Remarks on decay of correlations and Witten Laplacians II – Analysis of the dependence on the interaction –

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December 24, 1997

Abstract

This is the continuation of previous notes on the subject referred as [He5]and devoted to the analysis of Laplace integrals attached to the measure $\exp -\Phi(X)\ dX$ for suitable families of phase Φ appearing naturally in the context of statistical mechanics. The main application treated in Part I was a semi-classical one ($\Phi = \Psi/h$ and $h \to 0$) and the assumptions on the phase were related to weak non convexity.
We analyze here in the same spirit the case when the coefficient of the interaction $\mathcal{J}$ is possibly large and give rather explicit lower bounds for the lowest eigenvalue of the Witten Laplacian on 1-forms. We also analyze the case $\mathcal{J}$ small by discussing first an unpublished proof of [BaJeSj] and then an alternative approach based on the analysis of a family of 1-dimensional Witten Laplacians. We also compare with the results given by Sokal's approach. In part III of this serie [He5], we shall analyze, in a less explicit way but in a more general context, applications to the logarithmic Sobolev inequality.

1 Introduction

Our aim is to analyze Laplace integrals attached to the measure $\exp -\Phi(X)\ dX$ in the case when Φ (which is attached to a subset $\Lambda \subset \mathbb{Z}^d$) has the form, for $X \in \mathbb{R}^\Lambda$,

$$\Phi(X) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{j,k\ ,\ j\sim k} |x_j - x_k|^2 \tag{1.1}$$

where, in the case of our toy model, which actually plays an important role in Quantum Field Theory (see for example [GlJa]), the one particle phase ϕ takes the form

$$\phi^{TM}(x) = \frac{1}{12}\lambda x^4 + \frac{1}{2}\nu x^2\ . \tag{1.2}$$

The parameters λ and ν satisfy

$$\lambda > 0\ , \tag{1.3}$$

and we want mainly to analyze the possibility of having $\nu < 0$: the consideration of non convex phases is indeed our main interest. Here $j \sim k$ means that j and k are nearest neighbors in Λ considered as a discrete torus. The sum is over the non-oriented pairs of $\Lambda \times \Lambda$. Finally $\mathcal{J}$ is usually assumed to be positive but all the results we shall give under the condition that $\mathcal{J}$ is small enough could be extended to the case when $|\mathcal{J}|$ is small enough.
Our main problem is to analyze the properties of the measure

$$d\mu := \exp -\Phi(X)\ dX / \left(\int_{\mathbb{R}^\Lambda} \exp -\Phi(X)\ dX\right) \tag{1.4}$$

and more precisely the covariance associating to (f, g)

$$\mathrm{Cov}\,(f,g) = \langle (f - \langle f\rangle)(g - \langle g\rangle)\rangle \tag{1.5}$$

where $\langle \cdot \rangle$ denotes the mean value with respect to the measure $d\mu$.
As it is now established[1] , an elegant approach is to analyze a Witten Laplacian on 1-forms. The aim of this article, in continuation of [He5], is to show that this approach leads also to very explicit criteria in non convex case that we compare in particular with Sokal's approach.

[1]See [Sj2] or [He5] for historical comments on the origin of the method.

2 Lower bound for the spectrum of the Witten Laplacian

Let us recall that the Witten Laplacian on $C_0^\infty(\mathbb{R}^\Lambda)$ 1-forms is defined as[2],

$$W_1 := W_0 \otimes I + \text{ Hess}\,\Phi\ , \tag{2.1}$$

where W_0 is

$$W_0 := \left[\sum_{j\in\Lambda}\left(-\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\left(\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\right]\ . \tag{2.2}$$

The strict positivity of the Witten Laplacian W_1 was first observed by J. Sjöstrand [Sj2] for a somewhat narrow class, which does not contain the example above. Let us recall[3] for reference the extension recently obtained by J. Johnsen [Jo].

Proposition 2.1 *:*
Let us assume that the phase Φ satisfies the following assumptions[4].

$$(H1)\quad |D_x^\beta \nabla\Phi(x)| \leq C_\beta < |\nabla\Phi| >^{(1-\rho|\beta|)_+} \tag{2.3}$$

for some $\rho > 0$, and for all β.
There exists $\delta > 0$ and a constant C such that, for all x such that $|x| \geq C$, we have

$$(H2)\quad x\cdot\nabla\Phi(x) \geq \frac{1}{C}|x|^{(1+\delta)}\ . \tag{2.4}$$

Then W_1 is essentially self-adjoint and its selfadjoint extension[5] as an unbounded operator on L^2 $1-$forms with respect to the standard Lebesgue is with compact resolvent and strictly positive.

We observe that these assumptions are satisfied for the model (1.1) with ϕ satisfying, for some strictly positive constants δ, ρ, C, $(C_k)_{k\in\mathbb{N}}$,

$$(h1)\quad |\phi^{(k+1)}(t)| \leq C_k < \phi'(t) >^{(1-\rho k)_+}\ , \tag{2.5}$$

[2]It was denoted by $\Delta_\Phi^{(1)}$ in [Sj2].
[3]For simplicity, we give a stronger condition than in [Jo], which is sufficient for our purpose.
[4]For $u \in \mathbb{R}^m$, we use the notation : $< u >= \sqrt{1+||u||^2}$.
[5]Consequently, the Friedrichs extension.

and, for $|t| \geq C$,

$$(h2) \quad t\phi'(t) \geq \frac{1}{C}|t|^{(1+\delta)} \, . \tag{2.6}$$

It is immediate to see that the phase $\phi = \phi^{TM}$ introduced in (1.2) satisfies (h1) and (h2) with $\rho = \frac{1}{3}$ and $\delta = 3$. The aim of this section is the proof, as a typical example of the method, of the following

Theorem 2.2 *:*
Let $\Lambda \subset \mathbb{Z}^d$ and let $\Phi^{(\Lambda,TM)} = \Phi^{TM}$ be the phase on $\mathbb{R}^\Lambda$ given by

$$\Phi^{TM}(X) = \sum_{j\in\Lambda} \phi^{TM}(x_j) + \frac{\mathcal{J}}{2}\sum_{j\sim k}|x_j - x_k|^2 \, , \tag{2.7}$$

where ϕ^{TM} satisfies (1.2) and (1.3). Then if $(\lambda, \nu, \mathcal{J})$ satisfy

$$\lambda > 0 \; , \; \nu < 0 \; , \; \mathcal{J} \geq 0 \; , \tag{2.8}$$

$$\lambda^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\nu > 0 \tag{2.9}$$

and

$$\begin{array}{lll} \mathcal{J}\,d & < \frac{1}{\sqrt{2}}\lambda^{\frac{1}{2}} + \frac{\nu}{2} & \textit{if } -\frac{\nu}{\sqrt{2}} < \lambda^{\frac{1}{2}} \leq -\sqrt{2}\nu \\ \mathcal{J}\,d & < \frac{\lambda}{4|\nu|} & \textit{if } -\sqrt{2}\nu \leq \lambda^{\frac{1}{2}} \, , \end{array} \tag{2.10}$$

then there exists $\rho_1 > 0$ such that

$$\langle W_1 u \mid u\rangle_{L^2} \geq \rho_1 ||u||^2 \, , \tag{2.11}$$

for all Λ and all u in $C_0^\infty(\mathbb{R}^\Lambda \, ; \, \mathbb{R}^\Lambda)$.

The basic fact here, in comparison with Proposition 2.1, is to obtain a strictly positive lower bound which is independent of Λ.

Remark 2.3 *:*

- The method of proof is not limited to this example and we can surely analyze other cases (see [BaJeSj] or Antoniouk-Antoniouk [AA] for the type of assumptions which can be considered).

- We recall also that semi-classical results –that is when Φ is replaced by $\frac{\Phi}{h}$ and the limit $h \to 0$ is considered– were presented in [He5] (See also [HeSj], [Sj1], [Sj2] and [BaJeSj]).

- The strictly convex case $\nu > 0$ does not create any problem and no condition except $\mathcal{J} \geq 0$ appears.

- Finally it is interesting to compare this approach with the results by A. Sokal [Sok]. This will be done in Section 3.

This theorem has interesting consequences in statistical mechanics. We get for example the following

Corollary 2.4 :
Under the same assumptions, there exists C and $\kappa > 0$ such that, for any Λ and $(i, j) \in \Lambda \times \Lambda$ such that[6]

$$d(i,j) \leq \frac{1}{4} \inf \Big(d(i, \mathbb{Z}^d \setminus \Lambda), d(j, \mathbb{Z}^d \setminus \Lambda) \Big) \ , \tag{2.12}$$

then

$$\mathrm{Cov}\,(x_i, x_j) \leq C \exp -\kappa d(i,j) \ . \tag{2.13}$$

We say in this case that the pair correlations have exponential decay.
The proof of the corollary is similar to the proof given in [He5] (or Naddaf-Spencer [NaSp]) and based on a Combes-Thomas distorsion argument ([CT], see also [HeSj] for a study based on the Maximum Principle in the strictly convex case and [Sj1] in the weakly convex case).

Proof of Theorem 2.2:
This proof follows the same lines as in [He5] (which was considering the semi-classical case). Letting

$$X_j = \partial_j + \frac{1}{2} \partial_j \Phi \ , \tag{2.14}$$

we start from

$$\langle W_1 u \mid u \rangle_{L^2} = \sum_{j,k} ||X_k\, u_j||^2 + \sum_{j,k} \int \frac{\partial^2 \Phi}{\partial x_j\, \partial x_k} u_j\ u_k\ dX \ . \tag{2.15}$$

[6] d is the euclidean distance.

We first omit the terms $||X_k\, u_j||^2$ with $k \neq j$ and get

$$\begin{array}{ll}\langle W_1 u \mid u\rangle_{L^2} & \geq \sum_j \left(||X_j u_j||^2 + \int \frac{\partial^2\Phi}{\partial x_j\,\partial x_j} u_j^2 dX\right) \\ & + \sum_{j\neq k} \int \frac{\partial^2\Phi}{\partial x_j\,\partial x_k} u_j\; u_k\; dX\;.\end{array} \tag{2.16}$$

We analyze for fixed j the term

$$\langle w_j^{(1)} u_j \mid u_j\rangle := ||X_j u_j||^2 + \int \frac{\partial^2\Phi}{\partial x_j\,\partial x_j} u_j^2\; dX\;. \tag{2.17}$$

The operator $w_j^{(1)}$, initially defined as an unbounded operator on $L^2(I\!R^\Lambda)$, can also be seen as a family (indexed by the $(x_\ell)_{\ell\in\Lambda\setminus\{j\}}$) of one particle Witten Laplacians (relative to the effective phase ϕ_j at $j \in \Lambda$)

$$\phi_j(t) = \phi(t) + \mathcal{J}\sum_{k\sim j} |t - x_k|^2\;, \tag{2.18}$$

and defined on $L^2(I\!R)$ by

$$w_j^{(1)}(t, \frac{d}{dt}) := w_j^{(0)}(t, \frac{d}{dt}) + \phi_j''(t)\;, \tag{2.19}$$

and

$$w_j^{(0)}(t, \frac{d}{dt}) := -\frac{d^2}{dt^2} + \frac{1}{4}|\phi_j'(t)|^2 - \frac{1}{2}\phi_j''(t)\;. \tag{2.20}$$

We observe that $\phi_j = \phi_{j,X_{\hat{j}}}$ depends on the variables $X_{\hat{j}}$ (and actually only on the x_k $(k \sim j)$) which are now considered as parameters and this gives the following link between two points of view :

$$\phi'_{j,X_{\hat{j}}}(x_j) = (\partial_{x_j}\Phi)(X)\;, \tag{2.21}$$

and

$$\phi''_{j,X_{\hat{j}}}(x_j) = (\partial^2_{x_j}\Phi)(X)\;, \tag{2.22}$$

We met also this double point of view when applying Dobrushin's method in [He4].
We introduce the following decomposition of the phase Φ

$$\begin{array}{ll}\Phi & := \Phi_d + \Phi_i \\ \Phi_d(X) & := \sum_{j\in\Lambda} \phi(x_j) \\ \Phi_i(X) & := \frac{J}{2}\sum_{j\sim k} |x_j - x_k|^2\end{array} \tag{2.23}$$

Easy computations give (see [He5]) that, for any ϵ satisfying

$$0 < \epsilon \leq 1 \ , \tag{2.24}$$

$$\begin{aligned} &\langle w_j^{(1)} u_j \mid u_j \rangle \\ &\quad \geq \epsilon \left(||\partial_{x_j} u_j||^2 + \tfrac{2-\epsilon}{2\epsilon} \int \tfrac{\partial^2 \Phi_d}{\partial x_j \, \partial x_j} u_j^2 dX - \tfrac{1}{2} \int \tfrac{\partial^2 \Phi_i}{\partial x_j \, \partial x_j} u_j^2 \, dX \right) \\ &\quad\quad + \int \tfrac{\partial^2 \Phi_i}{\partial x_j \, \partial x_j} u_j^2 \, dX. \end{aligned} \tag{2.25}$$

Here we have introduced, for any ϵ, a new 1-particle operator

$$s_j^\epsilon := -\partial_{x_j}^2 + \frac{2-\epsilon}{2\epsilon} \frac{\partial^2 \Phi_d}{\partial x_j^2} - \frac{1}{2} \frac{\partial^2 \Phi_i}{\partial x_j^2} \ , \tag{2.26}$$

which is now independent of the variables $(x_\ell)_{\ell \in \Lambda \setminus \{j\}}$, and the inequality (2.25) takes the form

$$\langle w_j^{(1)} u_j \mid u_j \rangle \geq \epsilon \langle s_j^\epsilon u_j \mid u_j \rangle + \int \frac{\partial^2 \Phi_i}{\partial x_j \, \partial x_j} u_j^2 \, dX \tag{2.27}$$

Coming back to (2.16) and using the convexity of the interaction Φ_i, we get

$$\langle W_1 u \mid u \rangle_{L^2} \geq \epsilon \sigma_1(\epsilon, \mathcal{J}) (\sum_j ||u_j||^2) \ , \tag{2.28}$$

where $\sigma_1(\epsilon, \mathcal{J})$ is a lower bound for the spectrum of the operators s_j^ϵ seen as a selfadjoint operator on $L^2(I\!R^\Lambda)$ or a lower bound for the family of operators also denoted by s_j^ϵ defined on $L^2(I\!R)$ but depending on the parameters x_k with $j \sim k$.
The success comes[7] from the existence of a strictly positive (and independent of Λ)

$$\theta_1 := \sup_{\epsilon \in]0,1]} \sigma_1(\epsilon, \mathcal{J}) \ . \tag{2.29}$$

We have now to find suitable lower bounds for specific examples and to optimize over ϵ. The case when $\epsilon = 1$ corresponds to the approach presented in [He4].
We observe that for our specific example the possibility to work with the

[7]The lower bound ρ_1 whose existence is claimed in the theorem will actually be obtained, once the strict positivity of θ_1 is proved, as $\sup_{\epsilon \in]0,1]} (\epsilon \sigma_1(\epsilon, \mathcal{J}))$.

general Witten Laplacian leads to strict improvements.
Let us also observe that our operator s_j^ϵ depends actually only on the variable x_j and takes the simple form

$$s_j^\epsilon = s^\epsilon(x_j, \frac{d}{dx_j}) \,, \tag{2.30}$$

with

$$s^\epsilon(y, \frac{d}{dy}) := -\frac{d^2}{dy^2} + \frac{2-\epsilon}{2\epsilon}\phi''(y) - \mathcal{J} d \,. \tag{2.31}$$

Let us start the analysis of our specific model.

We have to consider

$$s^\epsilon(y, \frac{d}{dy}) := -\frac{d^2}{dy^2} + \lambda\frac{2-\epsilon}{2\epsilon}y^2 + \frac{2-\epsilon}{2\epsilon}\nu - \mathcal{J} d \,. \tag{2.32}$$

For fixed ϵ, we get that the lowest eigenvalue σ (which we take as our σ_1) of $s^epsilon$ is given by

$$\sigma(\mathcal{J}, \epsilon) = \lambda^{\frac{1}{2}}(\frac{2-\epsilon}{2\epsilon})^{\frac{1}{2}} + \nu\frac{2-\epsilon}{2\epsilon} - \mathcal{J}\, d \,. \tag{2.33}$$

Let us introduce the parameter $\eta = \sqrt{\frac{2-\epsilon}{2\epsilon}}$ which varies between $\frac{1}{\sqrt{2}}$ and $+\infty$. With this new variable, we get

$$\sigma(\epsilon, \mathcal{J}) = \hat{\sigma}(\eta, \mathcal{J}) = \lambda^{\frac{1}{2}}\eta + \nu\eta^2 - \mathcal{J}\, d \,.$$

Then the maximum (assuming $\nu < 0$) is obtained for

$$\eta = -\frac{\lambda^{\frac{1}{2}}}{2\nu} \,,$$

if

$$\lambda^{\frac{1}{2}} + \sqrt{2}\nu > 0 \,, \tag{2.34}$$

or for

$$\eta = 1/\sqrt{2} \,. \tag{2.35}$$

In the first case, we get

$$\sup_{\eta\in[\frac{1}{\sqrt{2}},+\infty]} \hat{\sigma}(\eta, \mathcal{J}) = \frac{\lambda}{4|\nu|} - \mathcal{J}\, d \,, \tag{2.36}$$

and the condition of strict positivity for this supremum is

$$\mathcal{J}\, d < \frac{\lambda}{4|\nu|}\ . \tag{2.37}$$

In the second case, we get

$$\sup_{\eta \in [\frac{1}{\sqrt{2}}, +\infty]} \hat{\sigma}(\eta, \mathcal{J}) = \frac{1}{\sqrt{2}} \lambda^{\frac{1}{2}} + \frac{\nu}{2}\ - \mathcal{J}\, d\ , \tag{2.38}$$

and the condition of strict positivity becomes

$$\mathcal{J}\, d < \frac{1}{\sqrt{2}} \lambda^{\frac{1}{2}} + \frac{\nu}{2}\ . \tag{2.39}$$

As necessary condition for the strict positivity, we have finally obtained the condition

$$\lambda^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \nu > 0 \tag{2.40}$$

and, combining the two cases $-\frac{\nu}{\sqrt{2}} < \lambda^{\frac{1}{2}} \leq -\sqrt{2}\nu$ and $-\sqrt{2}\nu \leq \lambda^{\frac{1}{2}}$, we get the conditions (2.10) for getting the strict positivity.

Remark 2.5 *:*
If we want to get a more abstract version of the previous result, we can see in the proof that our criterion says more generally that, if there exists $\epsilon \in]0, 1]$ such that the operator $s^\epsilon := -\frac{d^2}{dy^2} + \frac{2-\epsilon}{2\epsilon} \phi''(y) - \mathcal{J}\, d$ is strictly positive, then there exists $\rho_1 > 0$ such that (2.11) is verified for any $\Lambda \subset \mathbb{Z}^d$.
The direct one-dimensional approach leads to the criterion with $\epsilon = 1$. This is not optimal if $\int_{\mathbb{R}} \phi''(y) u_1(y)^2 dy > 0$ (where u_1 is the first eigenfunction of s^ϵ for $\epsilon = 1$). We can indeed analyze, by the Feynmann-Heilman formula, the sign of the first derivative with respect to η of the lowest eigenvalue of s^ϵ. This allows in principle the treatment of more general cases in which ϕ'' is negative in a sufficiently small region.
The question that we have treated very explicitly in the case of the harmonic oscillator can also be formulated as follows :
Given a real regular potential v tending to ∞ at ∞, a constant σ_0 and $\eta_0 > 0$, under which condition on v can we find $\eta \geq \eta_0$ such that the unbounded operator on $L^2(\mathbb{R})$, $P^\eta = -\frac{d^2}{dy^2} + \eta v(y) - \sigma_0$, is strictly positive ?

Remark 2.6 *:*
The proof is given in the case of periodic conditions but could be extended to other conditions after small modifications. By looking at these other conditions and verifying some uniformity, one would get the possibility of proving the Log-Sobolev inequalities as suggested by T. Bodineau referring to recent results by B. Zegarlinski [Ze] (Theorem 5.1). We shall analyze this point in detail in [He5].

3 Comparison with Sokal's results

Let us compare with the results proved by A. Sokal [Sok]. Under the assumption that the one particle phase ϕ satisfies some Lebowitz inequality, that is that ϕ is an even differentiable function such that

$$\phi' \text{ is convex on }]0,+\infty[\ , \tag{3.1}$$

(and this is the case for the above example), A. Sokal obtains the decay of the correlations in a rather general framework by introducing a kind of effective quadratic potential associated to ϕ. More precisely, starting from ϕ, we consider first

$$\phi_{\mathcal{J}}(t) = \phi(t) + \mathcal{J}\,d\,t^2\ . \tag{3.2}$$

We then consider the moment of order 2

$$c_{\mathcal{J}} = \frac{\int_{I\!R} t^2 \exp -\phi_{\mathcal{J}}(t)\ dt}{\int_{I\!R} \exp -\phi_{\mathcal{J}}(t)\ dt}\ . \tag{3.3}$$

Then A. Sokal proves

Theorem 3.1 *:*
If ϕ satisfies the Lebowitz inequality (3.1) and the condition

$$2\mathcal{J}\,d\,c_{\mathcal{J}} < 1\ , \tag{3.4}$$

then we have the exponential decay of the correlations in the sense of the Corollary 2.4.

Let us first verify that, when $\phi(t) = \frac{\alpha}{2}t^2$ with $\alpha > 0$ (convex case), we recover the property that there are no condition except $\mathcal{J} > 0$. We immediately get

$$c_{\mathcal{J}} = \frac{1}{\alpha + 2\mathcal{J}d}\ ,$$

and see that (3.4) is satisfied.
The main idea in Sokal [Sok] is that the correlation functions are controlled by the new potential

$$\phi_{\mathcal{J}}^{eff}(t) = \left(\frac{1}{2c_{\mathcal{J}}} - \mathcal{J}d\right) t^2 \tag{3.5}$$

The condition (3.4) simply means the strict convexity of $\phi_{\mathcal{J}}^{eff}$.
The first thing to observe is that the condition is always satisfied when $\mathcal{J}$ is small enough. We get indeed

$$\lim_{\mathcal{J}\to 0} c_{\mathcal{J}} = \frac{\int_{I\!R} t^2 \exp -\phi(t)\ dt}{\int_{I\!R} \exp -\phi(t)\ dt} > 0\ . \tag{3.6}$$

As we already said, this can be applied to our previous toy models. But it can be interesting to explicate the condition that we get and to understand the links between the criteria obtained by the various methods. In particular, what is going on when $\mathcal{J}$ is large ? Let us consequently consider

$$c(\mathcal{J},\lambda,\nu) = \frac{\int_{I\!R} t^2 \exp -\phi_{\mathcal{J},\lambda,\nu}(t)\ dt}{\int_{I\!R} \exp -\phi_{\mathcal{J},\lambda,\nu}(t)\ dt}\ , \tag{3.7}$$

with

$$\phi_{\mathcal{J},\lambda,\nu}(t) = \frac{1}{12}\lambda t^4 + (\frac{1}{2}\nu + \mathcal{J}d)t^2\ . \tag{3.8}$$

For fixed $\lambda > 0$ and $\nu < 0$, let us compute the asymptotic of $c(\mathcal{J},\lambda,\nu)$ as $\mathcal{J} \to +\infty$. An easy computation based on the Laplace integral method (in dimension 1) gives

Proposition 3.2 *:*
As $\mathcal{J}$ tends to $+\infty$, we have the following asymptotic behavior

$$c(\mathcal{J},\lambda,\nu) = \frac{1}{2d}\frac{1}{\mathcal{J}} - \frac{\nu}{4d^2}\frac{1}{\mathcal{J}^2} + \mathcal{O}(\frac{1}{\mathcal{J}^3})\ . \tag{3.9}$$

We first indeed observe that

$$(\frac{\nu}{2} + \mathcal{J}d)^{\frac{1}{2}} \int t^2 \exp -\phi_{\mathcal{J},\lambda,\nu}(t)\ dt = \frac{1}{\nu + 2\mathcal{J}d}(\int \exp -s^2\ ds) + \mathcal{O}((\frac{1}{\mathcal{J}})^3)\ . \tag{3.10}$$

Then the renormalized denominator of $c(\mathcal{J},\lambda,\nu)$ has the following asymptotical behavior

$$\begin{aligned}&(\tfrac{\nu}{2}+\mathcal{J}d)^{\frac{1}{2}}\int \exp -\phi_{\mathcal{J},\lambda,\nu}(t)\ dt\\ &\quad = (\int \exp -s^2\ ds)\ -\tfrac{\lambda}{12}(\tfrac{\nu}{2}+\mathcal{J}d)^{-2}(\int s^4 \exp -s^2\ ds)\ +\mathcal{O}((\tfrac{1}{\mathcal{J}})^3)\ ,\end{aligned}$$

and we get

$$\begin{aligned}&(\tfrac{\nu}{2}+\mathcal{J}d)^{\frac{1}{2}}\int \exp -\phi_{\mathcal{J},\lambda,\nu}(t)\ dt\\ &\quad = (\int \exp -s^2\ ds)\ \left(1-\tfrac{3\lambda}{48}(\tfrac{\nu}{2}+\mathcal{J}d)^{-2}\right)+\mathcal{O}((\tfrac{1}{\mathcal{J}})^3)\ .\end{aligned}\tag{3.11}$$

Taking the quotient, we obtain

$$c(\mathcal{J},\lambda,\nu)=\left(\frac{1}{\nu+2\mathcal{J}d}\right)\ (1+\mathcal{O}((\frac{1}{\mathcal{J}})^2))\ ,\tag{3.12}$$

which gives the proof of (3.9).
This shows that (3.4) becomes false for $\mathcal{J}$ large. This is what we expect, because we hope to see a phase transition when $\mathcal{J}$ is large enough.
On the other hand for fixed $\mathcal{J}>0$ and $\nu<0$ fixed, one easily gets, using the change of variable $t=\lambda^{-\frac{1}{4}}s$ in (3.3), the

Proposition 3.3 *:*
For fixed $\mathcal{J}>0$ and $\nu<0$, we have

$$\lim_{\lambda\to+\infty} c(\mathcal{J},\lambda,\nu)=0\ .\tag{3.13}$$

This gives one case when Sokal's theorem could be applied.

We can give more explicit sufficient conditions in the following way. We do not try to be optimal but just want to sketch how it can be done. We observe first that

$$t\phi'_{\mathcal{J},\lambda,\nu}(t)=(2\mathcal{J}d+\nu)t^2+\frac{\lambda}{3}t^4\ .$$

We can for example start of the inequality

$$t^2\le\frac{1}{4\mathcal{J}d}+\mathcal{J}d\ t^4\ ,\tag{3.14}$$

and get, under the condition that

$$2\mathcal{J}d+\nu>0\ ,\tag{3.15}$$

the estimate

$$t^2 \leq \frac{1}{4\mathcal{J}d} + \frac{3\mathcal{J}d}{\lambda} t\phi'_{\mathcal{J},\lambda,\nu}(t) \ . \tag{3.16}$$

This leads immediately, after an integration by parts, to the

Proposition 3.4 *:*
Under the condition (3.15), we have the

$$c(\mathcal{J},\lambda,\nu) \leq \frac{1}{4\mathcal{J}d} + \frac{3\mathcal{J}d}{\lambda} \ . \tag{3.17}$$

We then obtain as a corollary the exponential decay of the correlation pair function under the conditions (3.15) and

$$\lambda > 12\mathcal{J}^2 d^2 \ . \tag{3.18}$$

The inequality (3.18) implies, using (3.15) and (3.17), the condition (3.4).

Remark 3.5 *:*
This is of course very rough and can be surely improved by considering instead of (3.14) the inequality

$$t^2 \leq \frac{\alpha}{4\mathcal{J}d} + \frac{\mathcal{J}d}{\alpha} t^4 \ ,$$

for a strictly positive α, and using also an improvement of (3.16) leads to

$$\langle t^2 \rangle \leq \left(1 + \frac{3\mathcal{J}d}{\alpha} \cdot \frac{2\mathcal{J}d + \nu)}{\lambda}\right)^{-1} \left(\frac{\alpha}{4\mathcal{J}d} + \frac{3\mathcal{J}d}{\alpha\lambda}\right) \ .$$

The question is then to determine under which condition one can find α such that the r.h.s is strictly less than $\frac{1}{2\mathcal{J}d}$.
Then one could compare with the condition appearing in (2.10). This is left as an exercise to the reader.

4 The case $\mathcal{J}$ small: perturbative approach

Sokal's result allows one to prove that, for a large class of ferromagnetic examples that in any case –in particular also for highly non convex situations–

we have exponential decay of the correlations for $\mathcal{J}$ small enough. It is natural to try to understand also this property in the framework of the Witten Laplacians. This was one of the goals of a recent analysis presented by Bach-Jecko-Sjöstrand [BaJeSj]. Let us describe their approach in a rather more particular situation that we shall also analyze differently in the next section.

The basic idea is that when $\mathcal{J} = 0$, the Witten Laplacian on 1-forms is clearly strictly positive (uniformly with respect to the subset Λ of $\mathbb{Z}^d$). It is indeed diagonal.
We have consequently to justify a perturbation argument for $\mathcal{J}$ small. The only small difficulty is the control with respect to the dimension.
Let us start again from (2.15), where we now introduce explicitely $\mathcal{J}$ in the notations,

$$\langle W_1^{\mathcal{J}} u \mid u\rangle_{L^2} = \sum_{j,k} ||X_k\, u_j||^2 + \sum_{j,k} \int \frac{\partial^2 \Phi}{\partial x_j\, \partial x_k} u_j\, u_k\, dX\ . \tag{4.1}$$

We denote by $w_j^{(0;\mathcal{J})}$ and $w_j^{(1;\mathcal{J})}$ the one particle Witten Laplacians attached to the site j and the phase

$$\phi_{j;\mathcal{J}}(t) := \phi(t) + \mathcal{J} \sum_{k\sim j} |t - x_k|^2\ , \tag{4.2}$$

on respectively functions or 1-forms.
We analyze first the case $\mathcal{J} = 0$

$$\langle W_1^0 u \mid u\rangle_{L^2} = \sum_{j\neq k} \langle w_k^{(0)}\, u_j \mid u_j\rangle + \sum_j \langle w_j^{(1)} u_j \mid u_j\rangle\ . \tag{4.3}$$

If $\rho_1(0)$ is the lowest eigenvalue of $w_j^{(1;0)}$, which is strictly positive according to Proposition 2.1, we first get as before

$$\langle W_1^0 u \mid u\rangle_{L^2} \geq \sum_j \langle w_j^{(1;0)} u_j \mid u_j\rangle \geq \rho_1(0) ||u||^2\ . \tag{4.4}$$

The new point is to use the other term (which was neglected before by positivity, see when we came from (2.15) to (2.16)) in order to get some control of the perturbation when $\mathcal{J}$ becomes different from 0. The Witten Laplacian $w_j^{(0;0)}$ takes simply the form, with $t = x_j$, $w_j^{(0;0)} = w^{(0;0)}(t, \frac{d}{dt})$ with

$$w^{(0;0)}(t, \frac{d}{dt}) = -\frac{d^2}{dt^2} + \frac{1}{4}|\phi'(t)|^2 - \frac{1}{2}\phi''(t)\ , \tag{4.5}$$

and $w_j^{(1;0)}$ the form, with $t = x_j$, $w_j^{(1;0)} = w^{(1;0)}(t, \frac{d}{dt})$ with

$$w^{(1;0)}(t, \frac{d}{dt}) = -\frac{d^2}{dt^2} + \frac{1}{4}|\phi'(t)|^2 + \frac{1}{2}\phi''(t) \ . \tag{4.6}$$

Under conditions $(h1)$ and $(h2)$ on ϕ, we observe that (with $s = 0$ or 1)

$$\int_{I\!R} \phi'(t)^2 f(t)^2 dt \leq C \left(\int_{I\!R} (w^{(s;0)}(t, \frac{d}{dt}) f(t)) f(t)\ dt + \int_{I\!R} f(t)^2\ dt \right) \ . \tag{4.7}$$

We just apply this inequality for the pairs j, ℓ and get the following regularity property, for a suitable constant C,

$$||\phi'_{j;0}\, u_\ell\, ||^2 \leq C \left(\langle w_j^{(0;0)} u_\ell \mid u_\ell \rangle_{L^2} + ||u_\ell||^2 \right) \ . \tag{4.8}$$

We also obtain for another constant C, that, for $(j, k, \ell) \in \Lambda$ with $j \sim k$,

$$||(x_j - x_k) u_\ell||^2 \leq C \left(\langle w_j^{(0;0)} u_\ell \mid u_\ell \rangle_{L^2} \ + \ \langle w_k^{(0;0)} u_\ell \mid u_\ell \rangle_{L^2} + ||u_\ell||^2 \right) \ . \tag{4.9}$$

These estimates are not sufficient. The detailed proof is more tricky and is given in [BaJeSj]. One of the points is to introduce a parameter $0 < q < 1$ and to write the following inequality

$$\langle W_1^{\mathcal{J}} u \mid u \rangle_{L^2} \geq \sum_{j,k} q^{d_\Lambda(j,k)} ||X_k\, u_j||^2 + \sum_{j,k} \int \frac{\partial^2 \Phi}{\partial x_j\, \partial x_k} u_j\ u_k\ dX \ , \tag{4.10}$$

where $d_\Lambda(j, k)$ is the distance between j and k on the cube Λ. The introduction of this q transforms an error of order $|\Lambda| \cdot \mathcal{O}(J)$ into an error of order $\mathcal{O}(\mathcal{J}) \cdot (\sum_{k \in \Lambda} q^{d_\Lambda(j,k)})$, that is actually of order $\mathcal{O}(\mathcal{J})$. We then obtain, for a suitable C,

$$\begin{aligned} &\langle W_1^{\mathcal{J}} u \mid u \rangle_{L^2} \\ &\quad \geq (1 - C\mathcal{J}) \left(\textstyle\sum_{j \neq k} q^{d(j,k)} \langle w_k^{(0)}\, u_j |\, u_j \rangle + \sum_j \langle w_j^{(1)}\, u_j \mid u_j \rangle \right) - C\mathcal{J} \textstyle\sum_j ||u_j||^2 \ . \end{aligned} \tag{4.11}$$

So we have proved the existence of another Λ independent constant C,

$$\langle W_1^{\mathcal{J}} u \mid u \rangle_{L^2} \geq (\rho_1(0) - C\mathcal{J})(\sum_j ||u_j||^2) \ . \tag{4.12}$$

This is summarized in

Proposition 4.1 *:*
If ϕ satisfies $(h1)$ and $(h2)$ and Φ is defined by (1.1), there exists $\mathcal{J}_0 > 0$ such that, if $0 \leq \mathcal{J} \leq \mathcal{J}_0$, then there exists $\rho_1 > 0$ such that, for any cube $\Lambda \subset \mathbb{Z}^d$,

$$W_1^{\mathcal{J}} \geq \rho_1 \ . \tag{4.13}$$

Remark 4.2 *:*
This proof could probably work also for other boundary conditions (periodic, free,...). But the condition that $|\mathcal{J}|$ is smaller than some $\mathcal{J}_0$ may depend on the choice of these boundary conditions. This will be analyzed in [He5].
Let us also observe that the condition on the sign of $\mathcal{J}$ is unimportant in the proof. Sokal's approach is more related to ferromagnetic systems.

5 The case $\mathcal{J}$ small : an 1-dimensional approach

In the preceding section, we were suggesting that in the highly non convex case there was a problem to find a strictly positive lower bound uniformly with respect to the parameters x_k ($k \sim j$) for the operator $w_j^{(1;\mathcal{J})}$ seen as a family of operators on $L^2(\mathbb{R})$. We presented in Section 2 one way which was efficient in the case when the non convexity was not too strong. In Section 4, we explained the trick used by Bach-Jecko-Sjöstrand [BaJeSj] consisting in the use of the non diagonal terms for controlling the perturbation and also eliminating any assumption of weak convexity.
The study in this section will present another way, less perturbative in spirit, for treating the difficulty to have uniform control with respect to $\mathcal{J}$.
The problem is indeed the following. We want to analyze

$$w^{(1;\mathcal{J})}(t, \frac{d}{dt}) := -\frac{d^2}{dt^2} + \frac{1}{4}\left(\phi'(t) + 2\mathcal{J} d\, t - 2\mathcal{J} \sum_{0\sim k} x_k\right)^2 + \frac{1}{2}(\phi''(t) + 2\mathcal{J}\ d) \ . \tag{5.1}$$

The criterion given in Section 2 is (when taking $\epsilon = 1$) is actually related to the assumption that the operator

$$p^{(1;\mathcal{J})}(t, \frac{d}{dt}) := w^{(1;\mathcal{J})}(t, \frac{d}{dt}) - 2\mathcal{J} d \ ,$$

that is,

$$p^{(1;\mathcal{J})}(t,\frac{d}{dt}) := -\frac{d^2}{dt^2} + \frac{1}{4}\left(\phi'(t) + 2\mathcal{J}d\,t - 2\mathcal{J}\sum_{0\sim k} x_k\right)^2 + \frac{1}{2}(\phi''(t) - 2\mathcal{J}d)\ , \tag{5.2}$$

is uniformly stricty positive (with respect to x_k and $\mathcal{J}$).
The previous criteria correspond to taking the lower bound obtained by forgetting the term $\frac{1}{4}\left(\phi'(t) + 2\mathcal{J}d\,t - 2\mathcal{J}\sum_{0\sim k} x_k\right)^2$ and comparing with the lowest eigenvalue of $s(t,\frac{d}{dt})$ but this works only when this last operator is strictly positive. This is not satisfactory for $\mathcal{J}$ small. In this regim, it is sufficient (the two operators differ from $2\mathcal{J}d$), to consider $w^{(1;\mathcal{J})}(t,\frac{d}{dt})$ and, changing slightly the various parameters, to analyze the following family depending on $\mathcal{J}$ and α:

$$w(t,\frac{d}{dt},\alpha,\mathcal{J}) := -\frac{d^2}{dt^2} + \frac{1}{4}\left(\phi'(t) + 2\mathcal{J}d\,t - \alpha\right)^2 + \frac{1}{2}(\phi''(t) + 2\mathcal{J}d)\ , \tag{5.3}$$

with

$$\alpha = -2\mathcal{J}\sum_{0\sim k} x_k\ . \tag{5.4}$$

We will then analyze the uniform strict positivity of this family with respect to $(\alpha,\mathcal{J}) \in I\!\!R\times[-\mathcal{J}_0,+\mathcal{J}_0]$ which is better because we can forget the relation (5.4).

For any fixed pair $(\alpha,\mathcal{J})$, the operator is strictly positive by Proposition 2.1 and a standard perturbation argument permits to show (under assumptions $(h1)$, $(h2)$ and with $\delta \geq 1$) that the lowest eigenvalue of $w(t,\frac{d}{dt},\alpha,\mathcal{J})$ is continuous with respect to $\mathcal{J}$ and α.
So it is a problem of uniformity as $\alpha = \pm\infty$. Let us treat our model where $\phi(t) = \phi^{TM}$ using a semi-classical approach. Changing of ν, we can forget $\mathcal{J}$ although remembering that our argument has to be locally uniform in ν.
So we have finally to verify the following

Lemma 5.1 *:*
There exists $\tau > 0$ such that for $|\alpha| \geq \tau$ the operator

$$s(t,\frac{d}{dt},\alpha) := -\frac{d^2}{dt^2} + \frac{1}{4}(\frac{\lambda}{3}t^3 + \nu t - \alpha)^2 + \frac{1}{2}(\lambda t^2 + \nu) \tag{5.5}$$

satisfies

$$s(t,\frac{d}{dt},\alpha) \geq \frac{1}{\tau}\ |\alpha|^{\frac{2}{3}}\ . \tag{5.6}$$

The problem is symmetric in α. It is consequently sufficient to treat the case $\alpha > 0$.
Let us introduce the scaling $t = \alpha^{\frac{1}{3}} y$. We then obtain the following inequality

$$q(y, \frac{d}{dy}, \alpha) := -\alpha^{-\frac{2}{3}} \frac{d^2}{dy^2} + \frac{\alpha^2}{4} (\frac{\lambda}{3} y^3 + \nu \alpha^{-\frac{2}{3}} y - 1)^2 + \frac{1}{2} \alpha^{\frac{2}{3}} (\lambda y^2 + \alpha^{-\frac{2}{3}} \nu) \geq \frac{1}{\tau} |\alpha|^{\frac{2}{3}} \tag{5.7}$$

to be proved. In order to get a more semi-classical form we can write it as

$$q(y, \frac{d}{dy}, \alpha) := \alpha^2 \tilde{q}(y, h\frac{d}{dy}; h; \mu) \ , \tag{5.8}$$

with

$$\tilde{q}(y, h\frac{d}{dy}; h; \mu) := \left(-h^2 \frac{d^2}{dy^2} + \frac{1}{4} (\frac{\lambda}{3} y^3 + \mu y - 1)^2 + \frac{1}{2} h (\lambda y^2 + \mu) \right) \ , \tag{5.9}$$

where

$$h = \alpha^{-\frac{4}{3}} \ ; \ \mu = \nu \alpha^{-\frac{2}{3}} = \mathcal{O}(h^{\frac{1}{2}}) \ . \tag{5.10}$$

This is a non degenerate one-well semi-classical problem near the point $y = 1$ and this leads, for the operator $\tilde{q}(y, h\frac{d}{dy}; h; \mu)$, to a lowest eigenvalue which is asymptotically equal to $h\lambda$. We finally have obtained a behavior like $\lambda \alpha^{\frac{2}{3}}$ for the lowest eigenvalue of $q(y, \frac{d}{dy}, \alpha)$ as $\alpha \to +\infty$ uniformly with respect to ν and consequently of $s(t, \frac{d}{dt}, \alpha)$. This shows the lemma.

We have consequently given an alternative proof of Proposition 4.1, in the case of the toy model $\phi = \phi^{TM}$.

Remark 5.2 *:*
We shall discuss the uniformity of the argument with respect to other boundary conditions in [He5]. The price to pay is that the estimates given in this section will be much more implicit.

6 Conclusion

As we have seen, the lowest eigenvalue of the Witten Laplacian plays an important role. A natural question is the understanding of the phase transition

through this approach. The answer is not clear. Let us just observe here that the results obtained on the transition of phase by other methods like infrared estimates or Peierls argument (see [GlJa] for a presentation of these methods) will give an upper bound on the splitting between the two lowest eigenvalues of the Witten Laplacian on the functions or an upper bound of the smallest eigenvalue of the Witten Laplacian on the 1-forms.
We recall that in the infrared estimates (see [He2] for a recent presentation and a bibliography) one gets (but under the condition $d \geq 2$), a uniform[8] lower bound $\delta_0 > 0$ of the mean value of the variance $\mathrm{Cov}\,(f,f)$ of the function $x \mapsto f(x) = \frac{1}{|\Lambda|}(\sum_{j\in\Lambda} x_j)$ with respect to the measure $\exp -\Phi(X)\ dX$ and this gives an upper bound of the smallest eigenvalue, and consequently of the splitting, by $\frac{1}{\delta_0|\Lambda|}$.

Acknowledgements :
We are grateful to V. Bach, T. Jecko and J. Sjöstrand for useful discussions (and in particular for communicating before publication the results of [BaJeSj]). Many thanks to T. Bodineau for discussions around possible applications to logarithmic Sobolev inequalities and to J. Johnsen for useful comments on various versions of these notes.
We thank also the Fields Institute in Toronto where one part of this work was done and the European Union which partially supported this research through the TMR Programme FMRX-CT 960001 of the European Commission - Network Postdoctoral training programme in partial differential equations and application in quantum mechanics-.

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