Trace Formula for Quantum Integrable Systems, Lattice–Point Problem, and Small Divisors

Pavel Bleher

Indiana University – Purdue University
at Indianapolis

To be published in the Proceedings of the Summer Program on
"Emerging Applications of Number Theory", July 15–26, 1996

Institute for Mathematics and Its Applications,
University of Minnesota, Minneapolis

1996

Address:

Department of Mathematical Sciences

Indiana University – Purdue University at Indianapolis

402 N. Blackford Street

Indianapolis, IN 46202, USA

E-mail: bleher@math.iupui.edu

Abstract. We review rigorous results concerning almost periodicity by Besicovitch of the spectral function of quantum integrable systems, lattice point-problem in convex and nonconvex domains, trace formula, and a related problem of small divisors. This includes the following subjects:

(i) Semiclassical quantization by Einstein-Brillouin-Keller and the problem of completeness of semiclassical eigenfunctions.

(ii) The lattice-point problem.

(iii) The Gutzwiller-Berry-Tabor trace formula for quantum integrable systems.

(iv) The problem of small divisors.

(v) The saturation phenomenon.

The work is supported in part by the NSF Grant No. DMS–9623214.

Contents

1. The Weyl Law

Let M^d be a smooth closed compact Riemannian manifold, and let

$$0 = E_0 < E_1 \le E_2 \le \cdots \to \infty,$$

be eigenvalues of the Laplace–Beltrami operator on M^d. The Weyl law [Wey] gives the asymptotics of the counting function of the eigenvalues,

$$N(E) = \#\{n\colon E_n \le E\},$$

as

$$N(E) = CE^{d/2} + n(E), \qquad n(E) = o(E^{d/2}), \qquad E \to \infty, \tag{1.1}$$

with

$$C = \frac{\operatorname{Vol} M^d \operatorname{Vol} B^d}{(2\pi)^d}, \qquad B^d = \{x \in \mathbb{R}^d\colon |x| \le 1\}.$$

Hörmander [Hör1] proves the general estimate for the error function:

$$n(E) = O\left(E^{(d-1)/2}\right), \tag{1.2}$$

and he shows that this estimate is sharp for the d-dimensional sphere S^d, when the eigenvalues have high multiplicity. Duistermaat and Guillemin [DG] improve the Hörmander estimate to

$$n(E) = o\left(E^{(d-1)/2}\right),$$

under the assumption that the union of all closed geodesics has Lebesgue measure zero on the cotangent unit bundle. For manifolds of negative curvature Bérard [Bér] proves the estimate

$$n(E) = O\left(E^{(d-1)/2}/\log E\right),$$

and it is a hard problem to improve the exponent $(d-1)/2$ in this estimate in $\varepsilon > 0$.

In this paper we will be interested in the opposite case of completely integrable geodesic flows. Let us begin with a simple example of a two–dimensional torus $\mathbb{T}^2 = (2\pi)\mathbb{R}^2/\mathbb{Z}^2$ with the flat metric

$$dq^2 = dq_1^2 + dq_2^2. \tag{1.3}$$

Then the eigenvalues are

$$E_{n_1 n_2} = n_1^2 + n_2^2, \qquad n_1, n_2 \in \mathbb{Z}, \tag{1.4}$$

so that

$$N(E) = \#\{(n_1, n_2) \in \mathbb{Z}^2 \colon n_1^2 + n_2^2 \le E\}$$

is just the number of lattice points inside the circle $x_1^2 + x_2^2 = E$, and (1.1) reads

$$N(E) = \pi E + n(E).$$

In this case the problem of evaluating $n(E)$ is the classical circle problem which goes back to Gauss. Gauss proves the estimate

$$n(E) = O(E^{1/2}),$$

which is equivalent in this particular case to the Hörmander estimate (1.2). The Gauss estimate is rather obvious since $n(E)$ cannot grow faster than the length of the boundary. First nontrivial estimate,

$$n(E) = O(E^{1/3}),$$

is derived by Sierpinski [Sie]. Then the exponent 1/3 in this estimate has been improved due to the works by Hardy, Landau, Vinogradov, Walfisz, Titchmarsh, Hua, Kolesnik, Iwaniec and Mozzochi, Huxley, and others. Huxley [Hux] proves the estimate

$$n(E) = O\left(E^{23/73}(\log E)^{315/146}\right), \tag{1.5}$$

which is probably the best estimate at the present time. Hardy's careful conjecture [Har1]: "it is not unlikely that

$$n(E) = O\left(E^{(1/4)+\varepsilon}\right) \tag{1.6}$$

for all positive ε," remains open. On the other hand, Hardy [Har2] proves that

$$\begin{aligned} \limsup_{E\to\infty} \frac{n(E)}{E^{1/4}} &> 0, \\ \liminf_{E\to\infty} \frac{n(E)}{E^{1/4}(\log E)^{1/4}} &> 0, \end{aligned} \tag{1.7}$$

which shows that the exponent in (1.6) cannot be smaller than $(1/4)+\varepsilon$.

The Hardy conjecture (1.6) combined with (1.7) suggest that "typical" values of $n(E)$ are of the order of $E^{1/4}$, and it is confirmed by the classical theorem of Cramér [Cra]:

$$\lim_{T\to\infty} \frac{1}{T^{3/2}} \int_0^T |n(E)|^2 dE = C > 0. \tag{1.8}$$

Fig. 1 shows the error function $n(E)$ as a function of the radius $R = \sqrt{E}$. It is easy to see that $n(R^2)$ is irregularly oscillating around 0, and it looks like a random function. So it is quite natural to study probabilistic characteristics of $n(R^2)$, like average value, mean square deviation, limit distribution, correlations, etc. Since there is nothing random in the problem, the probabilistic characteristics of $n(R^2)$ are understood as ergodic averages. Let us introduce the normalized error function as

$$F(R) = \frac{n(R^2)}{\sqrt{R}}, \qquad R \geq 1, \tag{1.9}$$

and $F(R) = 0$ when $R < 1$. The result of Cramér (1.8) implies that the second moment of $F(R)$ exists and it is positive. Heath-Brown [H-B] proves the existence of a limit distribution of $F(R)$.

Theorem 1.1 (see [H-B]). *There exists a probability density $p(x)$ such that for every bounded continuous function $g(x)$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(F(R))dR = \int_{-\infty}^{\infty} g(x)p(x)dx. \tag{1.10}$$

In addition, the density $p(x)$ decays as $x \to \infty$ faster than polynomially, and it can be extended to an entire function on a complex plane.

2. General Lattice–Point Problem

Convex Domains. Let Λ be an open convex domain on a plane $\mathbb{R}^2$, and let $\Gamma = \partial\Lambda$ be its boundary. We will assume that

(i) Γ is C^7-smooth;

(ii) $0 \in \Lambda$;

(iii) Γ is strictly convex, i.e., the curvature $\varkappa(x) > 0$ for all $x \in \Gamma$.

Let $\alpha \in \mathbb{R}^2$ be an arbitrary fixed point on the plane. For $R > 0$, define the set

$$R\Lambda + \alpha = \{x \in \mathbb{R}^2 : (x - \alpha)/R \in \Lambda\}, \tag{2.1}$$

which is a dilation of Λ with the coefficient R and its shift in α. Let

$$N_\alpha(R) = \#\, \mathbb{Z}^2 \cap (R\Lambda + \alpha),$$

the number of lattice points in $R\Lambda + \alpha$, and

$$F_\alpha(R) = \frac{N_\alpha(R) - AR^2}{\sqrt{R}}, \qquad R \geq 1, \tag{2.2}$$

where

$$A = \text{Area}\,\Lambda,$$

and $F_\alpha(R) = 0$ for $R < 1$.

Theorem 2.1. (see [Ble1]) *If Λ satisfies the conditions (i)–(iii), then for all $\alpha \in \mathbb{R}^2$, $F_\alpha(R)$ is an almost periodic function in R from the Besicovitch space B^2, and its Fourier series in B^2 is*

$$F_\alpha(R) = \pi^{-1} \sum_{n \in \mathbb{Z}^2,\ n \neq 0} |n|^{-3/2} |\varkappa(x(n))|^{-1/2} \cos\big(\omega(n)R - \phi(n;\alpha)\big), \tag{2.3}$$

where $x(n) \in \Gamma$ is the unique point on Γ where the outer normal vector to Γ coincides with $n/|n|$, $\varkappa(x)$ is the curvature, and

$$\omega(n) = 2\pi(n, x(n)) > 0; \qquad \phi(n;\alpha) = 2\pi(\alpha, n) + (3\pi/4). \tag{2.4}$$

The meaning of the B^2-almost periodicity and (2.3) is that

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\Bigg|F_\alpha(R)-\pi^{-1}\sum_{n\in\mathbb{Z}^2,\ 0<|n|<N}|n|^{-3/2}|\varkappa(x(n))|^{-1/2} \times\cos\big(\omega(n)R-\phi(n;\alpha)\big)\Bigg|^2 dR=0 \tag{2.5}$$

(see [Bes] and [LZh]). Since in B^2 we have the Parceval identity, Theorem 2.1 has the following corollary, which is an extension of the Cramér theorem. Denote by $0<\omega_1<\omega_2<\dots$, all different values of $\omega(n)$, $n\in\mathbb{Z}^2$, $n\neq 0$.

Corollary.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F_\alpha(R)\,dR=0, \tag{2.6}$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |F_\alpha(R)|^2 dR=C, \tag{2.7}$$

where

$$\begin{aligned} C&=\sum_{k=1}^{\infty}\left(|a_k^+|^2+|a_k^-|^2\right),\\ a_k^\pm&=\frac{1}{2\pi}\sum_{n:\ \omega(n)=\omega_k}|n|^{-3/2}|\varkappa(x(n))|^{-1/2}\exp(\pm i\phi(n;\alpha)) \end{aligned} \tag{2.8}$$

The B^2-almost periodicity also implies the existence of a limit distribution of $F_\alpha(R)$. Namely, we have the following general result.

Theorem 2.2. (see [Ble1]). *If $f(R)\in B^2$ then there exists a probability distribution $\nu(dx)$ on a line such that for all bounded continuous functions $g(x)$,*

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T g(f(R))\,dR=\int_{-\infty}^{\infty}g(x)\,\nu(dx).$$

In addition,

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T f(R)^j\,dR=\int_{-\infty}^{\infty}x^j\,\nu(dx),\qquad j=1,2.$$

Corollary. *There exists a probability distribution $\nu_\alpha(dx)$ on a line such that for all bounded continuous functions $g(x)$,*

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T g(F_\alpha(R))\,dR=\int_{-\infty}^{\infty}g(x)\,\nu_\alpha(dx). \tag{2.9}$$

We would like to know the properties of $\nu_a(dx)$. For a shifted circle it is studied by Bleher, Cheng, Dyson, and Lebowitz [BCDL].

Theorem 2.3 (see [BCDL]). *Assume that* Γ *is a circle. Then for all* $\alpha \in \mathbb{R}^2$, $\nu_\alpha(dx)$ *is absolutely continuous with respect to the Lebesgue measure, and the density* $p_\alpha(x) = \nu_\alpha(dx)/dx$ *is an entire function such that for all* $\varepsilon > 0$,

$$\lim_{|x|\to\infty} \frac{\log p_\alpha(x)}{|x|^{4+\varepsilon}} = 0, \tag{2.10}$$

and

$$\lim_{x\to\infty} \frac{\log P_\alpha^\pm(x)}{|x|^{4-\varepsilon}} = \infty, \tag{2.11}$$

where

$$P_\alpha^\pm = \left| \int_{\pm x}^{\pm\infty} p_\alpha(x)\, dx \right|.$$

Roughly (2.10) and (2.11) mean that $p_\alpha(x)$ decays at infinity as $\exp(-cx^4)$. The variance of $p_\alpha(t)$ turns out to be sensitive to arithmetical properties of α. Bleher and Dyson [BD1] proves the following result.

Theorem 2.4 (see [BD1]). *Let*

$$D(\alpha) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |F_\alpha(R)|^2 dR = \int_{-\infty}^{\infty} x^2 p_\alpha(x)\, dx.$$

Then $D(\alpha)$ *is a continuous function of* α, *and for all rational* $\alpha \in \mathbb{Q}^2$,

$$\lim_{\beta\to\alpha} \frac{D(\alpha) - D(\beta)}{|\alpha - \beta| \big|\log|\alpha - \beta|\big|} = C(\alpha) > 0. \tag{2.12}$$

The equation (2.12) implies that $D(\alpha)$ has a sharp local maximum at every rational point. Properties of $\nu_\alpha(dx)$ for a generic Γ are studied in [Ble2] and [BKS]. Let

$$Z = \{n = (n_1, n_2) \in \mathbb{Z}^2 \colon n_1, n_2 \neq 0 \text{ are relatively prime}\} \cup (1,0) \cup (0,1). \tag{2.13}$$

Theorem 2.5 (see [Ble2] and [BKS]). *Assume that the numbers* $\{\omega(n),\ n \in Z\}$ *are linearly independent over* $\mathbb{Z}$, *i.e., if* $k_1\omega_1 + \cdots + k_m\omega_m = 0$ *for some* $k_i \in \mathbb{Z}$ *then all* $k_i = 0$. *Then* *for all* $\alpha \in \mathbb{R}^2$*, the distribution* $\nu_\alpha(dx)$ *has a density* $p_\alpha(x) = \nu_\alpha(dx)/dx$ *which is an entire function in* x*, and*

$$\lim_{x\to\pm\infty} -\frac{\log p_\alpha(x)}{x^4} = c_\pm(\alpha) > 0.$$

If Γ possesses a symmetry then $\omega(n)$ coincides for symmetric n. In this case Theorem 2.5 remains valid if $\omega(n_1),\dots,\omega(n_m)$ are linearly independent over $\mathbb{Z}$ for all $n_1,\dots,n_m \in Z$ such that n_i and n_j are not symmetric. For an ellipse $x_1^2+\mu^{-2}x_2^2=1$ the condition of linear independence of $\omega(n)$ over $\mathbb{Z}$ holds if μ^{-2} is a transcendental number (see [Ble2]), hence in this case we have Theorem 2.5. For the ellipse $x_1^2+\mu^{-2}x_2^2=1$ the normalized error function is

$$F_\alpha(R) = \frac{\#\left\{(n_1,n_2)\in\mathbb{Z}^2\colon n_1^2+\mu^{-2}n_2^2 \le R^2\right\} - \pi\mu}{R^{1/2}}. \tag{2.14}$$

Figs. 2–10 (taken from [Ble2]) show the density $p_\alpha(x)$ of the limit distribution of $F_\alpha(R)$ for ellipses with different value of μ and different α. It obviously varies and can be bimodal.

How to explain the non-Gaussian nature of $p_\alpha(x)$? Let us consider, as an example, an ellipse $x_1^2+\mu^{-2}x_2^2=1$ with some transcendental μ^{-2}, and let us take $\alpha=0$. We rewrite the Fourier series (2.3) in this case as

$$F_0(R) = (\mu/\pi) \sum_{n\in\mathbb{Z}^2\colon\, n\ne 0} |n|_\mu^{-3/2}\cos\bigl(2\pi|n|_\mu R-\phi\bigr),$$

where $|n|_\mu^2 = n_1^2+\mu^2 n_2^2$ (see [Ble2]). Now we can take some m from the set

$$Z_+ = \{m=(m_1,m_2)\in\mathbb{Z}^2\colon m_1,m_2>0 \text{ are relatively prime}\}\cup(1,0)\cup(0,1), \tag{2.15}$$

and first make a summation over all $n=km,\ k\in\mathbb{N}$, and symmetric points, and then over $m\in Z_+$. This gives

$$F_0(R) = \sum_{m\in Z_+} |m|_\mu^{-3/2} f_m(|m|_\mu R), \tag{2.16}$$

where

$$f_m(t) = (\mu/\pi)r(m)\sum_{k=1}^{\infty} k^{-3/2}\cos(2\pi kt-\phi), \tag{2.17}$$

where $r(m)$ is the symmetry factor: $r(m)=4$ if $m_1m_2\ne 0$, and $r(m)=2$ otherwise. The function $f_m(t)$ is periodic of period 1. The number μ is transcendental, hence the frequencies $|m|_\mu,\ m\in Z_+$, are linearly independent over $\mathbb{Z}$ (see [Ble2]). Therefore the numbers $|m|_\mu R \mod 1$ behave like independent random variables θ_m uniformly distributed on $[0,1]$. Thus the limit distribution of $F(R)$ is the distribution of the random series

$$\sum_{m\in Z_+} |m|_\mu^{-3/2} f_m(\theta_m). \tag{2.18}$$

This series is only conditionally convergent because Z_+ is a two-dimensional set and the exponent $3/2$ is less than 2. However the series of the variances,

$$\sum_{m\in Z_+} |m|_\mu^{-3} \int_0^1 f_m(\theta)\, d\theta,$$

converges, and hence, by the Kolmogorov theorem, the random series (2.18) is well defined and it is non-Gaussian. An analysis of the characteristic function of the random series (2.18) gives the tail behavior of its density as $\exp(-cx^4)$. The exponent 4 is related to the one $(3/2)$ in (2.18). As concerns uniform estimate of the error term $n(E)$, Huxley's estimate (1.5) is proved for arbitrary smooth domain with nonvanishing curvature and the estimate of the error function is uniform in translation and rotation (see [Hux]).

Nonconvex Domains. Let us consider a nonconvex domain Λ. We will assume that Λ is a star-like domain such that

(i) $\Gamma = \partial\Lambda$ is C^7-smooth;

(ii) $(n(x), x) > 0$ for all $x \in \Gamma$, where $n(x)$ is the outer normal vector to Γ at the point x.

In addition, we will assume the following Diophantine hypothesis.

Hypothesis D. *(i) The curvature $\varkappa(x) \neq 0$ everywhere on Γ except, maybe, a finite set $W = \{w_1, \dots, w_K\}$, and*

$$\frac{d\varkappa}{ds}(w_k) \neq 0, \qquad k = 1, \dots, K, \tag{2.19}$$

where s is the natural coordinate (the arc length) on Γ. (ii) For all $w_k \in W$ the outer normal vector $n(w_k)$ to Γ is either rational, i.e., $(m, n(w_k)) = 0$ for some $m \in \mathbb{Z}^2$, $m \neq 0$, or Diophantine in the sense that there exist $0 < \zeta < 1$ and $C > 0$ such that for all $m \in \mathbb{Z}^2$, $m \neq 0$,

$$|(m, n(w_k))| > C\, |m|^{-1-\zeta}. \tag{2.20}$$

For $x \in \Gamma$ we denote by $L(x)$ the line on a plane through 0 with slope $n(x)$. By Γ_r we denote the set of all points on Γ where the vector $n(x)$ is rational. The set Γ_r is countable.

Theorem 2.6 (see [Ble3]). *Assume that Λ satisfies (i), (ii), and that the Hypothesis D holds. Then for all $\alpha \in \mathbb{R}^2$,*

$$N_\alpha(R^2) = AR^2 + R^{2/3} \sum_{k:\ n(\omega_k) \text{ is rational}} \Phi_k(R;\alpha) + R^{1/2} F_\alpha(R), \tag{2.21}$$

where $\Phi_k(R;\alpha)$ are continuous periodic functions of R,

$$\Phi_k(R) = \frac{\Gamma(2/3)}{2\, 3^{2/3} \pi^{4/3}} \left| \frac{d\varkappa}{ds}(w_k) \right|^{-1/3} \sum_{n \in L(w_k),\ n \neq 0} \sin\bigl[2\pi(w_k, n(w_k))|n|R - 2\pi(n,\alpha)\bigr], \tag{2.22}$$

and $F_\alpha(R)$ is a B^2-almost periodic function of R. The B^2-Fourier series of $F_\alpha(R)$ is

$$F_\alpha(R) = \pi^{-1} \sum_{x \in \Gamma_r} |\varkappa(x)|^{-1/2} \sum_{n \in L(x),\ n \neq 0} |n|^{-3/2} \cos\bigl[2\pi(x, n(x))|n|R - \phi(x,n,\alpha)\bigr], \tag{2.23}$$

where $\phi(x,n,\alpha) = (\pi/2) + (\pi/4)\operatorname{sgn}\varkappa(x) + 2\pi(n,\alpha)$.

We can compare this result with the Colin de Verdière estimate [CdV2].

Theorem 2.7 (see [CdV2]). *Assume that Λ satisfies (i) and (ii). Then*

$$\bigl|N_\alpha(R^2) - AR^2\bigr| \le CR^{2/3} \tag{2.24}$$

Comparison of (2.21) with (2.24) shows that the Colin de Verdière estimate (2.24) is sharp when there is an inflection point w_k on Γ with rational normal vector $n(w_k)$. In a "typical" situation, however, there is no inflection point with rational $n(w_k)$, and the intermediate term with $R^{2/3}$ is missed in (2.21). The typical situation means that the normal vectors $n(w_k)$ at the inflection points w_k satisfy the Diophantine condition (2.20). This is a problem of small divisors, which can be seen from (2.23). Namely, the multiplier $|\varkappa(x)|^{-1/2}$ approaches ∞ as $x \to w_k$, and the Diophantine condition is needed to secure that the growth of this multiplier is compensated by the decrease of the multiplier $|n|^{-3/2}$ in (2.23). Along the lines of the proof of Theorem 2.6, it is possible to show that if for some w_k, $n(w_k)$ is a Liouville number, i.e., it is exceptionally well approximated by rationals, then the error function $n_\alpha(R^2)$ behaves eratically, with no powerlike asymptotics.

The importance of the inflection points for the trace formula was emphasized by Berry and Tabor [BT1]:

"Physically, the region [where the curvature] $K = 0$ would rise to a large peak in $n(E)$, a dense cluster of energy levels. The simplest region $K = 0$ is a point of inflexion on the energy contour for a two-dimensional case, and the corresponding feature in $n(E)$ could be called a 'bound state rainbow' (see [BM], section 6.3). More complicated coalescences would give rise to higher order 'catastrophes' (Thom 1975; Connor 1975; Duistermaat 1974). At present we know of no case involving smooth potentials where the energy contours have inflexions $K = 0$, and so we do not consider these coalescences any further here (Balian and Bloch [BaB] study an enclosure in the shape of a waisted Greek vase whose mode show a rainbow)."

In the next section we will consider a geodesic flow on a smooth surface of revolution, and we will see that the points of inflection on the energy contour are indeed possible in a generic situation.

3. Surfaces of Revolution

Surface of revolution is a simple example of integrable geodesic flow. Assume that M^2 is a smooth surface of revolution, which is diffeomorphic to a sphere. Let A be the axis of M^2, and let N and S be its north and south poles, respectively (see Fig. 11). The geodesic flow on M^2 is integrable due to the Clairaut integral,

$$I = r \sin\alpha = \text{ const}\,,$$

where r is the radial coordinate and α is the angle between the velocity vector and the meridian. Let

$$r = f(s), \qquad 0 \le s \le L,$$

be the equation of M^2, where s is the normal coordinate (the arc length) along meridian. Then

$$\Delta = f(s)^{-1}\frac{\partial f}{\partial s}\left(f(s)\frac{\partial f}{\partial s}\right) + f(s)^{-2}\frac{\partial f^2}{\partial \varphi^2},$$

where φ is the angular coordinate. We will assume that $f(s)$ has a simple structure, so that

$$f'(s) \neq 0, \quad s \neq s_{\max}; \qquad f''(s_{\max}) < 0,$$

where

$$f(s_{\max}) = \max_{0\le s\le L} f(s).$$

For normalization we put $f(s_{\max}) = 1$. We call the circle $s = s_{\max}$ on M^2 the equator.

Another assumption on M^2 is the twist hypothesis. Let x_0 be an arbitrary point on the equator. Consider a geodesic γ which goes out of x_0 at some angle $-(\pi/2) < \alpha_0 < (\pi/2)$

to the north direction on the meridian. The Clairaut integral is $I = \sin\alpha_0$ and we can parametrize γ by $-1 < I < 1$: $\gamma = \gamma(I)$. It follows from the Clairaut integral that $\gamma(I)$ oscillates between two parallels, $s = s_-$ and $s = s_+$ where $f(s_-) = f(s_+) = I$. Hence $\gamma(I)$ intersects the equator infinitely many times. Let x_n be the nth intersection of $\gamma(I)$ with the equator, $n \in \mathbb{Z}$. Define

$$\tau(I) = |\gamma[x_0, x_2]|$$

to be the length of $\gamma = \gamma(I)$ between x_0 and x_2, and

$$\omega(I) = (2\pi)^{-1}(\varphi(x_2) - \varphi(x_0))$$

to be the phase of $\gamma(I)$ between x_0 and x_2 (see Fig. 11). Observe that $\omega(I)$ is defined mod 1. To define $\omega(I)$ uniquely, we choose a continuous branch of $\omega(I)$ starting at $\omega(0) = 0$. Define $\tau(1) = \lim_{I\to 1-0}\tau(I)$ and $\omega(1) = \lim_{I\to 1-0}\omega(I)$. A finite geodesic $\gamma(I)$ on M^2 with $0 < I < 1$ is closed if and only if $\omega(I)$ is rational. More precisely, let $n(\gamma)$ denote the number of revolution of a closed geodesic γ around the axis A, and let $m(\gamma)$ denote the number of oscillations of γ along meridian. Then

$$\omega(I) = \frac{n(\gamma(I))}{m(\gamma(I))} - 1.$$

Twist Hypothesis. *$\omega'(I) \neq 0$ for all $0 \le I \le 1$.*

Theorem 3.1 (see [Ble3]). *Assume that M^2 is a surface of revolution of simple structure and that M^2 satisfies the Twist Hypothesis. Let $N(E) = \#\{n\colon E_n \le E\}$ be the counting function of eigenvalues of the Laplacian on M^2. Then for $E \ge 1$,*

$$N(E) = \frac{\operatorname{Vol} M^2}{4\pi}\, E + E^{1/4}F(E^{1/2}), \tag{3.1}$$

where $F(R)$ is a B^2-almost periodic function, and the Fourier series of $F(R)$ in B^2 is

$$F(R) = \sum_{\text{closed geodesics } \gamma} A(\gamma)\cos(|\gamma|R - \phi), \tag{3.2}$$

where summation goes over all closed (in general, mulptiple) oriented geodesics $\gamma \neq 0$ on M^2, $\phi = (\pi/2) + (\pi/4)\operatorname{sgn}\omega'(I)$ and

$$\begin{aligned} A(\gamma) &= \pi^{-1}(-1)^{m(\gamma)}|\omega'(I)|^{-1/2}m(\gamma)^{-3/2} \\ &= \pi^{-1}(-1)^{m(\gamma)}|\omega'(I)|^{-1/2}\tau(I)^{3/2}|\gamma|^{-3/2}, \qquad I = I(\gamma). \end{aligned} \tag{3.3}$$

The equation (3.2) is a trace formula which relates eigenvalues of the Laplace operator to closed geodesics, and it implies that the B^2-Fourier spectrum of $F(R)$ coincides with the geodesic spectrum of the manifold M^2. It is also to be noted that (3.2) gives the Fourier series of $F(R)$ in the Besicovitch space B^2 which means that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left| F(R) - \sum_{\text{closed geodesics } \gamma \text{ with } |\gamma|<N} A(\gamma)\cos(|\gamma|R - \phi) \right|^2 dR = 0.$$

Therefore (3.2) is actually an asymptotic trace formula describing the behavior of $F(R)$ for large R (cf. [Gui]). In addition, (3.2) can be viewed as a rigorous version of the general trace formula of Berry and Tabor (see [BT1] and [BT3]) for integrable quantum systems, applied to the surface of revolution.

Fig. 12 taken from [Ble3], shows the phase function $\omega(I)$ for three surfaces of revolution: (a) oblong ellipsoid, (b) oblate ellipsoid, and (c) bell-like surface. It is $\omega'(I) > 0$ for the oblong ellipsoid, $\omega'(I) < 0$ for the oblate ellipsoid, and $\omega(I)$ has a critical point for the bell-like surface. Hence Theorem 3.1 is applicable in the cases (a) and (b), and it is not applicable in the case (c). To extend Theorem 3.1 to the case (c), we introduce the following condition.

Diophantine Hypothesis. *Assume that $\omega(I)$ has at most finitely many critical points $0 < I_1 < \cdots < I_K < 1$, and $\omega''(I_k) \neq 0$ for all $k = 1, \ldots, K$. Assume, in addition, that for all $k = 1, \ldots, K$, $\omega(I_k)$ is either rational or Diophantine in the sense that there exist $0 < \zeta < 1$ and $C > 0$ such that*

$$\left|\omega(I_k) - \frac{p}{q}\right| \geq \frac{C}{q^{2+\zeta}}, \qquad \forall \frac{p}{q} \in \mathbb{Q}.$$

Theorem 3.2 (see [Ble3]). *Assume that M^2 is a surface of revolution of simple structure and that M^2 satisfies the Diophantine Hypothesis. Then*

$$N(E) = \frac{\operatorname{Vol} M}{4\pi} E + E^{1/3} \sum_{k:\ \omega(I_k)\in\mathbb{Q}} \Phi_k(E^{1/2}) + E^{1/4} F(E^{1/2}), \tag{3.4}$$

where $\Phi_k(R)$ are bounded periodic functions and $F(R)$ is B^2-almost periodic function. The Fourier series of $\Phi_k(R)$ is

$$\Phi_k(R) = \frac{\Gamma(2/3)\tau(I_k)^{4/3}}{2\pi^{4/3}} \sum_{\gamma:\ I(\gamma)=I_k} (-1)^{m(\gamma)} |\gamma|^{-4/3} \sin(|\gamma|R), \tag{3.5}$$

and the Fourier series of $F(R)$ in B^2 is

$$F(R) = \sum_{\gamma:\ I(\gamma)\neq I_1,\dots,I_K} A(\gamma)\cos(|\gamma|R - \phi(\gamma)), \tag{3.6}$$

where $\phi(\gamma) = (\pi/2) + (\pi/4)\,\mathrm{sgn}\,\omega'(I),\ I = I(\gamma)$ and $A(\gamma)$ is as in (3.2).

The proof of Theorem 3.2 (observe that Theorem 3.1 reduces to Theorem 3.2) uses the semiclassical result of Colin de Verdière.

Theorem 3.3 (see [CdV3]). *If M^2 is a surface of revolution of simple structure, then*

$$\mathrm{Spectrum}(-\Delta) = \{E_{kl} = Z(k + (1/2), l);\quad k, l \in \mathbb{Z},\quad |l| \le k\}, \tag{3.7}$$

with $Z(p) = Z(p_1, p_2) \in C^\infty(\mathbb{R}^2)$ such that

$$Z(p) = Z_2(p) + Z_0(p) + O(p^{-1}), \qquad |p| \to \infty,$$

where

$$Z_2(p),\ Z_0(p) \in C^\infty(\mathbb{R}^2\backslash\{0\}); \qquad Z_2(p) > 0, \quad p \neq 0,$$

and

$$Z_j(\lambda p) = \lambda^j Z_j(p), \qquad \forall\, \lambda > 0,\ p \in \mathbb{R}^2; \qquad j = 0, 2.$$

In addition, in the sector $\{p_1 \ge |p_2|$, $Z_2(p)$ satisfies the equation

$$\pi^{-1}\int_a^b \sqrt{Z_2(p) - p_2^2 f(s)^{-2}}\, ds = p_1 - |p_2|,$$

where a and b are the turning points, i.e.,

$$Z_2(p) - p_2^2 f(s)^{-2} = 0, \quad \text{for} \quad s = a, b.$$

We define semiclassical eigenvalues as

$$E_{kl}^{\mathrm{EBK}} = Z_2(k + (1/2), l).$$

This is the Einstein-Brillouin-Keller quantization formula, applied to the surface of revolution. The semiclassical counting function is then

$$N^{\mathrm{EBK}}(E) = \#\,\{(k, l)\colon Z_2(k + (1/2), l) \le E\}. \tag{3.8}$$

The following estimate shows that we can replace $N(E)$ by $N^{\mathrm{EBK}}(E)$.

Lemma 3.4 (see [Ble3]).

$$\lim_{T\to\infty}\int_0^T \frac{|N(R^2) - N^{\mathrm{EBK}}(R^2)|^2 dR}{R} = 0.$$

Since $Z_2(p)$ is a homogeneous function of order 2, the problem of finding asymptotics of $N^{\mathrm{EBK}}(E)$ reduces to the lattice-point problem. The condition $|l| \le k$ in (3.7) restricts lattice points to the sector between diagonals, and the additive term (1/2) in (3.7) shifts the lattice in (1/2) along the x-axis. Fig. 13 depicts the graph of the level set $Z_2(p) = 1$ for the three surfaces of revolution shown in Fig. 12. Observe that the critical point of $\omega(I)$ for the bell-like surface leads to the inflection point on the level set $Z_2(p) = 1$. The influence of the inflection points on the oscillatory asymptotics of the error function was observed by Berry and Tabor [BT1]. They called this phenomenon a 'bound state rainbow', with a reference to the paper of Balian and Bloch [BaB].

Theorem 2.6 allows an extension to the lattice points in the sector between diagonals (see [Ble3]), and this proves the B^2-almost periodicity for the semiclassical error function. Lemma 3.4 proves it then for the error function $F(R)$. Theorem 3.1 can be used to get a limit distribution of the normalized error function $F(R)$.

Theorem 3.5 (see [Ble3], [KMS], and [BKS]). *Assume that M^2 is a surface of revolution of simple structure and that M^2 satisfies the Twist Hypothesis. Then there exists a probability distribution $\nu(dx)$ on a line such that for all bounded continuous functions $g(x)$,*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(F(R))\,dR = \int_{-\infty}^{\infty} g(x)\,\nu(dx).$$

If, in addition, the lengths of all primitive closed geodesics on M^2 with non-negative Clairaut integral I are linearly independent over $\mathbb{Z}$, then the probability distribution $\nu(dx)$ is absolutely continuous, and the density $p(x) = \nu(dx)/dx$ is an entire function such that

$$\lim_{x\to\pm\infty} -\frac{\log p(x)}{x^4} = c_\pm > 0.$$

Similar result holds as well for the error function $F(R)$ in the case when M^2 has a simple structure and it satisfies the Diophantine Hypothesis (see [BKS]). It is to be noted that the limit distribution $\nu(dx)$ is not Gaussian. As we have discussed before, it is related to the convergence of the series of squared amplitudes in the trace formula. For chaotic systems, Aurich, Bolte, and Steiner [ABS] have found universally a Gaussian limit distribution of the error function. This can be explained as follows.

In the chaotic case we can apply the Gutzwiller trace formula and we can write it for $0 < R < T$ as

$$n(R^2) = \sum_{k:\ \gamma_k < l(T)} A_k f_k(\gamma_k R) + \text{error term},$$

where $0 < \gamma_1 < \gamma_2 < \dots$ are all the different lengths of primitive closed geodesics on the manifold, $l(T)$ is a cut-off, and $f_k(t)$ are periodic functions of period 1 with

$$\int_0^1 f_k(t)\, dt = 0, \qquad \int_0^1 |f_k(t)|^2 dt = 1.$$

Assume that the numbers $\{\gamma_k\}$ are incommensurable. Then the limit distribution of $n(R^2)$ is to be the one of the random series

$$\sum_{k:\ \gamma_k < l(T)} A_k f_k(\theta_k),$$

where θ_k are independent random variables uniformly distributed on $[0, 1]$. In the chaotic case the series

$$\sum_{k=1}^{\infty} |A_k|^2$$

diverges, therefore by the Lindeberg central limit theorem, the distribution of the random variable

$$\frac{\sum_{k=1}^{l(T)} A_k f_k(\theta_k)}{\left(\sum_{k=1}^{l(T)} |A_k|^2\right)^{1/2}}$$

approaches a standard Gaussian distribution in the limit $T \to \infty$. This gives a heuristic explanation of the Gaussian limit distribution of the error function in the chaotic case.

4. Liouville Surfaces

Let M^2 be a smooth compact closed Riemann surface diffeomorphic to a torus. It is called the Liouville surface if its metric has the form

$$dq^2 = [U_1(q_1) - U_2(q_2)](dq_1^2 + dq_2^2), \tag{4.1}$$

where $0 \le q_1, q_2 \le 1$ are coordinates on the torus and $U_1(q_1)$, $U_2(q_2)$ are smooth periodic functions of period 1 satisfying the inequality $U_1(q_1) > U_2(q_2)$ for all q_1, q_2. The Laplacian is then

$$\Delta = [U_1(q_1) - U_2(q_2)]^{-1}\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2}\right). \tag{4.2}$$

The Hamiltonian of the geodesic flow is

$$H(p,q) = \frac{1}{2(U_1(q_1) - U_2(q_2))}\,(p_1^2 + p_2^2). \tag{4.3}$$

The geodesic flow on the Liouville surface is integrable due to the integral

$$S(p,q) = \frac{1}{2(U_1(q_1) - U_2(q_2))}\,[U_2(q_2)p_1^2 + U_1(q_1)p_2^2]. \tag{4.4}$$

We associate with $S(p,q)$ the quantum operator

$$\sigma = [U_1(q_1) - U_2(q_2)]^{-1}\left(U_2(q_2)\frac{\partial^2}{\partial q_1^2} + U_1(q_1)\frac{\partial^2}{\partial q_2^2}\right).$$

A direct computation shows that Δ and σ commute. Consider the counting function $N(E)$ of $-\Delta$ and the normalized error function

$$F(R) = \frac{N(R^2) - CR^2}{R^{1/2}}, \qquad C = \frac{\operatorname{Area} M^2}{4\pi}. \tag{4.5}$$

The asymptotic behavior of $F(R)$ is studied in the paper of Kosygin, Minasov, and Sinai [KMS]. It is shown in [KMS] that under some condition of non-degeneracy of the Liouville surface, the function $F(R)$ is B^1-almost periodic. If, in addition, the Liouville surface is "generic" then the distribution density of $F(R)$ is an entire function, and it decays at infinity at least as $\exp(-|x|^{(16/9)-\varepsilon})$. These results are further extended by Bleher, Kosygin, and Sinai (see [BKS]). To describe the results we have to discuss some properties of the geodesic flow. Define

$$\begin{aligned} c_1 &= \max_{0\le x\le 1} U_1(x) = U_1(M_1), & c_2 &= \min_{0\le x\le 1} U_1(x) = U_1(m_1), \\ c_3 &= \max_{0\le x\le 1} U_2(x) = U_2(M_2), & c_4 &= \min_{0\le x\le 1} U_2(x) = U_2(m_2). \end{aligned} \tag{4.6}$$

Consider the energy surface

$$M_E = \{(p,q)\colon H(p,q) = E\}, \qquad E > 0,$$

and partition M into the invariant sets

$$M_{E,c} = \{(p,q) \in M_E\colon S(p,q) = cE\}.$$

By (4.3) and (4.4),

$$\frac{S(p,q)}{H(p,q)} = \frac{U_2(q_2)p_1^2 + U_1(q_1)p_2^2}{p_1^2 + p_2^2}, \tag{4.7}$$

hence on M_E the value $c = S(p,q)/H(p,q)$ varies from c_4 to c_1. If $c_4 < c < c_3$ or $c_2 < c < c_1$ the set $M_{E,c}$ is a union of two two-dimensional torii which correspond to two opposite directions of motion. In this case the projection of $M_{E,c}$ onto M^2 is a band, and a geodesic goes along the band oscillating between its two edges. If $c_3 < c < c_2$ then $M_{E,c}$ is a union of four two-dimensional torii which correspond to different directions of motion along the axes q_1 and q_2. In this case the projection of $M_{E,c}$ on M^2 is the whole manifold M^2 and a geodesic winds around M^2. At $c = c_4$ and $c = c_1$, $M_{E,c}$ degenerates into two circles, and a geodesic is a stable periodic orbit. At $c = c_2$ and $c = c_3$, $M_{E,c}$ consists of two unstable periodic orbits and four separatrix cylinders.

Let $I_1, I_2, \varphi_1, \varphi_2$ be the action-angle variables for the geodesic flow on the Liouville surface M^2. The action variables I_1, I_2 are constant on the invariant torii and they are calculated as follows:

$$I_1 = 2(2E)^{1/2} \int_{q_1\,:\; U_1(q_1) - c \geq 0} (U_1(q_1) - c)^{1/2} dq_1, \quad \text{when} \quad c_2 < c \leq c_1, \tag{4.8a}$$

and

$$I_1 = \pm(2E)^{1/2} \int_0^1 (U_1(q_1) - c)^{1/2} dq_1, \quad \text{when} \quad c_4 \leq c < c_2. \tag{4.8b}$$

The sign $\pm$ in the last formula corresponds to the invariant torii with positive and negative direction of motion along q_1. Similarly,

$$I_2 = 2(2E)^{1/2} \int_{q_2\,:\; c - U_2(q_2) \geq 0} (c - U_2(q_2))^{1/2} dq_2, \quad \text{when} \quad c_4 \leq c < c_3, \tag{4.9a}$$

and

$$I_2 = \pm(2E)^{1/2} \int_0^1 (c - U_2(q_2))^{1/2} dq_2, \quad \text{when} \quad c_3 < c \leq c_1. \tag{4.8b}$$

The angular variables are defined as

$$\omega_1 = \frac{\partial H}{\partial I_1}, \qquad \omega_2 = \frac{\partial H}{\partial I_2}. \tag{4.9}$$

Let

$$f_1(c) = \int_{q_1\,:\;U_1(q_1)-c\geq 0} (U_1(q_1)-c)^{1/2}dq_1,$$
$$f_2(c) = \int_{q_2\,:\;c-U_2(q_2)\geq 0} (c-U_2(q_2))^{1/2}dq_2, \tag{4.10}$$

and

$$\omega(c) = \frac{f_2'(c)}{f_1'(c)}. \tag{4.11}$$

Let, in addition,

$$a(c) = \det\left(\frac{\partial^2 H}{\partial I_k \partial I_l}\right) = \det\left(\frac{\partial \omega_k}{\partial I_l}\right). \tag{4.12}$$

The function $a(c)$ can be computed by the formula

$$a(c) = \frac{4\Delta_2}{m(c)\Delta_1^3}, \tag{4.13}$$

where $m(c)$ is the number of invariant torii in the set $M_{E,c}$ and

$$\Delta_1 = f_1 f_2' - f_2 f_1', \qquad \Delta_2 = f_1' f_2'' - f_2' f_1''. \tag{4.14}$$

We call a pair of real numbers (α_1, α_2) Diophantine if there exist $\tau > 1$ and $C > 0$ such that for all nonzero pairs of integers (k_1, k_2),

$$|k_1\alpha_1 + k_2\alpha_2| \geq \frac{C}{|k|(\log|k|)^\tau}, \qquad |k| = (k_1^2 + k_2^2)^{1/2}.$$

A number α is called Diophantine if the pair $(1, \alpha)$ is Diophantine. The complement to the set of Diophantine numbers has zero Lebesgue measure. Introduce the following condition.

Diophantine Hypothesis.

(i) Assume that $a(c)$ has at most finitely many zeros, $a(c_1), a(c_4) \neq 0$, and all these zeros are simple.

(ii) If c is a zero of $a(c)$ then the pair $(f_1'(c), f_2'(c))$ is Diophantine.

(iii) The pairs $(f_1(c_2), f_2(c_2))$ and $(f_1(c_3), f_2(c_3))$ are Diophantine.

(iv) The pairs $(f_1'(c_1), f_2'(c_1))$ and $(f_1'(c_4), f_2'(c_4))$ are Diophantine.

We will identify all closed geodesics which belong to the same invariant torus.

Theorem 4.1 (see [BKS]). *Assume that a Liouville surface M^2 satisfies the Diophantine Hypothesis. Then the error function $F(R)$ (see (4.5)) is a B^1-almost periodic function. The B^1-Fourier series of $F(R)$ is*

$$F(R) = (2\pi^3)^{-1/2} \sum_{\text{closed geodesics } \gamma} |\gamma|^{-3/2} \varkappa(\gamma)^{-1/2} \sin\left(|\gamma| R - \frac{\pi \operatorname{ind} \gamma}{2} - \frac{\pi \sigma(\gamma)}{4} \right),$$

where

$$\varkappa(\gamma) = \left| \det \left(\frac{\partial^2 H}{\partial I_k \partial I_l} \right)(c) \right|, \qquad \sigma(\gamma) = \operatorname{sgn} \det \left(\frac{\partial^2 H}{\partial I_k \partial I_l} \right)(c), \qquad \gamma \in M_{E,c},$$

and $\operatorname{ind}\gamma$ *is the Maslov index of* γ.

The function $F(R)$ is probably a B^2-almost periodic function. Still in the proof of Theorem 4.1 there is a problem of estimating the difference between the semiclassical counting function $N^{\text{EBK}}(E)$ and $N(E)$ (cf. Lemma 3.4 above). The usual Einstein–Brillouin–Keller semiclassical quantization formula for the eigenvalues,

$$E^{\text{EBK}}(n_1, n_2) = H\left(\pi\big(n_1 + \frac{\alpha_1}{4}\big), \pi\big(n_2 + \frac{\alpha_2}{4}\big) \right),$$

where $H(I_1, I_2)$ is the Hamiltonian in action variables and α_1, α_2 are Maslov's indices, turns out to be insufficient for this purpose because it does not work near unstable periodic orbits. An improved formula can be derived which gives all semiclassical eigenvalues, but it is difficult to obtain a good estimate of the error function in this formula (see [KMS] and [Ble4]).

Theorem 4.1 is used to get a limit probability distribution of $F(R)$.

Theorem 4.2 (see [BKS]). *Assume that a Liouville surface M^2 satisfies the Diophantine Hypothesis. Assume, in addition, that the lengths of all primitive geodesics on M^2 are linearly independent over $\mathbb{Z}$. Then the error function $F(R)$ has a limit distribution $p(x)\,dx$ where $p(x)$ is an entire function such that*

$$\lim_{x \to \pm\infty} -\frac{\log p(x)}{x^4} = c_{\pm} > 0.$$

5. Saturation Phenomenon

Let Λ be a convex domain on a plane, such that (i) $\Gamma = \partial\Lambda$ is C^7-smooth, (ii) $0 \in \Lambda$, and (iii) the curvature $\varkappa(x) > 0$ for all $x \in \Gamma$. For normalization we will assume that

$$\text{Area}\,\Lambda = 1. \tag{5.1}$$

Let

$$N_\alpha(E) = \#\,\mathbb{Z}^2 \cap (E^{1/2}\Lambda + \alpha), \qquad \alpha \in \mathbb{R}^2.$$

We are interested in the statistical properties of $N_\alpha(E+S) - N_\alpha(E)$ as $E \to \infty$. As was discovered by Casati, Chirikov, and Guarneri [CCG] and explained by Berry [Ber1], the answer depends on the relation between E and S. We should distinguish the four cases:

(i) $S \gg E^{1/2}$; (ii) $S \sim E^{1/2}$; (iii) $E^{1/2} \gg S \gg 1$, and (iv) $S \sim 1$.

We present some rigorous results in this direction by Bleher and Lebowitz [BL1, BL2].

Theorem 5.1 (see [BL1]). *Let*

$$F_\alpha(E,S) = \frac{N_\alpha(E+S) - N_\alpha(E) - S}{E^{1/4}}. \tag{5.2}$$

Then for all continuous bounded functions $g(x)$ on a line,

$$\lim_{S,T\to\infty:\ (S/T^{1/2})\to\infty,\ (S/T)\to 0} \frac{1}{T}\int_T^{2T} g\left(F_\alpha(E,S)\right)\, dE = \int_{-\infty}^{\infty} g(x)\mu_\alpha(dx), \tag{5.3}$$

where $\mu_a(dx)$ is the probability distribution of the difference $\xi_1 - \xi_2$ of two independent identically distributed random varibles ξ_1, ξ_2 whose distribution is the limit distribution $\nu_\alpha(dx)$ of

$$F_\alpha(E) = \frac{N_\alpha(E) - E}{E^{1/4}}$$

(see the formula (2.9) above).

Theorem 5.1 can be explained as follows. Actually the claim is that if $S, T \to \infty$ in such a way that $S/T^{1/2} \to \infty$ and $S/T \to 0$, then the normalized error functions $[N_\alpha(E+S) - (E+S)]/E^{1/4}$ and $[N(E) - E]/E^{1/4}$ are asymptotically independent. The proof uses ergodic averaging at two scales, and the large scale averaging secures the existence of a limit distribution of the two normalized error functions, while the small scale averaging secures their independence. This gives the theorem.

Theorem 5.2 (see [BL1]). *Let $F_\alpha(E,S)$ be as in (5.2). Then there exists a one-parameter family of probability measures $\{\mu_\alpha(dx;z),\ 0<z<\infty\}$ on a line such that for all $z>0$ and all continuous bounded functions $g(x)$,*

$$\lim_{S,T\to\infty:\ (S/T^{1/2})\to z} \frac{1}{T}\int_T^{2T} g\left(F_\alpha(E,S)\right)\, dE = \int_1^2 \int_{-\infty}^{\infty} g(x)\mu_\alpha(dx;z/c)\, dc. \tag{5.4}$$

If, in addition, the Fourier frequencies $\{\omega(n),\ n\in Z\}$ (see (2.4) and (2.13)) are linearly independent over $\mathbb{Z}$ then for all $z>0$, the distribution $\mu_a(dx;z)$ possesses a density $p_\alpha(x;z)=\mu_a(dx;z)/dx$ which is an entire function in x such that

$$\lim_{x\to\pm\infty} -\frac{\log p_\alpha(x;z)}{x^4} = c_\pm(\alpha;z) > 0.$$

Also in this case, if $\xi_{\alpha,z}$ is a random variable with the distribution $\mu_\alpha(dx;z)$, then as $z\to 0$ the distribution of the random variable $\xi_{\alpha,z}/(E\xi^2_{\alpha,z})^{1/2}$ approaches the standard Gaussian distribution, with zero mean and variance 1.

If Γ has a symmetry, Theorem 5.2 is valid if the numbers $\omega(n_1),\dots,\omega(n_m)$ are linearly independent for all $n_1,\dots,n_m$ such that all n_i and n_j are not symmetric. In particular, this holds for all ellipses $x_1^2+\mu^{-2}x_2^2=1$ with transcendental μ. A special consideration shows that an analog of Theorem 5.2 holds for the circle as well (see [BL1]). The second moment of the distribution $\mu_\alpha(dx;z)$,

$$D_\alpha(z) = \int_{-\infty}^{\infty} x^2\mu_\alpha(dx;z)$$

is the number variance, so that

$$\lim_{S,T\to\infty:\ (S/T^{1/2})\to z} \frac{1}{T}\int_T^{2T} F_\alpha(E,S)^2 dE = \int_1^2 D_\alpha(z/c)\, dc. \tag{5.5}$$

Figs. 14-16, taken from [BL1], show the number variance $D_\alpha(z)$ for the circle and $\alpha=0$, for the ellipse with $\mu=\pi$ and $\alpha=0$, and for the same ellipse and $\alpha=(0.1,0.1)$. The number variance approaches an almost periodic function at infinity, and it has different asymptotics at the origin, which depends on the symmetry of the problem (see [BL1]). For the circle $D_0(z)\sim Cz\,|\log z|$, and for the ellipse $D_0(z)\sim 4z$ and $D_\alpha(z)\sim z$ if $\alpha_1\alpha_2\neq 0$ (see Figs. 14–16).

Let us consider now the limit when $1\ll S\ll T^{1/2}$. In this limit we have results only for the number variance and only for the ellipse. We will call a number μ Diophantine if there exist $M,C>0$ such that for all rational numbers p/q,

$$\left|\mu-\frac{p}{q}\right| \geq \frac{C}{q^M}.$$

μ is a Liouville number if there exist a sequence of rationals p_i/q_i and a sequence of positive numbers c_i, $i = 1, 2, \ldots$, such that $\lim_{i\to\infty} c_i = \infty$ and

$$\left|\mu - \frac{p_i}{q_i}\right| \le e^{-c_i q_i}, \qquad i = 1, 2, \ldots.$$

Theorem 5.3 (see [BL2]). *Let*

$$\Gamma = \left\{ (x_1, x_2) \colon x_1^2 + \mu^{-2} x_2^2 = \frac{1}{\pi\mu} \right\}$$

be an ellipse of area 1, and let $S = T^\gamma$ where γ is an arbitrary fixed number such that $0 < \gamma < 1/2$. Then if μ is a Diophantine number then

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N(E+S) - N(E) - S|^2}{S} \, dE = 4 \tag{5.6}$$

(with the factor 4 coming from symmetry considerations). Contrariwise, if μ is rational then

$$\lim_{T\to\infty} \frac{1}{T\log T} \int_T^{2T} \frac{|N(E+S) - N(E) - S|^2}{S} \, dE = \left(\frac{1}{2} - \gamma\right) c(\mu), \tag{5.7}$$

with some $c(\mu) > 0$, and if μ is a Liouville number then

$$\limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} \frac{|N(E+S) - N(E) - S|^2}{S} \, dE = \infty. \tag{5.8}$$

The limit (5.6) is consistent with the conjecture that for every Diophantine μ, the numbers

$$E_{n_1,n_2} = n_1^2 + \mu^{-2} n_2^2, \qquad n_1, n_2 \in \mathbb{Z}, \qquad n_1, n_2 \ge 0,$$

are locally distributed as a Poisson random process. The limits (5.6) and (5.7) are not consistent with the Poisson distribution, and this is related to the fact that the Poisson conjecture violates for both rational and Liouville μ.

The last part of Theorem 5.2 concerning the Gaussian limit of $\xi_{\alpha,z}$ as $z \to \infty$, supports the following conjecture.

The Gaussian Conjecture. *Let*

$$F_\alpha(E, S) = \frac{N(E+S) - N(E) - S}{D(T,S)^{1/2}},$$

where

$$D(T,S) = \frac{1}{T}\int_T^{2T} |N(E+S) - N(E) - S|^2 dE,$$

and $S = T^\gamma$ *where* $0 < \gamma < 1/2$ *is fixed. Then for every bounded continuous function* $g(x)$*,*

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} g(F_\alpha(E,S))\, dE = (2\pi)^{-1/2}\int_{-\infty}^{\infty} g(x)e^{-x^2/2}dx.$$

Bleher and, independently, Bäcker have checked this conjecture numerically. This is not easy because the rate of convergence is very slow. Fig. 17 shows the distribution density found by Bleher for a circular annulus with the exponent $\gamma = 0.25$. The value of T is 4,000,000. The graph is close to a Gaussian density, with a small bias to negative values.

Let us consider now the case of fixed $S > 0$. In this case it is conjectured that in a "generic" situation, $N(E+S) - N(E)$ has a limit Poisson distribution with the parameter S. For an ellipse the conjecture can be formulated as follows.

The Poissonian Conjecture. *Let*

$$N(E) = \#\{(n_1,n_2)\colon n_1^2 + \mu^{-2}n_2^2 \le \frac{E}{\pi\mu}\},$$

and $N(E,S) = N(E+S) - N(E)$*. Assume that* μ *is a Diophantine number in the sense that for all* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that*

$$\left|\mu - \frac{p}{q}\right| \ge \frac{C(\varepsilon)}{q^{2+\varepsilon}}\,. \tag{5.9}$$

Then for all bounded continuous functions $g(x)$*,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} g(N(E,S))\, dE = \sum_{k=0}^{\infty} \frac{g(k)S^k e^{-S}}{k!}\,.$$

We formulate the Diophantine condition (5.9) in the conservative way, with the exponent $2+\varepsilon$. Probably this condition can be weakened. An analytical heuristic proof of the Poisson distribution in integrable systems is done in the paper [BT2] by Berry and Tabor (see also [Ber2] and [Tab]). Sinai [Sin1,Sin2] and Major [Maj] prove a limit Poisson distribution for $N(E,S)$ in the case of a random domain Λ. Major [Maj] also proves some other results in this direction, and he shows that a typical oval from the probability space, used in his and Sinai's papers, is not twice differentiable: the derivative of the oval equation behaves roughly as a Brownian trajectory. For smooth (say, analytic) ovals the Poisson conjecture remains open. Sarnak [Sar2] studies the distribution of the values at nonnegative integers of a positive binary quadratic form $P(x, y) = \alpha x^2 + \beta xy + \gamma y^2$. He proves that there is a set M in the space of the coefficients (α, β, γ) with $4\alpha\gamma > \beta^2$ of full Lebesgue measure such that for all $(\alpha, \beta.\gamma) \in M$, the second correlation function of the values of $P(m, n)$, $m, n \in \mathbb{Z}$, $m, n \geq 0$, has the Poisson asymptotics. To prove the individual asymptotics for the second correlation function, Sarnak actually estimates the fourth correlation function in the ensemble of the quadratic forms. Cheng, Lebowitz, and Major [CLM] proves the Poisson-type asymptotics for the second order correlation function in the lattice-point problem with a random shift.

The conjecture of local Poisson distribution for integrable quantum systems is a part of the universality conjecture for generic quantum systems: for integrable systems the local statistics is Poissonian, while for hyperbolic systems it is the Wigner–Dyson statistics of the ensemble of Gaussian matrices. This conjecture is based on a number of analytical, numerical and experimental results: see the papers and monographs by Bohigas, Giannoni, Schmidt [BGS], Bohigas [Boh], Casati, Chirikov, Guarneri [CCG], Berry and Tabor [BT], Berry [Ber2], Gutzwiller [Gut], Ozorio de Almeida [OdA], Tabor [Tab], Steiner [Ste], Gräf, Harney, Lengeler, Lewenkopf, Rangacharyulu, Richter, Schardt, and Weidenmüller [G-W], Stoffregen, Stein, Stöckmann, Kuś, and Haake [S-H], and many others.

6. Lattice-Point Problem in Dimension Greater Than 2

Let $\alpha \in \mathbb{R}^d$ be a fixed point. Define

$$N_\alpha(R) = \#\{\, n \in \mathbb{Z}^d \colon |n - \alpha| \leq R,$$

which is the number of lattice points in a d-dimensional ball of radius R centered at α, and

$$F_\alpha(R) = \frac{N_\alpha(R) - \Omega_d R^d}{R^{(d-1)/2}},$$

where Ω_d is the volume of a unit ball in R^d. We will assume that $d \geq 3$. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ satisfy the following Diophantine condition: for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for all $m = (m_0, m_1, \ldots, m_d) \in Z^{d+1}$, $m \neq 0$,

$$\left(\prod_{j=1}^{d} |m_j|\right)^{1+\varepsilon} \left| m_0 + \sum_{j=1}^{d} m_j \alpha_j \right| \geq C(\varepsilon). \tag{6.1}$$

This condition is fulfilled for almost all α with respect to the Lebesgue measure. Bleher and Bourgain [BB] prove the following result.

Theorem 6.1 (see [BB].

(1) Let α satisfy (6.1). Then $F_\alpha(R)$ has for $R \to \infty$ a limit distribution $\nu_\alpha(dx)$ satifying for all $\varepsilon > 0$ the tail estimate

$$\nu_\alpha[|x'| > x] < C_1(\varepsilon)\exp(-C_2(\varepsilon)x^{4-\varepsilon}).$$

(2) For almost all α, $\nu_\alpha(dx)$ is absolutely continuous with respect to the Lebesgue measure, with a smooth density $p_\alpha(x)$ satisfying for all $\varepsilon > 0$ the tail estimate

$$p_\alpha(x) < C_1(\varepsilon)\exp(-C_2(\varepsilon)x^{4-\varepsilon}).$$

(3) For $d = 3$ and almost all α, $p_\alpha(x)$ extends to an entire function and, in addition, one has for all $\varepsilon > 0$ the lower estimate

$$p_\alpha(x) > c_1(\varepsilon)\exp(-c_2(\varepsilon)x^{4+\varepsilon}).$$

The proof of Theorem 6.1 uses the Poisson summation formula, which yields essentially that

$$F_\alpha(R) = C_d \sum_{n\in\mathbb{Z}^d,\ n\neq 0} |n|^{-(d+1)/2}\cos(2\pi|n|R - \phi)e^{2\pi i(n,\alpha)}. \tag{6.2}$$

Writing $|n| = k\sqrt{m}$ where m is square free one may rewrite

$$F_\alpha(R) = \sum_{m \text{ square free}} f_m(R\sqrt{m}), \tag{6.3}$$

where

$$f_m(t) = C_d m^{-(d+1)/4}\sum_{k=1}^{\infty} k^{-(d+1)/2} r_{k^2m}(\alpha)\cos(2\pi kt - \phi) \tag{6.4}$$

and

$$r_m(\alpha) = \sum_{n\in\mathbb{Z}^d:\ |n|^2=m} e^{2\pi i(n,\alpha)}. \tag{6.5}$$

The functions $f_m(t)$ are periodic functions of period 1. The crucial role in the proof of Theorem 6.1 plays the following lemma.

Lemma 6.2 (see [BB]).

$$\sum_{m\le M} |r_m(\alpha)|^2 \ll M^{\frac{d}{2}+\varepsilon},$$

$$\sum_{m\le M} |r_m(\alpha)|^2 > cM^{\frac{d}{2}},$$

$$|r_m(\alpha)| \ll m^{\frac{d-1}{4}+\varepsilon}.$$

With the help of Lemma 6.2, it is proved that for typical α, $F_\alpha(R)$ is a B^1-almost periodic function and Theorem 6.1 holds. The Diophantine condition on α is essential. If α is rational then $N_\alpha(R)$ has big jumps. To see it let us consider $\alpha = 0$. For $d > 4$ the number of solutions of the equation

$$n_1^2 + \cdots + n_d^2 = m$$

is of the order of $m^{(d-2)/2}$, which gives rise to the jumps of the function $N_0(R)$.

For $d = 3$ Jarnik [Jar] proves the following result. Let

$$N_0(R) = \#\,\{n \in \mathbb{Z}^3 \colon |n| \le R\},$$

and

$$F_0(R) = \frac{N_0(R) - \frac{4\pi R^3}{3}}{R}, \qquad R \ge 1.$$

Put $F_0(R) = 0$ for $R < 1$.

Theorem 6.3 (see [Jar]).

$$\lim_{T\to\infty} \frac{1}{T\log T} \int_0^T |F_0(R)|^2 dR = K > 0.$$

Bleher and Dyson [BD3] find that

$$K = \frac{32\zeta(2)}{7\zeta(3)}\,,$$

and they conjecture that the limit distribution of $F_0(R)$ is Gaussian. Theorem 6.3 can be extended to $F_\alpha(R)$ with rational α.

Theorem 6.4. *If*

$$\alpha = \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right) \in \mathbb{Q}^3, \qquad \text{g.c.d.}\,(q_1, q_2, q_3) = 1,$$

then

$$\lim_{T\to\infty} \frac{1}{T\log T} \int_0^T |F_\alpha(R)|^2 dR = K(\alpha) > 0.$$

If q_1, q_2, q_3 are odd then

$$K(\alpha) = \frac{32\zeta(2)}{7q^2\zeta(3)} \prod_{p|q} \frac{p^2}{p^2+p+1}\,,$$

where q is the least common multiplier of q_1, q_2, q_3 and the product is taken over all prime divisors p of q. If all $q_i = 2 \mod 4$ then $K(\alpha) = K(\alpha/2)$. In all other cases

$$K(\alpha) = \frac{8\zeta(2)}{q^2\zeta(3)} \prod_{p|q} \frac{p^2}{p^2+p+1}\,.$$

The proof of Theorem 6.4 is an application of the formula (6.2) and the following lemma which is based on the Hardy-Littlewood circle method.

Lemma 6.5. *Assume that $d = 3$, $\alpha \in \mathbb{R}^3$, and $r_m(\alpha)$ is defined as in (6.5). Then*

$$\lim_{M\to\infty} M^{-2} \sum_{m=1}^{M} |r_m(\alpha)|^2 = L(\alpha) > 0,$$

with $L(\alpha) = (\pi^2/2)K(\alpha)$, where $K(\alpha)$ is defined in Theorem 6.4.

7. Quantum Linear Oscillators

The spectrum of a system of m linear oscillators is written as

$$E(n_1, \ldots, n_m) = E_0 + n_1\omega_1 + \cdots + n_m\omega_m, \qquad n_j \in \mathbb{Z}, \quad n_j \geq 0, \tag{7.1}$$

where E_0 is the ground state energy and ω_j are frequencies. When $m = 2$,

$$E(n_1, n_2) = E_0 + \omega_1(n_1 + \alpha n_2), \qquad \alpha = \frac{\omega_2}{\omega_1}\,,$$

hence the problem of finding the asymptotics of the eigenvalues $E(n_1, n_2)$ reduces to the same problem for the numbers

$$\lambda(n_1, n_2) = n_1 + \alpha n_2, \qquad n_1, n_2 \in \mathbb{Z}, \quad n_1, n_2 \geq 0. \tag{7.2}$$

We will assume that $\alpha > 0$ is irrational. A numerical and analytical study by Berry and Tabor [BT] indicates that the distribution of spacing between neighboring $\lambda(n_1, n_2)$ depends on the arithmetic properties of α. The results for the golden mean $\alpha = (\sqrt{5}-1)/2$ were obtained by Pandey, Bohigas, and Giannoni [PBG] and by Bleher [Ble5]. Let us order the numbers $\lambda(n_1, n_2)$ in the increasing order and denote them by

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty.$$

Consider the spacings

$$\Delta_j = \lambda_{j+1} - \lambda_j,$$

and the minimal spacing on $[0, \lambda]$,

$$\Delta_{\min}(\lambda) = \min\{\Delta_j \colon \lambda_j \le \lambda\}.$$

Define normalized spacings s_j on $[0, \lambda]$ as

$$s_j = \frac{\Delta_j}{\Delta_{\min}},$$

and the distribution function $P_\lambda(s)$ as

$$P_\lambda(s) = \frac{\#\{\lambda_j \le \lambda \colon s_j \le s\}}{\#\{\lambda_j \le \lambda\}}.$$

Theorem 7.1 (see [Ble5]). *Assume that $\alpha = (\sqrt{5}-1)/2$. Then there exists a periodic of period 1 family of distribution functions $P(s;\tau)$,*

$$P(s;\tau+1) = P(s;\tau),$$

such that

$$\lim_{\lambda\to\infty} \sup_{0\le s<\infty} |P_\lambda(s) - P(s;\log_\beta \lambda)| = 0,$$

where $\beta = \alpha^{-1}$. For every τ the probability distribution $dP(s;\tau)$ is an atomic measure, with atoms at the points β^k and with some weights $w_k(\tau)$, $k = 0, 1, 2, \dots$, which decay when $k \to \infty$ as $C\alpha^{2k}$.

In other words, when $\lambda \to \infty$ the spacing distribution function $P_\lambda(s)$ oscillates periodically as a function of $\log_\beta \lambda$. In particular, this implies that $P_\lambda(s)$ has no limit as $\lambda \to \infty$. An explicit formula for the limiting periodic family $P(s;\tau)$ is derived in [Ble5]. As a matter of fact, $dP_\lambda(s)$ is an atomic measure with atoms at the points β^k with some weights $w_{k,\lambda}$, $k = 0, 1, 2, \dots$. In the proof of Theorem 7.1 it is verified that the weights $w_{k,\lambda}$ approach $w_k(\log_\beta \lambda)$ when $\lambda \to \infty$. It is to be noted that the spectrum is very rigid, and locally the spacing takes only three values (see, e.g., [AA], [Sta], [PBG], and [Ble5]). This implies a hard core repulsion of energy levels with $P_\lambda(s) = 0$ for all $s < 1$. Along the same lines, a similar result can be obtained for every quadratic irrational α. The important point in the proof is that the continued fraction of any quadratic irrational is periodic.

The case of generic α is more complicated. In general, there exists no limit of spacing distribution, and the problem is to find, in what sense the limit can exist? This problem is studied by Bleher [Ble6]. Consider the continued fraction $[a_1, a_2, \dots]$ which represents α and the partial fractions

$$\frac{p_k}{q_k} = [a_1, \dots, a_k], \qquad k \geq 1.$$

Theorem 7.2 (see [Ble6]). *There exists a probability distribution function $P(s)$ such that for almost all α on $[0,1]$ with respect to Lebesgue measure,*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} P_{p_k}(s) = P(s).$$

The limit distribution function $P(s)$ is expressed in terms of the natural extension of the Gauss map $x \to \{1/x\}$ (see [Ble6]).

Acknowledgements. The author thanks Peter Sarnak and Martin Huxley for their comments to the preliminary version of this paper. The paper is based on the talk presented at the IMA Summer program on "Emerging Applications of Number Theory", July 15-20, 1996. The author is grateful to the organizers of the program for the invitation to give a talk at the meeting and to participate in this volume. He is also indebted to the Institute for Mathematics and its Applications for hospitality.

References

[AA] F. C. Auluck and H. S. Ahluwalia, On partitions into non-integral numbers and the nearest-neighbor spacing distribution function, *J. Math. Sci.* **1**, 9–11 (1966).

[ABS] R. Aurich, J. Bolte, and F. Steiner, Universal signatures of quantum chaos, *Phys. Rev. Lett.* **73**, 1356–1359 (1994).

[BaB] R. Balian and C. Bloch, Distribution of eigenfrequencies for the wave function in a finite domain: III. Eigenfrequency density oscillations, *Ann. Phys.* **69**, 76–160 (1972).

[Brd] P. H. Bérard, On the wave equation on a compact Riemannian manifold without conjugate points, *Math. Z.* **155**, 249–276 (1977).

[Ber1] M. V. Berry, Semiclassical theory of spectral rigidity, *Proc. R. Soc. Lond. A* **400**, 229–251 (1985).

[Ber2] M. V. Berry, Semiclassical mechanics of regular and irregular motion, In: Chaotic Behavior of Deterministic Systems. Les Houghes, Session XXXVI, 1981, 173–271, Eds. G. Iooss, R. H. G. Helleman, and R. Stora, North Holland Publ. Comp., 1983.

[BM] M. V. Berry and K. E. Mount, Semiclassical approximations in wave mechanics, *Rep. Progr. Phys.* **35**, 315–397 (1972).

[BT1] M. V. Berry, M.Tabor, Closed orbits and the regular bound spectrum, *Proc. R. Soc. Lond.* **A349**, 101–123 (1976).

[BT2] M. V. Berry and M. Tabor, Level clustering in the regular spectrum, *Proc. R. Soc. Lond. A* **356**, 375–394 (1977).

[BT3] M. V. Berry and M. Tabor, Calculating the bound spectrum by path summation in action-angle variables, *J. Phys. A: Math. Gen.* **10**, 371–379 (1977).

[Bes] A. S. Besicovitch, *Almost periodic functions*, Dover, New York, 1958.

[BS] D. Biswas and S. Sinha, Theory of fluctuations in pseudointegrable systems, *Phys. Rev. Lett.* **70**, 916–919 (1993).

[Ble1] P. M. Bleher, On the distribution of the number of lattice points inside a family of convex ovals, *Duke Math. Journ.* **67**, 461–481 (1992).

[Ble2] P. M. Bleher, Distribution of the error in the Weyl asymptotics for the Laplace operator on a two-dimensional torus and related lattice problems, *Duke Math. Journ.* **70**, 655–682 (1993).

[Ble3] P. M. Bleher, Distribution of energy levels of a quantum free particle on a surface of revolution, *Duke Math. Journ.* **74**, 45–93 (1994).

[Ble4] P. M. Bleher, Semiclassical quantization rules near separatrices, *Commun. Math. Phys.* **165**, 621–640 (1994).

[Ble5] P. M. Bleher, The energy level spacing for two harmonic oscillators with golden mean ratio of frequencies, *Journ. Statist. Phys.* **61**, 869–876 (1990).

[Ble6] P. M. Bleher, The energy level spacing for two harmonic oscillators with generic ratio of frequencies, *Journ. Statist. Phys.* **63**, 261–283 (1991).

[Ble7] P. M. Bleher, Quasiclassical expansion and the problem of quantum chaos, *Lect. Notes in Math.* **1469**, 60–89 (1991).

[BB] P. M. Bleher and J. Bourgain, Distribution of the error term for the number of lattice points inside a shifted ball, to appear in Proceeding of Halberstam's Conference.

[BCDL] P. M. Bleher, Zh. Cheng, F. J. Dyson, and J. L. Lebowitz, Distribution of the error term for the number of lattice points inside a shifted circle, *Commun. Math. Phys.* **154**, 433–469 (1993).

[BD1] P. M. Bleher and F. J. Dyson, The variance of the error function in the shifted circle problem is a wild function of the shift, *Commun. Math. Phys.*, **160**, 493–505 (1994)

[BD2] P. M. Bleher and F. J. Dyson, Mean square value of exponential sums related to representation of integers as sum of two squares, *Acta Arithm.*, **68**, 71–84 (1994).

[BD3] P. M. Bleher and F. J. Dyson, Mean square limit for lattice points in a sphere, *Acta Arithm.*, **68**, 383–393 (1994); Erratum: **73**, 199 (1995).

[BDL] P. M. Bleher, F. J. Dyson, and J. L. Lebowitz, Non–gaussian energy level statistics for some integrable systems, *Phys. Rev. Lett.* **71**, 3047–3050 (1993).

[BL1] P. M. Bleher and J. L. Lebowitz, Energy–level statistics of model quantum systems: universality and scaling in a lattice–point problem, *J. Stat. Phys.* **74**, 167–217 (1994).

[BL2] P. M. Bleher and J. L. Lebowitz, Variance of number of lattice points in random narrow elliptic strip, *Ann. Inst. Henri Poincaré: Probab. et Statist.* **31**, 27–58 (1995).

[Boh] O. Bohigas, Random matrix theory and chaotic dynamics, In: Chaos and Quantum Physics. Les Houghes, Session LII, 1989, 87–199, Eds. M.-J. Giannoni, A. Voros, and J. Zinn-Justin, Elsevier Sci. Publ., 1991.

[BGS] O. Bohigas, M.-J. Giannoni, and C. Schmidt, Characterization of chaotic quantum spectra and universality of level fluctuation laws, *Phys. Rev. Lett.*, **52**, 1–4 (1984).

[CCG] G. Casati, B. V. Chirikov, and I. Guarneri, Energy–level statistics of integrable quantum systems, *Phys. Rev. Lett.* **54**, 1350–1353 (1985).

[CL] Zh. Cheng and J. L. Lebowitz, Statistics of energy levels in integrable quantum systems, *Phys. Rev. A* **44**, R3399–3402 (1991).

[CLM] Zh. Cheng, J. L. Lebowitz, and P. Major, On the number of lattice points between two enlarged and randomly shifted copies of an oval, *Probab. Theory Related Fields* **100**, 253–268 (1994).

[CdV1] Y. Colin de Verdière, Spectre du laplacien et longueurs des geodesiques periodiques, *Comp. Math.* **27**, 159–184 (1973).

[CdV2] Y. Colin de Verdière, Nombre de points entiers dans une famille homothètique de domaines de $\mathbb{R}^n$, *Ann. Sci. École Norm Sup.* **10**, 559–576 (1977).

[CdV3] Y. Colin de Verdière, Spectra conjoint d'opérateurs pseudo-différentiels qui communtent. II. Le cas intégrable, *Math. Z.* **171**, 51–73 (1980).

[Cra] H. Cramér, Über zwei Sätze von Herrn G. H. Hardy, *Math. Z.* **15**, 201–210 (1922).

[DG] J. J. Duistermaat and V. Guillemin, Spectrum of elliptic operators and periodic bicharacteristics, *Invent. Math.* **29**, 39–79 (1975).

[Gui] V. Guillemin, Lectures on spectral theory of elliptic operators, *Duke Math. Journ.* **44**, 485–517 (1977).

[Gut] M. C. Gutzwiller, *Chaos in classical and quantum mechanics.* Springer–Verlag, N.Y. e.a., 1990.

[HOdA] J. H. Hannay and Ozorio de Almeida, Periodic orbits and a correlation function for the semiclassical density of states, *J. Phys. A: Math. Gen.* **17**, 3429–3440 (1984).

[Har1] G. H. Hardy, The average order of the arithmetical functors $P(x)$ and $\Delta(x)$, *Proc. London Math. Soc.* **15**, 192–213 (1916).

[Har2] G. H. Hardy, On Dirichlet's divisor problem, *Proc. London Math. Soc.* **15**, 1–25 (1916).

[HL] G. H. Hardy and E. Landau, The lattice points in a circle, Collected works of Hardy, Vol. 2, 329–356, Oxford, 1967.

[H-B] D. R. Heath-Brown, The distribution and the moments of the error term in the Dirichlet divisor problem, *Acta Arith.* **60**, 389–415 (1992).

[Hör1] L. Hörmander, The spectral function of an elliptic operator, *Acta Math.* **121**, 193–218 (1968).

[Hör2] L. Hörmander, Fourier integral operators I, *Acta Math.* **127**, 79–183 (1971).

[Hux] M. N. Huxley, Exponential sums and lattice points, *Proc. London Math. Soc.* **66**, 279–301 (1993).

[Jar] V. Jarnik, Über die Mittelwertsätze der Gitterpunktlehre, *Časopis Pěst. Mat. Fys.* **69**, 148–174 (1940).

[KMS] D. V. Kosygin, A. A. Minasov, and Ya. G. Sinai, Statistical properties of the Laplace–Beltrami operator on Liouville surfaces, *Uspekhi Mat. Nauk* **48**, 3–130 (1993).

[LPh] P. Lax and R. Phillips, *Scattering theory for automorphic functions*, Princeton University Press, Princeton, N.J., 1976.

[LZh] B. M. Levitan and V. V. Zhikov, *Almost periodic functions and differential equations*, Cambridge Univ. Press, 1968.

[LS] W. Luo and P. Sarnak, Number variance for arithmetic hyperbolic surfaces, *Commun. Math. Phys.* **161**, 419–432 (1994).

[Maj] P. Major, Poisson law for the number of lattice points in a random strip with finite area, *Prob. Theory Related Fields* **92**, 423–464 (1992).

[McK] H. B. McKean, Selberg's trace formula as applied to a compact Riemann surface *Comm. Pure Appl. Math.* **25**, 225–246 (1972).

[OdA] A. M. Ozorio de Almeida, *Hamiltonian systems: chaos and quantization*, Cambridge Univ. Press, New York, 1988.

[PBG] A. Pandey, O. Bohigas, and M.-J. Giannoni, Level repulsion in the spectrum of two-dimensional harmonic oscillators, *J. Phys. A: Math. Gen.* **22**, 4083–4088 (1989).

[RKC] K. Rezakhanlou, H. Kunz, and A. Crisanti, Stochastic behavior of the magnetization in a classically integrable quantum system, *Euriphys. Lett.* **16**, 629–634 (1991).

[RS] M. Rubinstein, P. Sarnak, Chebyshev's bias, *Experiment. Math.* **3**, 173–197 (1994).

[Sar1] P. Sarnak, Arithmetic quantum chaos, *Israel Math. Conf. Proc.* **8**, 183–236 (1995).

[Sar2] P. Sarnak, Values at integers of binary quadratic forms, Preprint, Princeton University, 1996.

[Sie] W. Sierpinski, Oeuvres Choisies, Vol. I, 73–108, P.W.N., Warsaw, 1974.

[Sin1] Ya. G. Sinai, Poisson distribution in a geometric problem, in *Advances in Soviet Mathematics*, Vol. 3, Ya. G. Sinai, ed. (AMS, Providence, Rhode Island, 1991), pp. 199–214.

[Sin2] Ya. G. Sinai, Mathematical problems in the theory of quantum chaos, *Lect. Notes in Math.*, **1469**, 41–59 (1991).

[Sla] N. B. Slater, Gaps and steps for the sequence $n\theta \mod 1$, *Proc. Camb. Phil. Soc.* **63**, 1115–1123 (1967).

[SKM] N. Srivastava, Ch. Kaufman, and G. Müller, Chaos in spin clusters: Quantum invariants and level statistics, *J. Appl. Phys.* **67**, 5627–5629 (1990).

[Ste] F. Steiner, Quantum chaos. In: Schlaglichter der Forschung. Zum 75. Jahrestag der Universität Hamburg, 1994, 543–564, Ed. R. Ansorge, Dietrich Reimer Verlag, Berlin and Hamburg, 1994.

[SB] V. Subrahmanyam and M. Barma, Commensurability effects on the spectra of integrable systems, *J. Phys. A: Math. Gen.* **24**, 4303–4313 (1991).

[Tab] M. Tabor, *Chaos and integrability in nonlinear dynamics. An introduction.* Wiley, New York, 1989.

[UZ] A. Uribe and S. Zelditch, Spectral statistics on Zoll surfaces, *Commun. Math. Phys.* **154**, 313–346 (1993).

[Wey] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte lineare partieller Diferentialgleichungen (mit einer Andwendung auf die Theorie der Hohlraumstrahlung), *Math. Annalen* **71** 441–479 (1912).

[Zel] S. Zelditch, Quantum transition amplitudes for ergodic and for completely integrable systems, *J. Funct. Anal.* **94**, 415–436 (1990).

Figures Captures

Fig. 1. The graph of the function $n(R^2) = \#\{n \in \mathbb{Z}^2 \colon |n| \leq R\} - \pi R^2$.

Fig. 2. The density $p_\alpha(x)$ of the limit distribution of the normalized error function $F_\alpha(R)$ given in (2.14) with $\mu = 1$ (circle) and $\alpha = (0,0)$.

Fig. 3. $\mu = \pi/10$, $\alpha = (0,0)$.

Fig. 4. $\mu = \pi/50$, $\alpha = (0,0)$.

Fig. 5. $\mu = 1$, $\alpha = (0.2, 0.2)$.

Fig. 6. $\mu = \pi/10$, $\alpha = (0.2, 0.2)$.

Fig. 7. $\mu = \pi/50$, $\alpha = (0.2, 0.2)$.

Fig. 8. $\mu = 1$, $\alpha = (0.3437, 0.4304)$.

Fig. 9. $\mu = \pi/10$, $\alpha = (0.3437, 0.4304)$.

Fig. 10. $\mu = \pi/50$, $\alpha = (0.3437, 0.4304)$.

Fig. 11. Geodesic on a surface of revolution.

Fig. 12. The phase function $\omega(I)$ for different surfaces of revolution. Cross sections of the surfaces of revolution are shown in the lower part of the figure.

Fig. 13. The energy level set for the surfaces of revolution shown on Figure 12.

Fig. 14. The number variance for the circle problem.

Fig. 15. The number variance for an ellipse with $\mu = \pi$, $\alpha = (0,0)$.

Fig. 16. The number variance for an ellipse with $\mu = \pi$, $\alpha = (0.1, 0.1)$.

Fig. 17. The density of distribution of the number of lattice points in a circular annulus with the parameter $\gamma = 0.25$.