PRECISE HIGH ENERGY ASYMPTOTICS FOR THE INTEGRATED DENSITY OF STATES OF AN UNBOUNDED RANDOM JACOBI MATRIX

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ABSTRACT. The purpose of this paper is to study the transition from the classical to the quantum asymptotics for the integrated density of states of an unbounded random Jacobi matrix. Therefore, we give precise results on the behavior of the tail of the integrated density of states near infinity. We study the evolution of these asymptotics when the decay of the tail of the distribution of the random potential increases.

RÉSUMÉ. Cet article est consacré à l'étude de la transition entre le régime classique et le régime quantique pour la densité d'états intégrée d'une matrice de Jacobi aléatoire non bornée. Pour cela nous donnons des asymptotiques précises du comportement de la densité d'états intégrée au voisinage de l'infini. De plus, nous étudions le comportement de cette asymptotique lorsque la décroissance à l'infini de la distribution du potentiel aléatoire augmente.

0. INTRODUCTION

Let H be a translational invariant Jacobi matrix with exponential off-diagonal decay that is $H = ((h_{k-k'}))_{k,k'\in\mathbb{Z}^d}$ such that,

- $h_{-k} = \overline{h_k}$ for $k \in \mathbb{Z}$ and for some $k \neq 0$, $h_k \neq 0$.
- there exists $C_0 > 0$ such that, for $k \in \mathbb{Z}^d$,

$$|h_k| \leq C_0 e^{-|k|/C_0}. \tag{0.1}$$

H defines a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. By Fourier transformation, H is unitarily equivalent to the multiplication by the function $\theta \mapsto \hat{H}(\theta)$ defined by $\hat{H}(\theta) = \sum_{k\in\mathbb{Z}} h_k e^{ik\theta}$ acting as an operator on $L^2([-\pi,\pi]^d)$ (here and in the rest of the paper, we identify the function L^2 over a cube $C = [-c,c]^d \subset \mathbb{R}^d$ with the functions that are locally L^2 and periodic of period the length of the cube C; moreover the L^2-norm is normalized so that the constant function 1 has norm 1). The spectrum of H is $\sigma(H) = \hat{H}([-\pi,\pi]^d)$. We define $[\hat{H}_-, \hat{H}_+] = \sigma(H)$.

We denote by $(\delta_k)_{k\in\mathbb{Z}^d}$ the vectors of the canonical basis of $\ell^2(\mathbb{Z}^d)$. More precisely, for $k \in \mathbb{Z}^d$, $\delta_k = (\delta_{jk})_{j\in\mathbb{Z}^d}$ (where δ_{jk} is Kronecker's symbol). Let us denote by Π_k the orthogonal projection on δ_k in $\ell^2(\mathbb{Z}^d)$.

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Consider now the d-dimensional random Jacobi matrix $H_\omega = H + V_\omega$ where V_ω is a diagonal matrix with independent identically distributed real entries denoted by $(\omega_k)_{k\in\mathbb{Z}^d}$ that is

$$V_\omega = \sum_{k\in\mathbb{Z}^d} \omega_k \Pi_k$$

The tail of the probability distribution of the $(\omega)_{k\in\mathbb{Z}^d}$ near $+\infty$ will be denoted by $F(\lambda) = \mathbb{P}(\{\omega_0 \geq \lambda\})$ (here $\mathbb{P}(\Omega)$ denotes the probability of event Ω). We will assume that the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ are unbounded from above i.e.

$$F(\lambda) > 0, \quad \forall \lambda \in \mathbb{R}. \tag{0.2}$$

Moreover we assume that F is absolutely continuous near $+\infty$ (see [9]).

It is well known that the operator H_ω has a density of states that we will denote by $d\nu$ ([2], [13], [1]). $d\nu$ is a positive measure that essentially measure the number of states of the operator per units of volume. It can be defined by

$$\int_{\mathbb{R}} \varphi d\nu = \mathbb{E}\left(\langle \delta_0, \varphi(H_\omega)\delta_0\rangle\right), \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}),$$

where $\mathbb{E}(\cdot)$ denotes the expectation with respect to the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$. Define

$$N(\lambda) = \int_\lambda^{+\infty} d\nu.$$

So, in this paper, N denotes the complementary to 1 of the usual integrated density of states.

Rough high energy asymptotics for N are classical: for any $p \in \mathbb{N}$, we have

$$\left(\exists a \in \mathbb{R}, \quad \frac{\ln^{(p)} F(\lambda + a)}{\ln^{(p)} F(\lambda)} \underset{\lambda\to+\infty}{\to} 1\right) \Rightarrow \ln^{(p)} N(\lambda) \underset{\lambda\to+\infty}{\sim} \ln^{(p)} F(\lambda) \tag{0.3}$$

where $\ln^{(p)}(\cdot) = \ln|\ln|\cdots||$ is the p times composed of $\ln|\cdot|$ with itself. The equivalence (0.3) is just a simple consequence of the boundedness of H: one has $F(\lambda + \|H\|) \leq N(\lambda) \leq F(\lambda - \|H\|)$.

The asymptotic (0.3) tells us that, to first order, N is given by F. The purpose of this paper is to get more precise asymptotics for $N(\lambda)$ when λ tends to $+\infty$. As a result, we will be able to study the transition between the classical and the quantum regime for N i.e between the regime when N is essentially given by F (the classical regime) even at second (or third) order and the regime when the first correction to the high energy behavior of N really results from tunneling (i.e the quantum regime), the main term still being given by F.

The parameter that will decide whether we are in one regime or the other is the decay rate of F at $+\infty$. This may understood heuristically in the following way. If one interprets N as the probability of H_ω (restricted to some box of size L) having an eigenvalue above level λ, then this eigenvalue will be created only if at least one of the random variables $(\omega_k)_k$ is larger than λ. We know that, if we just have a single ω_k that is larger than λ, then, to create an eigenvalue larger than λ, ω_k will have to be larger than $\lambda - h$ where h depends on H and satisfies $h < \hat{H}_+$). The probability of this event will roughly be of size

$F(\lambda - h)$. The other extreme option is that sufficiently many ω_k's are larger than $\lambda - \hat{H}_+$. The probability of this to happen is roughly $F(\lambda - \hat{H}_+)^N$ if N is the number of ω_k's larger than $\lambda - \hat{H}_+$. As N is assumed to be large, these two events are disjoint; so that the probability we want to estimate will be the sum of the two probabilities i.e. it will be the supremum of the two probabilities if one of the probabilities is much larger than the other one. We will be in the classical case if $F(\lambda - h)$ is much larger than $F(\lambda - \hat{H}_+)^N$ and in the quantum case if $F(\lambda - \hat{H}_+)^N$ is much larger than $F(\lambda - h)$. Here we did not say neither what N nor what L, the size of the box should be. All this will be made precise in the proofs.

1. The main results

It will be convenient to introduce the function $g = -\ln F$. It is increasing and tends to $+\infty$ at $+\infty$.

1.1. **The classical regime.** We will say that we are in the classical regime if,

$$\frac{g'(\lambda)}{g(\lambda)} \underset{\lambda\to+\infty}{\to} 0. \tag{1.1}$$

The precision of our results will then depend on the rate of increase of g and of the precision with which g is known. Under no more assumptions than the one made above, we show the

Theorem 1.1. *Let H_ω and N be defined as above. Then, for any $\delta > 0$, we have*

$$N(\lambda) = F(\lambda - h_0 + o(1)) + o(|\ln F(\lambda)|^{2d+\delta} F(\lambda)^{2-\delta}) \tag{1.2}$$

where $h_0 = \dfrac{1}{Vol(\mathbb{T})}\displaystyle\int_{\mathbb{T}} \hat{H}(\theta)d\theta$ is the zeroth Fourier coefficient of $\hat{H}$.

(1.2) is not very precise but we will see that, under the sole assumption (1.1), one cannot do better.

One can improve on the results of Theorem 1.1 if one assumes a minimal rate of decay for F at infinity. More precisely, if we assume that, for some $\eta > 0$,

$$F(\lambda) = o(\lambda^{-d-\eta})$$

then we get

Theorem 1.2. *For H_ω and N defined as above. Let $f : (0, +\infty) \to (0, +\infty)$ a function tending to 0 at $+\infty$ such that $f(\lambda)\cdot\lambda > 1$.*

Then there exists $\lambda_\delta > 0$ and $\delta : (\lambda_\delta, +\infty) \to (0, +\infty)$, a positive function such that

- $\delta(\lambda) \geq f(\lambda)$,
- $\lim_{\lambda\to+\infty} \delta(\lambda) = 0$,

and such that one has

$$N(\lambda) = F(\lambda + a_1 + a_2/\lambda + \delta(\lambda)/\lambda)\cdot(1 + o(1)) + E_1(\lambda) + E_2(\lambda) \tag{1.3}$$

where

- $a_1 = -h_0$ *is the zeroth Fourier coefficient of* $\hat{H}$,
- $$a_2 = -\sum_{k\in(\mathbb{Z}^d)^*} |h_k|^2 = \left|\frac{1}{Vol(\mathbb{T})}\int_{\mathbb{T}} \hat{H}(\theta)d\theta\right|^2 - \frac{1}{Vol(\mathbb{T})}\int_{\mathbb{T}} |\hat{H}(\theta)|^2 d\theta,$$
- $E_1(\lambda) = O(\lambda^{2d}F(\lambda)^{2-\nu})$,
- $$E_2(\lambda) = O\left(F(\lambda + a_1 + o(1))F\left(\frac{\delta(\lambda)\lambda}{\ln^d(\lambda/\delta(\lambda))}\right)\ln^d(\lambda/\delta(\lambda))\right)$$

Let us first comment on formula (1.3). For $\omega_0 > 0$, consider the operator $H + \omega_0\Pi_0$. Let $E(\omega_0)$ denote the supremum of its spectrum. Then, for ω_0 large, $E(\omega_0)$ is an eigenvalue of $H + \omega_0\Pi_0$. It is the unique solution of

$$\omega_0 \cdot I(E) = 1$$

where

$$I(E) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{1}{E - \hat{H}(\theta)} d\theta.$$

Hence $I(E)$ admits the following expansion

$$\frac{1}{I(E)} = E + a_1 + a_2/E + O(E^{-2}).$$

So we see that the principal term in (1.3) is essentially $F(\frac{1}{I(\lambda)})$ i.e. the probability that $E(\omega_0) \geq \lambda$. So, in this case, the density of states is essentially given by the probability that there is a single eigenvalue larger than λ. In this regime (at this level of precision), the density of states of H_ω does not at all feel the tunneling between the different sites; or it looks like as if one would be dealing with infinitely many i.i.d. copies of $H + \omega_0\Pi_0$ each one located at a site of $\mathbb{Z}^d$.

Remark 1.1. The analysis used to prove Theorem 1.2 can also be performed in the continuous case i.e. for a continuous Anderson model $H_\omega = H + V_\omega$ acting on $L^2(\mathbb{R}^d)$ where H is a periodic Schrödinger operator and V_ω a potential of the form e.g.

$$V_\omega = \sum_{\gamma\in\Gamma} \omega_\gamma V(\cdot - \gamma)$$

- Γ is a non degenerate lattice in $\mathbb{R}^d$,
- V is a localized potential,
- $(\omega_\gamma)_{\gamma\in\Gamma}$ a family of i.i.d. random variables that are not lower semi-bounded.

For such an operator the integrated density of states is well defined ([13]); let us call it $N(\lambda)$. Then the principal term of the behavior of N at $-\infty$ should be given by $P(E(\omega_0) \leq -\lambda)$ where $E(\omega_0)$ is the infimum of the spectrum of the operator $H + \omega_0 V$. This infimum will be an eigenvalue when ω_0 is large (see [6]).

In Theorem 1.2, we also give a lower bound on the error of the above described approximation i.e. the function $\delta(\lambda)$. We see that $\delta(\lambda)$ can not be taken in general of size $O(1/\lambda)$. This will be verified in Theorem 1.3 where we give more precise results (under

more restrictive assumptions on g). We will see that the size of this correction is growing with the increase of the rate of increase of g.
The two error terms E_1 and E_2 can be analyzed in the following way:

- the term E_1 comes from the fact that we neglected the possibility that the spectrum above energy λ can be created by many sites at the same time. That is from the fact that we neglected the possibility that more than a single random variable takes a value larger than λ.
- the second term E_2 comes from the fact that we neglected the tunneling between two sites k and k' at which the random variables take a value close to λ.

We can now apply the asymptotic (1.3) to different examples. E. g, if we assume that $F(\lambda) \underset{+\infty}{\sim} e^{-\lambda^\alpha}$ then, we get that

$$N(\lambda) \underset{\lambda\to+\infty}{\sim} e^{-(\lambda+a_1+a_2/\lambda+o(1/\lambda))^\alpha}. \tag{1.4}$$

This immediately of obtained from the fact that in this case

$$\begin{aligned} &\left|F(\lambda+a_1+o(1))F\left(\frac{\delta(\lambda)\lambda}{\ln^d(\lambda/\delta(\lambda))}\right)\ln^d(\lambda/\delta(\lambda))\right| + \\ &\qquad\qquad + |\lambda^{2d}F(\lambda)^{2-\nu}| = o(e^{-(\lambda+a_1+a_2/\lambda+o(1/\lambda))^\alpha}). \end{aligned} \tag{1.5}$$

Let us now assume that F has a quicker decay, say $F(\lambda) \underset{+\infty}{\sim} e^{-exp(\lambda^\alpha)}$, where $0<\alpha<1$ so that $g(\lambda) \underset{+\infty}{\sim} e^{\lambda^\alpha}$ still satisfies (1.1). Then the remainder term in (1.3) is not smaller than the principal term any more. Indeed, as $\delta(\lambda) \underset{+\infty}{\to} 0$, we have

$$F(\lambda+a_1+o(1))\cdot F\left(\frac{\delta(\lambda)\lambda}{\ln^d(\lambda/\delta(\lambda))}\right)\ln^d(\lambda/\delta(\lambda)) \gg F(\lambda+a_1+a_2/\lambda+\delta(\lambda)/\lambda)$$

In this case the tunneling between two sites at which the random variables take large values (close to λ) gives an important correction to the leading behavior. This may be understood in the following way: as g'/g tends to 0 in a slower way, the probability to have two random variables, say ω_k and $\omega_{k'}$ of size λ becomes much larger than that of having a single random variable of size $\lambda + C$. Hence, the tunneling between the sites k and k' becomes more probable and its effect is felt in the asymptotics of N. This will be seen very clearly in the results presented in Theorems 1.3 and 1.4.

1.1.1. *Asymptotic expansions in the classical regime.* Let us now assume that H is of finite range i.e. that $\hat{H}$ is a trigonometric polynomial. In this case, we will, under more restrictive assumption on H and F be able to get an asymptotic expansion of N. Therefore we need to introduce some geometric notions to control the tunneling induced by H (seen as a small perturbation of V_ω).

Pick $(k,k')\in\mathbb{Z}^d$. An H-path from k to k' is defined to be an n-tuple $\gamma(k,k') = (k_1,k_2,\cdots,k_n)$ such that $k_1=k$, $k_n=k'$ and for $j=1,\dots,n-1$, $h_{k_j-k_{j+1}}\neq 0$. For $\gamma(0,k)=(k_1,k_2,\cdots,k_n)$, an H-path from 0 to k, we define the length of $\gamma(0,k)$ to be $|\gamma(0,k)|=n$. For $k\in\mathbb{Z}^d$, we define $\Gamma(k)$ to be the set of all H-paths linking 0 to k. It may be empty. Let Λ be the set of k for which $\Gamma(k) \neq \emptyset$. Then, as $H \neq k\mathrm{Id}$ ($k \in \mathbb{R}$), $\Lambda \neq \emptyset$. For example, in the case of the discrete Laplace operator, we have $\Lambda = \mathbb{Z}^d$.

For $k \in \Lambda$, we define $l(k) = \inf\{|\gamma|;\ \gamma \in \Gamma(k)\}$, the length of the shortest H-path leading from 0 to k, and $\Gamma_-(k) = \{\gamma \in \Gamma(k);\ |\gamma| = l(k)\}$ the set of H-path from 0 to k being of shortest possible length. As H is of finite range, $\Gamma_-(k)$ is finite. For $l \in \mathbb{N}$, we also define $\Lambda_l = \{k \in \mathbb{Z}^d;\ l(k) = l\}$. If $h_0 \neq 0$ then $\Lambda_0 = \{0\}$ if not $\Lambda_0 = \emptyset$. Λ_l is finite and $\Lambda = \cup_l \Lambda_l$.

Let $k \in \mathbb{Z}^d$ such that $l(k) = l$. The tunneling between 0 and k through H will be controlled by the coefficient $h_k^{(l)}$ defined as

$$h_k^{(l)} = \sum_{\substack{\gamma=(k_1,\cdots,k_l)\\ k_1=0,\ k_l=k\\ |\gamma|=l}} h_{k_1-k_2}\cdot h_{k_2-k_3}\cdots h_{k_{l-2}-k_{l-1}}\cdot h_{k_{l-1}-k_l}$$

This coefficient is also the k-th Fourier coefficient of $\hat{H}^l$, the l-th power of the function $\hat{H}$ when the l-th power of $\hat{H}$ is the first power where the k-th Fourier coefficient may not vanish.

We assume that, for any $k \in \Lambda$, we have

$$h_k^{(l(k))} \neq 0 \tag{1.6}$$

This is for example the case if there is a unique path in $\Gamma_-(k)$ (e.g. if the dimension is 1). It is also the case is all the entries of H are non-negative e.g. for the discrete Laplace operator. Assumption (1.6) is an assumption ensuring the non-degeneracy of the tunneling a given distance.

To get an asymptotic expansion for N, we will, of course, also need a better knowledge of $F(\lambda)$. We assume

- all the moments of the i.i.d random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ are finite (i.e. $\forall n \in \mathbb{N}$, $\mathbb{E}(|\omega_0|^n) < +\infty$).
- g admits a twice differentiable asymptotic expansion near $+\infty$ of one of the following types:
  1. for some $\alpha > 1$ and $g_0 > 0$,

$$g(\lambda) \underset{\lambda\to+\infty}{\sim} \ln^\alpha \lambda \sum_{k\geq 0} g_k \lambda^{-k}. \tag{1.7}$$

  2. for some $\alpha > 0$ and $g_0^0 > 0$,

$$g(\lambda) \underset{\lambda\to+\infty}{\sim} \sum_{k\geq 0} g_k^0 \lambda^{\alpha(1-k)} + \ln \lambda \sum_{k\geq 0} g_k^1 \lambda^{-k}. \tag{1.8}$$

Then we have the

Theorem 1.3. *For H_ω and N defined as above, $\ln N(\lambda)$ has a complete asymptotic expansion in terms of λ. Its precise form depends on the type of the asymptotic known for g. More precisely,*

- *in the case of asymptotic of type (1.7), one has*

$$\ln N(\lambda) \underset{\lambda\to+\infty}{\sim} -\ln^\alpha\lambda \sum_{k\geq 0} g_k\lambda^{-k} + \sum_{j\geq 1} n_j\lambda^{-j}. \tag{1.9}$$

- *in the case of asymptotic of type (1.8),*
  - *if $\alpha \leq 3$ then, one has*

$$\begin{aligned}\ln N(\lambda) \underset{\lambda\to+\infty}{\sim} & -g_0^0\lambda^\alpha + g_0^0\alpha h_0\lambda^{\alpha-1} + g_0^0\alpha\left(\sum_{k\neq 0}|h_k|^2 - \frac{\alpha-1}{2}h_0^2\right)\lambda^{\alpha-2} + n^3 + \\ & - g_0^1\ln\lambda - g_1^0 + \sum_{\substack{i,j,m\geq 0\\ i+j+m\geq 1}} n_{i,j,m}\lambda^{-i-\alpha j}\left(\frac{\ln\lambda}{\lambda}\right)^m\end{aligned} \tag{1.10}$$

    where $n^3 = 0$ if $\alpha < 3$ (see (3.32)).
  - *if $\alpha > 3$ then, one has*

$$\begin{aligned}\ln N(\lambda) \underset{\lambda\to+\infty}{\sim} & \sum_{\substack{j,l\geq 0\\ 0\leq 2l+j\alpha\leq\alpha(\alpha-1)}} n_{l,j}\lambda^{\alpha-l-\frac{2j\alpha}{\alpha-1}} + \\ & + \sum_{\substack{i,j,l,m,p\geq 0\\ i+j+l+m+p\geq 0\\ \alpha(\alpha-1)<i\alpha+2l+j\alpha}} n_{i,j,l,m,p}\lambda^{\alpha(1-i)-l-\frac{2j}{\alpha-1}}\left(\frac{\ln\lambda}{\lambda^{\tilde\alpha}}\right)^{m+p\alpha}\end{aligned} \tag{1.11}$$

    where
    * $\tilde\alpha = \frac{2\alpha}{\alpha-1}\left(\frac{\alpha-1}{2} - \left[\frac{\alpha-1}{2}\right]\right)$ *if* $\alpha \notin 2\mathbb{N}+1$.
    * $\tilde\alpha = \frac{2\alpha}{\alpha-1}$ *if* $\alpha \in 2\mathbb{N}+1$.
    * $n_{0,0} = -g_0^0$, $n_{1,0} = g_0^0\alpha h_0$, $n_{2,0} = g_0^0\alpha\left(\sum_{k\neq 0}|h_k|^2 - \frac{\alpha-1}{2}h_0^2\right)$.
    * $n_{0,j} = n_{1,j} = 0$ *for any* j, $n_{2,1} = g_0^0(\alpha-1)\left(\sum_{k\neq 0}|h_k|^{2\frac{\alpha}{\alpha-1}}\right)$.

Remark 1.2. For the sake of simplicity, in the case of asymptotics of type 1.7, we assumed $\beta > 1$. As the proofs show, we could also deal with $\beta = 1$ if the coefficient g_0^1 is large enough, though the results we would get would not be as precise as the ones given below; indeed, we would not get a complete asymptotic expansion for N.
One can also deal with much more complicated asymptotic expansion than (1.7) or (1.8). The price to pay is an increasingly complicated asymptotic expansion for N.
In the case $\alpha < 2$, we can relax the assumption on the finite range of H. Off-diagonal exponential decay is sufficient.
One may also want to relax the assumption on the finiteness of the moments of the random variables $(\omega_k)_{k_in\mathbb{Z}^d}$. This may be done at the expense of changing the type of asymptotics that one gets for $N(\lambda)$. The tail at $-\infty$ has a direct influence on the terms of negative order. We will not study this possibility as the asymptotics one gets are quite complicated already.

Let us now comment on the results of Theorem 1.3. We can compare the different asymptotic expansions one gets when α is increasing. One sees that, with α increasing, new terms appear in the asymptotic expansion. These new terms carry the interaction (i.e. the tunneling) between the different sites in the lattice. The different terms of the principal part of the expansion may be analyzed in the following way:

- the terms in $\lambda^{\alpha-j}$ ($j \in \mathbb{N}$) come essentially from the interaction between the point 0 (that was privileged in our analysis) and itself through its H-neighbors, the nearest ones being considered first and the ordered by increasing order of distance i.e. by decreasing order of importance for tunneling. It is essentially the asymptotic of the probability distribution of the highest eigenvalue of the operator $H + \omega_0 \Pi_0$.
- the first new term, the term of order $\lambda^{\alpha-2-\frac{2}{\alpha-1}}$ comes from the larger probability of tunneling between one site and its nearest H-neighbors when the random variables located at these sites both take large values. The other terms in $\lambda^{\alpha-2-\frac{2j}{\alpha-1}}$ come from tunneling between one site and its more remote H-neighbors. The index j is counting how far away from each other the sites are.

As $\alpha \to +\infty$, one notices that $\alpha - 2 - \frac{2}{\alpha-1} \to \alpha - 2$. Hence the correction terms become more and more important. This underlines the already noticed fact that, as g'/g tends to 0 in a slower way, the possible tunneling between neighbors affects more and more the behavior of N. Hence N behaves more and more in a quantum way.

We can also compare the results of Theorem 1.3 and Theorem 1.2. We see that, if we rewrite (1.11) in the same form as (1.3), we get

$$\ln N(\lambda) = \ln F(\lambda + a_0 + a_1/\lambda + b_1/\lambda^{-(\alpha+1)/(\alpha-1)} + o(\lambda^{-(\alpha+1)/(\alpha-1)})).$$

Comparing this with (1.3) and the analysis of (1.3) given after Theorem 1.2, we see that the principal correction to $I^{-1}(\lambda)$ is of order $\lambda^{-(\alpha+1)/(\alpha-1)}$. This order is increasing with α i.e. the decay of F.

To finish the discussion of this regime, let us just give a concrete example where assumption (1.8) is satisfied: suppose the $(\omega_k)_{k\in\mathbb{Z}^d}$ are i.i.d. random variables with density $C_\alpha e^{-|t|^\alpha}$ ($\alpha > 0$). In this case, g has the following asymptotic expansion

$$g(\lambda) \underset{\lambda\to+\infty}{\sim} \lambda^\alpha + (\alpha - 1)\ln\lambda + \ln\alpha + \sum_{k\geq 1} g_k \lambda^{-k\alpha}.$$

Hence we may use Theorem 1.3 to compute the asymptotics of $N(\lambda)$ in this case.

The quantum phenomenon will be even stronger if the decay of g at $+\infty$ is faster than polynomial. In this case, we will see that the tunneling corrections already change the second term in the asymptotic. This is the object of our next result.
To make the results simple, we will assume that, for λ large enough, we have

$$g(\lambda) = g_0 e^{\lambda^\alpha} \tag{1.12}$$

for some $\alpha \in (0, 1)$. Then, we have the

Theorem 1.4. *For g as above, we have*

$$\ln N(\lambda) = -e^{g_0\lambda^\alpha}\left(1-\alpha h_0 g_0\lambda^{\alpha-1}+(\alpha g_0)^2\left(\sum_{k\neq 0}|h_k|^2\right)\lambda^{2(\alpha-1)}+o\left(\lambda^{2(\alpha-1)}\right)\right). \tag{1.13}$$

Let us rewrite (1.13) in the form (1.3). We get

$$\ln N(\lambda) = -e^{g_0(\lambda+h_0+ah^{\alpha-1}+o(\lambda^{\alpha-1}))^\alpha},$$

where

$$a = \alpha g_0\left(\sum_{k\neq 0}|h_k|^2\right)$$

It is clear from Theorem 1.4 why the asymptotics given by Theorem 1.2 breaks down in the classical case when g'/g tends to 0 too slowly: the correction term we saw propagating upwards in the asymptotics in Theorem 1.3 has now overcome the correction computed in Theorem 1.2 i.e it is the error terms in this expansion that now become principal.

1.2. **The quantum regime.** We will say that we are in the quantum regime if

$$\frac{g'(\lambda)}{g(\lambda)} \underset{\lambda\to+\infty}{\to} +\infty. \tag{1.14}$$

It will be convenient to introduce $h=\ln g$. Then h is increasing and tends to $+\infty$ at $+\infty$. (1.14) becomes $h'(\lambda)\to+\infty$ as $\lambda\to+\infty$. To prove our result we will need to assume that:

1. $\hat{H}$ has a single maximum and this maximum is non-degenerate.
2. for λ large enough, h' is increasing, differentiable and

$$-\left(\frac{1}{h'}\right)'(\lambda) = \frac{h''(\lambda)}{(h'(\lambda))^2} = \underset{\lambda\to+\infty}{\to} 0$$

.

We recall that $\hat{H}_+ = \sup_{\mathbb{T}}\hat{H}$. Then we have the following

Theorem 1.5. *In the quantum regime, under the above assumptions on $\hat{H}$, we have*

$$\ln|\ln N(\lambda)| = \ln|\ln F(\lambda-\hat{H}_+)| + \frac{d}{2}\ln\left(\frac{F'(\lambda-\hat{H}_+)}{F(\lambda-\hat{H}_+)\ln F(\lambda-\hat{H}_+)}\right)(1+o(1)) \tag{1.15}$$

when $\lambda\to+\infty$.

Remark 1.3. Let us comment on assumption 1. This assumption is clearly fulfilled if $\hat{H}$ is the free discrete Laplacian (i.e. $\hat{H}(\theta)=\cos\theta_1+\cdots+\cos\theta_d$). As in [8], we could have relaxed the assumption on $\hat{H}$ and have assumed that it reaches its maxima at isolated points. The technique we use here still works in that case.
As in the quantum case $h'\to+\infty$ at $+\infty$, assumption (2) just means that h' does not behave too wildly near $+\infty$.

The first correction to F in (1.15) is the shift of the energy λ by $-\hat{H}_+$. Notice that in the classical regime there was also a shift in energy but only by an amount $-\hat{H}_0$. The new shift increases the value of N as $-\hat{H}_+ < -\hat{H}_0$.

The second correction is due to the tunneling between many sites where the random variables take a value larger than $\lambda - \hat{H}_+$. This effect is of purely quantum nature: it is the same effect as the one discovered by I. Lifshits at the edges of the spectrum when the support of the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ is compact (see [10], [11], [12]).
The coefficient $d/2$ in (1.15) is nothing but the exponent giving the decay of the density of states of H at the upper edge of its spectrum. In case we replace $\hat{H}$ by some analytic function, say $\hat{H}_0$, that has not a non-degenerate critical point at its maximum, we expect the coefficient $d/2$ to be replaced by the exponent giving the decay of the density of states of H_0 at upper edge of its spectrum. This was proved in some simple cases for Lifshits tails ([8]).

Remark 1.4. As in the classical regime, in the quantum regime, one should be able to find a result similar to Theorem 1.5 for a continuous Anderson model that is not lower semi-bounded (cf Remark 1.1). To prove such a result, one should be able to use the techniques developed here and in [7].

2. Periodic approximations

We will first state an approximation result for the density of states of H_ω. Let $(\omega_j)_{j\in\mathbb{Z}^d}$ be a realization of the random variables defined above. Fix $n \in \mathbb{N}^*$. We define the following periodic operator acting on $\ell^2(\mathbb{Z}^d)$

$$H_\omega^n = H + V_\omega^n = H + \sum_{k\in\mathbb{Z}^d_{2n+1}} \omega_k \sum_{l\in(2n+1)\mathbb{Z}^d} \Pi_{l+k}.$$

where $\mathbb{Z}^d_{2n+1} = \mathbb{Z}^d/(2n+1)\mathbb{Z}^d$. In the sequel, we identify $\mathbb{Z}^d_{2n+1}$ with the cube $\{-n, -n+1, \cdots, n-1, n\}^d$. H_ω^n is $(2n+1)\mathbb{Z}^d$-periodic.
For H_ω^n, we define the density of states denoted by dN_ω^n (e.g. [14]) to be the positive Borel measure satisfying

$$(2.1)\qquad (\varphi, dN_\omega^n) = \int_{\mathbb{R}} \varphi(x)dN_\omega^n(x) = \frac{1}{(2n+1)^d}\sum_{k\in\mathbb{Z}^d_{2n+1}} \langle \delta_k, \varphi(H_\omega^n)\delta_k\rangle, \quad \varphi\in\mathcal{C}_0^\infty(\mathbb{R}).$$

Define

$$N_\omega^n(\lambda) = \int_\lambda^{+\infty} dN_\omega^n.$$

As in the case of H_ω, $N_\omega^n(\lambda)$ is the complementary to 1 of the usual density of states of H_ω^n.

We have the

Lemma 2.1. *For any $\nu > 1$, there exists $C_\nu > 1$ such that, for any $\varepsilon > 0$ and any $n \in \mathbb{N}^*$ such that $n\varepsilon > 1$, one has*

$$(2.2)\qquad \mathbb{E}(N_\omega^n(\lambda+\varepsilon)) - \delta(\nu,\varepsilon,n) \le N(\lambda) \le \mathbb{E}(N_\omega^n(\lambda-\varepsilon)) + \delta(\nu,\varepsilon,n)$$

where

$$0 \leq \delta(\nu,\varepsilon,n) \leq C_\nu (n+\frac{1}{\varepsilon})^3 \exp\left(-\left(\frac{n\varepsilon}{C_\nu}\right)^{1/\nu}\right). \tag{2.3}$$

Proof. The proof of this lemma is a direct consequence of Lemma 1.1 in [8] and the discussion following the proof of Lemma 1.1 in [8]. We will not repeat it here.

□

Let us now explain heuristically how we get the announced precise asymptotics for $N(\lambda)$. To use Lemma 2.2, we need to choose ε and n in a suitable way depending on λ. ε will essentially be set by the precision of the asymptotics we want to get. This tells us immediately that ε should be chosen so that $F(\lambda+\varepsilon)/F(\lambda)-1$ is of order the precision of the asymptotic we want to get. Nevertheless, in the case of (1.3), because of the precision we want to achieve in our asymptotics, we need to choose ε of order $o(\lambda^{-1})$; this introduces a new length scale that is rather arbitrary in the sense that it is not directly related to F or g. This explains why a minimla decay is needed for F.

ε being chosen, we will want to take n as small as is possible while keeping the error $\delta(\nu,\varepsilon,n)$ smaller than the error prescribed by the precision wanted for the asymptotic. The main advantage of keeping n small is the following: by Lemma 2.2, the rough asymptotic (0.3) will also be valid for $\mathbb{E}(N^n_\omega(\lambda))$ (at least if $\delta(\nu,\varepsilon,n)$ is much smaller than $F(\lambda)$); hence, if n is much smaller than $F(\lambda)$ then $N^n_\omega(\lambda)$ is well approximated by the probability that H^n_ω has some spectrum above energy λ. But this quantity is much easier to estimate than $N^n_\omega(\lambda)$ itself. This feature was already used (in different way) in [4], [5] or [8].

In terms of g, we get roughly that ε should be of order $g'(\lambda)^{-1}$ (or $o(\lambda^{-1})$ in the case of (1.3)). Then, as $\delta(\nu,\varepsilon,n)$ certainly will have to be of size less than $F(\lambda)$, we get that n should be taken at least of size $g(\lambda)^\nu\cdot g'(\lambda)^{-1}$. This is much smaller than $F(\lambda)$ (only if F decreases fast enough in the case of (1.3)). Actually, in the quantum regime, in Theorems 1.1, 1.3 and 1.4, any power of n will be infinitesimal compared to $F(\lambda)$.

2.1. **Some Floquet theory.** To analyze H^n_ω, we will need some Floquet theory. It is just a d-dimensional version of the Floquet theory developed in [8]. Let us first introduce some notations. By $\mathbb{T}_n$, we denote the d-dimensional torus $[-\frac{\pi}{2n+1},\frac{\pi}{2n+1}]^d$. $\mathbb{T}_0$ will be denoted by $\mathbb{T}=[-\pi,\pi]^d$. We denote by $\mathcal{F}:\ L^2(\mathbb{T})\to\ell^2(\mathbb{Z}^d)$ the usual d-dimensional Fourier series transform. Then, we have, for $u\in L^2(\mathbb{T})$,

$$\hat{H}_\omega u(\theta)=(\mathcal{F}^*H_\omega\mathcal{F}u)(\theta)=\hat{H}(\theta)u(\theta)+\sum_{j\in\mathbb{Z}^d}\omega_j(\Pi_j u)(\theta)$$

$$\text{where } (\Pi_j u)(\theta)=\frac{1}{(2\pi)^d}e^{ij\theta}\int_{\mathbb{T}}e^{-ij\theta}u(\theta)d\theta.$$

For $u\in L^2(\mathbb{T})$, we can decompose u in unique way

$$u(\theta)=\sum_{k\in\mathbb{Z}^d_{2n+1}}e^{ik\theta}u_k(\theta) \tag{2.4}$$

where the functions $(u_k)_{k\in\mathbb{Z}^d_{2n+1}}$ are $\frac{2\pi}{2n+1}$-periodic. Define the operator U acting on $L^2(\mathbb{T})$ by $(Uu)(\theta)=(u_k(\theta))_{k\in\mathbb{Z}^d_{2n+1}}$. Then U is a unitary equivalence from $L^2(\mathbb{T})$ to

$L^2(\mathbb{T}_n)\otimes\ell^2(\mathbb{Z}^d_{2n+1})$.
For $n>1$, we compute $U\mathcal{F}^*H^n_\omega\mathcal{F}U^*$ and get that it is the multiplication operator $M^n_\omega(\theta)=((\hat{H}_{j-j'}(\theta)+\omega_j\delta_{jj'}))_{(j,j')\in(\mathbb{Z}^d_{2n+1})^2}$ acting on $L^2(\mathbb{T}_n)\otimes\ell^2(\mathbb{Z}^d_{2n+1})$; here the functions $(\hat{H}_k)_{k\in\mathbb{Z}^d_{2n+1}}$ are the components of $\hat{H}$ decomposed according to (2.4).
The above discussion tells us that the Floquet eigenvalues and eigenvectors of H^n_ω with Floquet quasi-momentum θ (i.e. the solutions of the problem

$$\begin{cases} H^n_\omega u &= \lambda u \text{ (where } u=(u_j)_{j\in\mathbb{Z}^d})\\ u_{j+k} &= e^{-ik\theta}u_j \text{ for } j\in\mathbb{Z}^d,\ k\in(2n+1)\mathbb{Z}^d)\end{cases}$$

are nothing but the eigenvalues and eigenvectors (continued periodically) of the $(2n+1)^d\times(2n+1)^d$ "matrix" $M^n_\omega(\theta)$ acting on $\ell^2(\mathbb{Z}^d_{2n+1})$. This gives us that, for $\varepsilon>0$,

$$N^n_\omega(\lambda)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}_n}\sharp\{\text{eigenvalues of } M^n_\omega(\theta) \text{ larger than } \lambda\}d\theta \tag{2.5}$$

Considering H as being $(2n+1)\mathbb{Z}^d$-periodic on $\mathbb{Z}^d$, we get that the Floquet eigenvalues of H (for the quasi-momentum θ) are $(\hat{H}(\theta+\frac{2\pi k}{2n+1}))_{k\in\mathbb{Z}^d_{2n+1}}$ each of them being associated with the vector $(u_k(\theta))_{k\in\mathbb{Z}^d_{2n+1}}$ where

$$u_k(\theta)=\frac{1}{(2n+1)^{d/2}}(e^{-i(\theta+\frac{2\pi k}{2n+1})j})_{j\in\mathbb{Z}^d_{2n+1}}.$$

in the canonical basis $(\delta_j)_{j\in\mathbb{Z}^d_{2n+1}}$.

3. The classical regime

Our analysis of the classical regime will rely on

Lemma 3.1. *Assume that g satisfies assumption (1.1). Then, there exists $C>0$ such that, for any $n\geq 1$ and $K>0$, we have*

$$\left|\mathbb{E}(N^n_\omega(\lambda))-\frac{1}{Vol(\mathbb{T})}\int_{\mathbb{T}}\mathbb{E}\left(F\left(\frac{1}{I(\lambda,\omega,\theta)}\right)\mathbf{1}_{\Omega'_K}\right)d\theta\right|\leq Cn^{2d}F(\lambda-\hat{H}_+)F(\lambda-K-\hat{H}_+) \tag{3.1}$$

where

- $I(\lambda,\omega,\theta)=\langle\delta_0,(\lambda-M^n_{\omega'}(\theta))^{-1}\delta_0\rangle$,
- $M^n_{\omega'}(\theta)$ *is the "matrix"* $((\hat{H}_{j-j'}(\theta)+\omega'_j\delta_{jj'}))_{(j,j')\in(\mathbb{Z}^d_{2n+1})^2}$ *acting on* $\ell^2(\mathbb{Z}^d_{2n+1})$,
- δ_0 *is the vector* $(\delta_{j0})_{j\in\mathbb{Z}^d_{2n+1}}$ *and* $\langle\cdot,\cdot\rangle$ *denotes the scalar product in* $\ell^2(\mathbb{Z}^d_{2n+1})$,
- $\omega'_j=\omega_j$ *if* $j\neq 0$ *and* $\omega'_0=0$,
- $\Omega'_K=\{\omega';\ \forall j\in\mathbb{Z}^d_{2n+1}\setminus\{0\},\ \omega'_j<\lambda-K-\hat{H}_+\}$.

Remark 3.1. Though in (3.1), we integrate with respect to all the random variables $(\omega_k)_{k\neq 0}$, as they are i.i.d. and as $M^n_{\omega'}$ only depends only on the random variables $(\omega_k)_{k\mathbb{Z}^d_{2n+1}\setminus\{0\}}$, we could as well only have integrated with respect to these last random variables.

Proof. For $k\in\mathbb{Z}^d_{2n+1}$, define $\mathbb{Z}_{n,k}=\mathbb{Z}^d_{2n+1}\setminus\{k\}$. We define the following sets

- $\Omega_- = \{\omega;\ \forall k \in \mathbb{Z}^d_{2n+1},\ \omega_j < \lambda - \hat{H}_+\}$,
- $\Omega_k = \{\omega;\ \omega_k \geq \lambda;\ \forall j \in \mathbb{Z}_{n,k},\ \omega_j < \lambda - K - \hat{H}_+\}$,
- $\Omega_+ = \{\omega; \exists j, k \in \mathbb{Z}^d_{2n+1},\ j \neq k, \omega_j \geq \lambda - \hat{H}_+,\ \omega_k \geq \lambda - K - \hat{H}_+\}$.

These sets obviously form a partition of Ω. By (2.5), $N^n_\omega(\lambda)$ only depends on the random variables $(\omega_k)_{k\in\mathbb{Z}^d_{2n+1}}$. Hence, we have

$$\mathbb{E}(N^n_\omega(\lambda)) = \mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_-}) + \sum_{k\in\mathbb{Z}^d_{2n+1}} \mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_k}) + \mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_+}). \tag{3.2}$$

As $H \leq \hat{H}_+$, on Ω_-, $H^n_\omega(\theta) < \lambda$ for any θ; so that

$$\mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_-}) = 0. \tag{3.3}$$

By definition, $0 \leq N^n_\omega(\lambda) \leq 1$ hence

$$|\mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_+})| \leq \mathbb{E}(\mathbf{1}_{\Omega_+}) \leq Cn^{2d}F(\lambda - \hat{H}_+)F(\lambda - K - \hat{H}_+). \tag{3.4}$$

So that we only have to study the terms $\mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_k})$. For $k \in \mathbb{Z}^d_{2n+1}$, let us define the translation operator τ_k acting on $\ell^2(\mathbb{Z}^d_{2n+1})$ by

$$\forall u \in \ell^2(\mathbb{Z}^d_{2n+1}) \quad (\tau_k u)_j = u_{j-k}.$$

On $L^2(\mathbb{T}_n) \otimes \ell^2(\mathbb{Z}^d_{2n+1})$, we can consider the translation $\mathbf{1} \otimes \tau_k$ that we will also denote by τ_k. In both cases τ_k is a unitary transformation.
On Ω, we define the shift t_k by

$$\forall \omega \in \Omega, \quad (t_k\omega)_j = \omega_{j-k}.$$

$\{\tau_k;\ k \in \mathbb{Z}^d_{2n+1}\}$ and $\{t_k;\ k \in \mathbb{Z}^d_{2n+1}\}$ are abelian groups. Moreover the product measure defined on Ω is invariant with respect to the action of the group $\{t_k;\ k \in \mathbb{Z}^d_{2n+1}\}$.
One checks that

$$\tau_k^* M^n_\omega(\theta)\tau_k = M^n_{t_k(\omega)}(\theta).$$

This implies that

$$N^n_{t_{-k}(\omega)}(\lambda) = N^n_\omega(\lambda)$$

hence that

$$\mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_k}) = \mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{t_k(\Omega_0)}) = \mathbb{E}(N^n_{t_{-k}(\omega)}(\lambda)\mathbf{1}_{\Omega_0}) = \mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_0}). \tag{3.5}$$

We need only to study $\mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_0})$. By (2.5), we need to analyze the number of eigenvalues of the "matrix" $M^n_\omega(\theta)$ larger than λ. For $\omega \in \Omega_0$, there is only a single random variable ω_0 larger than λ, so that, by the variational principle, the number of eigenvalues of $M^n_\omega(\theta)$ larger than λ is bounded by 1. Hence, by (2.5),

$$\mathbb{E}(N^n_\omega(\lambda)\mathbf{1}_{\Omega_0}) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}_n} \mathbb{E}(\mathbf{1}_{\{\omega;\ M^n_\omega(\theta) \text{ has an eigenvalue larger than } \lambda\}}\mathbf{1}_{\Omega_0})d\theta. \tag{3.6}$$

Let $M_0^n(\theta)$ be the "matrix" $((\hat{H}_{j-j'}(\theta)))_{(j,j')\in(\mathbb{Z}^d_{2n+1})^2}$. We write

$$M_\omega^n(\theta) = M_0^n(\theta) + \sum_{k\in\mathbb{Z}^d_{2n+1}} \omega_k\Pi_k = \omega_0\Pi_0 + M_0^n(\theta) + \sum_{k\in\mathbb{Z}^d_{2n+1}\setminus\{0\}} \omega_k\Pi_k = \omega_0\Pi_0 + M_{\omega'}^n(\theta)$$

where ω' is defined in Lemma 3.1 and Π_k denotes the orthogonal projector (in $\ell^2(\mathbb{Z}^d_{2n+1})$) on the vector $\delta_k = (\delta_{jk})_{j\in\mathbb{Z}^d_{2n+1}}$.
For $\omega \in \Omega_0$, $M_{\omega'}^n(\theta)$ is smaller than $\lambda - K$. Hence, by the Birman-Schwinger principle (see [14]), $M_\omega^n(\theta)$ has an eigenvalue larger than λ if and only if

$$\omega_0\langle\delta_0, (\lambda - M_{\omega'}^n(\theta))^{-1}\delta_0\rangle \geq 1$$

that is

$$\omega_0 \geq \frac{1}{\langle\delta_0, (\lambda - M_{\omega'}^n(\theta))^{-1}\delta_0\rangle}.$$

By (3.6), as the random variables $(\omega_k)_k$ are independent, we have

$$\mathbb{E}(N_\omega^n(\lambda)\mathbf{1}_{\Omega_0}) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}_n} \mathbb{E}\left(F\left(\frac{1}{\langle\delta_0, (\lambda - M_{\omega'}^n(\theta))^{-1}\delta_0\rangle}\right)\mathbf{1}_{\Omega'_K}\right)d\theta \tag{3.7}$$

where Ω'_K is defined in Lemma 3.1.
Now, as $M_{\omega'}^n(\theta)$ is $2\pi/(2n+1)$-periodic, using (3.5) and (3.7), we may rewrite

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d_{2n+1}} \mathbb{E}(N_\omega^n(\lambda)\mathbf{1}_{\Omega_k}) &= \frac{1}{\mathrm{Vol}\mathbb{T}}\sum_{k\in\mathbb{Z}^d_{2n+1}}\int_{\mathbb{T}_n} \mathbb{E}\left(F\left(\frac{1}{\langle\delta_0, (\lambda - M_{\omega'}^n(\theta + \frac{2k\pi}{2n+1}))^{-1}\delta_0\rangle}\right)\mathbf{1}_{\Omega'_K}\right)d\theta \\ &= \frac{\mathrm{Vol}\mathbb{T}}{\int}_{\mathbb{T}} \mathbb{E}\left(F\left(\frac{1}{\langle\delta_0, (\lambda - M_{\omega'}^n(\theta))^{-1}\delta_0\rangle}\right)\mathbf{1}_{\Omega'_K}\right)d\theta.\end{aligned}$$

Plugging this into (3.2) and using (3.3) an (3.4), we get (3.1).

□

3.1. **The proofs of Theorem 1.1 and 1.2.** These theorems are now deduced quite easily from Lemma 3.1.
Proof of Theorem 1.1. To choose ε for Lemma 2.1, we pick $\beta > 0$ small. We set $\varepsilon = g(\lambda)^{-\beta}$. Let ν as in Lemma 2.1 and choose it arbitrarily close to 1. Set $n = g(\lambda)^{\beta+(2\beta+1)\nu}$. Then the error estimate in Lemma 2.1 is of size

$$\delta(\nu, \varepsilon, n) \leq Ce^{-g(\lambda)^{1+\beta}}.$$

Now we estimate $\mathbb{E}(N_\omega^n(\lambda\pm\varepsilon))$ using Lemma 3.1. To do that we only need to estimate $\mathbb{E}(N_\omega^n(\lambda))$. For $\theta\in\mathbb{T}$, as $\omega'_0 = 0$, we have

$$\begin{aligned}I(\lambda,\omega,\theta) &= \langle\delta_0, (\lambda - M_{\omega'}^n(\theta))^{-1}\delta_0\rangle \\ &= \frac{1}{\lambda} + \frac{1}{\lambda^2}\langle\delta_0, M_{\omega'}^n(\theta)\delta_0\rangle + \frac{1}{\lambda^2}\langle M_{\omega'}^n(\theta)\delta_0, (\lambda - M_{\omega'}^n(\theta))^{-1}M_{\omega'}^n(\theta)\delta_0\rangle \\ &= \frac{1}{\lambda}\left(1 + \frac{1}{\lambda}\langle\delta_0, M_0^n(\theta)\delta_0\rangle + \frac{1}{\lambda}m_0(\omega',\theta,n)\right)\end{aligned} \tag{3.8}$$

where

$$m_0(\omega',\theta,n) = \langle M_0^n(\theta)\delta_0, (\lambda - M_{\omega'}^n(\theta))^{-1} M_0^n(\theta)\delta_0\rangle.$$

Notice that $m_0(\omega',\theta,n) > 0$ as $M_{\omega'}^n(\theta) < \lambda$. By definition $\langle \delta_0, M_0^n(\theta)\delta_0\rangle = \hat{H}_0(\theta)$ (where $\hat{H}_0(\theta)$ is given by the decomposition (2.4) for the function $\hat{H}$). As the Fourier coefficients on $\hat{H}$ are exponentially decaying, writing the Fourier series for $h_0(\theta)$, we get

$$\langle \delta_0, M_0^n(\theta)\delta_0\rangle = h_0 + O(e^{-n/C_0}).$$

Fix $K > 0$ a large real number. For $\omega' \in \Omega'_K$ (see Lemma 3.1), we have, uniformly in θ,

$$m_0(\omega,\theta,n) \leq \|H\|^2/K.$$

So that, by Neuman's formula, for $\omega' \in \Omega'_K$ and all θ,

$$\frac{1}{I(\lambda,\omega,\theta)} = \lambda - h_0 + O(K^{-1} + e^{-n/C_0} + 1/\lambda)$$

Hence, as F is decreasing, we get that, for any $\eta > 0$, there exists $\lambda_\eta > 0$ and $K > 0$ large such that, for $\lambda > \lambda_\eta$,

$$F(\lambda - h_0 + \eta) \leq \frac{1}{\mathrm{Vol}(\mathbb{T})}\int_{\mathbb{T}} \mathbb{E}\left(F\left(\frac{1}{I(\lambda \pm \varepsilon,\omega,\theta)}\right)\mathbf{1}_{\Omega'_K}\right) d\theta \leq F(\lambda - h_0 - \eta). \tag{3.9}$$

As F is absolutely continuous near $+\infty$, g too is absolutely continuous near $+\infty$. In particular, we know that, for x, y large enough, $g(x) - g(y) = \int_y^x g'(t)dt$ ([9]). We know that g satisfies assumption (1.1). That is for any $\eta > 0$, there exists $\lambda_\eta > 0$ such that, for $\lambda > \lambda_\eta$, $g'(\lambda) \leq \eta g(\lambda)$. Hence, for $\lambda > \lambda_\eta$,

$$0 \leq g\left(\lambda + \frac{1}{\sqrt{\eta}}\right) - g(\lambda) = \int_\lambda^{\lambda+\frac{1}{\sqrt{\eta}}} g'(t)dt \leq \eta \int_\lambda^{\lambda+\frac{1}{\sqrt{\eta}}} g(t)dt \leq \sqrt{\eta}\, g\left(\lambda + \frac{1}{\sqrt{\eta}}\right)$$

as g is non-decreasing.

This shows that there exists a function $K(\lambda)$ that tends to $+\infty$ at $+\infty$ such that, for any $\delta > 0$,

$$F(\lambda - \hat{H}_+) \leq F(\lambda - K(\lambda) - \hat{H}_+) = o(F(\lambda)^{1-\delta}).$$

This remark, (3.9) and Lemma 3.1 immediately imply Theorem 1.1 if we choose $K = K(\lambda)$ in (3.9).

□

Proof of Theorem 1.2. Theorem 1.2 is proved along the same lines as Theorem 1.1. We just need to choose ε somewhat differently and to get a more precise expansion for $I(\lambda,\theta,\omega)$ to achieve the announced precision.

We now have assumed $F(\lambda) = o(\lambda^{-n-\eta})$. Then pick $\varepsilon = g(\lambda)^{-\beta/2}\min(\lambda^{-1-\eta/(4d)}, g(\lambda)^{-\beta/2})$ where $\beta > 0$ is small. Let ν as in Lemma 2.1 and choose it arbitrarily close to 1. Set $n = g(\lambda)^{\nu(1+\beta/2)+\beta/2}\max(\lambda^{1+\eta/(4d)}, g(\lambda)^{\beta/2})$. Then the error estimate in Lemma 2.1 is of size

$$\delta(\nu,\varepsilon,n) \leq Ce^{-g(\lambda)^{1+\beta/2}}.$$

We will now expand $I(\lambda,\omega,\theta)$ a little further than in (3.8). We recall

$$I(\lambda,\omega,\theta)=\frac{1}{\lambda}\left(1+\frac{\hat{H}_0(\theta)}{\lambda}+\frac{1}{\lambda}m_0(\omega',\theta,n)\right)$$

where

$$m_0(\omega',\theta,n)=\langle M_0^n(\theta)\delta_0,(\lambda-M_{\omega'}^n(\theta))^{-1}M_0^n(\theta)\delta_0\rangle.$$

Let us define $\Pi_{\omega'}^n=\sum_{k\in(\mathbb{Z}_{2n+1}^d)^*}\omega'_k\Pi_k$. For $\omega'\in\Omega'_K$, we have $\lambda-\Pi_{\omega'}^n>K$. Then we can rewrite $m_0(\omega',\theta,n)$ as

$$\begin{aligned}
m_0(\omega',\theta,n)&=\langle M_0^n(\theta)\delta_0,(\lambda-\Pi_{\omega'}^n)^{-1}(1-M_0^n(\theta)(\lambda-\Pi_{\omega'}^n)^{-1})^{-1}M_0^n(\theta)\delta_0\rangle\\
&=\langle M_0^n(\theta)\delta_0,(\lambda-\Pi_{\omega'}^n)^{-1}M_0^n(\theta)\delta_0\rangle+\\
&\quad+\langle M_0^n(\theta)\delta_0,(\lambda-\Pi_{\omega'}^n)^{-1}(1-M_0^n(\theta)(\lambda-\Pi_{\omega'}^n)^{-1})^{-1}(\lambda-\Pi_{\omega'}^n)^{-1}M_0^n(\theta)\delta_0\rangle\\
&=\|v_0(\omega',\theta,n)\|^2+\langle v_0(\omega',\theta,n),V(\omega',\theta,n)v_0(\omega',\theta,n)\rangle
\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}
v_0(\omega',\theta,n)&=(\lambda-\Pi_{\omega'}^n)^{-1/2}M_0^n(\theta)\delta_0,\\
V(\omega',\theta,n)&=(\lambda-\Pi_{\omega'}^n)^{-1/2}(1-M_0^n(\theta)(\lambda-\Pi_{\omega'}^n)^{-1})^{-1}(\lambda-\Pi_{\omega'}^n)^{-1/2}.
\end{aligned}$$

Then, for K large, uniformly for $\omega'\in\Omega'_K$, we have

$$m_0(\omega',\theta,n)=\|v_0(\omega',\theta,n)\|^2(1+O(K^{-1})).\tag{3.11}$$

On the other hand, we can expand $\|v_0(\omega',\theta,n)\|^2$ in

$$\begin{aligned}
\|v_0(\omega',\theta,n)\|^2&=\frac{1}{\lambda}|\hat{H}_0(\theta)|^2+\sum_{k\in(\mathbb{Z}_{2n+1}^d)^*}\frac{1}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\\
&=\frac{1}{\lambda}\sum_{k\in\mathbb{Z}_{2n+1}^d}|\hat{H}_k(\theta)|^2+\sum_{k\in(\mathbb{Z}_{2n+1}^d)^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\\
&=\frac{1}{\lambda}\sum_{k\in\mathbb{Z}^d}|h_k|^2+\sum_{k\in(\mathbb{Z}_{2n+1}^d)^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2+O(e^{-2n/C_0})
\end{aligned}\tag{3.12}$$

by the exponential decay of the Fourier coefficients of $\hat{H}$.
In view of the asymptotic we announced, it is natural to estimate now the probability that $\left|\sum_{k\in(\mathbb{Z}_{2n+1}^d)^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\right|$ is large. This is the purpose of

Lemma 3.2. *For any $\delta>0$, uniformly in θ, one has that*

$$Proba\left(\left\{\omega\in\Omega'_K;\ \sum_{k\in(\mathbb{Z}_{2n+1}^d)^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\le-\delta\right\}\right)\underset{\lambda\to+\infty}{\to}0\tag{3.13}$$

and that, for some $C>0$ and λ large enough,

$$(3.14)\quad Proba\left(\left\{\omega\in\Omega'_K;\ \sum_{k\in(\mathbb{Z}^d_{2n+1})^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\geq\delta\right\}\right)\leq C(\ln(\lambda/\delta))^dF\left(\frac{\delta\lambda}{C(\ln(\lambda/\delta))^d+\delta}\right).$$

Let us use this lemma to finish the proof of Theorem 1.2. Actually we will use the

Corollary 3.1. *Let $f:(0,+\infty)\to(0,+\infty)$ a function tending to 0 at $+\infty$. Then there exists a δ a positive function defined for $\lambda\geq\lambda_\delta$ large enough such that*

- *for $\lambda\geq\lambda_\delta$, $\delta(\lambda)\geq f(\lambda)$*
-

$$(3.15)\qquad Proba\left(\left\{\omega\in\Omega'_K;\ \sum_{k\in(\mathbb{Z}^d_{2n+1})^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\leq-\delta(\lambda)\right\}\right)\underset{\lambda\to+\infty}{\to}0$$

-

$$(3.16)\quad Proba\left(\left\{\omega\in\Omega'_K;\ \sum_{k\in(\mathbb{Z}^d_{2n+1})^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\geq\delta(\lambda)\right\}\right)\leq C(\ln(\lambda/\delta(\lambda)))^dF\left(\frac{\delta(\lambda)\lambda}{C(\ln(\lambda/\delta(\lambda)))^d+\delta(\lambda)}\right).$$

Let $\lambda>0$ be large. If $\omega\in\Omega'_K$ such that $\displaystyle\sum_{k\in(\mathbb{Z}^d_{2n+1})^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\geq\delta(\lambda)$, then, by (3.8), (3.11) and (3.12), we get that

$$I(\lambda,\omega,\theta)=\frac{1}{\lambda}\left(1+\frac{h_0}{\lambda}+\frac{1}{\lambda}O(K^{-1})\right).$$

If $\omega\in\Omega'_K$ such that $\displaystyle\sum_{k\in(\mathbb{Z}^d_{2n+1})^*}\frac{\omega_k}{\lambda-\omega_k}|\hat{H}_k(\theta)|^2\leq-\delta(\lambda)$, then, by (3.8), (3.11) and (3.12), we get that

$$I(\lambda,\omega,\theta)\leq\frac{1}{\lambda}\left(1+\frac{h_0}{\lambda}+\frac{1}{\lambda^2}\sum_{k\neq0}|h_k|^2+\frac{1}{\lambda^2}O(K^{-1})\right)$$

as, in this case, the last term in (3.12) is negative. Here we replaced $\hat{H}_0$ by h_0 without damage because of the exponential decay of the Fourier coefficients of $\hat{H}$.
Now, we pick $K(\lambda)$ as in the proof of Theorem 1.1; using the above estimates for I and plugging them into (3.1), we get that, for any $\delta>0$ and any $\nu>0$, there exists $\lambda_{\delta,\nu}$ such

that, for $\lambda > \lambda_{\delta,\nu}$, we have

$$N(\lambda) = F(\lambda + a_1 + (a_2 + O(\delta(\lambda)))/\lambda) \cdot (1 + o(1)) + O(\lambda^{2d} F(\lambda)^{2-\nu}) + O(F(\lambda + a_1 + o(1)) F(\delta(\lambda)\lambda/(C\ln(\lambda/\delta(\lambda)))^d)(C\ln(\lambda/\delta(\lambda)))^d) \tag{3.17}$$

This ends the proof of Theorem 1.2.

□

Proof of Lemma 3.2. Let us start with (3.13). Fix $\delta > 0$. Then, for some $k_\delta > 0$, for any θ,

$$\sum_{\substack{|k| \geq k_\delta \\ k \neq 0}} |\hat{H}_k(\theta)|^2 \leq \delta/2.$$

as $\sum_k \sup_\theta |\hat{H}_k(\theta)|^2 < +\infty$.
Pick $\omega \in \Omega'_K$ such that $\displaystyle\sum_{k \in (\mathbb{Z}^d_{2n+1})^*} \frac{\omega_k}{\lambda - \omega_k} |\hat{H}_k(\theta)|^2 \geq \delta$. Then, as for $\omega_k \leq 0$,

$$-1 \leq \frac{\omega_k}{\lambda - \omega_k} \leq 0,$$

for λ large enough, we have

$$\sum_{\substack{|k| < k_\delta \\ k \neq 0}} \frac{\omega_k}{\lambda - \omega_k} |\hat{H}_k(\theta)|^2 \leq -\delta - \sum_{\substack{k \in (\mathbb{Z}^d_{2n+1})^* \\ |k| \geq k_\delta \\ \omega_k \leq 0}} \frac{\omega_k}{\lambda - \omega_k} |\hat{H}_k(\theta)|^2 \leq -\delta/2.$$

But

$$\text{Proba}\left(\left\{\omega \in \Omega'_K;\ \sum_{\substack{|k| < k_\delta \\ k \neq 0}} \frac{\omega_k}{\lambda - \omega_k} |\hat{H}_k(\theta)|^2 \leq -\delta/2\right\}\right) \underset{\lambda \to +\infty}{\to} 0$$

as k_δ does not depend on λ and, for any $\eta > 0$,

$$\text{Proba}\left(\left\{\omega_k;\ \frac{\omega_k}{\lambda - \omega_k} \leq -\eta\right\}\right) \underset{\lambda \to +\infty}{\to} 0.$$

This proves (3.13).
For $\omega \in \Omega'_K$, $\dfrac{\omega_k}{\lambda - \omega_k} \leq \dfrac{\lambda - K}{K}$. We pick $C > 0$ large enough (independent of λ and δ) and set $k(\lambda, \delta) = C\ln(\lambda/\delta)$. Then, as $\sup_\theta |\hat{H}_k(\theta)| \leq e^{-c|k|}$ for some $c > 0$, we have

$$\sum_{\substack{|k| \geq k_\delta \\ k \neq 0}} \frac{\omega_k}{\lambda - \omega_k} |\hat{H}_k(\theta)|^2 \leq \delta/2.$$

Hence, noticing that, for λ large enough, $k(\lambda, \delta) = o(n)$, we see that we just need to estimate

$$P_0 = \text{Proba}\left(\left\{\omega \in \Omega'_K;\ \sum_{\substack{|k|\leq k(\lambda,\delta)\\ k\neq 0}} \frac{\omega_k}{\lambda - \omega_k}|h_k(\theta)|^2 \geq \delta\right\}\right).$$

But this probability is easily estimated by

$$P_0 \leq Ck(\lambda,\delta)^d F\left(\frac{\delta\lambda}{Ck(\lambda,\delta)+\delta}\right)$$

for some $C > 0$ large enough.
This ends the proof of (3.14) hence of Lemma 3.2.

$\square$

Proof of Corollary 3.1. By (3.14), (3.16) holds for any function δ. Obviously the two probabilities estimated in Lemma 3.2 are decreasing in δ. Hence, to prove Corollary 3.1, we just need to find some function $\delta(\lambda)$ that tends to 0 at $+\infty$ such that (3.15) is satisfied. Let us define the function

$$f(\delta,\lambda) = \text{Proba}\left(\left\{\omega \in \Omega'_K;\ \sum_{k\in(\mathbb{Z}^d_{2n+1})^*} \frac{\omega_k}{\lambda - \omega_k}|\hat{H}_k(\theta)|^2 \leq -\delta\right\}\right).$$

By Lemma 3.2, f satisfies

1. for fixed δ, $f(\delta,\lambda) \to 0$ when $\lambda \to +\infty$,
2. for fixed λ, $f(\delta,\lambda)$ is a decreasing function of δ,
3. for fixed λ, $f(\delta,\lambda)$ is a right continuous function of δ.

We define

$$\Delta_\lambda = \{\delta; \forall\lambda' \geq \lambda;\ 0 \leq f(\delta,\lambda) < \delta\}.$$

Then, by 1, $\Delta_\lambda \neq \emptyset$. It is lower bounded by 0. We define $\delta(\lambda) = \inf \Delta_\lambda$. So that

- for $\lambda \leq \lambda'$, $\delta(\lambda') \subset \delta(\lambda)$ (as $\Delta_\lambda \subset \Delta_{\lambda'}$),
- by 1, $\delta(\lambda) \to 0$ when $\lambda \to +\infty$,
- $f(\lambda, \delta(\lambda)) \leq \delta(\lambda)$ as f is right continuous and decreasing by (2) and (3).

This ends the proof of Corollary 3.1.

3.2. **The asymptotic expansion.** Under our assumptions on g, we know that the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ admit a continuous density near $+\infty$, and that this density is non vanishing. Let us call f the absolute value of the logarithm of this density so that $N(\lambda) = \displaystyle\int_\lambda^{+\infty} e^{-f(t)}dt$. We then know that $f(t) = g(t) - \ln(g'(t))$ and that $e^{-f(t)} = o(t^{-\infty})$ near $+\infty$.

Proof of (1.10.) To keep things short, we will not prove the result in the case when g admits an asymptotic of type (1.7). This proof is very similar to the one in the case g admits an asymptotic of type (1.8) when $\alpha < 3$ and the modification to be made to this proof are obvious.

We now assume that g admits an asymptotic of type (1.8) with $\alpha < 3$. Fix some arbitrary integer $M \in \mathbb{N}$. This will be the order of the asymptotic expansion we want to get for N. Hence we need to take ε (of Lemma 2.1) of order λ^{-M}. So that it will be enough to take n in Lemma 2.1 of order $\lambda^{M+\sigma(\alpha+1)}$. Now, in Lemma 3.1, set $K = \nu\lambda$ (with $0 < \nu < 1$). In this case, let us consider the rest given by Lemma 3.1. For any $\eta \in (0,1)$, we have

$$n^{2d} F(\lambda - \hat{H}_+) F(\lambda - K - \hat{H}_+) = o\left(e^{-a_1(1+(1-\nu)^\alpha)(1-\eta)\lambda^\alpha}\right). \tag{3.18}$$

(Here $o(\cdot)$ depends on η).
We now just need to get an asymptotic expansion of the principal term in (3.1). Therefore, for $\omega \in \Omega'_K$, we compute an asymptotic expansion of $I(\lambda,\omega,\theta)$; using the notations defined before (3.10), we have

$$\begin{aligned} I(\lambda,\omega,\theta) &= \langle \delta_0, (\lambda - M^n_{\omega'}(\theta))^{-1}\delta_0\rangle \\ &= \langle \delta_0, (\lambda - \Pi^n_\omega)^{-1}\left(1 - M_0^n(\theta)(\lambda - \Pi^n_\omega)^{-1}\right)^{-1}\delta_0\rangle \\ &= \sum_{p\geq 0}\langle \delta_0, (\lambda - \Pi^n_\omega)^{-1}\left(M_0^n(\theta)(\lambda - \Pi^n_\omega)^{-1}\right)^p\delta_0\rangle \\ &= \sum_{p\geq 0}\langle \delta_0, (\lambda - \Pi^n_\omega)^{-1}\left(M_0^n(\lambda - \Pi^n_\omega)^{-1}\right)^p\delta_0\rangle + O(e^{-n/C_0}) \end{aligned} \tag{3.19}$$

where $M_0^n = ((h_{j-j'}))_{(j,j')\in(\mathbb{Z}^d_{2n+1})^2}$ acts on $L^2(\mathbb{T}_n)\otimes \ell^2(\mathbb{Z}^d_{2n+1})$.
As $K = \nu\lambda$ $(0<\nu<1)$, for $\tilde{\omega} \in \Omega'_K$, we have $\|\lambda - \Pi^n_\omega\| \geq \nu\lambda$. Hence the p-th summand in (3.19) is of size at most λ^{-p-1} (uniformly in $\theta \in \mathbb{T}$ and $\tilde{\omega} \in \Omega'_K$). For $p \geq 0$, we compute

$$\begin{aligned} T_p(\lambda,\omega) &= \lambda^2\langle \delta_0, (\lambda - \Pi^n_\omega)^{-1}\left(M_0^n(\lambda - \Pi^n_\omega)^{-1}\right)^p\delta_0\rangle \\ &= \sum_{(k_1,\cdots,k_{p-1})\in(\mathbb{Z}^d_{2n+1})^{p-1}}\left(\prod_{j=1}^{p-1}\frac{1}{\lambda - \tilde{\omega}_{k_j}}\right) h_{-k_1}h_{k_1-k_2}\cdots h_{k_{p-2}-k_{p-1}}h_{k_{p-1}} \end{aligned} \tag{3.20}$$

(we recall that $\tilde{\omega}_0 = 0$ is constant). We have $T_0(\lambda,\omega) = \lambda$ and $T_1(\lambda,\omega) = h_0$. Notice that T_p is of size λ^{-p+1} (uniformly in $\theta \in \mathbb{T}$ and $\tilde{\omega} \in \Omega'_K$).
$I(\lambda,\omega,\theta)^{-1}$ admits an asymptotic expansion of the form

$$\frac{1}{I(\lambda,\omega,\theta)} = \lambda\left(1 + \sum_{j=1}^{M+[\alpha]+1}\frac{(-1)^j}{\lambda^j}\left(\sum_{p=1}^{M+[\alpha]+1} T_p\right)^j\right) + O\left(\lambda^{-M-[\alpha]-1}\right). \tag{3.21}$$

We define

$$T = \sum_{j=1}^{M+[\alpha]+1}\frac{(-1)^j}{\lambda^j}\left(\sum_{p=1}^{M+[\alpha]+1} T_p\right)^j = -\frac{1}{\lambda}(T_1+T_2) + \frac{1}{\lambda^2}T_1^2 + O\left(\frac{1}{\lambda^3}\right). \tag{3.22}$$

T is of size λ^{-1}. We compose expansion (3.21) with the expansion of g to obtain

$$
\begin{aligned}
g\left(\frac{1}{I(\lambda,\omega,\theta)}\right) &= \lambda^{\alpha}(1+T)^{\alpha}\sum_{j=0}^{[M/\alpha]+1} g_j^0\lambda^{-j\alpha}(1+T)^{-j\alpha}+\\
&\qquad + (\ln\lambda+\ln(1+T))\sum_{j=0}^{M+1} g_j^1\lambda^{-j}(1+T)^{-j}+\\
&\qquad\qquad + O\left(\lambda^{-M-1}\right)\\
&= g_0^0\lambda^{\alpha}(1+T)^{\alpha}+g_1^0+g_0^1\ln\lambda+\\
&\qquad + R_0(\lambda,\omega)+R_1(\lambda,\omega)+R_2(\lambda,\omega)+O\left(\lambda^{-M-1}\right)
\end{aligned}
\tag{3.23}
$$

where

$$
\begin{aligned}
R_0(\lambda,\omega) &= \sum_{j=2}^{[M/\alpha]+1} g_j^0\lambda^{(1-j)\alpha}(1+T)^{(1-j)\alpha} = O(\lambda^{-\alpha})\\
R_1(\lambda,\omega) &= \ln(1+T)\sum_{j=0}^{M+1} g_j^1\lambda^{-j}(1+T)^{-j} = O(\lambda^{-1})\\
R_2(\lambda,\omega) &= \ln\lambda\sum_{j=1}^{M+1} g_j^1\lambda^{-j}(1+T)^{-j} = O(\lambda^{-1}\ln\lambda).
\end{aligned}
$$

Moreover, by (3.22), we have

$$
g_0^0\lambda^{\alpha}(1+T)^{\alpha} = \sum_{l=0}^{[\alpha]} g_{0,l}\lambda^{\alpha}T^l + R_4(\lambda,\omega) + O\left(\lambda^{-M-1}\right)
\tag{3.24}
$$

where

$$
R_4(\lambda,\omega) = \sum_{l=[\alpha]+1}^{M+[\alpha]+1} g_{0,l}\lambda^{\alpha}T^l = O(\lambda^{\alpha-[\alpha]-1}),
$$

$$
g_{0,0}=g_0^0,\ \ g_{0,1}=\alpha g_0^0 \text{ and } g_{0,2}=\frac{\alpha}{2}(\alpha+1)g_0^0.
$$

Using (3.22), we get that

$$
\begin{aligned}
\sum_{l=0}^{[\alpha]} g_{0,l}\lambda^{\alpha}T^l &= g_{0,0}\lambda^{\alpha} - g_{0,1}\lambda^{\alpha-1}T_1 + g_{0,2}\lambda^{\alpha-2}T_1^2 - g_{0,1}\lambda^{\alpha-1}T_2 +\\
&\qquad + R_5(\lambda,\omega) + O\left(\lambda^{-M-1}\right)
\end{aligned}
\tag{3.25}
$$

where

$$
R_5(\lambda,\omega) = O(\lambda^{\alpha-4}) + O(\lambda^{-\alpha}).
$$

We compose (3.23), (3.24) and (3.25) with the asymptotic expansion of e^{-x} to get

$$
\begin{aligned}
(3.26)\qquad \exp\left(-g(I^{-1}(\lambda,\omega,\theta))\right) &= e^{-g_{0,0}\lambda^{\alpha} - g_0^1 \ln\lambda - g_1^0 + g_{0,1}\lambda^{\alpha-1}T_1 - g_{0,2}\lambda^{\alpha-2}T_1^2} \cdot \\
&\cdot e^{g_{0,1}\lambda^{\alpha-1}T_2} \cdot \left(1 + R(\lambda,\omega) + O\left(\lambda^{-M-1}\right)\right)
\end{aligned}
$$

where R may be written

$$
\begin{aligned}
(3.27)\quad R(\lambda,\omega) = \sum_{j=0}^{M}\sum_{l=0}^{M}\lambda^{-l} \sum_{p_1+\dots+p_j\le M} c^0_{p_1,\dots,p_j,l} T_{p_1}\cdots T_{p_j} + \\
+ \sum_{j=0}^{M}\sum_{m,l=0}^{M}\lambda^{-l-m}\ln^m\lambda \sum_{p_1+\dots+p_j\le M} c^1_{p_1,\dots,p_j,l} T_{p_1}\cdots T_{p_j}.
\end{aligned}
$$

The coefficients $(c^0_{p_1,\dots,p_j,l}, c^1_{p_1,\dots,p_j,l})$ only depend on the coefficients of the asymptotic expansion of g.
Now, we want to integrate (3.26) in θ over the torus $\mathbb{T}$ and take its expectation in ω over Ω'_K. The principal terms in (3.26) do not depend on θ: only the rest i.e. the term $O(\lambda^{-M-1})$ depends on θ. Hence, the θ-integration is performed trivially. Among the exponentials coming into (3.26), only the last one depends on the random variables; it depends on T_2 where, by (3.20)

$$
T_2(\lambda,\omega) = \sum_{k\in\mathbb{Z}^d_{2n+1}} \frac{|h_k|^2}{\lambda-\tilde{\omega}_k} = \frac{1}{\lambda}\sum_{k\in\mathbb{Z}^d_{2n+1}} |h_k|^2 + \frac{1}{\lambda}\tilde{T}_2(\lambda,\omega)
$$

where

$$
\tilde{T}_2(\lambda,\omega) = \sum_{k\in(\mathbb{Z}^d_{2n+1})^*} \frac{|h_k|^2\omega_k}{\lambda-\omega_k}
$$

Hence we get that

$$
\begin{aligned}
(3.28)\quad \frac{1}{\mathrm{Vol}(\mathbb{T})}\int_{\mathbb{T}} \mathbb{E}\left(F\left(\frac{1}{I(\lambda,\omega,\theta)}\right)\mathbf{1}_{\Omega'_K}\right)d\theta = e^{-g_{0,0}\lambda^{\alpha} + -g_0^1\ln\lambda - g_1^0 + g_{0,1}\lambda^{\alpha-1}T_1} \cdot \\
\cdot \exp\left(-g_{0,2}\lambda^{\alpha-2}T_1^2 + g_{0,1}\lambda^{\alpha-2}\sum_{k\in\mathbb{Z}^d_{2n+1}}|h_k|^2\right)\cdot J(\lambda)
\end{aligned}
$$

where

$$
(3.29)\qquad J(\lambda) = \mathbb{E}\left(e^{g_{0,1}\lambda^{\alpha-2}\tilde{T}_2(\lambda,\omega)}\cdot\left(1+R(\lambda,\omega)+O\left(\lambda^{-M-1}\right)\right)\mathbf{1}_{\Omega'_K}\right).
$$

We now distinguish three cases:

1. if $\alpha<2$, then $\lambda^{\alpha-2}\tilde{T}_2(\lambda,\omega)$ tends to 0 uniformly on Ω'_K hence (3.29) reduces to

$$
J(\lambda) = \mathbb{E}\left(\left(1+\sum_{l=0}^{[M/(\alpha-2)]}\frac{1}{l!}\left(g_{0,1}\lambda^{\alpha-2}\tilde{T}_2(\lambda,\omega)\right)^l + R(\lambda,\omega) + O\left(\lambda^{-M-1}\right)\right)\mathbf{1}_{\Omega'_K}\right).
$$

2. if $\alpha = 2$, then $\tilde{T}_2$ is bounded by $\frac{1}{1-\nu}\sum_{k\in\mathbb{Z}^d}|h_k|^2$ on Ω'_K. Hence the exponential in (3.29) is bounded. Using the fact that all the moments of the random variables $(\omega_k)_k$ are finite and the fact that n is polynomially bounded in λ, we get that

$$J(\lambda) = \mathbb{E}\left(e^{g_{0,1}\tilde{T}_2(\lambda,\omega)}\cdot(1+R(\lambda,\omega)+O\left(\lambda^{-M-1}\right))\mathbf{1}_{\Omega^*_{\lambda^{1/2}}}\right).$$

where $\Omega^*_K = \{\omega;\ \forall k\in(\mathbb{Z}^d_{2n+1})^*,\ -K\le\omega_k\le K\}$
But on $\Omega^*_{\lambda^{1/2}}$, $|\tilde{T}_2(\lambda,\omega)|\le\lambda^{-1/2}$. In this region, we again expand $\exp\left(g_{0,1}\tilde{T}_2(\lambda,\omega)\right)$ according to its Taylor series as above and cut the expansion at order $2M$ getting thus a rest of size λ^{-M}.

3. if $2<\alpha<3$, we pick some exponent β such that $3-\alpha>\beta>\dfrac{3-\alpha}{\alpha-1}$. Then, as all the moments of the random variables $(\omega_k)_k$ are finite, we have

$$\int_{\lambda^\delta}^{\nu\lambda} e^{-f(t)+\frac{t}{\lambda-t}}dt = \lambda e^{-\lambda(\lambda^\delta-\lambda^{\alpha-3}/(1-\nu))} = o(\lambda^{-\infty}) \text{ and } \mathbb{E}(e^{-\frac{\omega_k}{\lambda-\omega_k}}\mathbf{1}_{\omega_k\le-\lambda^\delta}) = o(\lambda^{-\infty}).$$

As n is polynomial in λ, we have

$$J(\lambda) = \mathbb{E}\left(e^{g_{0,1}\lambda^{\alpha-2}\tilde{T}_2(\lambda,\omega)}\cdot(1+R(\lambda,\omega)+O\left(\lambda^{-M-1}\right))\mathbf{1}_{\Omega^*_{\lambda^\delta}}\right). \tag{3.30}$$

But, on $\Omega^*_{\lambda^\delta}$, $\tilde{T}_2 = O(\lambda^{\delta+\alpha-3})$, so that we can again expand the exponential inside the equation (3.30) defining J using its Taylor series and cutting this series at order $M/(\delta+\alpha-3)$ getting thus a rest of size λ^{-M}.

The outcome of this discussion is that we will be finished if we show that expressions of the following form admit an infinite asymptotic expansion in λ:

$$\mathbb{E}\left((\tilde{T}_2)^l T_{p_1}\cdots T_{p_j}\mathbf{1}_{\Omega^*_{\lambda^\delta}}\right) \text{ or } \mathbb{E}\left((\tilde{T}_2)^l T_{p_1}\cdots T_{p_j}\mathbf{1}_{\Omega'_K}\right).$$

This follows immediately from the definition of the $(T_p)_{p\in\mathbb{N}}$ and of $\tilde{T}_2$ and the fact that all the moments of the random variables $(\omega_k)_{k\in\mathbb{Z}^d}$ are finite.

When $\alpha=3$, we just have to change slightly the last part of the analysis. We write

$$T_2(\lambda,\omega) = \sum_{k\in\mathbb{Z}^d_{2n+1}}\frac{|h_k|^2}{\lambda-\tilde{\omega}_k} = \frac{1}{\lambda}\sum_{k\in\mathbb{Z}^d_{2n+1}}|h_k|^2 + \frac{1}{\lambda^2}\sum_{k\in(\mathbb{Z}^d_{2n+1})^*}|h_k|^2\omega_k + \frac{1}{\lambda^2}T'_2(\lambda,\omega) \tag{3.31}$$

where

$$T'_2(\lambda,\omega) = \sum_{k\in(\mathbb{Z}^d_{2n+1})^*}\frac{|h_k|^2\omega_k^2}{\lambda-\omega_k}.$$

As the function $x\mapsto x^3-\dfrac{\lambda x}{\lambda-x}$ is increasing on any segment $[\lambda^\delta,\nu\lambda]$ $(0<\delta<1)$ and as n is polynomially bounded in λ, we see that (3.30) still holds. So that, if we choose

$\delta \leq \eta$, we get

$$J(\lambda) = \mathbb{E}\left(e^{\displaystyle g_{0,1} \sum_{k\in(\mathbb{Z}^d)^*} |h_k|^2\omega_k} \cdot \left(1 + R(\lambda,\omega) + O\left(\lambda^{-M-1}\right)\right) \cdot \mathbf{1}_{\Omega^*_{\lambda\delta}} \right).$$

Notice that $\sum_{k\in(\mathbb{Z}^d)^*} |h_k|^2\omega_k$ is a finite sum as H is of finite range.
So we get the same result as in the case $\alpha < 3$ (R however does not have the same value but only the same form as in the case $\alpha < 3$) except that we integrate the expansion against the measure $\exp\left(g_{0,1}\sum_{k\in(\mathbb{Z}^d)^*}|h_k|^2\omega_k\right) dP(\omega)$ (instead of the measure $dP(\omega)$ in the case $\alpha < 3$). This will just change the form of the obtained asymptotic expansion by a constant equal to

$$n^3 = \ln\left(\mathbb{E}\left(\exp\left(\alpha g_0^0 \sum_{k\in(\mathbb{Z}^d)^*} |h_k|^2\omega_k\right)\right)\right). \tag{3.32}$$

□

Proof of (1.11). We now assume that g admits an asymptotic expansion of the form (1.8) with $\alpha > 3$. In this case, our method to get a full asymptotic heavily relies on the finite range of H. Let us recall and introduce some new definitions that will be useful in this respect. Pick $(k,k') \in \mathbb{Z}^d$. We recall that an H-path from k to k' is an n-tuple $\gamma(k,k') = (k_1,k_2,\cdots,k_n)$ such that $k_1 = k$, $k_n = k'$ and for $j = 1,\ldots,n-1$, $h_{k_j-k_{j+1}} \neq 0$. Of course, if $\gamma(0,k) = (k_1,k_2,\cdots,k_n)$ is an H-path from 0 to k, then $k' + \gamma(0,k) = (k'+k_1, k'+k_2,\cdots,k'+k_n)$ is an H-path from k' to $k+k'$.

On the set of H-paths, one can naturally define an addition in the following way: if $\gamma(k,k') = (k_1,k_2,\cdots,k_n)$ and $\gamma(k',k'') = (k_n,k_{n+1},\cdots,k_{n+m})$ then $\gamma(k,k') + \gamma(k',k'') = (k_1,k_2,\cdots,k_n,k_{n+1},\cdots,k_{n+m})$. Any H-path then admits an opposite.

Let γ be an H-path. If $\gamma = (k_1,k_2,\cdots,k_n)$, then we define $\sigma(\gamma)$, the support of γ to be the set $\{k_1,k_2,\cdots,k_n\}$ and h_γ to be the complex number

$$h_\gamma = h_{k_1-k_2}h_{k_2-k_3}\cdots h_{k_{n-2}-k_{n-1}}h_{k_{n-1}-k_n}$$

Let us now start with the proof of (1.11). Let $M \in \mathbb{N}^*$ be the order of the expansion we want to prove. By (3.19) and (3.20), uniformly for $\tilde{\omega} \in \Omega'_K$ (here $K = \nu\lambda$ for some $\nu \in (0,1)$), we have

$$I(\lambda,\omega,\theta) = \lambda^{-2}\sum_{j=0}^{M+[\alpha]+1} T_p(\lambda,\omega) + O(\lambda^{-M-2-[\alpha]}) \tag{3.33}$$

where $T_0(\lambda,\omega) = \lambda$, $T_1(\lambda,\omega) = h_0$ and we rewrite (3.20)

$$T_p(\lambda,\omega) = \sum_{\substack{\gamma=(k_1,\cdots,k_p)\in\Gamma(0)\\ |\gamma|=p}} \left(\prod_{j=2}^{p-1}\frac{1}{\lambda - \tilde{\omega}_{k_j}}\right) h_\gamma. \tag{3.34}$$

We now replace T_p by its expression (3.34) in (3.33) and group the thus obtained terms differently. Namely, we first sum over all the path that never leave the point 0, then over all the paths with support contained in $\{0\}\cup\Lambda_1$, then over all the paths with support contained in $\{0\}\cup\Lambda_1\cup\Lambda_2$, and so on (we recall that Λ_l was the set of points in $\mathbb{Z}^d$ such that the shortest path from such a point to 0 has length exactly l). We get

$$I(\lambda,\omega,\theta)=\lambda^{-2}\sum_{j=0}^{[M+\alpha+1]}S_p(\lambda,\omega)+O(\lambda^{-M-2-[\alpha]}) \tag{3.35}$$

where

$$S_0(\lambda,\omega)=S_0(\lambda)=\lambda+h_0\sum_{j=0}^{M+[\alpha]+1}\left(\frac{h_0}{\lambda}\right)^p=\frac{\lambda^2}{\lambda-h_0}+O(\lambda^{-M-2-[\alpha]}), \tag{3.36}$$

and for $p\geq 1$,

$$S_p(\lambda,\omega)=\sum_{n=2p-1}^{[(M+\alpha)/2]}\sum_{\substack{\gamma=(k_1,\cdots,k_n)\in\Gamma(0)\\ \sigma(\gamma)\subset\{0\}\cup\Lambda_1\cup\cdots\cup\Lambda_p\\ \sigma(\gamma)\not\subset\{0\}\cup\Lambda_1\cup\cdots\cup\Lambda_{p-1}}}\left(\prod_{j=2}^{n-1}\frac{1}{\lambda-\tilde\omega_{k_j}}\right)h_\gamma. \tag{3.37}$$

Notice that each of the summands in S_p contains at least one factor $(\lambda-\tilde\omega_k)^{-1}$ for some $k\in\Lambda_p$. S_p is of order λ^{-2p+1} and the term of smallest order in S_p (i.e. of order λ^{-2p+1}) is the term

(3.38)

$$\begin{aligned}S_p^0(\lambda,\omega)&=\sum_{k\in\Lambda_p}\frac{1}{\lambda-\tilde\omega_k}\sum_{\substack{\gamma=(k_1,\cdots,k_p)\in\Gamma(k)\\ \gamma'=(k'_1,\cdots,k'_p)\in\Gamma(k)\\ |\gamma|=|\gamma'|=p}}\left(\prod_{j=2}^{p-1}\frac{1}{\lambda-\tilde\omega_{k_j}}\right)\left(\prod_{j=2}^{p-1}\frac{1}{\lambda-\tilde\omega_{k'_j}}\right)h_\gamma h_{-\gamma'}\\ &=\sum_{k\in\Lambda_p}\frac{1}{\lambda-\tilde\omega_k}\left|\sum_{\substack{\gamma=(k_1,\cdots,k_p)\in\Gamma(k)\\ |\gamma|=p}}\left(\prod_{j=2}^{p-1}\frac{1}{\lambda-\tilde\omega_{k_j}}\right)h_\gamma\right|^2\underset{\lambda\to+\infty}{\sim}\sum_{k\in\Lambda_p}\frac{1}{\lambda-\tilde\omega_k}\left|h_k^{(p)}\right|^2\end{aligned}$$

as $h_{-k}=\overline{h_k}$ and λ and $(\tilde\omega_k)$ are all real. Notice that in (3.38), for $k\in\Lambda_p$, $\tilde\omega_k$ never appears in the square in the right hand side as the H-paths over which one is summing are of length exactly p.
Using the asymptotic expansion known for g, we write

$$\begin{aligned}e^{-g(I(\lambda,\omega,\theta)^{-1})}&=\exp\left(g\left(\frac{\lambda^2}{S_0(\lambda)}\frac{1}{1+\sum_{p=1}^{[(M+\alpha)/2]}\frac{S_p(\lambda,\omega)}{S_0(\lambda)}}\right)\right)+O(\lambda^{-M-1})\\ &=e^{-g(J(\lambda,\omega))}\cdot(1+\tilde R(\lambda,\omega)+O(\lambda^{-M-1}))\\ &=e^{-g_0^0J^\alpha(\lambda,\omega)-g_1^0-\ln J(\lambda,\omega)}\cdot(1+R(\lambda,\omega)+O(\lambda^{-M-1}))\end{aligned} \tag{3.39}$$

where

$$J(\lambda,\omega)=\frac{\lambda^2}{S_0(\lambda)}\frac{1}{1+\displaystyle\sum_{p=1}^{[(\alpha-1)/2]}\frac{S_p(\lambda,\omega)}{S_0(\lambda)}} \tag{3.40}$$

and, both $\tilde{R}$ and R have an expansion of the type (3.27) where the terms $(T_p)_p$ are replaced by the terms $(S_p)_p$.

To compute $\mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega'_K})$, we first note that $J(\lambda,\omega)$ only depends on the random variables $(\omega_k)_{k\in\Lambda^\alpha}$ where we have defined $\Lambda^\alpha=\cup_{j=0}^{[(\alpha-1)/2]}\Lambda_j$. So that the principal term in $e^{-g(I(\lambda,\omega,\theta)^{-1})}$ only depends on $(\omega_k)_{k\in\Lambda^\alpha}$. We will first integrate with respect to the random variables $(\omega_k)_{k\in\Lambda^\alpha}$. We recall that $dP(\omega_k)$ denotes the probability law of the random variable ω_k and that f was defined by $dP(\omega_k)=e^{-f(\omega_k)}d\omega_k$. Hence the expectation we want to compute can be written as

$$\mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega'_K})=\int\cdots\int_{\Omega'_K}e^{-g(I(\lambda,\omega,\theta)^{-1})}e^{\sum_k f(\omega_k)}\prod_k d\omega_k. \tag{3.41}$$

To perform this integration, we note that $g(I(\lambda,\omega,\theta)^{-1})$ puts most of its weight on large ω's. So that, if we cut the integral into two pieces one for ω_k large and one for ω_k small, the second piece should be negligible with respect to the first one.
In the first piece, as we know an asymptotic expansion for g, we also know an asymptotic expansion for f. We will then be able to use Laplace's method to get an asymptotic expansion for this piece.
We shall follow this idea. However we will not be able to do this in such a direct way. Instead, we will, step by step inductively, integrate out, first all the random variables $(\omega_k)_{k\in\Lambda_1}$, then all the random variables $(\omega_k)_{k\in\Lambda_2}$, and so on, until we have integrated out all the random variables $(\omega_k)_{k\in\Lambda_{[(\alpha-1)/2]}}$. Moreover, as in the case when $\alpha\leq 3$, we will have to distinguish between the case when α is an odd integer and the case when it is not.
To implement this procedure, it will be convenient to enumerate the points in Λ^α. We define the points $\{k_0,k_1,\dots,k_M\}=\Lambda^\alpha$ in such a way that, if $j\leq j'$ then $l(k_j)\leq l(k_{j'})$. We also define the sets $\Lambda^\alpha_n=\{k_0,k_1,\dots,k_n\}$. We will need some $\delta>0$ small (the size of which will be chosen later on) to control the cut off between large and small ω_k.
As the random variables are i.i.d, we write

$$\mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega'_K})=\mathbb{E}(\mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega'_K}|(\omega_k)_{k\neq k_0})).$$

We define

$$I_1(\lambda,(\omega_k)_{k\neq k_0})=\mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega'_K}|(\omega_k)_{k\neq k_0})=\int e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega'_K}dP(\omega_{k_0}). \tag{3.42}$$

Let us define $\Omega_K^+(k_0) = \{\omega \in \Omega'_K;\ \omega_{k_0} \geq \lambda\delta\}$ and $\Omega_K^-(k_0) = \{\omega \in \Omega'_K;\ \omega_{k_0} < \lambda\delta\}$. We write

(3.43)
$$\begin{aligned} I_1(\lambda, (\omega_k)_{k\neq k_0}) &= \mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega_K^+(k_0)}|(\omega_k)_{k\neq k_0}) + \mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})}\mathbf{1}_{\Omega_K^-(k_0)}|(\omega_k)_{k\neq k_0}) \\ &= I_1^+(\lambda, (\omega_k)_{k\neq k_0}) + I_1^-(\lambda, (\omega_k)_{k\neq k_0}). \end{aligned}$$

For $\omega \in \Omega_K^-(k_0)$, we have $dP(\omega_{k_0}) = e^{-f(\omega_{k_0})}d\omega_{k_0}$ where $f(\omega_{k_0}) = g(\omega_{k_0}) + \ln g'(\omega_{k_0})$. Hence, f admits an asymptotic expansion of the same form (1.8) with different coefficients. More precisely

$$f(\lambda) \underset{\lambda\to+\infty}{\sim} \sum_{k\geq 0} f_k^0 \lambda^{\alpha(1-k)} + \sum_{k\geq 0} f_k^3 \lambda^{-\alpha-k} + \ln\lambda \left(\sum_{k\geq 0} f_k^1\lambda^{-k} + \sum_{k\geq 0} f_k^2 \lambda^{-\alpha-k}\right)$$

where, in particular, $f_0^0 = g_0^0$, $f_1^0 = g_1^0 + \ln\alpha$, $f_0^1 = g_0^1 + \alpha - 1$.
Using the asymptotic expansion for f and (3.39), we write

(3.44)
$$\begin{aligned} I_1^+(\lambda, (\omega_k)_{k\neq k_0}) &= \int_{\lambda^\delta \leq \omega_{k_0}\leq \nu\lambda} \exp\left(-g(I(\lambda,\omega,\theta)^{-1}) - f(\omega_{k_0})\right) d\omega_{k_0} \\ &= \int_{\lambda^\delta \leq \omega_{k_0}\leq \nu\lambda} e^{-\psi((\omega_k)_{k\in\Lambda^\alpha})}[(1 + R_f(\omega))(1 + R(\lambda,\omega) + O(\lambda^{-M})]d\omega_{k_0} \end{aligned}$$

where

(3.45)
$$\begin{aligned} \psi((\omega_k)_{k\in\Lambda^\alpha}) &= \psi_0((\omega_k)_{k\in\Lambda^\alpha}) + f_0(\omega_{k_0}) \\ \psi_0((\omega_k)_{k\in\Lambda^\alpha}) &= g_0^0(J(\lambda,\omega))^\alpha + g_1^0 + g_0^1 \ln J(\lambda,\omega) \\ f_0(\lambda) &= f_0^0\lambda^\alpha + f_1^0 + f_0^1\ln\lambda, \end{aligned}$$

and $R_f(\omega)$ has an expansion of the following form

(3.46)
$$R_f(\omega) = \sum_{\substack{1\leq n^0+n^1+\alpha(n^2+n^3)+\leq[M/\delta] \\ n^0,n^1,n^2,n^3\in\mathbb{N}}} c_{n^0,n^1,n^2,n^3}\omega_{k_0}^{-n^0-\alpha n^3}\left(\frac{\ln\omega_{k_0}}{\omega_{k_0}}\right)^{n^1+\alpha n^2}.$$

The function ψ is analytic in $(\omega_k)_{k\in\Lambda^\alpha}$ for $\omega_{k_0} \in \Omega_K^+(k_0)$. We are then interested in the minima of ψ with respect to the variable ω_{k_0}, the $(\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}$ being considered as parameters. We compute

(3.47)
$$\begin{aligned} \frac{\partial\psi}{\partial\omega_{k_0}}((\omega_k)_{k\in\Lambda^\alpha}) = &\left(g_0^0\alpha(J(\lambda,\omega))^{\alpha-1} + \frac{g_0^1}{J(\lambda,\omega)}\right)\frac{\partial J}{\partial\omega_{k_0}}(\lambda,\omega) + \\ &+ g_0^0\alpha(\omega_{k_0})^{\alpha-1} + (g_0^1+\alpha-1)\frac{1}{\omega_{k_0}}. \end{aligned}$$

and

(3.48)

$$\frac{\partial J}{\partial \omega_{k_0}}(\lambda,\omega) = -\frac{\lambda^2}{(S_0(\lambda))^2}\frac{1}{\left(1+\sum_{p=1}^{[(\alpha-1)/2]}\frac{S_p(\lambda,\omega)}{S_0(\lambda)}\right)^2}\sum_{p=1}^{[(\alpha-1)/2]}\frac{\partial S_p}{\partial \omega_{k_0}}(\lambda,\omega)$$

$$= -\frac{\lambda^2}{(S_0(\lambda))^2}\frac{1}{\left(1+\sum_{p=1}^{[(\alpha-1)/2]}\frac{S_p(\lambda,\omega)}{S_0(\lambda)}\right)^2}\sum_{p=1}^{[(\alpha-1)/2]}\left(\frac{\partial S_p^0}{\partial \omega_{k_0}}(\lambda,\omega)+\frac{\partial(\Delta S_p)}{\partial \omega_{k_0}}(\lambda,\omega)\right).$$

where $\Delta S_p(\lambda,\omega) = (S_p - S_p^0)(\lambda,\omega)$.
By (3.37) and (3.38), uniformly in $\Omega_K^+(k_0)$, for $p \geq 1$, when $\lambda \to +\infty$, we have

$$\frac{\partial(S_p - S_p^0)}{\partial \omega_{k_0}}(\lambda,\omega) = O(\lambda^{-1-2p}).$$

Moreover, using (1.6), for $k \in \Lambda^\alpha$, we have

- if $p > 1$ then, uniformly in $\Omega_K^+(k_0)$, when $\lambda \to +\infty$,

$$\frac{\partial S_p^0}{\partial \omega_{k_0}}(\lambda,\omega) = O(\lambda^{-3})$$

- if $p = 1$ then, as $k_0 \in \Lambda_1$,

$$\frac{\partial S_1^0}{\partial \omega_{k_0}}(\lambda,\omega) = h_{k_0}^1\frac{1}{(\lambda-\omega_{k_0})^2} \underset{\lambda\to+\infty}{\sim} h_{k_0}^1\lambda^{-2}.$$

So that the solution of $\partial_{\omega_{k_0}}\psi = 0$ should roughly satisfy

$$\omega_{k_0} \underset{\lambda\to+\infty}{\sim} (h_{k_0}^1)^{\frac{1}{\alpha-1}}\lambda^{\frac{\alpha-3}{\alpha-1}}.$$

Let us perform the following change of variables: set

- $\omega_{k_0} = u_{k_0}\cdot(h_{k_0}^1)^{\frac{1}{\alpha-1}}\cdot\lambda^{\frac{\alpha-3}{\alpha-1}}$,
- $u_k = \dfrac{1}{\lambda-\omega_k}$ for $k \in \Lambda^\alpha\setminus\{k_0\}$,
- $v = \dfrac{1}{\lambda}$, $w = \lambda^{-\frac{2}{\alpha-1}}$ and $t_1 = \lambda^{-\alpha\frac{\alpha-3}{\alpha-1}}$.

Then, we have $\dfrac{1}{\lambda-\omega_{k_0}} = \dfrac{v}{1-w\cdot u_{k_0}}$ and

$$\frac{\partial \psi}{\partial \omega_{k_0}}((\omega_k)_{k\in\Lambda^\alpha}) = h_{k_0}^1 g_0^0\alpha\lambda^{\alpha-3}U_{k_0}(u_{k_0},u',v,w,t)$$

where U_{k_0} satisfies

(3.49)

$$U_k(u_{k_0}, u', v, w, t) = -\frac{v}{1 - w \cdot u_{k_0}} \cdot (1 + w \cdot g(u_{k_0}, u', v, w, t_1)) + u_{k_0}^{\alpha-1}\left(1 + \frac{t_1 \cdot w}{u_{k_0}}\right).$$

where $u' = (u_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}$ and g is analytic in some neighborhood of $(1,0,0,0,0)$. We compute

$$U_{k_0}(1,0,0,0,0) = 0,$$
$$\frac{\partial U_{k_0}}{\partial u_{k_0}}(1,0,0,0,0) = (\alpha - 1) \neq 0.$$

So we can apply the analytic version of the local inverse mapping theorem to the mapping $(u_{k_0}, u', v, w, t) \to U_{k_0}$ near the point $(1,0,0,0,0)$. This gives us that, for (u', t, v, w) in some neighborhood of 0, the equation $U_{k_0} = 0$ admits a unique solution $u(u', t, v, w)$ that is analytic in (u', t, v, w). This solution satisfies $u(0,0,0,0) = 1$. Hence, for λ large enough, ψ will have a unique critical point in the variable ω_{k_0} in $\Omega_K^+(k_0)$; we will denote this point by $\omega_{k_0}^0((\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}, \lambda)$. Moreover, this critical point can be represented in following form

$$\omega_{k_0}^0 = \omega_{k_0}^0((\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}, \lambda) = (h_{k_0}^1)^{1/(\alpha-1)}\lambda^{(\alpha-3)/(\alpha-1)}(1+\phi) \tag{3.50}$$

where ϕ admits a convergent expansion

$$\phi = \sum_{\substack{(n_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}, l, m, n \\ \sum n_k + l + m + n \geq 1}} a_{(n_k)_k, l, m, n}\left(\prod_{k\in\Lambda^\alpha\setminus\{k_0\}}\frac{1}{(\lambda - \omega_k)^{n_k}}\right)\lambda^{-l-2m/(\alpha-1)-\alpha n(\alpha-3)/(\alpha-1)} \tag{3.51}$$

Let us point out that, for $k \in \Lambda^\alpha$, in (3.51), the term of lowest order in λ containing $\dfrac{1}{\lambda - \omega_k}$ contains at least the term $S_{l(k)}^0/S_0$. This in particular implies that, for $k \in \Lambda^\alpha \setminus \{k_0\}$, we have

$$\frac{\partial \omega_{k_0}^0}{\partial \omega_k} = \lambda^{\frac{\alpha-3}{\alpha-1}} O\left(\frac{\partial S_{l(k)}^0}{\partial \omega_k}\right). \tag{3.52}$$

We compute the Hessian of ψ at $\omega^0(\lambda)$ to get that

$$\psi''(\omega_{k_0}^0, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}) = \alpha(\alpha-1)g_0^0(h_{k_0}^1)^{(\alpha-2)/(\alpha-1)}\lambda^{(\alpha-3)(\alpha-2)/(\alpha-1)}(1+\phi)$$

where ϕ admits an asymptotic expansion of the type (3.51). So $\omega^0(\lambda)$ is the unique critical of ψ in $\Omega_K^+(k_0)$ and it is a non degenerate minimum. The Hessian of ψ tends to $+\infty$ with λ uniformly in $\Omega_K^+(k_0)$. Hence, to compute (3.42), as announced, we will use the Laplace method. Indeed, in $\Omega_K^+(k_0)$, outside an interval $I_\rho = [\omega_{k_0}^0 - \rho\lambda^{\frac{\alpha-3}{\alpha-1}}, \omega_{k_0}^0 + \rho\lambda^{\frac{\alpha-3}{\alpha-1}}]$ $(\rho > 0)$, we know that

$$\psi(\omega) \geq \psi(\omega_{k_0}^0, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}) + \rho^2\lambda^{\frac{(\alpha-2)(\alpha-1-2[(\alpha-1]/2])}{\alpha-1}}.$$

Inside the interval I_ρ (if ρ is small enough), we write

$$\begin{aligned}\psi(\omega) &= \psi(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}) + \\ &\qquad + \psi''(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}})(\omega_{k_0} - \omega^0_{k_0})^2(1+\phi_1) \\ &= \psi(\omega^0(\lambda), (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}) + \\ &\qquad + \psi''(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}) \left[(\omega_{k_0} - \omega^0_{k_0})(1+\phi)\right]^2\end{aligned} \tag{3.53}$$

where ϕ_1 and ϕ admit an asymptotic expansion of the type (3.51).
We split the integral I_1^+ (given by (3.44)) into two parts $I^1 + I^2$: in I^1, we integrate over I_ρ and, in I^2, we integrate over the complementary of I_ρ. Then, obviously

$$I^2 \leq \lambda e^{-\psi(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}) - \rho^2 \lambda^{\frac{(\alpha-2)(\alpha-1-2[(\alpha-1)/2])}{\alpha-1}}}. \tag{3.54}$$

To compute I^1, we use (3.53) and perform the change of variables $\omega_{k_0} \to (\omega_{k_0} - \omega^0_{k_0})(1+\phi)$ to get the following asymptotic expansion for I^1

$$\begin{aligned}I^1 = \frac{\sqrt{\pi} e^{-\psi(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}})}}{\sqrt{\psi''(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}})}} \cdot [(1 + R_G(\lambda)) \cdot (1 + R_f(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}})) \cdot \\ \cdot (1 + R(\lambda, \omega^0_{k_0}, (\omega_k)_{k\notin\Lambda^\alpha\setminus\{k_0\}})) + O(\lambda^{-M})\right].\end{aligned} \tag{3.55}$$

Hence, adding (3.54) and (3.55) and averaging with respect to the rest of the random variables i.e. $(\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}$, we get

$$\mathbb{E}(I_1^+(\lambda, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}})) = \sqrt{\frac{\pi}{\alpha(\alpha-1) g_0^0 (h^{(1)}_{k_0})^{2\frac{\alpha-2}{\alpha-1}} \lambda^{\frac{(\alpha-3)(\alpha-2)}{\alpha-1}}}}(1 + R_G(\lambda)) \cdot E \tag{3.56}$$

where

$$\begin{aligned}E = \mathbb{E}\Big[e^{-\psi(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}, \lambda)} \cdot (1+\phi) \cdot (1 + R_f(\omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}})) \cdot \\ \cdot (1 + R(\lambda, \omega^0_{k_0}, (\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}})) + O(\lambda^{-M})\Big].\end{aligned} \tag{3.57}$$

Here ϕ admits an asymptotic expansion of the type (3.51) and the asymptotic expansions of R_f and R are respectively given by (3.46) and (3.27).

We now have to estimate I_1^- defined in (3.43) by

$$I_1^-(\lambda, (\omega_k)_{k\neq k_0}) = \mathbb{E}(e^{-g(I(\lambda,\omega,\theta)^{-1})} \mathbf{1}_{\Omega_K^-(k_0)} | (\omega_k)_{k\neq k_0}).$$

Therefore, let us notice that, by (3.47) and (3.48), $g(I(\lambda,\omega,\theta)^{-1})$ is decreasing in ω_{k_0} on $\Omega_K^-(k_0)$. Using the estimate on the derivative of $g(I(\lambda,\omega,\theta)^{-1})$ with respect to ω_{k_0} given by (3.47) and (3.48), we know that, uniformly in $\Omega_K^-(k_0)$,

$$g(I(\lambda,\omega,\theta)^{-1})_{\restriction \omega_{k_0} = \lambda^\delta} \underset{\lambda\to+\infty}{\sim} g(I(\lambda,\omega,\theta)^{-1})_{\restriction \omega_{k_0} = \omega^0_{k_0}} + \alpha g_0^0 (h^1_{k_0})^{\frac{\alpha}{\alpha-1}} \lambda^{\alpha - 3 + \frac{\alpha-3}{\alpha-1}}.$$

Using (3.45), we estimate the difference

$$\psi(\omega)_{\restriction \omega_{k_0} = \omega^0_{k_0}} - g(I(\lambda,\omega,\theta)^{-1})_{\restriction \omega_{k_0} = \omega^0_{k_0}} \underset{\lambda\to+\infty}{\sim} g_0^0 (h^{(1)}_{k_0})^{\frac{2\alpha}{\alpha-1}} \lambda^{\alpha\frac{\alpha-3}{\alpha-1}}.$$

This estimate is uniform in $\Omega_K^-(k_0)$ (or $\Omega_K^+(k_0)$ as the functions used above only depend on $(\omega_k)_{k\in\Lambda^\alpha\setminus\{k_0\}}$). As $\alpha-1>0$, we have that, for λ large enough, uniformly in $\Omega_K^-(k_0)$,

$$e^{-g(I(\lambda,\omega,\theta)^{-1})} \leq e^{-\psi(\omega)_{\restriction\omega_{k_0}=\omega^0_{k_0}}} \cdot e^{-\frac{\alpha-1}{2}g_0^0(h_{k_0}^{(1)})^{\frac{2\alpha}{\alpha-1}}\lambda^{\alpha\frac{\alpha-3}{\alpha-1}}}. \tag{3.58}$$

So that $I_1^-(\lambda,(\omega_k)_{k\neq k_0})$ is negligible with respect to $I_1^+(\lambda,(\omega_k)_{k\neq k_0})$.

To continue our scheme, we have to estimate the expectation defined in (3.57). This expectation does not contain the random variable ω_{k_0} anymore. But it has the same form as (3.41) except for the fact that $-g(I(\lambda,\omega,\theta)^{-1})$ has to be replaced with $-\psi(\omega)_{\restriction\omega_{k_0}=\omega^0_{k_0}}$. Let us denote this last function by $\psi^0((\omega)_{k\neq k_0})$. We want to reapply the same procedure as above to successively eliminate all the random variables $(\omega_k)_{k\in\Lambda^\alpha}$.
Let us point out the main characteristics of $\psi^0((\omega)_{k\neq k_0})$ that will enable us to continue the induction process i.e. integrate with respect to k_1. By (3.45), (3.47) and (3.48), we have that

$$\begin{aligned}\frac{\partial\psi^0}{\partial\omega_{k_1}} = \left(g_0^0\alpha J^{\alpha-1}+\frac{g_0^1}{J}\right)\left(\frac{\partial S_1^0}{\partial\omega_{k_1}}(1+o(1))+\frac{\partial S_1^0}{\partial\omega_{k_0}}\frac{\partial\omega^0_{k_0}}{\partial\omega_{k_1}}(1+o(1))\right)+\\ +\left(g_0^0\alpha(\omega_{k_0})^{\alpha-1}+(g_0^1+\alpha-1)\frac{1}{\omega_{k_0}}\right)\frac{\partial\omega^0_{k_0}}{\partial\omega_{k_1}}\end{aligned}$$

and, by (3.50), (3.51) and (3.52), we have that

$$\begin{aligned}\omega^0_{k_0} &= O(\lambda^{\frac{\alpha-3}{\alpha-1}})\\ \frac{\partial\omega^0_{k_0}}{\partial\omega_{k_1}} &= \lambda^{\frac{\alpha-3}{\alpha-1}}O\left(\frac{\partial S_1^0}{\partial\omega_{k_1}}\right).\end{aligned}$$

Hence, as $\alpha-1\geq\alpha\frac{\alpha-3}{\alpha-1}$, by (3.38), we have

$$\frac{\partial\psi^0}{\partial\omega_{k_1}} = -g_0^0\alpha J^{\alpha-1}\frac{\partial S_1^0}{\partial\omega_{k_1}}(1+o(1)) = -g_0^0\alpha\lambda^{\alpha-1-2l(k_1)}\left|h_{k_1}^{(l(k_1))}\right|^2(1+o(1))$$

where $o(1)\to 0$ as $\lambda\to+\infty$. So that, if we define $\psi((\omega)_{k\neq k_0})=\psi^0((\omega)_{k\neq k_0})+f_0(\omega_{k_1})$, we may apply the same procedure as above to compute the expectation of $e^{-\psi^0}$ with respect to the random variable ω_{k_1}.
Let us assume the procedure has been applied to all $(k_j)_{1\leq j\leq n-1}$ and that the minima (in the range $[\lambda^\delta,\nu\lambda]$) that have been successively found applying the procedure are denoted by $\omega^0_{k_j}$; our inductive assumption is that, for $1\leq j\leq n-1$,

$$\begin{aligned}\omega^0_{k_j} &= O(\lambda^{\frac{\alpha-1-2l(k_j)}{\alpha-1}})\\ \frac{\partial\omega^0_{k_j}}{\partial\omega_k} &= \lambda^{\frac{\alpha-1-2l(k_j)}{\alpha-1}}O\left(\frac{\partial S^0_{l(k_j)}}{\partial\omega_k}\right).\end{aligned} \tag{3.59}$$

We also assume that these minima admit asymptotic expansions described by (3.50) and (3.51), and that the remark following (3.51) holds for any $1 \leq j \leq n-1$. We set

$$\psi((\omega_{k_j})_{j\geq n}) = \psi^{n-1}((\omega^0_{k_j})_{1\leq j\leq n-1}, (\omega_{k_j})_{j\geq n}) + f_0(\omega_{k_n})$$
$$\psi^{n-1}((\omega^0_{k_j})_{1\leq j\leq n-1}, (\omega_{k_j})_{j\geq n}) = \psi_0((\omega^0_{k_j})_{1\leq j\leq n-1}, (\omega_{k_j})_{j\geq n}) + \sum_{j=0}^{n-1} f_0(\omega^0_{k_j}).$$

Then we compute

$$\frac{\partial\psi^{n-1}}{\partial\omega_{k_n}} = \left(g^0_0\alpha J^{\alpha-1} + \frac{g^1_0}{J}\right)\left(\frac{\partial S^0_{l(k_n)}}{\partial\omega_{k_n}}(1+o(1)) + \sum_{j=0}^{n-1}\frac{\partial S^0_{l(k_j)}}{\partial\omega_{k_j}}\frac{\partial\omega^0_{k_j}}{\partial\omega_{k_n}}(1+o(1))\right) +$$
$$+ \sum_{j=0}^{n-1}\left(g^0_0\alpha(\omega_{k_j})^{\alpha-1} + (g^1_0+\alpha-1)\frac{1}{\omega_{k_j}}\right)\frac{\partial\omega^0_{k_j}}{\partial\omega_{k_n}}$$

Then, as, for $1 \leq j \leq n-1$, we have $\alpha - 1 \geq \alpha\frac{\alpha-1-2l(k_j)}{\alpha-1}$, (3.38) gives us

$$\frac{\partial\psi^{n-1}}{\partial\omega_{k_n}} = -g^0_0\alpha J^{\alpha-1}\frac{\partial S^0_{l(k_n)}}{\partial\omega_{k_n}}(1+o(1)) = -g^0_0\alpha\lambda^{\alpha-1-2l(k_n)}\left|h^{(l(k_n))}_{k_n}\right|^2(1+o(1))$$

where $o(1) \to 0$ as $\lambda \to +\infty$.
We compute then the minimum for ψ to obtain that it is unique non-degenerate and satisfies (3.59) at the n-th order. We also compute the Hessian of ψ at the minimum to get that it is of order $\lambda^{(\alpha-2)\frac{\alpha-1-2l(k_n)}{\alpha-1}}$. So that we can again use Laplace's method to estimate the expectation with respect to ω_{k_n} at least in the interval $[\lambda^\delta, \nu\delta]$. Outside of this interval, we use the same method as above. ψ^{n-1} is a decreasing function of ω_{k_n} and we estimate

$$\psi^{n-1}_{\restriction\omega_{k_n}=\lambda^\delta} - \psi^{n-1}_{\restriction\omega_{k_n}=\omega^0_{k_n}} \underset{\lambda\to+\infty}{\sim} \alpha g^0_0(h^{l(k_n)}_{k_n})^{\frac{\alpha}{\alpha-1}}\lambda^{\alpha\frac{\alpha-2l(k_n)-1}{\alpha-1}}.$$

We also estimate the difference

$$\psi(\omega)_{\restriction\omega_{k_n}=\omega^0_{k_n}} - \psi^{n-1}_{\restriction\omega_{k_n}=\omega^0_{k_n}} \underset{\lambda\to+\infty}{\sim} g^0_0(h^{l(k_n)}_{k_n})^{\frac{\alpha}{\alpha-1}}\lambda^{\alpha\frac{\alpha-2l(k_n)-1}{\alpha-1}}.$$

As above, as $\alpha - 1 > 0$, this proves that the expectation of $e^{-\psi}$ on the set $\{\omega_{k_n} \leq \lambda^\delta\}$ is exponentially small compared to the expectation of $e^{-\psi}$ on the set $[\lambda^\delta, \nu\lambda]$.
To end the proof of Theorem 1.3 at least in the case when α is not an odd integer, we just need to count what powers of λ come into the asymptotics that we have computed. We also notice that, for any $0 \leq l \leq [\alpha - 1/2]$, there exists $k \in \mathbb{N}$ such that $\frac{\alpha-1-2l}{\alpha-1} = \frac{\alpha-1-2[\alpha-1/2]}{\alpha-1} + \frac{2k}{\alpha-1}$.

If $\alpha = 2n+1$, we can use the above procedure for all the sites k such that $l(k) \leq n-1$ i.e. $k \in \Lambda_1 \cup \cdots \cup \Lambda_{n-1}$. To eliminate the random variables $(\omega_k)_{k\in\Lambda_n}$, we will use the same idea as in the case $\alpha = 3$. More precisely, by (3.40), we have

$$
\begin{aligned}
(J(\lambda,\omega))^\alpha &= (J(\lambda,\omega))^{2n+1} \\
&= \left(\frac{\lambda^2}{S_0(\lambda)} \frac{1}{1+\displaystyle\sum_{p=1}^{n-1} \frac{S_p(\lambda,\omega)}{S_0(\lambda)}} \right)^{2n+1} - (2n+1)\left(\frac{\lambda^2}{S_0(\lambda)}\right)^{2n+1} \frac{S_n(\lambda,\omega)}{S_0(\lambda)} \\
&\qquad + R_n(\lambda,\omega) \\
&= (\tilde{J}(\lambda,\omega))^{2n+1} - (2n+1)J^*(\lambda,\omega) + R_n(\lambda,\omega)
\end{aligned}
$$

where R_n is a remainder term that admits an expansion of the form (3.27) and $R_n = O(\frac{1}{\lambda})$ (uniformly in $\omega \in \Omega'_K$).
The term $(\tilde{J}(\lambda,\omega))^{2n+1}$ is dealt with in the way described above. By (3.38), the new term $J^*(\lambda,\omega)$ satisfies

$$
J^*(\lambda,\omega) = \sum_{k\in\Lambda_n} |h_k^n|^2 \frac{\lambda^2}{\lambda-\omega_k} + R_n^*(\lambda,\omega),
$$

where R_n^* is a remainder term that admits an expansion of the form (3.27) and $R_n^* = O(\frac{1}{\lambda})$ (uniformly in $\omega \in \Omega'_K$).
Now, when we take the expectation of e^{-g} in the random variables $(\omega_k)_{k_i n\Lambda_n}$, we may restrict the range where we take the expectation in ω_k to $[\lambda^\delta, \lambda^\beta]$ (as the density is exponentially decaying). We then decompose J^* in the same way as in (3.31) and use the method described in the case $\alpha = 3$ to conclude.

□

Proof of Theorem 1.4. The method used to prove this theorem is the same as the one used to prove the asymptotics (1.11). Hence we will not give all the details. First, because of the precision we ask for our asymptotic of N, we pick $\varepsilon = \lambda^{-2}$ and $n = e^{2(1+\nu)\lambda}$ in Lemma 2.1 (ν chosen as in Lemma 2.1). We fix $K = \lambda^\delta$ (where $0 < \delta$ will be chosen later on). Hence

$$
n^{2d} F(\lambda - \hat{H}_+) F(\lambda - K - \hat{H}_+) = o\left(e^{-e^{g_0\lambda}}\right).
$$

We recall that by (3.8), (3.10) and (3.11), uniformly in Ω'_K and $\mathbb{T}$, we have

$$
\frac{1}{I(\lambda,\omega,\theta)} = \frac{\lambda^2}{\lambda + h_0 + \sum_k \frac{|h_k|^2}{\lambda-\omega_k}} + \delta I(\lambda,\omega,\theta),
$$

where δI is analytic in θ and ω, and, for some $C > 0$, it satisfies a uniform estimate

$$
|\delta I(\lambda,\omega,\theta)| \le C\lambda^{-2\delta}.
$$

Hence, in Ω'_K and $\mathbb{T}$, we have

$$g((I(\lambda,\omega,\theta))^{-1}) = e^{\frac{g_0\lambda^{2\alpha}}{\left(\lambda+h_0+\sum_k \frac{|h_k|^2}{\lambda-\omega_k}\right)^\alpha}} + \delta g(\lambda,\omega,\theta), \tag{3.60}$$

where δg satisfies a uniform estimate

$$|\delta g(\lambda,\omega,\theta)| \leq C\left(\lambda^{\alpha-1-2\delta}+\lambda^{2(\alpha-1)-\delta}\right).e^{g_0\lambda^\alpha}.$$

So that

$$\ln \mathbb{E}(e^{-g((I(\lambda,\omega,\theta))^{-1})}\mathbf{1}_{\Omega'_K}) = \ln E(\lambda) + O\left(\left(\lambda^{\alpha-1-2\delta}+\lambda^{2(\alpha-1)-\delta}\right)e^{g_0\lambda^\alpha}\right).$$

where

$$E(\lambda) = \mathbb{E}\left(\exp\left(-e^{\frac{g_0\lambda^{2\alpha}}{\left(\lambda+h_0+\sum_k \frac{|h_k|^2}{\lambda-\omega_k}\right)^\alpha}}\right)\mathbf{1}_{\Omega'_K}\right).$$

To get the asymptotic behavior of E, as above, we split the expectation into two parts:

1. E_1 where one takes the expectation over $\omega_k \leq \lambda^{\delta'}$ (where $0<\delta'$ is chosen small enough)
2. E_2 where one takes the expectation over $\lambda^{\delta'} \leq \omega_k \leq \lambda - \lambda^\delta$ (where $0<\delta'$ is chosen small enough).

E_2 may be rewritten as

$$E_2(\lambda) = \int\cdots\int_{[\lambda^{\delta'},\lambda-\lambda^\delta]^{\Lambda_1}} e^{-\psi(\omega)}\prod_{k\in\Lambda_1} d\omega_k$$

where

$$\psi(\omega) = e^{\frac{g_0\lambda^{2\alpha}}{\left(\lambda+h_0+\sum_k \frac{|h_k|^2}{\lambda-\omega_k}\right)^\alpha}} + \sum_{k\in\Lambda_0} e^{g_0\omega_k^\alpha}.$$

Then ψ admits a single minimum in $[\lambda^{\delta'},\lambda-\lambda^\delta]^{\Lambda_1}$; let us denote it by ω^0. We have the following asymptotics

$$\omega_k^0 = \lambda - \frac{2\lambda^{1-\alpha}}{\alpha g_0}\ln\left(\frac{\lambda^{1-\alpha}}{\alpha g_0|h_k|}\right)(1+o(1)).$$

Notice that this tells us that δ has to be chosen such that $(1-\alpha)/2\delta < 1-\alpha$.
The minimum ω^0 is non-degenerate as we have the following asymptotic for the Hessian of ψ

$$\frac{\partial^2\psi}{\partial\omega_k^2}(\omega^0) = (\alpha g_0)^2\lambda^{2(\alpha-1)}e^{g_0\lambda^\alpha}(1+o(1))$$
$$\frac{\partial^2\psi}{\partial\omega_k\omega_j}(\omega^0) = o\left(\lambda^{2(\alpha-1)}e^{g_0\lambda^\alpha}\right)$$

where $(k, j) \in \Lambda_1^2$ and $k \neq j$.
We can then use the Laplace method to get the following asymptotics for E

$$\begin{aligned}\ln E(\lambda) &= -\psi(\omega^0) + o\left(\lambda^{2(\alpha-1)} e^{g_0 \lambda^\alpha}\right) \\ &= -e^{g_0 \lambda^\alpha}\left(1 - \alpha h_0 g_0 \lambda^{\alpha-1} + (\alpha g_0)^2 \left(\sum_{k \neq 0} |h_k|^2\right) \lambda^{2(\alpha-1)} + o\left(\lambda^{2(\alpha-1)}\right)\right).\end{aligned}$$

By what we have said above, $\ln E$ and $\ln N$ satisfy the same asymptotics at $+\infty$. This ends the proof of Theorem 1.4.

□

4. The quantum regime

In this case, we recall that we have defined $h(\lambda) = \ln g(\lambda)$. We have assumed that h is increasing, twice differentiable and tends to $+\infty$ at $+\infty$. In terms of h, assumption (1.14) becomes

$$h'(\lambda) \underset{\lambda \to +\infty}{\rightarrow} +\infty. \tag{4.1}$$

We also recall that $\hat{H}$ is assumed to have a single maximum and that this maximum is non degenerate. Without restricting the generality of our discussion, we may assume that this maximum is attained at the point $\theta = 0$. We may also assume that $\hat{H}(0) = 0$. If this is not the case, we just have to shift H and λ by $-\hat{H}(0)$. Hence, from now on, we assume that $\hat{H} \leq 0$ and that, near 0, we have

$$-C|\theta|^2 \leq \hat{H}(\theta) \leq -\frac{1}{C}|\theta|^2 \tag{4.2}$$

where $C > 1$.

To prove (1.15), we will use Lemma 2.1. Hence we need to analyze $\mathbb{E}(N^n_\omega(\lambda))$. This is done in the

Lemma 4.1. *For any $\eta \in (0, \frac{1}{4d})$, there exists $\lambda_\eta > 0$ and $C_\eta > 0$ such that, for $n \in \mathbb{N}^*$ and $\delta > 0$ such that $n > \delta^{-d/2}$, and for $\lambda \geq \lambda_\eta$, we have*

$$F(\lambda + C_\eta \delta)^{\delta^{-d/2-\eta}} \leq \mathbb{E}(N^n_\omega(\lambda)). \tag{4.3}$$

For any $\eta \in (0, \frac{1}{4d})$, there exists $\lambda_\eta > 0$ and $\delta_\eta \in (0,1)$ such that, for $\delta \in (0, \delta_\eta)$, for $m > \delta^{-2d}$ ($m \in \mathbb{N}^*$) and for $\lambda \geq \lambda_\eta$, if we define

$$(2n+1) = (2m+1)[\delta^{-\eta}]_o [\delta^{-1/2+2\eta}]_o$$

(where $[\cdot]_o$ denotes the largest odd integer smaller than $\cdot$)
then, we have

$$\begin{aligned}\mathbb{E}(N^n_\omega(\lambda)) \leq (3n)^{d+1} F(\lambda - \delta)^{3^{d-2}\delta^{-d/2(1+E(\lambda,\delta,\eta))}} \cdot \left(1 + \frac{F(\lambda + \delta^{1-\eta/2})}{2F(\lambda-\delta)^4}\right)^{3^{d-2}\delta^{-d/2}} \\ + 2(3n)^{d+1} F(\lambda + \delta^{1-\eta/2}).\end{aligned} \tag{4.4}$$

where, uniformly in λ, δ and η, $E(\lambda, \delta, \eta)$ satisfies

$$E(\lambda, \delta, \eta) = O\left(\frac{\ln \delta}{\ln F(\lambda - \delta)} + \delta^{\eta/2} + \frac{F(\lambda + \delta^{1-\eta/2})}{F(\lambda - \delta)} + \eta\right).$$

Lemma 4.1 will be proved later. We now finish the proof of Theorem 1.5. To do this we need to:

- pick the precision $\varepsilon(\lambda)$,
- pick the scale $n(\lambda)$,
- pick the correction $\delta(\lambda)$

such that

- the $\ln \ln$ of the upper and lower bounds given by Lemma 4.1 are asymptotically equivalent as $\lambda \to +\infty$,
- the error estimate $\delta(\nu, \varepsilon, n)$ given by Lemma 2.1 becomes much smaller than the lower bound given by Lemma 4.1.

We notice that, by assumption (1.14) and (2) of section 4, we know that h' is increasing and $h''/(h')^2 \to 0$ at $+\infty$. Integrating this last relation, we get, for λ large enough and $\delta > 0$,

$$\int_{\lambda}^{\lambda+\delta} \frac{h''(t)}{(h'(t))^2} dt = \frac{1}{h'(\lambda)} - \frac{1}{h'(\lambda + \delta)} = \delta o_\lambda(1), \quad o_\lambda(1) \underset{\lambda \to +\infty}{\to} 0, \tag{4.5}$$

(uniformly in δ).
This gives

$$\frac{1}{h'(\lambda)} h'\left(\lambda + \frac{1}{h'(\lambda)}\right) = 1 + o_\lambda(1). \tag{4.6}$$

We set

$$\delta(\lambda) = \frac{1}{h'(\lambda) \ln h'(\lambda)}$$

As $h' \to +\infty$, $\delta(\lambda) \to 0$ as $\lambda \to +\infty$. By (4.5), for any $C > 0$, we have

$$\frac{1}{\delta(\lambda) h'(\lambda)} - o_\lambda(1) = \frac{1}{\delta(\lambda) h'(\lambda + C\delta(\lambda))}$$

so that

$$\delta(\lambda) h'(\lambda + C\delta(\lambda)) \underset{\lambda \to +\infty}{\to} 0.$$

As h' is increasing, this implies that, for any $C > 0$,

$$0 \le h(\lambda + C\delta(\lambda)) - h(\lambda - C\delta(\lambda)) \underset{\lambda \to +\infty}{\to} 0. \tag{4.7}$$

Let $\eta \in (0,1)$. By (4.5) and (4.6), we have

$$\begin{aligned} h(\lambda+\delta(\lambda)^{1-\eta})-h(\lambda) &= \sum_{k=0}^{N} h\left(\lambda+\delta(\lambda)^{1-\eta}-k\delta(\lambda)\right)-h\left(\lambda+\delta(\lambda)^{1-\eta}-(k+1)\delta(\lambda)\right) \\ &\qquad + h\left(\lambda+\delta(\lambda)^{1-\eta}-(N+1)\delta(\lambda)\right)-h(\lambda) \\ &\geq \sum_{k=0}^{N} \delta(\lambda)h'\left(\lambda+\delta(\lambda)^{1-\eta}-(k+1)\delta(\lambda)\right) \\ &\geq \sum_{k=0}^{N} \delta(\lambda)h'\left(\lambda+\delta(\lambda)\right) \geq \frac{N}{2} \end{aligned}$$

for $N \leq \delta(\lambda)^{\eta}/2$ and λ large enough.
This proves that, for any $\eta \in (0,1)$,

$$h(\lambda+\delta(\lambda)^{1-\eta})-h(\lambda) \underset{\lambda\to+\infty}{\to} +\infty. \tag{4.8}$$

This implies

$$\frac{F(\lambda+\delta(\lambda)^{1-\eta})}{F(\lambda-\delta(\lambda))} \underset{\lambda\to+\infty}{\to} 0. \tag{4.9}$$

Moreover, by (4.7), we know that

$$\frac{\ln\delta(\lambda)}{\ln F(\lambda-\delta(\lambda))} = -e^{-h(\lambda)}\ln h'(\lambda)(1+o(1)).$$

By assumption (2) of section 4, for any $\alpha > 0$, $(h'e^{-\alpha h})' = -\alpha(h')^2e^{-h}(1+o(1))$. So $h'e^{-\alpha h}$ is decreasing and positive. Hence

$$e^{-\alpha h(\lambda)}h'(\lambda) \underset{\lambda\to+\infty}{\to} 0. \tag{4.10}$$

So that

$$\frac{\ln\delta(\lambda)}{\ln F(\lambda-\delta(\lambda))} \underset{\lambda\to+\infty}{\to} 0. \tag{4.11}$$

We set

$$\varepsilon(\lambda) = \delta(\lambda),$$
$$n(\lambda) = C_\nu\varepsilon(\lambda)e^{(d+1)\nu h(\lambda)}.$$

(where ν and C_ν are fixed in Lemma 2.1).
Then, (4.10) shows that, for any $p > 0$,

$$\delta(\lambda)^p n(\lambda) \underset{\lambda\to+\infty}{\to} +\infty.$$

Hence, for lambda large enough, $n(\lambda)$ and $\delta(\lambda)$ satisfy

$$n(\lambda) > \delta(\lambda)^{-2d}.$$

We compute

$$\delta(\nu,\varepsilon,n)\le e^{-e^{(d+1)h(\lambda)}}.$$

Let us compare this estimate to the one we have for the lower bound in Theorem 4.1: for $C>0$ and $\eta\in(0,d/2)$ fixed, by (4.7), we have

$$\begin{aligned}\ln\left(\frac{F(\lambda+C\delta(\lambda))^{\delta(\lambda)^{-d/2-\eta}}}{\delta(\nu,\varepsilon,n)}\right)&=e^{h(\lambda)}\left(e^{dh(\lambda)}-\delta(\lambda)^{-d/2-\eta}\right)(1+o(1)\\&\ge 2e^{(d+1)h(\lambda)}\left(1-(\delta(\lambda)e^{h(\lambda)})^{-d}\right).\end{aligned}$$

Hence

$$\delta(\nu,\varepsilon,n)=o\left(F(\lambda+C\delta(\lambda))^{\delta(\lambda)^{-d/2-\eta}}\right).\tag{4.12}$$

At last, for $\lambda\to+\infty$, we have

$$\frac{\ln n(\lambda)}{\ln F(\lambda)}=(d+1)\nu h(\lambda)e^{-h(\lambda)}(1+o(1))\to 0.$$

Now, by (4.3), (4.7), Lemma 2.1 and (4.12), we have

$$\ln|\ln N(\lambda)|\ge h(\lambda)+\frac{d}{2}\ln h'(\lambda)(1+o(1)).$$

By (4.4), (4.7), (4.9), (4.11), Lemma 2.1 and (4.12), we have

$$\ln|\ln N(\lambda)|\le h(\lambda)+\frac{d}{2}\ln h'(\lambda)(1+o(1)).$$

As $\ln|\ln F|=h$ and $h'=F'/(F\ln F)$, we get Theorem 1.5.

4.1. **The proof of Lemma 4.1.** To prove (1.15), we will prove a lower and an upper bound on $\ln|\ln N(\lambda)|$. Many of the ideas used to prove the lower and upper bound on N are similar to the ones developed in ([8]). Therefore we will not give all the details and refer to ([8]) for additional material.

We now want to analyze $\mathbb{E}(N^n_\omega(\lambda))$. We will use the Floquet theory described in subsection 2.1 to do this. In the sequel, it will sometimes be convenient to give vectors by their components in the basis of eigenvectors of $M^n_0(\theta)$ that we called $(u_k)_{k\in\mathbb{Z}^d_{2n+1}}$. So that, if we denote the vectors of the canonical basis by $(\delta_l)_{l\in\mathbb{Z}^d_{2n+1}}$, their components in the basis $(u_k)_{k\in\mathbb{Z}^d_{2n+1}}$ are

$$\delta_l=\frac{1}{(2n+1)^{d/2}}(e^{i(\theta+\frac{2\pi k}{2n+1})l})_{k\in\mathbb{Z}^d_{2n+1}}.$$

We also define the vectors $(v_l)_{l\in\mathbb{Z}^d_{2n+1}}$ by

$$v_l=e^{-il\theta}\delta_l=\frac{1}{(2n+1)^{d/2}}(e^{i\frac{2\pi kl}{2n+1}})_{k\in\mathbb{Z}^d_{2n+1}}.$$

Remark 4.1. The canonical basis plays here a special role as it is the basis of eigenvectors for V^n_ω.

4.1.1. *The lower bound.* By (2.5), we just need to prove the right lower bound for the probability that $M^n_\omega(\theta)$ has an eigenvalue larger than λ. We will do this by explicitly constructing such an eigenvalue for a sufficiently large set of ω's. Therefore we start with fixing θ.

Let $a \in \ell^2(\mathbb{Z}^d_{2n+1})$ be expressed by its coordinates in the basis $(\delta_l)_{l\in\mathbb{Z}^d_{2n+1}}$ i.e $a = \sum_{l\in\mathbb{Z}_{2n+1}} a_l\delta_l$. Then

$$\langle M^n_\omega(\theta)a, a\rangle = \sum_{k\in\mathbb{Z}^d_{2n+1}} \hat{H}\left(\theta + \frac{2k\pi}{2n+1}\right)|A_k|^2 + \sum_{l\in\mathbb{Z}^d_{2n+1}} \omega_l|a_l|^2 \tag{4.13}$$

where

$$A_k = \frac{1}{(2n+1)^d}\sum_{l\in\mathbb{Z}^d_{2n+1}} a_l e^{-i\theta l}e^{-2i\pi kl/(2n+1)}.$$

Both bases being orthonormal, we have $\sum_{k\in\mathbb{Z}^d_{2n+1}}|A_k|^2 = \|a\|^2$.
Let $b \in \mathcal{C}^\infty_0((-1/4, 1/4)^d) \subset L^2([-1/2, 1/2])$, b non negative such that $\|b\|_{L^2([-1/2,1/2])} = 1$. Pick $\eta \in (0, 1/2)$ small. Set

$$a^\delta_l = e^{il\theta}\delta^{d/2+d\eta}b\left(\delta^{1/2+\eta}l\right).$$

Let a^δ be the vector $\sum_{l\in\mathbb{Z}_{2n+1}} a^\delta_l\delta_l$. Notice that this vector has its support in the set $\{k \in \mathbb{Z}^d_{2n+1};\ |k| \le \delta^{-1/2-\eta}\}$. Using the smoothness of b and the fact that b is normalized, one obtains

$$\|a^\delta\|^2 = \sum_{l\in\mathbb{Z}^d_{2n+1}} |a^\delta_l|^2 = 1 + O(\delta^{1/2+\eta}).$$

As in [8], one can use discrete integration by parts to estimate A^δ_k; doing this in the variable k_j $(1 \le j \le d)$, one gets

$$|A^\delta_k|^2 \le \frac{\delta^{q+2q\eta}}{(2n+1)^{d+2q}}\frac{1}{\left|e^{-2i\pi k_j/(2n+1)} - 1\right|^{2q}}\left\|\frac{\partial^q b}{\partial x^q_j}\right\|^2.$$

Hence, if $|k_j| \ge \delta^{1/2}(2n+1)$, we have

$$|A^\delta_k|^2 \le C\frac{\delta^{2q\eta-q}}{(2n+1)^{d+2q}} \le C\frac{\delta^{2q\eta}}{(2n+1)^d}.$$

as $n \ge \delta^{-d/2}$. So that

$$-C\delta^{2q\eta} \le \sum_{\substack{k\in\mathbb{Z}^d_{2n+1}\\ |k|\ge\delta^{1/2}}} \hat{H}\left(\theta + \frac{2k\pi}{2n+1}\right)|A^\delta_k|^2 \le 0$$

and

$$-C\delta \le \sum_{\substack{k\in\mathbb{Z}^d_{2n+1}\\ |k|\le\delta^{1/2}}} \hat{H}\left(\theta + \frac{2k\pi}{2n+1}\right)|A^\delta_k|^2 \le 0$$

as, by (4.2), we know that, for $|k|\leq\delta^{1/2}$, $\hat H(\theta+\frac{2k\pi}{2n+1})\geq -C\delta$.
Compiling all this, we see that, there exists $C>0$ such that, if, for $|l|\leq\delta^{-1/2-\eta}$, we have $\omega_l\geq\lambda+C\delta$, then $\langle M^n_\omega(\theta)a^\delta,a^\delta\rangle\geq\lambda$. This tells us that, for any $0<\eta<d/2$, there exists $C_\eta>0$ such that

$$\mathbb{E}(N^n_\omega(\lambda))\geq F(\lambda+C_\eta\delta)^{C\delta^{-d/2-\eta}}.$$

This proves (4.3).

$\square$

4.1.2. *The upper bound.* We want to estimate $\mathbb{E}(N^n_\omega(\lambda))$ from above. By (2.5), we have

$$\mathbb{E}(N^n_\omega(\lambda))\leq\sup_\theta P(\{M^n_\omega(\theta)\text{ has an eigenvalue larger than }\lambda\}).$$

So we need to upper bound the probability that there exists $a=(a_k)_{k\in\mathbb{Z}^d_{2n+1}}$ (the coordinates are given in the basis $(u_k)_{k\in\mathbb{Z}^d_{2n+1}}$) normalized such that $\langle a,M^n_\omega(\theta)a\rangle\geq\lambda$, and we need to do this uniformly in θ. Pick $0<\eta'<\eta<\frac{1}{2d}$ and write

$$\begin{aligned}\{(\omega_k)_{k\in\mathbb{Z}^d_{2n+1}}\}&=\{(\omega_k)_{k\in\mathbb{Z}^d_{2n+1}};\ \forall k\in\mathbb{Z}^d_{2n+1},\ \omega_k<\lambda+\delta^{1-\eta'}\}\\&\qquad\cup\{(\omega_k)_{k\in\mathbb{Z}^d_{2n+1}};\ \exists k\in\mathbb{Z}^d_{2n+1},\ \omega_k\geq\lambda+\delta^{1-\eta'}\}\\&=\Omega_-\cup\Omega_+.\end{aligned}$$

Then, one has

$$P(\Omega_+)\leq(3n)^dF(\lambda+\delta^{1-\eta'}).$$

We define

$$\Omega_-(\delta,n,\lambda)=\{(\omega_k)_{k\in\mathbb{Z}^d_{2n+1}}\in\Omega_-;\ M^n_\omega(\theta)\text{ has an eigenvalue larger than }\lambda\},$$

and the following auxiliary random variables: for $l\in\mathbb{Z}^d$, let

$$\tilde\omega_l=\begin{cases}-\delta & \text{if }\omega_l\leq\lambda-\delta\\ \omega_l-\lambda & \text{if }\lambda-\delta\leq\omega_l\leq\lambda+\delta^{1-\eta'}\\ \delta^{1-\eta'} & \text{if }\omega_l\geq\lambda+\delta^{1-\eta'}.\end{cases}$$

Then, for $\omega\in\Omega_-(\delta,n,\lambda)$, we have $\omega_l-\lambda\leq\tilde\omega_l$ so that, if $\langle a,M^n_\omega(\theta)a\rangle\geq\lambda$ then

$$\langle a,M^n_0(\theta)a\rangle+\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde\omega_l|A_l|^2\geq 0\tag{4.14}$$

where

$$A_l=\frac{1}{(2n+1)^d}\sum_{k\in\mathbb{Z}^d_{2n+1}}a_ke^{-2i\pi kl/(2n+1)}.$$

We used the fact that the map $a\mapsto A=(A_l)_{l\in\mathbb{Z}^d_{2n+1}}$ is the discrete Fourier transform; hence $\sum_{l\in\mathbb{Z}^d_{2n+1}}|A_l|^2=\|a\|^2=1$. As a conclusion, we see that we only need to upper bound the probability of the event

$$\tilde{\Omega}_-(\delta,n,\lambda)=\left\{(\omega_k)_{k\in\mathbb{Z}^d_{2n+1}}\in\Omega_-;\ \exists a\in\ell(\mathbb{Z}^d_{2n+1});\ \langle M_0^n(\theta)a,a\rangle+\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A_l|^2\geq 0\right\}.$$

Therefore, we define

- $2L'_\delta+1=[\delta^{-1/2+2\eta}]_o$,
- $2K'_\delta+1=(2m+1)[\delta^{-\eta}]_o$,

Then, by Lemma 4.1, we defined $2n+1=(2K'_\delta+1)(2L'_\delta+1)$. We have the

Lemma 4.2. *For any $0<\eta'<\eta<\frac{1}{4d}$, there exists $\delta_0>0$ such that, for $0<\delta<\delta_0$ and $m\geq\delta^{-d}$, we have, for $|\theta|\leq\frac{\pi}{2n+1}$,*

$$\tilde{\Omega}_-(\delta,n,\lambda)\subset\bigcup_{|k'|\leq K'_\delta}\tilde{\Omega}^k_-(\delta,n,\lambda)\bigcup\Delta\tilde{\Omega}_-(\delta,n,\lambda)\tag{4.15}$$

where, for $|k'|\leq K'_\delta$, we define the event

$$\Omega^k_-(\delta,n,\lambda)=\left\{\omega;\ \frac{1}{(2L'_\delta+1)^d}\sum_{|l'|\leq L'_\delta}\left(\tilde{\omega}_{k'(2L'_\delta+1)+l'}+\frac{3}{4}\delta\right)\geq 0\right\}.$$

and we have the estimate

$$P(\Delta\tilde{\Omega}_-(\delta,n,\lambda))\leq(3n)^dP(\omega_0\geq\lambda+\delta^{1-\eta}).\tag{4.16}$$

Before proving Lemma 4.2, let us indicate how to derive the upper bound (4.4) from it. We want to estimate the probability of $\Omega^k_-(\delta,n,\lambda)$. Therefore, we compute the expectation of $\tilde{\omega}_0+3\delta/4$ to get that $\mathbb{E}(\tilde{\omega}_0)+3\delta/4\to-\delta/4$ as $\lambda\to+\infty$: we see that, in the limit $\lambda\to+\infty$, when we estimate $P(\Omega^k_-(\delta,n,\lambda))$, we are in a large deviation regime. Using Cràmer's ideas ([3]), we have

$$P(\Omega^k_-(\delta,n,\lambda))\leq\exp\left((2L'_\delta+1)^d\ln\mathbb{E}(e^{t(\tilde{\omega}_0+3\delta/4)})\right)\tag{4.17}$$

for any $t\geq 0$. Hence, we only have to estimate $(2L'_\delta+1)^d\ln\mathbb{E}(e^{t(\tilde{\omega}_0+3\delta/4)})$ and minimize the thus obtained estimate in t over $t\geq 0$. We compute

$$\begin{aligned}\mathbb{E}(e^{t(\tilde{\omega}_0+3\delta/4)})&=e^{-t\delta/4}P(\omega_0\leq\lambda-\delta)+\mathbb{E}(e^{t(\omega_0-\lambda+3\delta/4)}\mathbf{1}_{\lambda-\delta\leq\omega_0\leq\lambda+\delta^{1-\eta'}})+\\&\qquad\qquad e^{t\delta^{1-\eta'}}P(\omega_0\geq\lambda+\delta^{1-\eta'})\\&\leq e^{-t\delta/4}+e^{t\delta^{1-\eta'}}P(\omega_0\geq\lambda+\delta^{1-\eta'})+(e^{t\delta^{1-\eta'}+3t\delta/4}-1)P(\omega_0\geq\lambda-\delta)\\&=f(t,\lambda,\delta).\end{aligned}$$

Then $\partial_tf(t,\lambda,\delta)=0$ gives

$$\delta^{\eta'}e^{-t\delta^{1-\eta'}(1+\delta^{\eta'})}=4P(\omega_0\geq\lambda-\delta)\left(1+\frac{3}{4}\delta^{\eta'}+e^{-t3\delta/4}\frac{P(\omega_0\geq\lambda+\delta^{1-\eta'})}{P(\omega_0\geq\lambda-\delta)}\right).$$

A short computation gives that the minimum of $f(t,\lambda,\delta)$ is attained in a point t_0 given by

$$t_0 = -\frac{\delta^{\eta'-1}}{1+\delta^{\eta'}} \ln P(\omega_0 \geq \lambda - \delta)(1 + E(\lambda, \delta, \eta'))$$

where uniformly in λ large enough and δ and η small enough, we have

$$E(\lambda,\delta,\eta') = O\left(\frac{\ln \delta}{\ln P(\omega_0 \geq \lambda - \delta)} + \delta^{\eta'} + \frac{P(\omega_0 \geq \lambda + \delta^{1-\eta'})}{P(\omega_0 \geq \lambda - \delta)}\right).$$

Then we compute

$$\mathbb{E}(e^{t_0(\tilde{\omega}_0 + 3\delta/4)}) \leq 2P(\omega_0 \geq \lambda - \delta)^{\frac{\delta^{\eta'}}{4+4\delta^{\eta'}}(1+E(\lambda,\delta,\eta'))}\left(1 + \frac{P(\omega_0 \geq \lambda + \delta^{1-\eta'})}{2P(\omega_0 \geq \lambda - \delta)^{\frac{1}{1+\delta^{\eta'}}(1+E(\lambda,\delta,\eta))}}\right).$$

Using the fact that $P(\omega_0 \geq \lambda)$ is decreasing in λ, for δ small enough (depending on η'), we get

$$\mathbb{E}(e^{t_0(\tilde{\omega}_0 + 3\delta/4)}) \leq 2P(\omega_0 \geq \lambda - \delta)^{\frac{\delta^{\eta'}}{8}(1+E(\lambda,\delta,\eta'))}\left(1 + \frac{P(\omega_0 \geq \lambda + \delta^{1-\eta'})}{2P(\omega_0 \geq \lambda - \delta)^4}\right).$$

Plugging this into (4.17) and picking $\eta' = \eta/2$, we get that

$$P(\Omega_-^k(\delta,n,\lambda)) \leq 2P(\omega_0 \geq \lambda - \delta)^{3^{d-2}\delta^{-d/2(1+E(\lambda,\delta,\eta))}} \cdot \left(1 + \frac{P(\omega_0 \geq \lambda + \delta^{1-\eta/2})}{2P(\omega_0 \geq \lambda - \delta)^4}\right)^{3^{d-2}\delta^{-d/2}}$$

where, uniformly in λ large enough and δ and η small enough, we have

$$E(\lambda,\delta,\eta) = O\left(\frac{\ln \delta}{\ln P(\omega_0 \geq \lambda - \delta)} + \delta^{\eta/2} + \frac{P(\omega_0 \geq \lambda + \delta^{1-\eta/2})}{P(\omega_0 \geq \lambda - \delta)} + \eta\right).$$

This ends the proof of Lemma 4.1

$\square$

Proof of Lemma 4.2. Let $a \in \ell(\mathbb{Z}^d_{2n+1})$ normalized such that (4.14) holds. We define the vectors a^1 and a^2 by

$$a^1 = \sum_{|k| \leq \frac{1}{4}\delta^{1/2-\eta}n} a_k u_k, \quad a^2 = \sum_{|k| > \frac{1}{4}\delta^{1/2-\eta}n} a_k u_k.$$

So that $a = a^1 + a^2$, $a_1 \perp a^2$, and

$$\langle M_0^n(\theta)a, a\rangle = \langle M_0^n(\theta)a^1, a^1\rangle + \langle M_0^n(\theta)a^2, a^2\rangle. \tag{4.18}$$

We notice that, by (4.2),

$$\langle M_0^n(\theta)a^2, a^2\rangle \leq -C\delta^{1-2\eta}\|a^2\|^2. \tag{4.19}$$

On the other hand, as $A^1+A^2=A$ (here, as above, A^1 (resp. A^2) denotes the discrete Fourier transform of a^1 (resp. a^2)), we have

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A_l|^2 &= \sum_{\substack{l\in\mathbb{Z}^d_{2n+1}\\ \tilde{\omega}_l\geq 0}}\tilde{\omega}_l|A^1_l+A^2_l|^2+\sum_{\substack{l\in\mathbb{Z}^d_{2n+1}\\ \tilde{\omega}_l\leq 0}}\tilde{\omega}_l|A^1_l+A^2_l|^2\\ &\leq 2\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l(|A^1_l|^2+|A^2_l|^2)+\sum_{\substack{l\in\mathbb{Z}^d_{2n+1}\\ \tilde{\omega}_l\leq 0}}-\tilde{\omega}_l(2(|A^1_l|^2+|A^2_l|^2)-|A^1_l+A^2_l|^2).\end{aligned}$$

as $|A^1_l+A^2_l|^2\leq 2(|A^1_l|^2+|A^2_l|^2)$. Hence, as $-\tilde{\omega}_l\leq\delta$, we have

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A_l|^2 &\leq 2\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l(|A^1_l|^2+|A^2_l|^2)+\delta\sum_{l\in\mathbb{Z}^d_{2n+1}}(2(|A^1_l|^2+|A^2_l|^2)-|A^1_l+A^2_l|^2)\\ &= 2\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l(|A^1_l|^2+|A^2_l|^2)+\delta\|a^1\|^2+\delta\|a^2\|^2.\end{aligned}$$

So that (4.14) and (4.18) give

$$\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A^1_l|^2+\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A^2_l|^2\geq -\frac{1}{2}\langle(M^n_0(\theta)+\delta)a^1,a^1\rangle-\frac{1}{2}\langle(M^n_0(\theta)+\delta)a^2,a^2\rangle.$$

Then, either of two things happen:

1. either

$$\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A^1_l|^2\geq -\frac{1}{2}\langle(M^n_0(\theta)+\delta)a^1,a^1\rangle,$$

2. or

$$\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A^2_l|^2\geq -\frac{1}{2}\langle(M^n_0(\theta)+\delta)a^2,a^2\rangle.$$

In case 2, we know that that there exists some $l\in\mathbb{Z}^d_{2n+1}$ such that $\tilde{\omega}_l\geq(C\delta^{1-2\eta}-\delta)/2$. Indeed, if this were not the case then, by (4.19), we would have

$$(C\delta^{1-2\eta}+\delta)/2\|a^2\|^2\leq\sum_{l\in\mathbb{Z}^d_{2n+1}}\tilde{\omega}_l|A^2_l|^2<(C\delta^{1-2\eta}-\delta)/2\|a^2\|^2$$

which is absurd. So that the probability that we are in case 2 is upper bounded by $(3n)^dP(\omega_0\geq\lambda+(C\delta^{1-2\eta}-\delta)/2)$.
If we now define $\Delta\tilde{\Omega}_-(\delta,n,\lambda)$ as the event of all the ω for which we are in case 2, then the above analysis immediately gives us (4.16).

We now turn to case 1. In this case, as $M^n_0(\theta)\leq 0$, we have

$$\sum_{l\in\mathbb{Z}^d_{2n+1}}(\tilde{\omega}_l+\delta/2)|A^1_l|^2\geq 0. \tag{4.20}$$

A^1 is the discrete Fourier transform of a^1, and a^1 has its support in $\{|k| \leq \delta^{1/2-\eta'} n\}$. We define

- $2L_\delta + 1 = [\delta^{-1/2+2\eta}]_o[\delta^{-\eta}]_o$,
- $2K_\delta + 1 = (2m+1)$.

By an easy extension of Lemma 3.1 of [8], we know that there exists $\tilde{a}^1 \in \ell^2(\mathbb{Z}^d_{2n+1})$ such that

1. $\|a^1 - \tilde{a}^1\|_{\ell^2(\mathbb{Z}^d_{2n+1})} \leq C\delta^\eta \|a^1\|_{\ell^2(\mathbb{Z}^d_{2n+1})}$,
2. for $l' \in \mathbb{Z}^d_{2L'_\delta+1}$ and $k' \in \mathbb{Z}^d_{2K'_\delta+1}$, we have
$$\tilde{A}^1_{l'+k'(2L'_\delta+1)} = \tilde{A}^1_{k'(2L'_\delta+1)}.$$

Remark 4.2. This lemma roughly says that if a is a vector in $\ell^2(\mathbb{Z}^d_{2N+1})$ with coefficients concentrated near 0 in a region of width $K < N$, then up to a small error in ℓ^2-norm, a can be considered to have Fourier coefficients that are constant over cubes of side length N/K.

Then, as $\tilde{\omega}_l \leq \delta^{1-\eta'}$,

$$\begin{aligned}
\left| \sum_{l \in \mathbb{Z}^d_{2n+1}} (\tilde{\omega}_l + \delta/2)|\tilde{A}^1_l|^2 - |A^1_l|^2 \right| &\leq \sum_{l \in \mathbb{Z}^d_{2n+1}} |\tilde{\omega}_l + \delta/2| \left(2|A^1_l - \tilde{A}^1_l||A^1_l| + |A^1_l - \tilde{A}^1_l|^2 \right) \\
&\leq (\delta^{1-\eta'} + \delta/2)(2\|a^1 - \tilde{a}^1\|\|a^1\| + \|a^1 - \tilde{a}^1\|^2) \\
&\leq 8\delta^{1-\eta'+\eta}\|\tilde{a}^1\|^2.
\end{aligned}$$

Hence, as $\eta' < \eta$ and as $\sum_{l \in \mathbb{Z}^d_{2n+1}} |\tilde{A}^1_l|^2 = \|\tilde{a}^1\|^2$, for δ small enough, (4.20) gives

$$\sum_{l \in \mathbb{Z}^d_{2n+1}} (\tilde{\omega}_l + 3\delta/4)|\tilde{A}^1_l|^2 \geq 0$$

But by 2.,

$$\sum_{l \in \mathbb{Z}^d_{2n+1}} (\tilde{\omega}_l + 3\delta/4)|\tilde{A}^1_l|^2 = \sum_{k' \in \mathbb{Z}^d_{2K'_\delta+1}} B_{k'}(2L'_\delta + 1)^d |\tilde{A}^1_{k'(2L'_\delta+1)}|^2.$$

where

$$B_{k'} = \frac{1}{(2L'_\delta+1)^d} \left(\sum_{l' \in \mathbb{Z}^d_{2L'_\delta+1}} (\tilde{\omega}_{l'+k'(2L'_\delta+1)} + 3\delta/4) \right)$$

Hence, for some $k' \in \mathbb{Z}^d_{2K'_\delta+1}$, we have

$$\frac{1}{(2L'_\delta+1)^d} \sum_{l' \in \mathbb{Z}^d_{2L'_\delta+1}} (\tilde{\omega}_{l'+k'(2L'_\delta+1)} + 3\delta/4) \geq 0.$$

This ends the proof of Lemma 4.2.

$\square$

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