\}^M\{w_0^{(j)}(\xi)\chi_\lambda(\sigma^0(\xi))\}dtd\xi|$$

$$\leq \lambda^{n-M}C_M\int_{\mathbf{R}}(1-\chi(t))|t|^{-M+j}|\hat{f}(t)|dt, \tag{V.24}$$

and as for (V.21), (with C_M is as in (V.21)),

$$|\sum_{j=0}^{M-1}\frac{i^j}{j!}\int_{\mathbf{R}\times\mathbf{R}^n}\chi(t)\hat{f}(t)e^{it(\sigma^0(\xi)-\lambda)}\{\partial_\xi\frac{\xi}{|\xi|^2}\sigma^0(\xi)\}^j\{w_0^{(j)}(\xi)\chi_\lambda(\sigma^0(\xi))\}dtd\xi- \tag{V.25}$$

$$\sum_{j=0}^{M-1}\frac{i^j}{j!}\int_{\mathbf{R}\times\mathbf{R}^n}\chi(t)\hat{f}(t)e^{it(\sigma^0(\xi)-\lambda)}(1-\chi(|\xi|))\{\partial_\xi\frac{\xi}{|\xi|^2}\sigma^0(\xi)\}^j\{w_0^{(j)}(\xi)\}dtd\xi|$$

$$\leq \lambda^{n-M}C_M\sum_{j=0}^{M}\parallel\partial_t^j\hat{f}(t)\parallel_{L^1(\mathbf{R})}.$$

Then theorem (V.3) follows from (V.21) - (V.25). The properties (V.15) come from (IV.4) - (IV.8) and (IV.41).

Theorem V.4 *Let* $\widetilde{N}(\lambda;Q)$ *be the integrate state density of* Q :

$$\widetilde{N}(\lambda;Q) = \widetilde{\mathrm{T}}\mathrm{r}(\chi_{]-\infty,\lambda[}(\mathrm{Q})). \tag{V.26}$$

Then, for any $\delta\in]0,1[$,

$$\widetilde{N}(\lambda;Q) = \widetilde{N}(\lambda;Q_0) + \mathcal{O}(\lambda^{n-1-\delta}),\ \text{as}\ \lambda\to+\infty. \tag{V.27}$$

(If I is an interval, $\chi_I(\tau)$ denotes the characteristic function of I.)

Remind that

$$\widetilde{N}(\lambda; Q_0) = (2\pi)^{-n} \int_{\sigma^0(\xi)<\lambda} d\xi = (2\pi)^{-n} \frac{|S_n|}{n} \lambda^n + \mathcal{O}(\lambda^{n-2}). \tag{V.28}$$

Proof of the Theorem Let $\delta \in]0,1[$ be fixed. With $\chi(\tau)$ as (V.18), define the function

$$f_{\lambda,\delta}(\tau) = \chi(\lambda^{\delta}\tau). \tag{V.29}$$

As $\delta < 1$, the estimate (V.17), with $N \geq \frac{1+\delta}{1-\delta}$, shows that when $\lambda \to +\infty$

$$\widetilde{\mathrm{Tr}}(\mathrm{f}_{\lambda,\delta}(\mathrm{Q} - \lambda \mathrm{I_d})) = \mathcal{O}(\lambda^{\mathrm{n}-1-\delta}). \tag{V.30}$$

Define the function $\varphi_{\lambda,\delta}(\tau)$ by

$$\varphi_{\lambda,\delta}(\tau) = \begin{cases} 1, \text{ if } 0 < \tau < \lambda, \\ \chi(\lambda^{\delta}(\tau - \lambda)), \text{ if } \tau > \lambda \\ \chi(\tau), \text{ if } \tau < 0. \end{cases} \tag{V.31}$$

Then $\varphi_{\lambda,\delta}(\tau) \in C_0^{\infty}(\mathbf{R})$ and its fourier-transform is

$$\hat{\varphi}_{\lambda,\delta}(t) = -it^{-1}\hat{\chi}_{1_-}(t) - it^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)e^{-it\lambda}, \tag{V.32}$$

with $\chi_{1_\pm}(\tau) = \partial_\tau \chi(\tau)$, if $\pm\tau \geq 0$,
so, by (V.18), $\chi_{1_\pm}(\tau) \in C_0^{\infty}(\mathbf{R})$.

According to the proof of (V.21), for any integer M, we have

$$\int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}(1-\chi(t))t^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)(1-\chi_\lambda(\sigma^0(\xi)))w_0(t,\xi)dtd\xi$$
$$= i^M \int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}(\sigma^0(\xi)-\lambda)^{-M}(1-\chi_\lambda(\sigma^0(\xi)))\times$$
$$\partial_t^M\{(1-\chi(t))t^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)w_0(t,\xi)\}dtd\xi.$$

So, if $M > n$, taking into a count (IV.39) and (IV.4), then as $\lambda \to +\infty$,

$$\int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}(1-\chi(t))t^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)(1-\chi_\lambda(\sigma^0(\xi)))w_0(t,\xi)dtd\xi = \mathcal{O}(1), \tag{V.33}$$

Integrating by parts in $|\xi|$, we get as for (V.22) that,

$$\int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}(1-\chi(t))t^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)\chi_\lambda(\sigma^0(\xi))w_0(t,\xi)dtd\xi =$$
$$i^M \int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}(1-\chi(t))t^{-1-M}\hat{\chi}_{1_+}(\lambda^{-\delta}t)\{\partial_\xi \frac{\xi}{|\xi|^2}\sigma^0(\xi)\}^M\{\chi_\lambda(\sigma^0(\xi))w_0(t,\xi)\}dtd\xi.$$

So using (IV.39), we found that, for any $M > 0$, there exists a constant C_M, depending only on M, such that

$$|\int_{\mathbf{R}\times\mathbf{R}^n} e^{it(\sigma^0(\xi)-\lambda)}(1-\chi(t))t^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)\chi_\lambda(\sigma^0(\xi))w_0(t,\xi)dtd\xi| \tag{V.34}$$

$$\leq \lambda^{n-M} C_M \int_{\mathbf{R}} |t|^{M-1}|\hat{\chi}_{1_+}(\lambda^{-\delta}t)|dt$$

$$\leq \lambda^{n-M(1-\delta)} C_M \int_{\mathbf{R}} |t|^{M-1} |\hat{\chi}_{1_+}(t)| dt.$$

Now, from (V.32) - (V.34), we obtain that for any $\delta < 1$, as $\lambda \to +\infty$,

$$\int_{\mathbf{R}\times\mathbf{R}^n} (1-\chi(t))\hat{\varphi}_{\lambda,\delta}(t) w_0(t,\xi) e^{it(\sigma^0(\xi)-\lambda)} dt d\xi = \mathcal{O}(1). \tag{V.35}$$

Following the proof of the Theorem (V.3) and the one of (V.35), we get easily that, there exists a constant C_δ, depending only on δ, such that

$$|\widetilde{\mathrm{Tr}}(\varphi_{\lambda,\delta}(\mathrm{Q})) - \sum_{\mathrm{j}=0}^{\mathrm{n}} \int_0^{+\infty} \mathrm{a}_{\mathrm{n}-1-\mathrm{j}}(\mathrm{r})\varphi_{\lambda,\delta}((\mathrm{r}^2+\lambda_0)^{1/2})\mathrm{dr}|$$

$$\leq C_\delta[1 + \int_{\mathbf{R}} |\hat{\varphi}_{\lambda,\delta}(t)| dt].$$

So by (V.32), as $\lambda \to +\infty$,

$$\widetilde{\mathrm{Tr}}(\varphi_{\lambda,\delta}(\mathrm{Q})) = \sum_{\mathrm{j}=0}^{\mathrm{n}} \int_0^{+\infty} \mathrm{a}_{\mathrm{n}-1-\mathrm{j}}(\mathrm{r})\varphi_{\lambda,\delta}((\mathrm{r}^2+\lambda_0)^{1/2})\mathrm{dr} + \mathcal{O}(\lambda). \tag{V.36}$$

The definitions (V.31) of $\varphi_{\lambda,\delta}(\tau)$, (V.26) of $\widetilde{N}(\lambda, Q)$ and (V.29) of $f_{\lambda,\delta}(\tau)$ lead to

$$|\widetilde{N}(\lambda, Q) - \widetilde{\mathrm{Tr}}(\varphi_{\lambda,\delta}(\mathrm{Q}))| \leq \widetilde{\mathrm{Tr}}(\mathrm{f}_{\lambda,\delta}(\mathrm{Q})), \tag{V.37}$$

we use that the density of state of Q, is a positive measure.

The Theorem (V.4) follows from (V.30), (V.36), (V.37), (V.15) and (V.16) •

VI Proof of Theorem (1.1) and Theorem (2.1)

Proof of Theorem (1.1)

Taking into account (I.17)–(I.19) and (V.28), one can see easily that Theorem (I.1) follows from Theorem (V.4). As a matter of fact, if we take $\lambda = (\mu+\lambda_0)^{1/2}$, with $\mu >> 1$, we get that $N(\mu; P_{V,A}) = \widetilde{N}(\lambda; Q)$ and $\widetilde{N}(\lambda; Q_0) = a_n \mu^{n/2}$, with the a_n of Theorem (I.1).

Proof of Theorem(2.1)

We need the Proposition bellow.

Proposition VI.1 *Assume the assumptions of Theorem (I.1), (I.6) and (I.20).*

If $b \in \Gamma$ and $b \neq 0$, then for every $\delta \in [0,1[$, there exists a constant $C_{b,\delta}$, depending only on b and δ, such that

$$|\widetilde{N}_b(\mu; P_{V,A}) - \frac{\mu^{\frac{n-1}{4}}}{|b|^{\frac{n+1}{2}}} \times \frac{2}{(2\pi)^{\frac{n+1}{2}}|\mathcal{K}|} sin(\mu^{\frac{1}{2}} - \frac{n-1}{2}\pi) \int_{\mathcal{K}} e^{-i\phi_b(x)} dx| \tag{VI.1}$$

$$\leq C_{b,\delta}(1 + \mu^{\frac{n-3}{4}} + \mu^{\frac{n-1-\delta}{2}}).$$

$\phi_b(x)$ *is the regular, periodic and real function defined by*

$$\phi_b(x) = \sum_{\gamma\in[b]^\perp} bA_\gamma e^{ix\gamma}, \tag{VI.2}$$

where $[b]^\perp \subset \Gamma^\star$ *is the Bragg hyper-plane sub-lattice orthogonal to* b,

$$[b]^\perp = \{\gamma \in \Gamma^\star;\ \gamma b = 0\},$$

and the A_γ *are the Fourier coefficients of* $A(x)$:

$$A(x) = \sum_{\gamma\in\Gamma^\star} A_\gamma e^{ix\gamma}.$$

The Theorem (II.1) follows from Proprosition (VI.1) as we will see below.
Noting (I.3), we choose

$$b = \sum_{j=1}^{n} k_j e_j, \quad \text{with} \;\; 1 < k_1 < \ldots < k_n, \tag{VI.3}$$

the k_j are chosen to be prime integers. Then by (I.9),

$$\gamma \in [b]^\perp,\ \gamma \neq 0 \implies \gamma = \sum_{j=1}^{n} l_j e_j^\star \;\; \text{with} \;\; \sum_{j=1}^{n} |l_j| > k_1. \tag{VI.4}$$

As $A_0 = 0$ by (I.6), noting (VI.2) and (IV.44) we can found a constant $N_0 > 0$ such that, if in (VI.3) $k_1 > N_0$, then

$$|\phi_b(x)| + |\phi_{2b}(x)| \leq \frac{\pi}{4}. \tag{VI.5}$$

For a such b, if $\frac{n-1}{4} \notin \mathbf{N}$, (which is always the case if $n = 2$), (VI.2) and (VI.5) show that (II.10) is satisfied.

Remark (II.3) follows in the same way: multiplying A by ϵ leads to multiply $\phi_b(x)$ by ϵ.

Proof of Proposition(VI.1)

The formulas (II.4) and (II.5) show that, for any $f \in C_0^\infty(\mathbf{R})$,

$$\varrho^b_{V,A}(f) = (2\pi)^{-n}|\mathcal{K}|^{-1}\int_{\mathcal{K}\times\mathbf{R}^n} e^{ib\xi}\sigma_f(x,\xi)dxd\xi = \widetilde{\mathrm{Tr}}_\mathrm{b}(\mathrm{f}(\mathrm{P}_{\mathrm{V,A}})), \tag{VI.6}$$

where $\sigma_f(x,\xi)$ is the symbol of $f(P_{V,A})$: $\sigma_f(x,D) = f(P_{V,A})$.

If $\mu >> 1$, and $\lambda = (\mu+\lambda_0)^{1/2}$, then by (II.7)

$$N_b(\mu; P_{V,A}) = \varrho^b_{V,A}(\chi_{]-\infty,\mu[}(P_{V,A})) = \widetilde{\mathrm{Tr}}_\mathrm{b}(\chi_{]-\infty,\lambda[}(\mathrm{Q})). \tag{VI.7}$$

Now if $\delta \in]0,1[$ and if $\varphi_{\lambda,\delta}(\tau)$ is the function defined by (V.31), then, using (II.9), we get

$$|\widetilde{N}_b(\lambda; Q) - \widetilde{\mathrm{Tr}}_\mathrm{b}(\varphi_{\lambda,\delta}(\mathrm{Q}))| \leq \widetilde{\mathrm{Tr}}(\chi(\lambda^\delta(\mathrm{Q}-\lambda \mathrm{I_d})). \tag{VI.8}$$

If $\delta \in]0,1[$, it follows from (V.30) and (VI.8) that

$$|\widetilde{N}_b(\lambda; Q) - \widetilde{\mathrm{Tr}}_\mathrm{b}(\varphi_{\lambda,\delta}(\mathrm{Q}))| = \mathcal{O}(\lambda^{\mathrm{n}-1-\delta}), \quad \text{as } \lambda \to +\infty. \tag{VI.9}$$

But by (VI.6), if $w_0(t,\xi)$ is the symbol defined by (IV.36) and (IV.37), we may write

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}(\varphi_{\lambda,\delta}(\mathrm{Q})) = (2\pi)^{-\mathrm{n}-1}\int_{\mathbf{R}\times\mathbf{R}^{\mathrm{n}}} \hat{\varphi}_{\lambda,\delta}(\mathrm{t})\mathrm{w}_0(\mathrm{t},\xi)\mathrm{e}^{\mathrm{i}(\mathrm{t}\sigma^0(\xi)+\mathrm{b}\xi)}\mathrm{dtd}\xi. \tag{VI.10}$$

Now the proof remains the same as the one of Theorem (V.4) with some minor alterations. Let $\delta\in]0,1[$ be fixed. Changing $\chi(t)$ into $\chi(C_b^{-1}t)$, with $C_b>0$ a large constant depending only on $|b|$, and taking (V.32) into account , we may found that, for any $\tau\in]0,1[$, as $\lambda\to+\infty$,

$$\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(C_b^{-1}t))\hat{\varphi}_{\lambda,\delta}(t)w_0(t,\xi)\chi(\lambda^{-\tau}(\sigma^0(\xi)-\lambda))e^{i(t\sigma^0(\xi)+b\xi)}dtd\xi = \mathcal{O}(1), \tag{VI.11}$$

providing that we integrate by parts in ξ variable using

$$(1-\chi(C_b^{-1}t))[t+\sigma^0(\xi)|\xi|^{-2}b\xi]^{-1}|\xi|^{-2}\sigma^0(\xi)\xi\partial_\xi e^{i(t\sigma^0(\xi)+b\xi)} = ie^{i(t\sigma^0(\xi)+b\xi)}, \tag{VI.12}$$

(instead of $(1-\chi(t))t^{-1}\sigma^0(\xi)|\xi|^{-2}\xi\partial_\xi e^{it\sigma^0(\xi)} = ie^{it\sigma^0(\xi)}$).
In the same way we get also that for any $\tau\in]0,1[$, as $\lambda\to+\infty$,

$$\int_{\mathbf{R}\times\mathbf{R}^n}(1-\chi(C_b^{-1}t))t^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)e^{-it\lambda}w_0(t,\xi)e^{i(t\sigma^0(\xi)+b\xi)} \tag{VI.13}$$

$$\times(1-\chi(\lambda^{-\tau}(\sigma^0(\xi)-\lambda)))dtd\xi = \mathcal{O}(\lambda^{-\infty}).$$

So (V.32), (VI.11) and (VI.13) show that, as $\lambda\to+\infty$,

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = (2\pi)^{-\mathrm{n}-1}\int_{\mathbf{R}\times\mathbf{R}^{\mathrm{n}}}\chi(\mathrm{C}_{\mathrm{b}}^{-1}\mathrm{t})\hat{\varphi}_{\lambda,\delta}(\mathrm{t})\times \tag{VI.14}$$

$$w_0(t,\xi)e^{i(t\sigma^0(\xi)+b\xi)}dtd\xi \ + \ \mathcal{O}(1).$$

We recall that from (IV.39) we have

$$\chi(C_b^{-1}t)w_0(t,\xi) \ \in S(1;g_t^0+<\xi>^{-2}g_\xi^0). \tag{VI.15}$$

We let

$$t_0(\xi) = \sigma^0(\xi)|\xi|^{-2}b\xi, \tag{VI.16}$$

so

$$(1-\chi(|\xi|))t_0(\xi) \ \in S(1;g_\xi^0). \tag{VI.17}$$

By Taylor formula, we may write that

$$w_0(t,\xi)-\sum_{j=0}^{n}\frac{(t+t_0(\xi))^j}{j!}\partial_t^j w_0(-t_0(\xi),\xi) = \frac{(t+t_0(\xi))^{n+1}}{(n+1)!}c(t,\xi), \tag{VI.18}$$

and by (VI.15), we get that

$$\chi(C_b^{-1}t)(1-\chi(|\xi|))c(t,\xi) \ \in S(1;g_t^0+<\xi>^{-2}g_\xi^0). \tag{VI.19}$$

Integrating by parts using (VI.12), we found

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = (2\pi)^{-\mathrm{n}-1}\int_{\mathbf{R}\times\mathbf{R}^{\mathrm{n}}}\chi(\mathrm{C}_{\mathrm{b}}^{-1}\mathrm{t})\hat{\varphi}_{\lambda,\delta}(\mathrm{t})\mathrm{e}^{\mathrm{i}(\mathrm{t}\sigma^0(\xi)+\mathrm{b}\xi)} \tag{VI.20}$$

$$\sum_{j=0}^{n} \frac{i^j}{j!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^j \partial_t^j w_0(-t_0(\xi),\xi)(1-\chi(|\xi|))dtd\xi$$

$$+(2\pi)^{-n-1}\int_{\mathbf{R}\times\mathbf{R}^n} \chi(C_b^{-1}t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(t\sigma^0(\xi)+b\xi)}[w_0(t,\xi)\chi(|\xi|)$$

$$+\frac{i^{n+1}}{(n+1)!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^{n+1}c(t,\xi)(1-\chi(|\xi|))]dtd\xi.$$

Now we use the following properties, $t\hat{\varphi}_{\lambda,\delta}(t)\in L^\infty(\mathbf{R})$, (this comes from (V.32)), $w_0(0,\xi)=1$ and the one (VI.19), to see that when $\lambda\to+\infty$

$$\int_{\mathbf{R}\times\mathbf{R}^n} \chi(C_b^{-1}t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(t\sigma^0(\xi)+b\xi)}[w_0(t,\xi)\chi(|\xi|)$$

$$+\frac{i^{n+1}}{(n+1)!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^{n+1}c(t,\xi)(1-\chi(|\xi|))]dtd\xi$$

$$=\int_{\mathbf{R}\times\mathbf{R}^n} \chi(C_b^{-1}t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(t\sigma^0(\xi)+b\xi)}[\chi(|\xi|)$$

$$+\frac{i^{n+1}}{(n+1)!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^{n+1}c(0,\xi)(1-\chi(|\xi|))]dtd\xi+\mathcal{O}(1)$$

$$=\int_{\mathbf{R}\times\mathbf{R}^n} \hat{\varphi}_{\lambda,\delta}(t)e^{i(t\sigma^0(\xi)+b\xi)}[\chi(|\xi|)$$

$$+\frac{i^{n+1}}{(n+1)!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^{n+1}c(0,\xi)(1-\chi(|\xi|))]dtd\xi+\mathcal{O}(1)$$

$$=2\pi\int_{\mathbf{R}^n} \varphi_{\lambda,\delta}(\sigma^0(\xi))e^{ib\xi}[\chi(|\xi|)$$

$$+\frac{i^{n+1}}{(n+1)!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^{n+1}c(0,\xi)(1-\chi(|\xi|))]d\xi+\mathcal{O}(1).$$

So, as $\lambda\to+\infty$,

$$\int_{\mathbf{R}\times\mathbf{R}^n} \chi(C_b^{-1}t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(t\sigma^0(\xi)+b\xi)}[w_0(t,\xi)\chi(|\xi|)$$

$$+\frac{i^{n+1}}{(n+1)!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^{n+1}c(t,\xi)(1-\chi(|\xi|))]dtd\xi=\mathcal{O}(1).$$

By (VI.20) we get also that, as $\lambda\to+\infty$,

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q}))=(2\pi)^{-\mathrm{n}}\int_{\mathbf{R}^{\mathrm{n}}} \varphi_{\lambda,\delta}(\sigma^0(\xi))\mathrm{e}^{\mathrm{i}\mathrm{b}\xi}\times \tag{VI.21}$$

$$\sum_{j=0}^{n} \frac{i^j}{j!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^j \partial_t^j w_0(-t_0(\xi),\xi)(1-\chi(|\xi|))d\xi+\mathcal{O}(1).$$

But by (VI.15), for $j\geq 1$,

$$\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^j \partial_t^j w_0(-t_0(\xi),\xi)(1-\chi(|\xi|))\ \in S(<\xi>^{-j};g_\xi^0), \tag{VI.22}$$

then (V.31), (VI.21) and (VI.22) gives that, as $\lambda\to+\infty$,

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q}))=(2\pi)^{-\mathrm{n}}\int_{\{\xi\in\mathbf{R}^{\mathrm{n}},\ \sigma^0(\xi)<\lambda\}} \mathrm{e}^{\mathrm{i}\mathrm{b}\xi}\times \tag{VI.23}$$

$$\sum_{j=0}^{n} \frac{i^j}{j!}\{\partial_\xi \sigma^0(\xi)|\xi|^{-2}\xi\}^j \partial_t^j w_0(-t_0(\xi),\xi)(1-\chi(|\xi|))d\xi + \mathcal{O}(1+\lambda^{n-1-\delta}).$$

We need the Lemma bellow.

Lemma VI.2 *If* $0 \leq m$ *and* $c(\xi) \in S(<\xi>^{-m}; <\xi>^{-2}\, g^0_\xi)$, *then, as* $\tau \to +\infty$,

$$\int_{\{\xi\in\mathbf{R}^n,\ |\xi|<\tau\}} e^{ib\xi} c(\xi)d\xi = \mathcal{O}(1+\tau^{\frac{n-1-2m}{2}}). \tag{VI.24}$$

Proof of the Lemma If $m > n$, (VI.24) is obvious, ($c(\xi)$ is integrable on $\mathbf{R}^n$).

If $m \leq n$, and if $k \in \mathbf{N}$ is such that $m+k > n$, as $b \neq 0$, we may integrate by parts k times to obtain

$$\int_{\{\xi\in\mathbf{R}^n,\ |\xi|<\tau\}} e^{ib\xi} c(\xi)d\xi = \int_{\{\xi\in\mathbf{R}^n,\ |\xi|<\tau\}} e^{ib\xi}\{\frac{ib}{|b|^2}\partial_\xi\}^k c(\xi)d\xi \tag{VI.25}$$

$$+\tau^{n-1}\sum_{j=0}^{k-1}\int_{\mathbf{S}^{n-1}} e^{i\tau b\omega}\frac{b\omega}{i|b|^2}\{\frac{ib}{|b|^2}\partial_\xi\}^j c(\tau\omega)d\Omega(\omega)\ ,$$

($\mathbf{S}^{n-1}$ is the unit sphere and $d\Omega(\omega)$ its usual measure).

As for any $\alpha \in \mathbf{N}^n$, $\tau^{m+|\alpha|}\partial_\xi^\alpha c(\tau\omega) \in L^\infty(\mathbf{R}_+ \times \mathbf{S}^{n-1})$, the stationary phase theorem[16] shows that as $\tau \to +\infty$,

$$\int_{\mathbf{S}^{n-1}} e^{i\tau b\omega}\frac{b\omega}{i|b|^2}\{\frac{ib}{|b|^2}\partial_\xi\}^j c(\tau\omega)d\Omega(\omega) = \mathcal{O}(\tau^{-m-j-\frac{n-1}{2}}). \tag{VI.26}$$

Then (VI.24) follows from (VI.25) and (VI.26).

Continuation of the proof of Proposition(VI.1).

The Theorem (IV.5) and the properties (IV.46), (IV.48)- -(IV.50), (VI.22) and (VI.23) lead to

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = \frac{1}{(2\pi)^{\mathrm{n}}|\mathcal{K}|}\int_{\{\xi\in\mathbf{R}^{\mathrm{n}},\ \sigma^0(\xi)<\lambda\}}\int_{\mathcal{K}} \mathrm{e}^{\mathrm{i}\mathrm{b}\xi}\mathrm{e}^{\mathrm{i}\psi(-\mathrm{t}_0(\xi),\mathrm{x},\xi)}\mathrm{dx}\mathrm{d}\xi + \mathcal{O}(\lambda^{\rho_\delta}), \tag{VI.27}$$

as $\lambda \to +\infty$, $(\rho_\delta = \mathrm{Max}\{0, \frac{n-3}{2}, n-1-\delta\})$.

Taking (IV.4) into account, we let

$$\tau = \sqrt{\lambda^2 - \lambda_0}, \tag{VI.28}$$

then from the Lemma (VI.2), (VI.25) and (VI.27), when $\lambda \to +\infty$

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = \frac{\tau^{\mathrm{n}-1}}{(2\pi)^{\mathrm{n}}|\mathcal{K}|}\int_{\mathbf{S}^{\mathrm{n}-1}}\int_{\mathcal{K}} \frac{\mathrm{b}\omega}{\mathrm{i}|\mathrm{b}|^2}\mathrm{e}^{\mathrm{i}\tau\mathrm{b}\omega}\mathrm{e}^{\mathrm{i}\psi(-\mathrm{t}_0(\tau\omega),\mathrm{x},\tau\omega)}\mathrm{dx}\mathrm{d}\Omega(\omega) + \mathcal{O}(\lambda^{\rho_\delta}). \tag{VI.29}$$

Noting (IV.4), (IV.47) and (VI.16) we get

$$\psi(-t_0(\tau\omega),x,\tau\omega) = \psi_b(x,\omega) := -(b\omega)\sum_{\gamma\in\Gamma^\star}(\omega A_\gamma)e^{ix\gamma}\zeta(-i(b\omega)(\gamma\omega)). \tag{VI.30}$$

The stationary phase theorem (see[16]), (VI.29) and (VI.30) show that as $\lambda \to +\infty$

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = \frac{\tau^{\frac{\mathrm{n}-1}{2}}}{|\mathrm{b}|^{\frac{\mathrm{n}-1}{2}}} \times \frac{1}{(2\pi)^{\frac{\mathrm{n}+1}{2}}|\mathcal{K}|} \times \tag{VI.31}$$

$$\int_{\mathcal{K}} \frac{1}{i|b|}[e^{i(\tau|b|-\frac{n-1}{4}\pi)}e^{i\psi_b(x,\frac{b}{|b|})} - e^{-i(\tau|b|-\frac{n-1}{4}\pi)}e^{i\psi_b(x,-\frac{b}{|b|})}]dx + \mathcal{O}(\lambda^{\rho_\delta}).$$

But by (VI.30)

$$\psi_b(x, \frac{b}{|b|}) = \psi_b(x, -\frac{b}{|b|}) = -\sum_{\gamma\in\Gamma^\star} bA_\gamma e^{ix\gamma}\zeta(-ib\gamma), \tag{VI.32}$$

as $\zeta(z) = (e^z - 1)/z$ and $b\gamma \in 2\pi\mathbf{Z}$, $\forall\gamma \in \Gamma^\star$,
we get from (VI.32)

$$\psi_b(x, \frac{b}{|b|}) = \psi_b(x, -\frac{b}{|b|}) = -\sum_{\gamma\in[b]^\perp} bA_\gamma e^{ix\gamma}, \tag{VI.33}$$

where $[b]^\perp \subset \Gamma^\star$ is the Bragg hyper-plane, sub-lattice orthogonal to b.

The Proposition (VI.1) follows if we notice that, as $\lambda \to +\infty$,

$$N_b(\lambda; Q) - \widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = \widetilde{\mathrm{Tr}}_{\mathrm{b}}(\chi_{]-\infty,\lambda[}(\mathrm{Q})) - \widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = \mathcal{O}(\lambda^{\mathrm{n}-1-\delta})$$

which comes from (II.9) (V.30) and (V.31) •

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